SEVERAL NOTES ON EXISTENCE THEOREM OF PEANO

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Abstract. An ODE with non-Lipschitz right hand side has been considered. A family of solutions with Borel measurable dependence of the initial data has been obtained.

1. Introduction

Consider a system of ordinary differential equations of the following form

$$\dot{x} = v(t, x), \quad x \in \mathbb{R}^m. \tag{1.1}$$

The vector-function v is defined in the cross product of some interval $[-T, T]$ and a domain $D \subseteq \mathbb{R}^m$.

The simplest and often occurred situation is when the vector field v is continuous and fulfills the Lipschitz condition in the second variable:

$$\|v(t, x') - v(t, x'')\| \le c\|x' - x''\|. \tag{1.2}$$

In such a case problem (1.1) has a unique solution $x(t)$ that satisfies the initial condition $x(0) = x_0 \in D$. This result is known as Cauchy-Picard existence theorem. (All the classical facts we mention without reference are contained in [7].)

In general, the solution $x(t)$ is defined not in the whole interval $[-T, T]$ but in its smaller subinterval. In the described above conditions the solution $x(t)$ depends continuously on the initial data x_0.

The Cauchy-Picard existence theorem as well as its proof transmit literally from the case $x \in \mathbb{R}^m$ to the case when x belongs to an infinite dimensional Banach space.

If we refuse Lipschitz hypothesis (1.2) then our problem becomes widely complicated. Particularly, it is known that in an infinite dimensional Banach space problem (1.1) may have no solutions [15], [6]. In the finite dimensional case the existence is guaranteed by Peano's theorem.

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So, when the function v is only continuous in $[-T,T]\times D$ then for the same initial datum x_0 there may be several solutions. Nevertheless if by some reason for any initial condition x_0 the solution is unique then it depends continuously on the initial data.

There are a lot of works devoted to investigating of different types of the uniqueness conditions. As far as the author knows this activity has been started from Kamke [8] and Levy [11]. Their results have been generalized in different directions. See for example [12], [1] and references therein. Anther approach is contained in [10], [2].

The problem of existence of individual solutions to ODE with measurable in t and continuous in x right-hand side has been considered by Caratheodory in [3].

The case when the vector field belongs to Sobolev spaces (at least $H^{1,1}$) has been studied in [5] in connection with the Navier-Stokes equation. In this article the results on existence and dependence on the initial data have been obtained.

If problem (1.1) admits non-uniqueness then for some initial datum x_0 there are many ways to pick up a solution $x(t)$ such that $x(0)=x_0$. Actually we even do not know how many ways to do this we have and how many such points x_0 are there. An attempt to clarify the last question has been done in [14]. The main result of that article is as follows: the initial data with non-unique solution form a Borel set of the class $F_{\sigma\delta}$.

Anyway for each x_0 we can choose one of the solutions $x(t)$ such that $x(0)=x_0$ and write

$$x(t)=x(t,x_0),\quad x(0,x_0)=x_0.$$

At this moment our argument is heavily rested on the Axiom of Choice.

From analysis we know that the Axiom of Choice is the best device to produce very queer functions. It is sufficient to recall that non-measurable functions exist due to the Choice Axiom.

Thus a priori we should not expect anything good from the function $x(t,x_0)$.

The aim of this article is to show that one can choose the function $x(t,x_0)$ to be measurable.

2. Main Theorems

Equip the space $\mathbb{R}^m=\{x=(x^1,\dots,x^m)\}$ with a norm

$$\|x\|=\max_{k=1,\dots,m}|x^k|.$$

Let $Q\subset\mathbb{R}^m$ be an open domain. By I_τ denote an interval

$$I_\tau=[0,\tau).$$

Introduce a vector-function $f(t,x)=(f^1,\dots,f^m)(t,x)\in C(I_\tau\times\overline{Q},\mathbb{R}^m)$. Suppose that

$$M=\sup_{(t,x)\in I_\tau\times\overline{Q}}\|f(t,x)\|<\infty.$$

Let $F\subset Q$ be a compact set and assume that

$$d=\inf\{\|x-y\|\mid x\in F,\quad y\in\partial Q\}>0.$$

We investigate the set of solutions to the following IVP.

$$u_t(t,x)=f(t,u(t,x)),\quad u(0,x)=x\in F. \tag{2.1}$$

Suppose all the solutions to this problem are defined on the interval I_T with some $T\in(0,\tau]$. From the basic ODE theory we know that $T\ge\min\{\tau,d/M\}$. Constants T,τ can take infinite values.

Equip the space $C(I_T,\mathbb{R}^m)$ with compact convergence topology.

Let $\mathcal{B}(V,W)$ stands for the set of Borel measurable functions of topological space V to topological space W.

Theorem 1. *1) Problem (2.1) has a general solution $w(t,x)$ such that the functions $x\mapsto w(t,x),\quad x\mapsto w_t(t,x)$ belong to $\mathcal{B}(F,C(I_T,\mathbb{R}^m))$. We shall write $w(t,x),w_t(t,x)\in\mathcal{B}(F,C(I_T,\mathbb{R}^m))$.*

2) Let $h(t,x)\in\mathcal{B}(F,C(I_T,\mathbb{R}^m))$ be a general solution to (2.1). Then the mapping $t\mapsto h(t,x)$ belongs to the space $C^1(I_T,(L^\infty(F))^m)$.

Remark 1. *To use this theorem it is convenient to keep in mind that if $u(t,x)\in\mathcal{B}(F,C(I_T,\mathbb{R}^m))$ then for any t the mapping $x\mapsto u(t,x)$ belongs to the set $\mathcal{B}(F,\mathbb{R}^m)$.*

Indeed, let $\delta_t:C(I_T,\mathbb{R}^m)\to\mathbb{R}^m$ stands for the $\delta-$function: $\delta_t(v(\cdot))=v(t)$. This is a continuous function. Thus $\delta_t\circ u$ is measurable.

Theorem 1 opens possibility for dynamical studying of system (2.1). For instance, let μ be a Borel measure in Q.Then we shall say that the general solution $w(t,x)$ preserves the measure μ if for any Borel set B and for any $t\in I_T$ one has $\mu(B)=\mu(w^{-1}(t,B))$.

Suppose that f does not depend on t and let $T=+\infty$. Then the set of mappings $\{w(t,\cdot)\}_{t\ge0}$ possesses natural semigroup structure: by definition put

$$w(t',\cdot)*w(t'',\cdot)=w(t'+t'',\cdot),\quad t',t''\ge0.$$

3. Proof of Theorem 1

Prove the first assertion of the Theorem.

Consider a set

$$K=\{u(\cdot)\in C^1(I_T,\mathbb{R}^m)\mid u_t(t)=f(t,u(t)),\quad u(0)\in F\}.$$

We regard K as a topological space with inducted from $C(I_T,\mathbb{R}^m)$ topology.

First, we intent to show that K is a compact set.

The functions from K satisfy the integral equation

$$u(t) = u(0) + \int_0^t f(s, u(s))\, ds. \tag{3.1}$$

Thus the set K is uniformly continuous: for every $t', t'' \in I_T$ one has

$$\|u(t') - u(t'')\| \le M|t' - t''|.$$

The set $K(t) = \{u(t) \mid u(\cdot) \in K\}$ is bounded for each t, indeed, by formula (3.1) it follows that

$$\|u(t)\| \le \max_{x \in F} \|x\| + tM. \tag{3.2}$$

Thus by Ascoli theorem [13] the set K is relatively compact in $C(I_T, \mathbb{R}^m)$.

It remains to note that K is closed in $C(I_T, \mathbb{R}^m)$. Indeed, if a sequence $\{u_n(t)\} \subseteq K$ and this sequence is convergent to the function $u(t)$ then from standard theorems of analysis we know that $u \in C(I_T, \mathbb{R}^m)$ and u satisfies equation (3.1). Thus $u \in K$.

The following proposition is a consequence from the Measurable Selection Theorem [9].

Proposition 1. *Let K be a compact metric space and let Y be a separable Hausdorff topological space. Then for any continuous mapping $g : K \to Y$ there exists a Borel set $B \subseteq K$ such that $g(B) = g(K)$ and $g\mid_B$ is an injection and $g^{-1} : g(K) \to B$ is Borel measurable.*

On a role Y we take F and let $g(u(\cdot)) = u(0)$. By Proposition 1 we obtain the Borel function $x \mapsto w(t, x)$ that solves problem (2.1).

Denote the mapping $x \mapsto w(t, x)$ by $q : F \to C(I_T, \mathbb{R}^m)$. To show that the mapping $x \mapsto w_t(t, x) = f(t, w(t, x))$ is Borel measurable introduce a continuous function $\psi : C(I_T, \overline{Q}) \to C(I_T, \mathbb{R}^m)$ by the following formula $\psi(y(\cdot)) = f(t, y(t))$. Now the mapping $f(t, w(t, x)) = \psi \circ q$ is a measurable function as a composition of measurable functions.

Prove the second part of the Theorem.

By Remark 1 for any fixed t the function $h(t, x)$ is a measurable function of F with values in $\mathbb{R}^m$. Since $h(t, x)$ is bounded (by the same argument as expressed in formula (3.2)) we have $h(t, x) \in (L^\infty(F))^m$ for any $t \in I_T$.

To make sure that the mapping $t \mapsto h(t, x)$ belongs to the space $C(I_T, (L^\infty(F))^m)$ it is sufficient to observe that

$$\|h(t', x) - h(t'', x)\| \le M|t' - t''|, \quad t', t'' \in I_T.$$

But the mapping $t \mapsto h(t, x)$ also belongs to the space $C^1(I_T, (L^\infty(F))^m)$.

Indeed, say for definiteness $\xi > 0$ then

$$\begin{aligned}
&\Big\| \frac{h(t+\xi,\cdot)-h(t,\cdot)}{\xi} - f(t,h(t,\cdot))\Big\|_{(L^\infty(F))^m} \\
&= \Big\| \frac{1}{\xi}\int_t^{t+\xi} f(s,h(s,\cdot)) - f(t,h(t,\cdot))\,ds\Big\|_{(L^\infty(F))^m} \\
&\leq \sup_{t\leq s\leq t+\xi} \|f(s,h(s,\cdot)) - f(t,h(t,\cdot))\|_{(L^\infty(F))^m}. \qquad (3.3)
\end{aligned}$$

We have already shown that $\|h(t,\cdot)-h(s,\cdot)\|_{(L^\infty(F))^m} \to 0$ as $s \to t$.The function $f(t,y)$ is uniformly continuous in a compact set

$$[t-\varepsilon, t+\varepsilon] \times \{y \in \mathbb{R}^m \mid \|y\| \leq \max_{x\in F}\|x\| + (t+\varepsilon)M\} \cap \overline{Q}$$

provided $\varepsilon > 0$ is small enough. Consequently the final expression in formula (3.3) tends to zero as $\xi \to 0$.

Theorem 1 is proved.

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