ANALYTIC SOLUTIONS OF MOMENT PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract. We consider the Cauchy problem for linear moment partial differential equations with constant coefficients in two complex variables. We construct an integral representation of the solution of this problem and study its analyticity. As a result we derive a characterisation of multisummable formal solutions of the Cauchy problem.

1. Introduction

We study the initial value problem for a general linear moment partial differential equation with constant coefficients in two complex variables t, z

$$(1)\qquad P(\partial_{m_1,t},\partial_{m_2,z})u(t,z)=0,\quad \partial^j_{m_1,t}u(0,z)=\varphi_j(z)\quad \text{for } j=0,\dots,n-1,$$

where $P(\lambda,\zeta)$ is a polynomial in variables (λ,ζ) of degree n with respect to λ, $\partial_{m_1,t}$ and $\partial_{m_2,z}$ denote the formal moment differentiations, and the Cauchy data $\varphi_j(z)$ are analytic functions in a complex neighbourhood of the origin.

The formal m-moment differentiation $\partial_{m,z}$ was introduced recently by Balser and Yoshino [8] as the linear operator on the space of power series defined by

$$\partial_{m,z}\Big(\sum_{j=0}^{\infty}\frac{u_j z^j}{m(j)}\Big):=\sum_{j=0}^{\infty}\frac{u_{j+1}z^j}{m(j)},$$

where $m(u)$ is a moment function (see Definitions 5 and 6).

This concept generalises the usual and fractional differentiation. Indeed, for $m(u)=\Gamma(1+u)$ the operator $\partial_{m,z}$ coincides with the usual differentiation ∂_z. Hence for $m_1(u)=m_2(u)=\Gamma(1+u)$, (1) is the initial value problem for a linear partial differential equation with constant coefficients. Moreover, for $p\in\mathbb{N}$ and $m(u)=\Gamma(1+u/p)$ the operator $\partial_{m,z}$ is closely related to the $1/p$-fractional differentiation $\partial_z^{1/p}$ (see Remark 3).

Such a general approach to PDEs is especially useful in the theory of Borel summability, since the formal power series $\hat{u}(t,z)=\sum_{j=0}^{\infty}\frac{u_j(z)}{m_1(j)}t^j$ satisfies (1) if and only if its k-Borel transform $v(s,z)=\sum_{j=0}^{\infty}\frac{u_j(z)}{m_1(j)\Gamma(1+j/k)}s^j$ satisfies the same equation with the m_1-moment differentiation $\partial_{m_1,t}$ replaced by the $\tilde{m}_1$-moment differentiation $\partial_{\tilde{m}_1,s}$, where $\tilde{m}_1(u)=m_1(u)\Gamma(1+u/k)$. In that way the question about summability of formal solution of (1) is reduced to the question about analytic continuation properties of solution of the same equation with $m_1(u)$ replaced by

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$\tilde{m}_1(u)$. For that reason we are concerned with the study of analytic continuation properties of solutions to general moment partial differential equations.

In the paper we construct a formal solution $\hat{u}$ of (1) and study its Gevrey asymptotic properties. In the case when the formal solution $\hat{u}$ is convergent, its sum u is an analytic solution of (1) defined in a complex neighbourhood of the origin. The main result establishes the relation between analytic continuation properties of u and the Cauchy data φ_j ($j = 0, \dots, n-1$). As a corollary we characterise the multisummable formal solution of (1) in terms of analytic continuation properties and growth estimates of the Cauchy data.

We proceed as follows. We represent $P(\lambda, \zeta)$ in the form

$$P(\lambda, \zeta) = P_0(\zeta)(\lambda - \lambda_1(\zeta))^{n_1} \cdots (\lambda - \lambda_l(\zeta))^{n_l},$$

where $P_0(\zeta)$ is a polynomial and $\lambda_1(\zeta), \dots, \lambda_l(\zeta)$ are the characteristic roots of multiplicity $n_1, \dots, n_l$ ($n_1 + \cdots + n_l = n$) respectively. However a formal solution of (1) may be not uniquely defined. To avoid this inconvenience, we choose the normalised formal solution $\hat{u}$, which satisfies also

$$(\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} \cdots (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l} \hat{u} = 0,$$

where $\lambda_1(\partial_{m_2,z}), \dots, \lambda_l(\partial_{m_2,z})$ are the moment pseudodifferential operators (see Definition 8).

Next we show that $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$ with $\hat{u}_{\alpha\beta}$ being the formal solution of

$$(\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^{\beta} \hat{u}_{\alpha\beta} = 0. \tag{2}$$

We prove that the Gevrey order of $\hat{u}_{\alpha\beta}$ depends on the order q_α of the pole of $\lambda_\alpha(\zeta)$ at infinity and depends on the orders k_1, k_2 of moment functions m_1, m_2 respectively.

In the case when $1/k_1 = q/k_2$, the formal solution $\hat{u}_{\alpha\beta}$ of (2) is convergent. Hence its sum $u_{\alpha\beta}$ is an analytic solution of (2) defined in a complex neighbourhood of the origin. Using an integral representation of solution $u_{\alpha\beta}$ we find the connection between analytic continuation properties of $u_{\alpha\beta}$ and the Cauchy data.

In the case when $1/k_1 < q/k_2$, we characterise $(q/k_2 - 1/k_1)^{-1}$-summable solutions of (2) in terms of analytic continuation properties of the Cauchy data.

Finally, returning to the formal solution $\hat{u}$ of (1), we describe multisummable solutions of (1) in terms of φ_j ($j = 0, \dots, n-1$).

In the last section the above results are extended to the inhomogeneous moment equations

$$P(\partial_{m_1,t}, \partial_{m_2,z})\hat{u} = \hat{f}, \quad \partial_{m_1,t}^{j} \hat{u}(0,z) = \varphi_j(z) \quad \text{for } j = 0, \dots, n-1, \tag{3}$$

where the inhomogeneity $\hat{f}(t,z)$ is a formal power series with respect to t.

In general, a formal solution of (3) may also be not unique, but it is uniquely determined by every formal power series $\hat{g}$ satisfying $P_0(\partial_{m_2,z})\hat{g} = \hat{f}$, since there is exactly one formal solution of (3) satisfying also the moment pseudodifferential equation

$$(\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} \cdots (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l} \hat{u} = \hat{g}.$$

We find the formal solution $\hat{u}$ of (3) determined by $\hat{g}$. As in the homogeneous case we show that $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$, where $\hat{u}_{\alpha\beta}$ satisfies

$$(\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^{\beta} \hat{u}_{\alpha\beta} = \hat{g}_{\alpha\beta} \tag{4}$$

for some formal series $\hat{g}_{\alpha\beta}$ connected with $\hat{g}$. Expressing the formal solution $\hat{u}_{\alpha\beta}$ in terms of $\hat{g}$, we calculate the Gevrey order of $\hat{u}_{\alpha\beta}$. We also get the characterisation of analytic continuation properties and summability of $\hat{u}_{\alpha\beta}$ in terms of $\hat{g}$. And lastly, the multisummable solutions of (3) are also expressed in terms of $\hat{g}$.

The present paper is a generalisation of [14, 16], where the characterisation of multisummable solutions of homogeneous and inhomogeneous linear PDEs with constant coefficients was given. The inspiration for our study was the paper of Balser and Yoshino [8], where the notion of moment differentiation was introduced and the Gevrey order of formal solutions of general inhomogeneous linear moment PDEs with constant coefficients was determined. Finally, let us point out that the summability of formal solutions of homogeneous linear PDEs with constant coefficients was studied by Balser [1, 3], Balser and Miyake [7], Ichinobe [9], Lutz, Miyake and Schäfke [10], Malek [11], Michalik [12, 14, 15] and Miyake [17]. The inhomogeneous case was investigated by Balser [4], Balser and Loday-Richaud [6], Balser, Duval and Malek [5] and Michalik [13, 16].

2. Notation, Gevrey formal power series and Borel summability

We use the following notation. The complex disc in $\mathbb{C}^n$ with a centre at the origin and a radius $r > 0$ is denoted by $D^n_r := \{z \in \mathbb{C}^n : \ |z| < r\}$. To simplify notation, we write D_r instead of D^1_r. If the radius r is not essential, then we write it D^n (resp. D) for short.

A sector in a direction $d \in \mathbb{R}$ with an opening $\varepsilon > 0$ in the universal covering space $\widetilde{\mathbb{C}}$ of $\mathbb{C} \setminus \{0\}$ is defined by

$$S(d, \varepsilon) := \{z \in \widetilde{\mathbb{C}} : \ z = re^{i\theta}, \ d - \varepsilon/2 < \theta < d + \varepsilon/2, \ r > 0\}.$$

Moreover, if the value of opening angle ε is not essential, then we write S_d for short. We denote by $\hat{S}_d$ the set $S_d \cup D$.

By $\mathcal{O}(G)$ we understand the space of analytic functions on a domain $G \subseteq \mathbb{C}^n$. The Banach space of analytic functions on D_r, continuous on its closure and equipped with the norm $\|\varphi\|_r := \max_{|z|\leq r} |\varphi(z)|$ is denoted by $\mathbb{E}(r)$.

The space of formal power series

$$\hat{u}(t, z) = \sum_{j=0}^{\infty} u_j(z) t^j \quad \text{with} \quad u_j(z) \in \mathbb{E}(r)$$

is denoted by $\mathbb{E}(r)[[t]]$. Moreover, we set $\mathbb{E}[[t]] := \bigcup_{r>0} \mathbb{E}(r)[[t]]$.

In this section we also recall some definitions and fundamental facts about the Gevrey formal power series and Borel summability. For more details we refer the reader to [2].

Definition 1. A function $u \in \mathcal{O}(S(d, \varepsilon) \times D_r)$ is of *exponential growth of order at most $K > 0$ as $t \to \infty$ in $S(d, \varepsilon)$* if and only if for any $r_1 \in (0, r)$ and any $\varepsilon_1 \in (0, \varepsilon)$ there exist $A, B < \infty$ such that

$$\max_{|z|\leq r_1} |u(t, z)| < Ae^{B|t|^K} \quad \text{for every} \quad t \in S(d, \varepsilon_1).$$

The space of such functions is denoted by $\mathcal{O}^K(S(d, \varepsilon) \times D_r)$. We also write $\mathcal{O}^K(\hat{S}_d \times D)$ for the space $\mathcal{O}^K(S_d \times D) \cap \mathcal{O}(\hat{S}_d \times D)$.

Analogously, a function $\varphi \in \mathcal{O}(S(d,\varepsilon))$ is of *exponential growth of order at most* $K > 0$ *as* $z \to \infty$ *in* $S(d,\varepsilon)$ if and only if for any $\varepsilon_1 \in (0,\varepsilon)$ there exist $A, B < \infty$ such that

$$|\varphi(z)| < Ae^{B|z|^K} \quad \text{for every} \quad z \in S(d,\varepsilon_1).$$

The space of such functions is denoted by $\mathcal{O}^K(S(d,\varepsilon))$. We also set $\mathcal{O}^K(\hat{S}_d) := \mathcal{O}^K(S_d) \cap \mathcal{O}(\hat{S}_d)$.

Definition 2. Let $s \geq 0$. A formal power series

$$\hat{u}(t,z) := \sum_{j=0}^{\infty} u_j(z)t^j \quad \text{with} \quad u_j(z) \in \mathbb{E}(r) \tag{5}$$

is called a *Gevrey formal power series in* t *of order* s if its coefficients satisfy

$$\max_{|z|\leq r} |u_j(z)| \leq AB^j\Gamma(1+sj) \quad \text{for} \quad j = 0, 1, \dots$$

with some positive constants A and B.

The set of Gevrey formal power series in t of order s over $\mathbb{E}(r)$ is denoted by $\mathbb{E}(r)[[t]]_s$. We also set $\mathbb{E}[[t]]_s := \bigcup_{r>0} \mathbb{E}(r)[[t]]_s$.

Definition 3. Let $k > 0$ and $d \in \mathbb{R}$. A formal series $\hat{u} \in \mathbb{E}[[t]]_{1/k}$ defined by (5) is called *k-summable in a direction d* if and only if its *k-Borel transform* v satisfies

$$v(s,z) := \sum_{j=0}^{\infty} u_j(z)\frac{s^j}{\Gamma(1+j/k)} \in \mathcal{O}^k(\hat{S}_d \times D).$$

The *k-sum of $\hat{u}(t,z)$ in the direction d* is represented by the Laplace transform of v

$$u^\theta(t,z) := \frac{1}{t^k}\int_0^{\infty(\theta)} e^{-(s/t)^k} v(s,z)\, ds^k,$$

where the integration is taken over any ray $e^{i\theta}\mathbb{R}_+ := \{re^{i\theta} : \ r \geq 0\}$ with $\theta \in (d-\varepsilon/2, d+\varepsilon/2)$.

We are now ready to define multisummability in some multidirection.

Definition 4. Let $k_1 > \dots > k_n > 0$. We say that a real vector $(d_1,\dots,d_n) \in \mathbb{R}^n$ is an *admissible multidirection* if and only if

$$|d_j - d_{j-1}| \leq \pi(1/k_j - 1/k_{j-1})/2 \quad \text{for} \quad j = 2,\dots,n.$$

Let $\mathbf{k} = (k_1,\dots,k_n) \in \mathbb{R}^n_+$ and let $\mathbf{d} = (d_1,\dots,d_n) \in \mathbb{R}^n$ be an admissible multidirection. We say that a formal power series $\hat{u}$ given by (5) is **k**-*summable in the multidirection* **d** if and only if $\hat{u} = \hat{u}_1 + \dots + \hat{u}_n$, where $\hat{u}_j$ is k_j-summable in the direction d_j for $j = 1,\dots,n$.

3. Moment methods

In this section we recall the notion of moment summability methods introduced by Balser [2].

Definition 5 (see [2, Section 5.5])**.** A pair of functions $e_m(z)$ and $E_m(z)$ is said to be *kernel functions of order k* ($k > 1/2$) if they have the following properties:

1. $e_m(z) \in \mathcal{O}(S(0,\pi/k))$, $e_m(z)/z$ is integrable at the origin, $e_m(x) \in \mathbb{R}_+$ for $x \in \mathbb{R}_+$ and $e_m(z)$ is exponentially flat of order k in $S(0,\pi/k)$ (i.e. $\forall_{\varepsilon>0}\exists_{A,B>0}$ such that $|e_m(z)| \leq Ae^{-(|z|/B)^k}$ for $z \in S(0,\pi/k-\varepsilon)$).
2. $E_m(z) \in \mathcal{O}^k(\mathbb{C})$ and $E_m(1/z)/z$ is integrable at the origin in $S(\pi, 2\pi-\pi/k)$.
3. The connection between $e_m(z)$ and $E_m(z)$ is given by the *corresponding moment function $m(u)$ of order k* as follows. The function $m(u)$ is defined in terms of $e_m(z)$ by

$$m(u) := \int_0^\infty x^{u-1} e_m(x)dx \quad \text{for} \quad \operatorname{Re} u \geq 0 \tag{6}$$

and the kernel function $E_m(z)$ has the power series expansion

$$E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} \quad \text{for} \quad z \in \mathbb{C}. \tag{7}$$

Observe that in case $k \leq 1/2$ the set $S(\pi, 2\pi - \pi/k)$ is not defined, so the second property in Definition 5 can not be satisfied. It means that we must define the kernel functions of order $k \leq 1/2$ and the corresponding moment functions in another way.

Definition 6 (see [2, Section 5.6])**.** A function $e_m(z)$ is called *a kernel function of order $k > 0$* if we can find a pair of kernel functions $e_{\tilde{m}}(z)$ and $E_{\tilde{m}}(z)$ of order $pk > 1/2$ (for some $p \in \mathbb{N}$) so that

$$e_m(z) = e_{\tilde{m}}(z^{1/p})/p \quad \text{for} \quad z \in S(0,\pi/k).$$

For a given kernel function $e_m(z)$ of order $k > 0$ we define the *corresponding moment function $m(u)$ of order $k > 0$* by (6) and the *kernel function $E_m(z)$ of order $k > 0$* by (7).

Remark 1. Observe that by Definitions 5 and 6 we have

$$m(u) = \tilde{m}(pu) \quad \text{and} \quad E_m(z) = \sum_{j=0}^{\infty} \frac{z^j}{m(j)} = \sum_{j=0}^{\infty} \frac{z^j}{\tilde{m}(jp)}.$$

Remark 2 (see [2, Section 5.5]). If $m(u)$ is a moment function of order k then the moments $m(n)$ are of the same order as $\Gamma(1 + n/k)$. It means that there exist constants $c, C > 0$ such that

$$c^j\Gamma(1 + j/k) \leq m(j) \leq C^j\Gamma(1 + j/k) \quad \text{for} \quad j \in \mathbb{N}.$$

The most important examples of kernel functions of order $k > 0$ with corresponding moment functions are given by

- $e_m(z) = kz^k e^{-z^k}$
- $m(u) = \Gamma(1 + u/k)$
- $E_m(z) = \sum_{j=0}^{\infty} z^j/\Gamma(1 + j/k) =: \mathbf{E}_{1/k}(z)$, where $\mathbf{E}_{1/k}$ is the Mittag-Leffler function of index $1/k$.

The next proposition provides a method for the construction of new moment functions.

Proposition 1 (see [2, Theorem 31])**.** *Let two kernel functions $e_{m_j}(z)$ of orders k_j, with corresponding moment functions $m_j(u)$, be given. Then there is a unique kernel function $e_m(z)$ of order $k = (1/k_1 + 1/k_2)^{-1}$ with corresponding moment function $m(u) = m_1(u)m_2(u)$.*

By Remark 2 and by the general theory of moment summability (see [2, Section 6.5 and Theorem 38]) we may characterise the Gevrey order and the Borel summability of formal power series as follows

Proposition 2. *Let $m(u)$ be a moment function of order k, $\hat{u}(t,z) = \sum_{n=0}^{\infty} u_n(z)t^n$ be a formal power series with coefficients $u_n(z) \in \mathcal{O}(D)$ and $v(s,z) := \sum_{n=0}^{\infty} \frac{u_n(z)}{m(n)} s^n$. Then*

- *$\hat{u}$ is a Gevrey series of order $1/k$ if and only if $v \in \mathcal{O}(D^2)$.*
- *$\hat{u}$ is k-summable in a direction d ($d \in \mathbb{R}$) if and only if $v \in \mathcal{O}^k(\hat{S}_d \times D)$.*

4. Moment operators

In this section we recall the notion of moment differential operators constructed recently by Balser and Yoshino [8]. We also introduce the concept of moment pseudodifferential operators, which generalise the pseudodifferential operators defined in [14, 15].

Definition 7. For every moment functions $m_1(u)$ and $m_2(u)$ the linear operators $\partial_{m_1,t}, \partial_{m_2,z} \colon \mathbb{C}[[t,z]] \to \mathbb{C}[[t,z]]$ defined by

$$\partial_{m_1,t}\Big(\sum_{j=0}^{\infty} \frac{u_j(z)}{m_1(j)} t^j\Big) := \sum_{j=0}^{\infty} \frac{u_{j+1}(z)}{m_1(j)} t^j$$

and

$$\partial_{m_2,z}\Big(\sum_{j=0}^{\infty} \frac{\tilde{u}_j(t)}{m_2(j)} z^j\Big) := \sum_{j=0}^{\infty} \frac{\tilde{u}_{j+1}(t)}{m_2(j)} z^j$$

are called the *moment differential operators* $\partial_{m_1,t}$ and $\partial_{m_2,z}$.

Moreover, the right-inversion operators $\partial^{-1}_{m_1,t}, \partial^{-1}_{m_2,z} \colon \mathbb{C}[[t,z]] \to \mathbb{C}[[t,z]]$ given by

$$\partial^{-1}_{m_1,t}\Big(\sum_{j=0}^{\infty} \frac{u_j(z)}{m_1(j)} t^j\Big) := \sum_{j=1}^{\infty} \frac{u_{j-1}(z)}{m_1(j)} t^j$$

and

$$\partial^{-1}_{m_2,z}\Big(\sum_{j=0}^{\infty} \frac{\tilde{u}_j(t)}{m_2(j)} z^j\Big) := \sum_{j=1}^{\infty} \frac{\tilde{u}_{j-1}(t)}{m_2(j)} z^j$$

are called the *moment integration operators* $\partial^{-1}_{m_1,t}$ and $\partial^{-1}_{m_2,z}$.

Remark 3. Observe that

- For $m_1(j) = \Gamma(1+j)$, the operator $\partial_{m_1,t}$ coincides with the usual differentiation ∂_t.
- For $m_1(j) = \Gamma(1+j/k)$ ($k>0$), the operator $\partial_{m_1,t}$ satisfies

  $$(\partial_{m_1,t}u)(t^{1/k},z) = \partial_t^{1/k}(u(t^{1/k},z)),$$

  where $\partial_t^{1/k}$ is the Caputo fractional derivative of order $1/k$ (see also [16, Definition 5 and Remark 1]) defined by

  $$\partial_t^{1/k}\Big(\sum_{j=0}^{\infty} \frac{u_j(z)}{\Gamma(1+j/k)} t^{j/k}\Big) := \sum_{j=0}^{\infty} \frac{u_{j+1}(z)}{\Gamma(1+j/k)} t^{j/k}.$$

By the end of this section we assume that $e_m(z)$ and $E_m(z)$ are kernel functions of order $k>0$ with a corresponding moment function $m(u)$.

The moment differential operator $\partial_{m,z}$ is well-defined for every $\varphi(z)\in\mathcal{O}(D)$. In addition, we have the following integral representation of $\partial_{m,z}\varphi(z)$.

Proposition 3. *Let $\varphi\in\mathcal{O}(D_r)$. Then for every $|z|<\varepsilon<r$ and $n\in\mathbb{N}$ we have*

$$\partial^n_{m,z}\varphi(z)=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^n E_m(z\zeta)\frac{e_m(w\zeta)}{w\zeta}\,d\zeta\,dw, \tag{8}$$

where $\theta\in(-\arg w-\frac{\pi}{2k},-\arg w+\frac{\pi}{2k})$.

Proof. Since $\varphi\in\mathcal{O}(D_r)$, we see that

$$\varphi(z)=\sum_{j=0}^{\infty}\frac{\varphi^{(j)}(0)}{j!}z^j=\sum_{j=0}^{\infty}\frac{\partial^j_{m,z}\varphi(0)}{m(j)}z^j\quad\text{for}\quad |z|<r.$$

Hence, by the Cauchy integral formula

$$\partial^j_{m,z}\varphi(0)=\frac{m(j)}{j!}\varphi^{(j)}(0)=\frac{m(j)}{2\pi i}\oint_{|w|=\varepsilon}\frac{\varphi(w)}{w^{j+1}}\,dw\quad\text{for}\quad \varepsilon<r.$$

By the definition of moment functions we have

$$\frac{m(j)}{w^{j+1}}=\int_0^{\infty}y^{j-1}\frac{e_m(y)}{w^{j+1}}\,dy\overset{y=\zeta w}{=}\int_0^{\infty(\theta)}\zeta^j\frac{e_m(\zeta w)}{\zeta w}\,d\zeta\quad\text{with}\quad \theta=-\arg w.$$

Moreover, since $e_m(z)$ is exponentially flat of order k for $\arg z\in(-\frac{\pi}{2k},\frac{\pi}{2k})$, we may replace the direction $\theta=-\arg w$ by any direction $\theta\in(-\arg w-\frac{\pi}{2k},-\arg w+\frac{\pi}{2k})$.

It means that

$$\partial^j_{m,z}\varphi(0)=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\zeta^j\frac{e_m(\zeta w)}{\zeta w}\,d\zeta\,dw$$

and consequently for $|z|<\varepsilon$ we have

$$\begin{aligned}\varphi(z)&=\sum_{j=0}^{\infty}\frac{\partial^j_{m,z}\varphi(0)}{m(j)}z^j=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}\frac{e_m(\zeta w)}{\zeta w}\sum_{j=0}^{\infty}\frac{\zeta^j z^j}{m(j)}\,d\zeta\,dw\\&=\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_0^{\infty(\theta)}E_m(\zeta z)\frac{e_m(\zeta w)}{\zeta w}\,d\zeta\,dw.\end{aligned}$$

Since

$$\partial^n_{m,z}E_m(\zeta z)=\partial^n_{m,z}\sum_{j=0}^{\infty}\frac{\zeta^j z^j}{m(j)}=\sum_{j=0}^{\infty}\frac{\zeta^{j+n}z^j}{m(j)}=\zeta^n E_m(\zeta z), \tag{9}$$

we finally obtain (8). □

The formula (8) motivates the introduction of moment pseudodifferential operators on the space of analytic functions. To this end, let $\lambda(\zeta)$ be an analytic function for $|\zeta|>|\zeta_0|$ of polynomial growth at infinity. By (9) we may define

$$\lambda(\partial_{m,z})E_m(\zeta z):=\lambda(\zeta)E_m(\zeta z).$$

Hence we have

Definition 8. A *moment pseudodifferential operator* $\lambda(\partial_{m,z})$ is defined by

$$\lambda(\partial_{m,z})\varphi(z) := \frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}\lambda(\zeta)E_m(\zeta z)\frac{e_m(\zeta w)}{\zeta w}\,d\zeta\,dw \tag{10}$$

for every $\varphi \in \mathcal{O}(D)$, where $\theta \in (-\arg w - \frac{\pi}{2k}, -\arg w + \frac{\pi}{2k})$.

Since $\lambda(\zeta)$ is a holomorphic function for $|\zeta| > |\zeta_0|$ and is of polynomial growth at infinity, the left-hand side of (10) is a well-defined analytic function in a complex neighbourhood of the origin.

Definition 9. We define a *pole order* $q \in \mathbb{Q}$ and a *leading term* $\lambda \in \mathbb{C}\setminus\{0\}$ of $\lambda(\zeta)$ as the numbers satisfying the formula

$$\lim_{\zeta\to\infty}\frac{\lambda(\zeta)}{\zeta^q} = \lambda.$$

Sometimes we write it also $\lambda(\zeta) \sim \lambda\zeta^q$.

We have the following estimation

Lemma 1. *Let $\varphi \in \mathcal{O}(D)$. Then there exist $r > 0$ and $A, B < \infty$ such that for every moment pseudodifferential operator $\lambda(\partial_{m,z})$ we have*

$$\sup_{|z|<r}|\lambda(\partial_{m,z})\varphi(z)| \le |\lambda|AB^q\Gamma(1+q/k), \quad \textit{where} \quad \lambda(\zeta) \sim \lambda\zeta^q.$$

Proof. Since $\lambda(\zeta) \sim \lambda\zeta^q$, we may assume that $|\lambda(\zeta)| \le 2|\lambda||\zeta|^q$ for $|\zeta| > |\zeta_0|$. Hence, by the definition of kernel functions we have

$$\begin{aligned}
&\Big|\int_{\zeta_0}^{\infty(\theta)}\lambda(\zeta)E_m(\zeta z)\frac{e_m(\zeta w)}{\zeta w}\,d\zeta\Big| \le \int_{|\zeta_0|}^{\infty}2|\lambda|s^qA_1e^{b_1|z|^ks^k}\frac{A_2e^{-b_2|w|^ks^k}}{s|w|}\,ds \\
\le\;& \frac{2|\lambda|A_1A_2}{|w|}\int_0^\infty s^{q-1}e^{(b_1|z|^k-b_2|w|^k)s^k}\,ds \\
\overset{\sigma=s^k}{\le}\;& \frac{2|\lambda|A_1A_2}{k|w|}\int_0^\infty \sigma^{q/k-1}e^{(b_1|z|^k-b_2|w|^k)\sigma}\,d\sigma \le |\lambda|\tilde{A}\tilde{B}^q\frac{\Gamma(1+q/k)}{|w|(b_2|w|^k-b_1|z|^k)^{q/k}}.
\end{aligned}$$

We choose $r > 0$ such that $b_2\varepsilon^k - b_1r^k > b_2\varepsilon^k/2$. Then for $z \in D_r$ we have

$$\begin{aligned}
|\lambda(\partial_{m,z})\varphi(z)| &\le \frac{1}{2\pi}\oint_{|w|=\varepsilon}|\varphi(z)||\lambda|\tilde{A}\tilde{B}^q\frac{\Gamma(1+q/k)}{|w|(b_2|w|^k-b_1|z|^k)^{q/k}}\,d|w| \\
&\le |\lambda|\tilde{A}\tilde{B}^q\frac{\Gamma(1+q/k)}{\varepsilon(b_2\varepsilon^k/2)^{q/k}}\frac{1}{2\pi}\oint_{|w|=\varepsilon}|\varphi(z)|\,d|w| \le |\lambda|AB^q\Gamma(1+q/k).
\end{aligned}$$

□

5. Formal solutions

In this section we study formal solutions of the initial value problem for a general linear moment partial differential equation with constant coefficients

$$\begin{cases} P(\partial_{m_1,t}, \partial_{m_2,z})u = 0 \\ \partial_{m_1,t}^j u(0,z) = \varphi_j(z) \in \mathcal{O}(D) \quad \text{for} \quad j = 0,\dots,n-1, \end{cases} \tag{11}$$

where

$$P(\lambda,\zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^n P_j(\zeta)\lambda^{n-j}$$

is a general polynomial of two variables, which is of order n with respect to λ.

If $P_0(\zeta) \neq$ const. then a formal solution of (11) is not uniquely determined. To avoid this inconvenience we shall choose some special solution which is called a *normalised formal solution* (see also Balser [3] and Michalik [14]). To this end we factorise the moment differential operator $P(\partial_{m_1,t}, \partial_{m_2,z})$ as follows

$$\begin{aligned} P(\partial_{m_1,t}, \partial_{m_2,z}) &= P_0(\partial_{m_2,z})(\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} \cdots (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l} \\ &=: P_0(\partial_{m_2,z})\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z}) \end{aligned}$$

where $\lambda_1(\zeta), \dots, \lambda_l(\zeta)$ are the characteristic roots of $P(\lambda, \zeta) = 0$ with multiplicity $n_1, \dots, n_l$ ($n_1 + \dots + n_l = n$) respectively.

Since $\lambda_j(\zeta)$ are algebraic functions, they are also holomorphic for sufficiently large ζ (say, for $|\zeta| > |\zeta_0|$) and of polynomial growth at infinity. It means that the moment pseudodifferential operators $\lambda_j(\partial_{m_2,z})$ are well defined.

Now we are ready to define the uniquely determined normalised solution of (11).

Definition 10. A formal solution $\hat{u}$ of (11) is called a *normalised formal solution* if and only if $\hat{u}$ is also a solution of the pseudodifferential equation $\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})\hat{u} = 0$.

Our aim is to study the normalised formal solution of (11). We begin by describing the formal solution of simple moment pseudodifferential equation

$$(12) \qquad \begin{cases} (\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^{\beta} u = 0 \\ \partial^j_{m_1,t} u(0,z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial^{\beta-1}_{m_1,t} u(0,z) = \lambda^{\beta-1}(\partial_{m_2,z})\varphi(z) \in \mathcal{O}(D). \end{cases}$$

The trivial verification shows that

Lemma 2. *A formal solution $\hat{u}$ of (12) is given by*

$$(13) \qquad \hat{u}(t,z) = \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} \frac{\lambda^j(\partial_{m_2,z})\varphi(z)}{m_1(j)} t^j.$$

Next, we have

Proposition 4. *If $\hat{u}$ is a formal solution of*

$$(14) \qquad \begin{cases} \tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})\hat{u} = 0 \\ \partial^j_{m_1,t} \hat{u}(0,z) = 0 \quad (j = 0, \dots, n - 2) \\ \partial^{n-1}_{m_1,t} \hat{u}(0,z) = \varphi(z) \in \mathcal{O}(D), \end{cases}$$

then $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$ with $\hat{u}_{\alpha\beta}$ being a formal solution of

$$(15) \qquad \begin{cases} (\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^{\beta} \hat{u}_{\alpha\beta} = 0 \\ \partial^j_{m_1,t} \hat{u}_{\alpha\beta}(0,z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial^{\beta-1}_{m_1,t} \hat{u}_{\alpha\beta}(0,z) = \lambda_\alpha^{\beta-1}(\partial_{m_2,z})\varphi_{\alpha\beta}(z), \end{cases}$$

where $\varphi_{\alpha\beta}(z) := d_{\alpha\beta}(\partial_{m_2,z})\varphi(z) \in \mathcal{O}(D)$ and $d_{\alpha\beta}(\zeta)$ is some holomorphic function of polynomial growth.

Proof. Observe that the formal solution $\hat{u}$ of (14) is given by

$$(16) \qquad \hat{u}(t,z) = \sum_{j=0}^{\infty} q_j(\partial_{m_2,z})\varphi(z) \frac{t^j}{m_1(j)},$$

where $q_j(\zeta)$ are the solutions of the difference equations

$$P_0(\zeta)q_j(\zeta)=\sum_{k=1}^{n}P_k(\zeta)q_{j-k}(\zeta)\quad\text{for}\quad j\geq n$$

with the initial conditions $q_0(\zeta)=q_1(\zeta)=\cdots=q_{n-2}(\zeta)=0$ and $q_{n-1}(\zeta)=1$. Since the solutions $q_j(\zeta)$ are rational functions, it follows that the moment pseudodifferential operators $q_j(\partial_{m_2,z})$ are well defined. Moreover, according to the theory of difference equations, we have

$$q_j(\zeta)=\sum_{\alpha=1}^{l}\sum_{\beta=1}^{\min\{j+1,n_\alpha\}}c_{\alpha\beta}(\zeta)\frac{j!}{(j+1-\beta)!}\lambda_\alpha^j(\zeta), \tag{17}$$

where $c_{\alpha\beta}(\zeta)$ are holomorphic functions of polynomial growth for sufficiently large $|\zeta|$ and $\lambda_\alpha(\zeta)$ are the characteristic roots of multiplicity n_α. Combining (16) and (17) we obtain

$$\hat{u}(t,z)=\sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}c_{\alpha\beta}(\partial_{m_2,z})\sum_{j=\beta-1}^{\infty}\frac{j!}{(j+1-\beta)!}\lambda_\alpha^j(\partial_{m_2,z})\frac{t^j}{m_1(j)}\varphi(z).$$

It means that $\hat{u}=\sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}\hat{u}_{\alpha\beta}$, where

$$\hat{u}_{\alpha\beta}(t,z)=c_{\alpha\beta}(\partial_{m_2,z})\sum_{j=\beta-1}^{\infty}\frac{j!}{(j+1-\beta)!}\lambda_\alpha^j(\partial_{m_2,z})\frac{t^j}{m_1(j)}\varphi(z).$$

Bt Lemma 2, the formal power series $\hat{u}_{\alpha\beta}$ is a solution of (15) with $d_{\alpha\beta}(\zeta)=(\beta-1)!c_{\alpha\beta}(\zeta)$. □

We generalise the above result as follows

Theorem 1. *If $\hat{u}$ is a normalised formal solution of (11) then $\hat{u}=\sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}\hat{u}_{\alpha\beta}$ with $\hat{u}_{\alpha\beta}$ being a formal solution of*

$$\begin{cases}(\partial_{m_1,t}-\lambda_\alpha(\partial_{m_2,z}))^\beta\hat{u}_{\alpha\beta}=0\\ \partial_{m_1,t}^j\hat{u}_{\alpha\beta}(0,z)=0\quad(j=0,\dots,\beta-2)\\ \partial_{m_1,t}^{\beta-1}\hat{u}_{\alpha\beta}(0,z)=\lambda_\alpha^{\beta-1}(\partial_{m_2,z})\varphi_{\alpha\beta}(z),\end{cases}$$

where $\varphi_{\alpha\beta}(z):=\sum_{j=0}^{n-1}d_{\alpha\beta j}(\partial_{m_2,z})\varphi_j(z)\in\mathcal{O}(D)$ and $d_{\alpha\beta j}(\zeta)$ are some holomorphic functions of polynomial growth.

Proof. Applying the principle of superposition of solutions of linear equation in the same way as in [14, Remark 2] and repeating the proof of Proposition 4, we obtain the assertion. □

The next lemma allows us the study of solutions of moment equations in the case when the moment function $m_1(u)$ is of order $k_1\leq 1/2$. We have

Lemma 3. *Let $m_1(u)$ and $m_2(u)$ be moment functions of orders $k_1>0$ and $k_2>0$ respectively, $p\in\mathbb{N}$ and $\tilde{k}_1:=k_1p>1/2$. Then $\tilde{m}_1(u):=m_1(u/p)$ is a moment function of order $\tilde{k}_1$. Moreover, $\hat{u}=\hat{u}(t,z)$ is a formal solution of*

$$\begin{cases}P(\partial_{m_1,t},\partial_{m_2,z})\hat{u}=0\\ \partial_{m_1,t}^k\hat{u}(0,z)=\varphi_k(z),\quad k=0,\dots,n-1.\end{cases} \tag{18}$$

if and only if $\hat{v}(t,z) := \hat{u}(t^p, z)$ *is a formal solution of*

$$\left\{\begin{array}{l} P(\partial^p_{\tilde{m}_1,t}, \partial_{m_2,z})\hat{v} = 0, \\ \partial^j_{\tilde{m}_1,t}\hat{v}(0,z) = \varphi_k(z) \text{ for } j = kp,\ k = 0,\dots,n-1, \\ \partial^j_{\tilde{m}_1,t}\hat{v}(0,z) = 0 \text{ for } j = 1,\dots,np-1 \text{ and } p \nmid j. \end{array}\right. \tag{19}$$

Proof. By Definition 6 and Remark 1 we see that $\tilde{m}_1(u) = m_1(u/p)$ is a moment function of order $\tilde{k}_1$. Moreover, observe that the formal power series $\hat{u}(t,z) = \sum_{j=0}^{\infty} \frac{u_j(z)}{m_1(j)} t^j$ is a formal solution of (18) if and only if the coefficients $u_j(z)$ satisfy conditions $u_j(z) = \varphi_j(z)$ for $j = 0,\dots,n-1$ and

$$P_0(\partial_{m_2,z})u_j(z) = \sum_{k=1}^{n} P_k(\partial_{m_2,z})u_{j-k}(z) \quad \text{for} \quad j \geq n.$$

On the other hand

$$\hat{v}(t,z) = \sum_{k=0}^{\infty} \frac{u_k(z)}{m_1(k)} t^{kp} = \sum_{n=0}^{\infty} \frac{u_k(z)}{\tilde{m}_1(kp)} t^{kp} = \sum_{j=0}^{\infty} \frac{v_j(z)}{\tilde{m}_1(j)} t^j,$$

where $v_j(z) = \left\{\begin{array}{ll} 0 & j \neq kp \\ u_k(z) & j = kp \end{array}\right.$. Hence the coefficients $u_j(z)$ satisfy the above conditions if and only if $v_j(z) = \left\{\begin{array}{ll} 0 & j \neq kp,\ j < np \\ \varphi_k(z) & j = kp,\ j < np \end{array}\right.$ and

$$P_0(\partial_{m_2,z})v_j(z) = \sum_{k=1}^{n} P_k(\partial_{m_2,z})v_{j-kp}(z) \quad \text{for} \quad j \geq np.$$

Moreover, the coefficients $v_j(z)$ satisfy the above conditions if and only if $\hat{v}$ is a formal solution of (19). □

6. Gevrey estimates

In this section we study the Gevrey order of formal solution $\hat{u}$ of (11), which depends on the orders k_1, k_2 of moment functions $m_1(u)$, $m_2(u)$ respectively, and depends on the pole orders q_α of the characteristic roots $\lambda_\alpha(\zeta)$ ($\alpha = 1,\dots,l$). First, we consider the simple moment pseudodifferential equation (12). We have

Proposition 5. *We assume that* $\hat{u}$ *is a formal solution of (12) and* q *is a pole order of* $\lambda(\zeta)$. *We have*

- *If* $1/k_1 < q/k_2$ *then* $\hat{u}$ *is a Gevrey series of order* $q/k_2 - 1/k_1$ *with respect to* t.
- *If* $1/k_1 = q/k_2$ *then* $u \in \mathcal{O}(D^2)$.
- *If* $1/k_1 > q/k_2$ *then* $u \in \mathcal{O}^{\frac{k_1k_2}{k_2-qk_1}}(\mathbb{C} \times D)$.

Proof. We estimate the coefficients of the formal solution $\hat{u}(t,z) = \sum_{j=0}^{\infty} u_j(z)t^j$ of (12). By Lemmas 1 and 2 we have

$$|u_j(z)| = \binom{j}{\beta - 1} \frac{|\lambda^j(\partial_{m_2,z})\varphi(z)|}{m_1(j)} \leq AB^j \frac{\Gamma(1 + jq/k_2)}{\Gamma(1 + j/k_1)} \quad \text{for} \quad z \in D.$$

Hence $\hat{u}$ is a Gevrey series of order $q/k_2 - 1/k_1$ for $1/k_1 < q/k_2$, a convergent series in a complex neighbourhood of the origin for $1/k_1 = q/k_2$ and an entire function for $1/k_1 > q/k_2$. In the last case, by the properties of the Mittag-Leffler function (see [2, p. 234]), we have

$$
\begin{aligned}
|u(t,z)| &\leq \sum_{j=0}^{\infty} AB^j \frac{\Gamma(1+jq/k_2)}{\Gamma(1+j/k_1)}|t|^j \leq \sum_{j=0}^{\infty} AB^j \frac{|t|^j}{\Gamma(1+j(1/k_1-q/k_2))} \\
&\leq A\mathbf{E}_{1/k_1-q/k_2}(B|t|) \leq \tilde{A}e^{\tilde{B}|t|^{\frac{k_1k_2}{k_2-qk_1}}},
\end{aligned}
$$

where $\mathbf{E}_{1/k_1-q/k_2}$ is the Mittag-Leffler function of index $1/k_1 - q/k_2$. □

Combining Theorem 1 and Proposition 5 we obtain

Theorem 2. *Let $\hat{u}$ be a normalised formal solution of (11) with the decomposition $\hat{u} = \sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}\hat{u}_{\alpha\beta}$ constructed in Theorem 1 and let q_α be a pole order of $\lambda_\alpha(\zeta)$ for $\alpha = 1,\dots,l$. Then a formal series $\hat{u}_{\alpha\beta}$ is characterised as follows:*

- *If $1/k_1 < q_\alpha/k_2$ then $\hat{u}_{\alpha\beta}$ is a Gevrey series of order $q_\alpha/k_2 - 1/k_1$ with respect to t.*
- *If $1/k_1 = q_\alpha/k_2$ then $u_{\alpha\beta} \in \mathcal{O}(D^2)$.*
- *If $1/k_1 > q_\alpha/k_2$ then $u_{\alpha\beta} \in \mathcal{O}^{\frac{k_1k_2}{k_2-q_\alpha k_1}}(\mathbb{C}\times D)$.*

7. Analytic solutions

In this section we study the convergent solutions u of (12). Therefore by Proposition 5 we assume that $1/k_1 \geq q/k_2$, where k_1, k_2 are orders of moment functions $m_1(u)$, $m_2(u)$ respectively, and q is a pole order of $\lambda(\zeta)$. We find a characterisation of analytic continuation property of u in terms of the Cauchy data φ.

First, we introduce the following integral representation of solution.

Lemma 4. *Let u be a solution of (12) and $1/k_1 \geq q/k_2$. Then u is analytic in some complex neighbourhood of the origin and is given by*

(20)
$$
u(t,z) = \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_{\zeta_0}^{\infty(\theta)} E_{m_1}(t\lambda(\zeta))E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw
$$

with $\theta \in (-\arg w - \frac{\pi}{2k_2}, -\arg w + \frac{\pi}{2k_2})$.

Proof. Since $1/k_1 \geq q/k_2$, by Proposition 5 the solution u is analytic in some complex neighbourhood of the origin. To show that the inner integral on the right-hand side of (20) is convergent, observe that by Definitions 5 and 6 there exist constants A_i and b_i ($i = 1,2,3$) such that $|E_{m_1}(t\lambda(\zeta))| \leq A_1e^{b_1|t|^{k_1}|\zeta|^{k_1q}}$, $|E_{m_2}(\zeta z)| \leq A_2e^{b_2|\zeta|^{k_2}|z|^{k_2}}$ and $|e_{m_2}(\zeta w)| \leq A_3e^{-b_3|\zeta|^{k_2}|w|^{k_2}}$. Hence, for fixed $w \in \mathbb{C}\setminus\{0\}$ such that $|z|$ is small relative to $|w|$ and for $|t| < a|w|^q$ with some fixed $a > 0$, we have

$$
\Big|\int_{\zeta_0}^{\infty(\theta)} E_{m_1}(t\lambda(\zeta))E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\Big| \leq \int_{|\zeta_0|}^{\infty}\tilde{A}e^{-\tilde{b}s^{k_2}|w|^{k_2}}\,ds < \infty.
$$

It means that the right-hand side of (20) is a well-defined holomorphic function in some complex neighbourhood of the origin. To show (20), observe that by Lemma

2 and by (10) we have

$$\begin{aligned}
u(t,z) &= \sum_{j=\beta-1}^{\infty}\binom{j}{\beta-1}\frac{\lambda^j(\partial_{m_2,z})\varphi(z)}{m_1(j)}t^j = \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}\sum_{j=0}^{\infty}\frac{\lambda^j(\partial_{m_2,z})\varphi(z)}{m_1(j)}t^j\\
&= \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}\sum_{j=0}^{\infty}\frac{t^j}{m_1(j)}\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}\lambda^j(\zeta)E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw\\
&= \frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}\frac{1}{2\pi i}\oint_{|w|=\varepsilon}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}\sum_{j=0}^{\infty}\frac{t^j\lambda^j(\zeta)}{m_1(j)}E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw.
\end{aligned}$$

□

In the next crucial lemma we use the integral representation (20) of solution u to find its analytic continuation.

Lemma 5. *Let $1/k_1 = q/k_2$, $q = \mu/\nu$ with relatively prime numbers $\mu,\nu \in \mathbb{N}$, $K > k_1$ and let u be a solution of*

$$\begin{cases} (\partial_{m_1,t}-\lambda(\partial_{m_2,z}))^{\beta}u = 0\\ \partial_{m_1,t}^{j}u(0,z) = \varphi_j(z)\in\mathcal{O}(D) \quad (j=0,\dots,\beta-1).\end{cases}$$

If $\varphi_j \in \mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu})$ for $k = 0,\dots,\mu-1$ and $j = 0,\dots,\beta-1$, then $u \in \mathcal{O}^K(\hat{S}_{d+2n\pi/\nu}\times D)$ for $n = 0,\dots,\nu-1$.

Proof. First, we consider case $k_1 > 1/2$. By the principle of superposition of solutions of linear equations we may assume that u satisfies (12) with $\varphi \in \mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu})$ for $k = 0,\dots,\mu-1$. So, by Lemma 4, u is given by (20). Next, observe that the function

$$t \mapsto \int_{\zeta_0}^{\infty(\theta)} E_{m_1}(t\lambda(\zeta))E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta, \tag{21}$$

which is holomorphic on $\{t\in\mathbb{C}\colon |t| < a|w|^q\}$, can be also analytically continued to the set

$$\{t\in\widetilde{\mathbb{C}}\colon (\arg t + 2k\pi + \arg\lambda)/\mu \neq (\arg w + 2n\pi)/\nu \quad \text{for every } k,n\in\mathbb{Z}\}. \tag{22}$$

Indeed, we may replace a direction θ in (21) by $\tilde{\theta}$ satisfying the following conditions

- $$\arg t + 2k\pi + \arg\lambda + q\arg\tilde{\theta} \in \Big(\frac{\pi}{2k_1}, 2\pi - \frac{\pi}{2k_1}\Big) \text{ for some } k\in\mathbb{Z}$$
  (in this case, by Definition 5, we have $E_{m_1}(t\lambda(\zeta)) \to 0$ as $\zeta\to\infty$, $\arg\zeta = \tilde{\theta}$),
- $$\arg w + 2n\pi + \arg\tilde{\theta} \in \Big(-\frac{\pi}{2k_2}, \frac{\pi}{2k_2}\Big) \text{ for some } n\in\mathbb{Z}$$
  (in this case, by Definitions 5 and 6, there exists $\varepsilon > 0$ such that
  $$\Big|\frac{E_{m_2}(\zeta z)e_{m_2}(\zeta w)}{\zeta w}\Big| \leq e^{-\varepsilon|\zeta|^{k_2}} \quad \text{as} \quad \zeta\to\infty, \quad \arg\zeta = \tilde{\theta}).$$

Since $q = k_2/k_1 = \mu/\nu$, these requirements may be together satisfied under the condition that $(\arg t + 2k\pi + \arg\lambda)/\mu \neq (\arg w + 2n\pi)/\nu$ for every $k,n\in\mathbb{Z}$. Therefore the function (21) can be analytically continued to the sectors (22) and has the exponential growth of order at most k_1 there.

To estimate u, fix z such close to the origin, that $\arg(w-z)\approx\arg w$ along a circle $|w|=\varepsilon$. Repeating the proof of Theorem 3.1 in [7], we split this circle into 2μ arcs γ_{2k} and γ_{2k+1} ($k=0,\dots,\mu-1$), where γ_{2k} extends between points of argument $(d+\arg\lambda)/q+2k\pi/\mu\pm\tilde{\delta}/3$ and γ_{2k+1} extends between $(d+\arg\lambda)/q+2k\pi/\mu+\tilde{\delta}/3$ and $(d+\arg\lambda)/q+2(k+1)\pi/\mu-\tilde{\delta}/3 \mod 2\pi$. Finally, since $\varphi\in\mathcal{O}(S((d+\arg\lambda)/q+2k\pi/\mu,\tilde{\delta}))$, we may deform γ_{2k} into a path γ_{2k}^R along the ray $\arg w=(d+\arg\lambda)/q+2k\pi/\mu-\tilde{\delta}/3$ to a point with modulus R (which can be chosen arbitrarily large), then along the circle $|w|=R$ to the ray $\arg w=(d+\arg\lambda)/q+2k\pi/\mu+\tilde{\delta}/3$ and back along this ray to the original circle. So, we have

$$u(t,z)=\frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}u_1(t,z)+\frac{t^{\beta-1}}{(\beta-1)!}\partial_t^{\beta-1}u_2(t,z),$$

where

$$u_1(t,z):=\sum_{k=0}^{\mu-1}\frac{1}{2\pi i}\oint_{\gamma_{2k}^R}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}E_{m_1}(t\lambda(\zeta))E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw$$

and

$$u_2(t,z):=\sum_{k=0}^{\mu-1}\frac{1}{2\pi i}\oint_{\gamma_{2k+1}}\varphi(w)\int_{\zeta_0}^{\infty(\theta)}E_{m_1}(t\lambda(\zeta))E_{m_2}(\zeta z)\frac{e_{m_2}(\zeta w)}{\zeta w}\,d\zeta\,dw.$$

Note that R may be chosen arbitrarily large and the function (21) is analytic on $\{t\in\mathbb{C}\colon |t|<a|w|^q\}$ (we assume that $|z|$ is small relative to $|w|$). Hence, one can find $\delta>0$ such that u_1 is analytically continued to $S(d+2n\pi/\nu,\delta)\times D_r$ for $n=0,\dots,\nu-1$. Estimating this integral we see that it is of exponential growth of order at most K as $t\to\infty$.

Moreover, since the function (21) is analytically continued into the region (22), we see that u_2 is also analytically continued to $S(d+2n\pi/\nu,\delta)\times D_r$ for $n=0,\dots,\nu-1$ and is of exponential growth of order at most k_1 as $t\to\infty$.

Hence also u is analytically continued to $S(d+2n\pi/\nu,\delta)\times D_r$ for $n=0,\dots,\nu-1$ and is of exponential growth of order at most K as $t\to\infty$.

In case $k_1\leq 1/2$ there exists $p\in\mathbb{N}$ such that $\tilde{k}_1:=pk_1>1/2$. By Lemma 3, $v(t,z):=u(t^p,z)$ is a solution of

$$\begin{cases}(\partial_{\tilde{m}_1,t}^p-\lambda(\partial_{m_2,z}))^\beta v=0,\\ \partial_{\tilde{m}_1,t}^{np}v(0,z)=\varphi_n(z)\in\mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu})\text{ for }n=0,\dots,\beta-1\\ \partial_{\tilde{m}_1,t}^{j}v(0,z)=0\text{ for }j=1,\dots,\beta p-1\text{ and }p\not|\, j,\end{cases}$$

where $\tilde{m}_1(u)=m_1(u/p)$ is a moment function of order $\tilde{k}_1>1/2$.

By Theorem 1 we have $v=v_0+\dots+v_{p-1}$, where v_j $(j=0,\dots,p-1)$ satisfy

$$\begin{cases}(\partial_{\tilde{m}_1,t}-e^{i2j\pi/p}\lambda^{1/p}(\partial_{m_2,z}))^\beta v_j=0,\\ \partial_{\tilde{m}_1,t}^{n}v_j(0,z)=\tilde{\varphi}_{jn}(z)\in\mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu})\text{ for }n=0,\dots,\beta-1.\end{cases}$$

Applying the first part of the proof to the above equation we obtain $v_j(t,z)\in\mathcal{O}^{pK}(\hat{S}_{(d+2j\pi)/p+2n\pi/(p\nu)}\times D)$ for $j=1,\dots,p$. It means that $u(t,z)=v(t^{1/p},z)\in\mathcal{O}^K(\hat{S}_{d+2n\pi/\nu}\times D)$. □

To show that the sufficient condition for analytic continuation of u given in Lemma 5 is also necessary, we need the following auxiliary lemmas.

Lemma 6. *We assume that* $1/k_1 = q/k_2$. *Then* $u \in \mathcal{O}(D^2)$ *satisfies the equation*

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))u = 0 \tag{23}$$

if and only if u *is a solution of the equation*

$$(\partial_{m_2,z} - \lambda^{-1}(\partial_{m_1,t}))u = 0. \tag{24}$$

Proof. Since the equations (23) and (24) are symmetric, we only need the implication one way. To this end, we assume that u satisfies (23). It means, by Lemma 4, that

$$u(t,z) = \frac{1}{2\pi i}\oint_{|w|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} E_{m_1}(t\lambda(\zeta))E_{m_2}(\zeta z)\frac{e(\zeta w)}{\zeta w}\,d\zeta\,dw$$

for some $\varphi \in \mathcal{O}(D)$. Hence, by the definition of the pseudodifferential operator $\lambda^{-1}(\partial_{m_1,t})$ we have

$$\lambda^{-1}(\partial_{m_1,t})u(t,z) = \frac{1}{2\pi i}\oint_{|w|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} \lambda^{-1}(\lambda(\zeta))E_{m_1}(t\lambda(\zeta))E_{m_2}(\zeta z)\frac{e(\zeta w)}{\zeta w}\,d\zeta\,dw$$
$$= \frac{1}{2\pi i}\oint_{|w|=\varepsilon} \varphi(w) \int_{\zeta_0}^{\infty(\theta)} E_{m_1}(t\lambda(\zeta))\zeta E_{m_2}(\zeta z)\frac{e(\zeta w)}{\zeta w}\,d\zeta\,dw = \partial_{m_2,z}u(t,z).$$

So u satisfies also (24). □

Repeating the proof of Lemma 6 in [15], we generalise the last result as follows

Lemma 7. *We assume that* $1/k_1 = q/k_2$. *Then* $u \in \mathcal{O}(D^2)$ *satisfies the equation*

$$(\partial_{m_1,t} - \lambda(\partial_{m_2,z}))^\beta u = 0$$

if and only if u *satisfies the equation*

$$(\partial_{m_2,z} - \lambda^{-1}(\partial_{m_1,t}))^\beta u = 0.$$

Now we can state the main result of the paper

Theorem 3. *Let us assume that* u *is a solution of (12),* $1/k_1 = q/k_2$ *and* $q = \mu/\nu$ *with relatively prime numbers* $\mu, \nu \in \mathbb{N}$. *Then for every* $K > k_1$ *and* $d \in \mathbb{R}$ *we have*

$$\varphi \in \mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu}) \ for\ k = 0,\dots,\mu-1$$

if and only if

$$u \in \mathcal{O}^K(\hat{S}_{d+2n\pi/\nu} \times D) \ for\ n = 0,\dots,\nu-1.$$

Proof. ($\Longrightarrow$) The implication is given by Lemma 4.

($\Longleftarrow$) By Lemma 7, the function u satisfies

$$(\partial_{m_2,z} - \lambda^{-1}(\partial_{m_1,t}))^\beta u = 0$$

with the initial conditions $\partial^j_{m_2,z}u(t,0) = \psi_j(t) \in \mathcal{O}^K(\hat{S}_{d+2n\pi/\nu})$ for $n = 0,\dots,\nu-1$ and $j = 0,\dots,\beta-1$. Observe that, if $\lambda(\zeta) \sim \lambda\zeta^q$ then $\lambda^{-1}(\tau) \sim \lambda^{-1/q}\tau^{1/q}$. By Lemma 5 with replaced variables, we conclude that if $\psi_j(t) \in \mathcal{O}^K(\hat{S}_{d+2n\pi/\nu})$ for $n = 0,\dots,\nu-1$ and $j = 0,\dots,\beta-1$ then $u(t,z) \in \mathcal{O}^{qK}(D \times \hat{S}_{\theta_k})$, where $\theta_k = (d + 2k\pi/\nu)/q - \arg(\lambda^{-1/q}) = (d+\arg\lambda)/q + 2k\pi/\mu$ for $k = 0,\dots,\mu-1$. In consequence, $\varphi(z) = u(0,z) \in \mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu})$ for $k = 0,\dots,\mu-1$. □

8. Summable and multisummable solutions

In this section we characterise the summable formal solutions $\hat{u}$ of (12) in terms of the Cauchy data φ. Next, we also give a similar characterisation of multisummable normalised formal solutions of general equation (11).

Proposition 6. *Let us assume that $\hat{u}$ is a formal solution of (12), $1/k_1 < q/k_2$, $q = \mu/\nu$ with relatively prime numbers $\mu, \nu \in \mathbb{N}$, $K = (q/k_2 - 1/k_1)^{-1}$ and $d \in \mathbb{R}$. Then $\hat{u}$ is K-summable in directions $d + 2n\pi/\nu$ for $n = 0, \dots, \nu - 1$ if and only if $\varphi \in \mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu})$ for $k = 0, \dots, \mu - 1$.*

Proof. By Lemma 2 and Proposition 2, $\hat{u}$ is K-summable in directions $d + 2n\pi/\nu$ $(n = 0, \dots, \nu - 1)$ if and only if

$$v(t,z) := \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} \frac{\lambda^j(\partial_{m_2,z})\varphi(z)}{m_1(j)m(j)} t^j \in \mathcal{O}^K(\hat{S}_{d+2n\pi/\nu} \times D) \ (n = 0, \dots, \nu - 1)$$

with some moment function $m(u)$ of order K. By Proposition 1, we see that $\tilde{m}_1(u) := m_1(u)m(u)$ is a moment function of order $\tilde{k}_1 := (1/k_1 + 1/K)^{-1} = k_2/q$. It means, by Lemma 2, that v is a solution of

$$\begin{cases} (\partial_{\tilde{m}_1,t} - \lambda(\partial_{m_2,z}))^\beta v = 0 \\ \partial^j_{\tilde{m}_1,t} v(0,z) = 0 \quad (j = 0, \dots, \beta - 2) \\ \partial^{\beta-1}_{\tilde{m}_1,t} v(0,z) = \lambda^{\beta-1}(\partial_{m_2,z})\varphi(z) \in \mathcal{O}(D). \end{cases}$$

Hence, by Theorem 3, $v \in \mathcal{O}^K(\hat{S}_{d+2n\pi/\nu} \times D)$ for $n = 0, \dots, \nu - 1$ if and only if $\varphi \in \mathcal{O}^{qK}(\hat{S}_{(d+\arg\lambda)/q+2k\pi/\mu})$ for $k = 0, \dots, \mu - 1$. □

Now we return to the general equation (11). For convenience we assume that

$$P(\lambda,\zeta) = P_0(\zeta) \prod_{\alpha=1}^{\tilde{n}} \prod_{\beta=1}^{l_\alpha} (\lambda - \lambda_{\alpha\beta}(\zeta))^{n_{\alpha\beta}}, \tag{25}$$

where $\lambda_{\alpha\beta}(\zeta)$ is the characteristic root with a pole orders $q_\alpha \in \mathbb{Q}$ for $\alpha = 1, \dots, \tilde{n}$ and $\beta = 1, \dots, l_\alpha$. Without loss of generality we may assume that there exist exactly N pole orders of the characteristic roots, which are greater than k_2/k_1, say $k_2/k_1 < q_1 < \dots < q_N < \infty$ and let $K_\alpha > 0$ be defined by $K_\alpha := (q_\alpha/k_2 - 1/k_1)^{-1}$ for $\alpha = 1, \dots, N$.

By Theorem 1, the normalised formal solution $\hat{u}$ of (11) is given by

$$\hat{u} = \sum_{\alpha=1}^{\tilde{n}} \sum_{\beta=1}^{l_\alpha} \sum_{\gamma=1}^{n_{\alpha\beta}} \hat{u}_{\alpha\beta\gamma}$$

with $\hat{u}_{\alpha\beta\gamma}$ satisfying

$$\begin{cases} (\partial_{m_1,t} - \lambda_{\alpha\beta}(\partial_{m_2,z}))^\gamma \hat{u}_{\alpha\beta\gamma} = 0 \\ \partial^j_{m_1,t} \hat{u}_{\alpha\beta\gamma}(0,z) = 0 \quad \text{for} \quad j = 0, \dots, \gamma - 2 \\ \partial^{\gamma-1}_{m_1,t} \hat{u}_{\alpha\beta\gamma} = \lambda_{\alpha\beta}(\partial_{m_2,z})\varphi_{\alpha\beta\gamma}(z), \end{cases}$$

where $\varphi_{\alpha\beta\gamma}(z) = \sum_{j=0}^{n-1} d_{\alpha\beta\gamma j}(\partial_{m_2,z})\varphi_j(z) \in \mathcal{O}(D)$ and $d_{\alpha\beta\gamma j}(\zeta)$ are holomorphic functions of polynomial growth at infinity.

Hence, by Proposition 6, we have

Theorem 4. *Under the above conditions with $q_\alpha = \mu_\alpha/\nu_\alpha$ with relatively prime numbers $\mu_\alpha, \nu_\alpha \in \mathbb{N}$ for $\alpha = 1\dots,N$, the normalised formal solution $\hat{u}$ of (11) is $(K_1,\dots,K_N)$-summable in multidirections $(d_1 + 2n_1\pi/\nu_1,\dots,d_N + 2n_N\pi/\nu_N)$ $(n_\alpha = 0,\dots,\nu_\alpha - 1,\ \alpha = 0,\dots,N)$ if and only if*

$$\varphi \in \mathcal{O}^{q_\alpha K_\alpha}(\hat{S}_{(d_\alpha+\arg\lambda_{\alpha\beta})/q_\alpha+2n_\alpha\pi/\mu_\alpha})$$

for every $n_\alpha = 0,\dots,\mu_\alpha - 1$, $\beta = 1,\dots,l_\alpha$ and $\alpha = 1,\dots,N$.

9. Inhomogeneous equations

In the last section we generalise the above results to the inhomogeneous case. To this end we consider the Cauchy problem for general inhomogeneous linear moment partial differential equation with constant coefficients

$$\begin{cases} P(\partial_{m_1,t},\partial_{m_2,z})\tilde{u} = \hat{\tilde{f}} \\ \partial^j_{m_1,t}\tilde{u}(0,z) = \varphi_j(z) \in \mathcal{O}(D) \quad \text{for} \quad j = 0,\dots,n-1, \end{cases} \tag{26}$$

where $\hat{\tilde{f}}(t,z) \in \mathbb{E}[[t]]$ and $P(\lambda,\zeta)$ is a polynomial in (λ,ζ), of degree $n \in \mathbb{N}$ with respect to λ. In other words

$$P(\lambda,\zeta) = P_0(\zeta)\lambda^n - \sum_{j=1}^n P_j(\zeta)\lambda^{n-j} = P_0(\zeta)\Big(\lambda^n - \sum_{j=1}^n \tilde{P}_j(\zeta)\lambda^{n-j}\Big),$$

where $P_0(\zeta),\dots,P_n(\zeta)$ are polynomials and $\tilde{P}_j(\zeta) := P_j(\zeta)/P_0(\zeta)$ $(j = 1,\dots,n)$ are rational functions.

Following [16], without loss of generality we may assume that the Cauchy data φ_j vanish. Indeed, after substitution

$$u(t,z) := \tilde{u}(t,z) - \sum_{j=0}^{n-1}\frac{\varphi_j(z)}{m_1(j)}t^j$$

we reduce the Cauchy problem (26) to

$$\begin{cases} P(\partial_{m_1,t},\partial_{m_2,z})u(t,z) = \hat{f}(t,z) \\ \partial^j_{m_1,t}u(0,z) = 0 \qquad\qquad \text{for} \quad j = 0,\dots,n-1, \end{cases} \tag{27}$$

where

$$\hat{f}(t,z) := \hat{\tilde{f}}(t,z) - P(\partial_{m_1,t},\partial_{m_2,z})\sum_{j=0}^{n-1}\frac{\varphi_j(z)}{m_1(j)}t^j \in \mathbb{E}[[t]].$$

Using the pseudodifferential operators defined by (10) we have

$$\begin{aligned} P(\partial_{m_1,t},\partial_{m_2,z}) &= P_0(\partial_{m_2,z})\Big(\partial^n_{m_1,t} - \sum_{j=1}^n \tilde{P}_j(\partial_{m_2,z})\partial^{n-j}_{m_1,t}\Big) \\ &=: P_0(\partial_{m_2,z})\tilde{P}(\partial_{m_1,t},\partial_{m_2,z}). \end{aligned}$$

Observe that, if $P_0(\partial_{m_2,z}) \neq \text{const.}$ then the Cauchy problem (27) is not uniquely determined. In the homogeneous case this problem was solving by the choice of normalised formal solution. In the inhomogeneous case the formal solution is determined by the formal power series $\hat{g}(t,z) \in \mathbb{E}[[t]]$ (see also [8] and [16]), which satisfies the equation

$$P_0(\partial_{m_2,z})\hat{g} = \hat{f}.$$

For a given $\hat{g}$ there is exactly one formal solution $\hat{u}$ of the Cauchy problem

$$(28)\qquad \begin{cases} \tilde{P}(\partial_{m_1,t},\partial_{m_2,z})\hat{u}=\hat{g} & \\ \partial^j_{m_1,t}\hat{u}(0,z)=0 & \text{for}\quad j=0,\dots,n-1, \end{cases}$$

which is also a solution of (27) and is called a *formal solution of (27) determined by* $\hat{g}$.

In the next proposition we find the formal solution $\hat{u}$. We have

Proposition 7. *A formal solution $\hat{u}$ of (27) determined by $\hat{g}$ is given by*

$$(29)\qquad \hat{u}(t,z)=\sum_{j=0}^{\infty}(\partial^{-1}_{m_1,t})^{j+1}q_j(\partial_{m_2,z})\hat{g}(t,z),$$

where $q_j(\zeta)$ are solutions of the difference equation

$$q_j(\zeta)=\sum_{k=1}^{n}\tilde{P}_k(\zeta)q_{j-k}(\zeta)\quad \textit{for}\quad j\geq k$$

with the initial conditions $q_0(\zeta)=\dots=q_{n-2}(\zeta)=0$ and $q_{n-1}(\zeta)=1$

Proof. Since $q_j(\zeta)$ are rational functions, they are of polynomial growth at infinity and they are holomorphic for sufficient large $|\zeta|$. Hence the pseudodifferential operators $q_j(\partial_{m_2,z})$ are well defined.

To finish the proof, it is sufficient to show that the formal series given by (29) is a solution of (28). To this end observe that $(\partial_{m_1,t})^j\hat{u}(0,z)=0$ for $j=0,\dots,n-1$ and

$$\begin{aligned} &\tilde{P}(\partial_{m_1,t},\partial_{m_2,z})\hat{u}(t,z)=\tilde{P}(\partial_{m_1,t},\partial_{m_2,z})\Big(\sum_{j=n-1}^{\infty}(\partial^{-1}_{m_1,t})^{j+1}q_j(\partial_{m_2,z})\hat{g}(t,z)\Big)\\ &=\sum_{j=n-1}^{\infty}(\partial^{-1}_{m_1,t})^{j-n+1}q_j(\partial_{m_2,z})\hat{g}(t,z)-\sum_{k=1}^{n}\sum_{j=n-1}^{\infty}(\partial^{-1}_{m_1,t})^{j-n+1+k}\tilde{P}_k(\partial_{m_2,z})q_j(\partial_{m_2,z})\hat{g}(t,z)\\ &=\sum_{j=n-1}^{\infty}(\partial^{-1}_{m_1,t})^{j-n+1}q_j(\partial_{m_2,z})\hat{g}(t,z)-\sum_{j=n}^{\infty}\sum_{k=1}^{n}(\partial^{-1}_{m_1,t})^{j-n+1}\tilde{P}_k(\partial_{m_2,z})q_{j-k}(\partial_{m_2,z})\hat{g}(t,z)\\ &=\sum_{j=n-1}^{\infty}(\partial^{-1}_{m_1,t})^{j-n+1}q_j(\partial_{m_2,z})\hat{g}(t,z)-\sum_{j=n}^{\infty}(\partial^{-1}_{m_1,t})^{j-n+1}q_j(\partial_{m_2,z})\hat{g}(t,z)\\ &=q_{n-1}(\partial_{m_2,z})\hat{g}(t,z)=\hat{g}(t,z). \end{aligned}$$

□

Now we consider the simple inhomogeneous moment pseudodifferential equation

$$(30)\qquad \begin{cases} (\partial_{m_1,t}-\lambda(\partial_{m_2,z}))^{\beta}u(t,z)=\hat{g}(t,z) & \\ \partial^j_{m_1,t}u(0,z)=0, & j=0,\dots,\beta-1. \end{cases}$$

The direct calculation shows that

Lemma 8. *The formal solution $\hat{u}$ of (30) is given by*

$$\hat{u}(t,z)=\sum_{j=\beta-1}^{\infty}\binom{j}{\beta-1}(\partial^{-1}_{m_1,t})^{j+1}\lambda^{j-\beta+1}(\partial_{m_2,z})\hat{g}(t,z).$$

Next, using Proposition 7 and the factorisation of operator $\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z})$, we obtain the following decomposition of solution of (27) determined by $\hat{g}$.

Theorem 5. *Let $\hat{u}$ be a formal solution of (27) determined by $\hat{g}$ and*

$$\tilde{P}(\partial_{m_1,t}, \partial_{m_2,z}) = (\partial_{m_1,t} - \lambda_1(\partial_{m_2,z}))^{n_1} \cdots (\partial_{m_1,t} - \lambda_l(\partial_{m_2,z}))^{n_l}.$$

Then $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$, where $\hat{u}_{\alpha\beta}$ is a formal solution of

$$\left\{ \begin{array}{l} (\partial_{m_1,t} - \lambda_\alpha(\partial_{m_2,z}))^{\beta} \hat{u}_{\alpha\beta}(t,z) = \hat{g}_{\alpha\beta}(t,z) \\ \partial^{j}_{m_1,t} \hat{u}_{\alpha\beta}(0,z) = 0 \text{ for } j = 0, \ldots, \beta - 1 \end{array} \right. \tag{31}$$

and $\hat{g}_{\alpha\beta}(t,z) = d_{\alpha\beta}(\partial_{m_2,z})\hat{g}(t,z)$ for some holomorphic function $d_{\alpha\beta}(\zeta)$ of polynomial growth.

Proof. By Proposition 7 the formal solution of (27) determined by $\hat{g}$ is given by

$$\hat{u}(t,z) = \sum_{j=0}^{\infty} (\partial^{-1}_{m_1,t})^{j+1} q_j(\partial_{m_2,z}) \hat{g}(t,z),$$

where

$$q_j(\zeta) = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{\min\{j+1,n_\alpha\}} c_{\alpha\beta}(\zeta) \frac{j!}{(j+1-\beta)!} \lambda^{j}_{\alpha}(\zeta),$$

$c_{\alpha\beta}(\zeta)$ are holomorphic functions of polynomial growth for sufficiently large $|\zeta|$ (see Section 5 in [14] for more details) and $\lambda_\alpha(\zeta)$ are the characteristic roots of multiplicity n_α.

It means that $\hat{u} = \sum_{\alpha=1}^{l} \sum_{\beta=1}^{n_\alpha} \hat{u}_{\alpha\beta}$, where

$$\hat{u}_{\alpha\beta} = \sum_{j=\beta-1}^{\infty} \frac{j!}{(j+1-\beta)!} (\partial^{-1}_{m_1,t})^{j+1} \lambda^{j}_{\alpha}(\partial_{m_2,z}) c_{\alpha\beta}(\partial_{m_2,z}) \hat{g}(t,z).$$

By Lemma 8, $\hat{u}_{\alpha\beta}$ is a formal solution of (31) with

$$\hat{g}_{\alpha\beta}(t,z) = d_{\alpha\beta}(\partial_{m_2,z})\hat{g}(t,z) \quad \text{and} \quad d_{\alpha\beta}(\zeta) = (\beta-1)! c_{\alpha\beta}(\zeta) \lambda^{\beta-1}_{\alpha}(\zeta).$$

□

Now we are ready to study the Gevrey order of formal solution to inhomogeneous equation. First we consider the simple equation (30). We have

Proposition 8. *Let $\hat{u}$ be a formal solution of (30) determined by a Gevrey series $\hat{g} \in \mathbb{E}[[t]]_s$ of order $s \geq 0$ and let q be a pole order of $\lambda(\zeta)$. Then $\hat{u}$ is a Gevrey series of order $\max\{\frac{qk_1-k_2}{k_1k_2}, s\}$ with respect to t.*

Proof. Since

$$\hat{g}(t,z) = \sum_{n=0}^{\infty} \frac{g_n(z)}{m_1(n)} t^n \quad \text{with} \quad g_n(z) \in \mathbb{E}(r)$$

is a Gevrey series of order $s \geq 0$, there exist $A, B < \infty$ such that

$$\max_{|z|\leq r} |g_n(z)| \leq AB^n \Gamma(1 + (s + 1/k_1)n) \quad \text{for} \quad n = 0, 1, \ldots \tag{32}$$

By Lemma 8 we have

$$\begin{aligned}
\hat{u}(t,z) &= \sum_{j=\beta-1}^{\infty}\binom{j}{\beta-1}(\partial_{m_1,t}^{-1})^{j+1}\lambda^{j-\beta+1}(\partial_{m_2,z})\hat{g}(t,z)\\
&= \sum_{j=\beta-1}^{\infty}\binom{j}{\beta-1}\lambda^{j-\beta+1}(\partial_{m_2,z})\sum_{n=0}^{\infty}\frac{g_n(z)}{m_1(n+j+1)}t^{n+j+1}\\
&\overset{k=n+j+1}{=} \sum_{k=\beta}^{\infty}\frac{t^k}{m_1(k)}\sum_{j=\beta-1}^{k-1}\binom{j}{\beta-1}\lambda^{j-\beta+1}(\partial_{m_2,z})g_{k-j-1}(z)\\
&=: \sum_{k=\beta}^{\infty}\frac{u_k(z)t^k}{m_1(k)}.
\end{aligned}$$

Now, by Lemma 1 and by (32), we obtain the estimate

$$\begin{aligned}
|u_k(z)| &\leq \sum_{j=\beta-1}^{k-1}\binom{j}{\beta-1}|\lambda^{j-\beta+1}(\partial_{m_2,z})g_{k-j-1}(z)|\\
&\leq \tilde{A}\tilde{B}^k\sum_{j=\beta-1}^{k-1}\Gamma(1+jq/k_2)\Gamma(1+(k-j-1)(s+1/k_1))
\end{aligned}$$

for $z \in D_r$. Hence there exist $C, D < \infty$ such that

$$\sup_{|z|<r}|u_k(z)| \leq CD^K\Gamma(1+k(\tilde{s}+1/k_1)) \quad \text{for} \quad k=0,1,\ldots,$$

where $\tilde{s} = \max\{\frac{qk_1-k_2}{k_1k_2}, s\}$. □

By Theorem 5 and Proposition 8, we obtain the Gevrey estimates for solutions of (27), which improve the result of Balser and Yoshino [8]. Namely we have

Theorem 6. *Let $\hat{u}$ be a formal solution of (27) determined by a Gevrey series $\hat{g} \in \mathbb{E}[[t]]_s$ of order $s \geq 0$ and let $\hat{u} = \sum_{\alpha=1}^{l}\sum_{\beta=1}^{n_\alpha}\hat{u}_{\alpha\beta}$ be a decomposition of solution constructed in Theorem 5. Then $\hat{u}_{\alpha\beta}$ is a Gevrey series of order $\max\{\frac{q_\alpha k_1-k_2}{k_1k_2}, s\}$ with respect to t, where $q_\alpha \in \mathbb{Q}$ is a pole order of the characteristic root $\lambda_\alpha(\zeta)$.*

Immediately by Lemma 8 and Proposition 8 we obtain the characterisation of analytic continuation properties of convergent solutions of (30) in terms of inhomogeneity $\hat{g}$.

Proposition 9. *We assume that $1/k_1 = q/k_2$, $\hat{g} \in \mathbb{E}[[t]]_0$ is a convergent power series and u is a solution of (30). Then u is analytic in some complex neighbourhood of the origin. Moreover, for every $K > k_1$ and $d \in \mathbb{R}$ we have $\hat{u} \in \mathcal{O}^K(\hat{S}_d \times D)$ if and only if $\hat{g}$ satisfies condition*

$$\sum_{j=\beta-1}^{\infty}\binom{j}{\beta-1}(\partial_{m_1,t}^{-1})^{j+1}\lambda^{j-\beta+1}(\partial_{m_2,z})\hat{g}(t,z) \in \mathcal{O}^K(\hat{S}_d \times D).$$

Using Lemma 8 and Proposition 9 we characterise the summable formal solutions $\hat{u}$ of (30) in a similar way to Proposition 6.

Proposition 10. *We assume that $\hat{u}$ is a formal solution of (30), $1/k_1 < q/k_2$, $\hat{g}(t,z) = \sum_{n=0}^{\infty} \frac{g_n(z)}{m_1(n)} t^n \in \mathbb{E}[[t]]_s$ for some $s \in [0, q/k_2 - 1/k_1]$, $K = (q/k_2 - 1/k_1)^{-1}$ and $d \in \mathbb{R}$. Then $\hat{u}$ is K-summable in a direction d if and only if the coefficients $g_n(z)$ ($n \in \mathbb{N}$) of $\hat{g}$ satisfy condition*

$$\sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} (\partial_{\tilde{m}_1,t}^{-1})^{j+1} \lambda^{j-\beta+1}(\partial_{m_2,z}) \sum_{n=0}^{\infty} \frac{g_n(z)}{\tilde{m}_1(n)} t^n \in \mathcal{O}^K(\hat{S}_d \times D),$$

where $\tilde{m}_1(u) = m_1(u)m(u)$ and $m(u)$ is a moment function of order K.

Proof. By Lemma 8 an Proposition 2, $\hat{u}$ is K-summable in a direction d if and only if $v(t,z) \in \mathcal{O}^K(\hat{S}_d \times D)$, where

$$\begin{aligned} v(t,z) &= \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} \lambda^{j-\beta+1}(\partial_{m_2,z}) \sum_{n=0}^{\infty} \frac{g_n(z)}{m_1(n+j+1)m(n+j+1)} t^{n+j+1} \\ &= \sum_{j=\beta-1}^{\infty} \binom{j}{\beta-1} (\partial_{\tilde{m}_1,t}^{-1})^{j+1} \lambda^{j-\beta+1}(\partial_{m_2,z}) \sum_{n=0}^{\infty} \frac{g_n(z)}{\tilde{m}_1(n)} t^n \end{aligned}$$

with $\tilde{m}_1(u) = m_1(u)m(u)$ and $m(u)$ being a moment function of order K. □

We generalise the last result to formal solutions of (27). To this end, in a similar way to the homogeneous case we assume that $P(\lambda,\zeta)$ is given by (25) and there exist exactly N pole orders of the characteristic roots of $P(\lambda,\zeta)$, which are greater than k_2/k_1, say $k_2/k_1 < q_1 < \cdots < q_N < \infty$ and let $K_\alpha = (q_\alpha/k_2 - 1/k_1)^{-1}$ for $\alpha = 1, \ldots, N$.

By Theorem 5, the normalised formal solution $\hat{u}$ of (27) is given by

$$\hat{u} = \sum_{\alpha=1}^{\tilde{n}} \sum_{\beta=1}^{l_\alpha} \sum_{\gamma=1}^{n_{\alpha\beta}} \hat{u}_{\alpha\beta\gamma},$$

with $\hat{u}_{\alpha\beta\gamma}$ satisfying

$$\begin{cases} (\partial_{m_1,t} - \lambda_{\alpha\beta}(\partial_{m_2,z}))^{\gamma} \hat{u}_{\alpha\beta\gamma} = \hat{g}_{\alpha\beta\gamma} \\ \partial_{m_1,t}^j \hat{u}_{\alpha\beta\gamma}(0,z) = 0 \quad \text{for} \quad j = 0, \ldots, \gamma - 1, \end{cases}$$

where $\hat{g}_{\alpha\beta\gamma} = d_{\alpha\beta\gamma}(\partial_{m_2,z})\hat{g}(t,z)$ for some holomorphic functions $d_{\alpha\beta\gamma}(\zeta)$ of polynomial growth at infinity.

Hence, by Proposition 10, we have

Proposition 11. *Under the above conditions the formal solution $\hat{u}$ of (27) determined by a convergent formal series $\hat{g}(t,z) = \sum_{n=0}^{\infty} \frac{g_n(z)}{m_1(n)} t^n \in \mathbb{E}[[t]]_0$ is $(K_1, \ldots, K_N)$-summable in a multidirection $(d_1, \ldots, d_N)$ if and only if the coefficients $g_n(z)$ ($n \in \mathbb{N}$) of $\hat{g}$ satisfy conditions*

$$\sum_{j=\gamma-1}^{\infty} \binom{j}{\gamma-1} (\partial_{m_{1\alpha},t}^{-1})^{j+1} \lambda_{\alpha\beta}^{j-\gamma+1}(\partial_{m_2,z}) \sum_{n=0}^{\infty} \frac{g_n(z)}{\tilde{m}_{1\alpha}(n)} t^n \in \mathcal{O}^{K_\alpha}(\hat{S}_{d_\alpha} \times D),$$

where $\tilde{m}_{1\alpha}(u) = m_1(u)\tilde{m}_\alpha(u)$ and $\tilde{m}_\alpha(u)$ is a moment function of order K_α, for every $\beta = 1, \ldots, l_\alpha$ and $\alpha = 1, \ldots, N$.

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