Bourbaki, Nicolas
Integration. I. Chapters 1-6.
Translated from the French fascicles of *Éléments de mathématique*
(Fascs. XIII, XXI, XXV) by Sterling K. Berberian.
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Integration. II. Chapters 7-9.
Translated from the French fascicles of *Éléments de mathématique*
(Fascs. XXIX, XXXV) by Sterling K. Berberian.
Springer–Verlag, Berlin, 2004. [ISBN 3-540-20585-3]

The volumes under review (cf. MR 2004i:28001 and MR 2005f:28001) complete the English translations of the first six "Books" that are the core of the author's *Éléments de mathématique* series:

Book I:	Theory of Sets		
Book II:	Algebra	(Vol. I, Chs. 1–3;	Vol. II, Chs. 4–7)
Book III:	General Topology	(Vol. I, Chs. 1–4;	Vol. II, Chs. 5–10)
Book IV:	Functions of a Real Variable		
Book V:	Topological Vector Spaces		
Book VI:	Integration	(Vol. I, Chs. 1–6;	Vol. II, Chs. 7–9)

The 6-Book set will be referred to as the "core" of the series; it serves as the foundation for the Books that follow. Their titles are abbreviated, for convenient reference, as S, A, GT, FRV, TVS and INT. The corresponding French versions are abbreviated E, A, TG, FVR, EVT and INT (for *Théorie des ensembles*, *Algèbre*, *Topologie générale*, *Fonctions d'une variable réelle*, *Espaces vectoriels topologiques*, and *Intégration*).

At the end of S there is a "Summary of Results", abbreviated S, R; since S, IV abbreviates Chapter IV of S, we may think of S, R as the concluding chapter of S. It is a succinct summary of the definitions, symbols and results (stated without proof) deduced in S from the axioms assumed there, that are to serve as the set-theoretical foundation for subsequent Books. It is, in effect, a primer for the language in which the rest of the series is written.

Claude Chevalley: "It took us about four years to bring out the first fascicle, the one on results in the theory of sets. The writing of the complete *text* on the theory of sets had been put back to later. The first fascicle had been published so that readers would understand the ideas of the theory that would be employed constantly by Bourbaki." ([DG], p. 20).

None of the other 5 Books of the "core" includes a "Summary of Results", although a partial Summary for *Topologie générale* was published as a separate fascicle (in 1953, decades before the publication of the definitive 'bound edition' of Vol. 2 of TG); and a Summary for *Espaces vectoriels topologiques* was published in 1955, based on the first edition of all five chapters, rendered obsolete by the changes in EVT. In a sense, this leaves these Books 'open-ended', subject to revision or even to the addition of new chapters, whereas Book I is intended to be immutable.

Books that have followed: *Algèbre commutative* (AC), *Théories spectrales* (TS), *Variétés différentielles et analytiques* (VAR), *Groupes et algèbres de Lie* (LIE).

There is a summary of results VAR, R, published (in 1971) in two 'volumes' (Fascs. XXXIII and XXXVI) without an antecedent VAR, that provided provisional references for use in existing chapters of LIE (as indicated in the footnote on the first page of LIE, Ch. II, published in 1972).

Is the "core" a textbook? Yes. It is a textbook for self-study, not a textbook for a course in the usual sense.

From the "*To the Reader*" reproduced at the beginning of each bound volume, "The Elements of Mathematics series takes up mathematics at their beginning, and gives complete proofs." And later, "... every statement in the text assumes as known only those results which have already been discussed in the same chapter, or in the previous chapters ... " (of the same or earlier Books).

Pierre Cartier: "You can think of the first books of Bourbaki as an encyclopedia. If you consider it as a textbook, it's a disaster ([PC], p. 24). ... When I was a student, every time that Bourbaki published a new book, I would just buy it or borrow it from the library, and learn it. For me, for people in my generation, it was a textbook. But the misunderstanding is that it should be a textbook for everybody ([PC], p. 25)."

Jean Dieudonné: " ... [we wanted to produce] a demonstrative text, from beginning to end." ([D1], p. 138)

Self-contained (no references to external sources), essentially complete proofs (L. Schwartz and J.-P. Serre in [PP]). Minor steps are omitted from proofs, presumably to maintain the flow of the exposition and to engage the reader's participation. Clues to the author's mindset may be gleaned from the paragraph "How detailed should a proof be?" in Dieudonné's contribution to *How to write mathematics* ([SHSD], pp. 63–64]): "For textbooks ... all the details must be filled in with only the exception of the completely trivial ones. In my opinion, a textbook where a lot of proofs are 'left to the reader' or relegated to exercises, is entirely useless for a beginner." An opinion of particular weight, as its author was responsible for the 'final draft' of the Bourbaki fascicles ([PC], p. 28); I suspect that a large part of the task was to prune out omissible steps. A grain of salt: the meaning of 'completely trivial' must be judged through the eyes of the beholder. {Some of my own experiences in navigating such gaps while studying *Intégration* are documented in "intnotes.pdf", posted at the University of Texas web site for archiving mathematical documents (www.ma.utexas.edu/mp_arc/) as item 08–193 in the folder for 2008.}

Target readership and prerequisites

From *To the Reader*: "In principle, it [the 'Elements' series] requires no particular knowledge of mathematics on the reader's part, but only a certain familiarity with mathematical reasoning and a certain capacity for abstract thought. Nevertheless, it is directed especially to those who have a good knowledge of at least the content of the first year or two of a university mathematics course."

Here, "university" must be given its meaning in French higher education. According to Dieudonné ([PP]) the work is written for readers at least at the level of the beginning of the 'third cycle', having had at least 4 years of university study, for "future Doctors"; judging from the contents of Dixmier's textbooks [JD] for the two years of the 'first cycle' (more demanding than the first two undergraduate years of any university of my experience), I would guess that the student Dieudonné describes is at the threshold of a doctoral thesis. On the same wave length, Pierre Cartier (in 1997): "The first six books of Bourbaki comprise the basic background knowledge of a modern graduate student." ([PC], p. 24, col. 3). Yet, strictly speaking, a student who has gone through the first cycle as conceived in Dixmier's course qualifies for the prerequisites of *To the Reader*.

The design of the exposition makes it possible for a reader to go directly to the topic targeted for study, and work through earlier results as needed, guided by the frequent back-references and the index of terminology.

The unique procedure by which the work is written is described in detail by Dieudonné ([D1], p. 141 and [MM], p. 57), Chevalley ([DG], p. 20, col. 1), Cartier ([PC], p. 23, col. 3), and, in [PP], by all of the above plus Cartan and Samuel; the benefits of the intense self-criticism it entails are evident in the advances made in successive editions of chapters of the Books.

Reflections on the author's objectives

To some extent, this entails attempting to read the author's mind ... but the author has left many clues.

1. At the outset (1934–35) the objective was to write an updated *Traité d'analyse* to replace the outdated books then in use (A. Weil, in [PP]). Chevalley: "The project, at that time, was extremely naive: the basis for teaching the differential calculus was Goursat's *Traité*, very insufficient on a number of points. The idea was to write another to replace it. This, we thought, would be a matter of one or two years. Five years later, we had still published nothing." ([DG], p. 19).

2. Somewhere along the way, the goal posts were moved. Dieudonné: "The idea was that it would be finished in three years, and in this time we should draft the basic essentials of mathematics. Events and history decided differently ([D1], p. 136)." The foundation was to be the theory of sets; the method, axiomatic; the novelty, the concept of a mathematical 'structure' ([DG], p. 19).

3. The term 'structure' is best illustrated by an example (for an informal definition, see S, R, §8). From Bourbaki's perspective, a topological group is an 'algebraic structure' equipped with a 'topological structure' such that the two structures are 'compatible' via suitable axioms (that the algebraic operations of composition and inversion are continuous functions) (GT, Ch. III, §1, No. 1, Def. 1). It is an astonishing consequence of compatibility that there exists an abundant supply of continuous real-valued functions on the group, thanks to its associated (left or right) 'uniform structure' ([W1]; GT, Ch. III, §3, No. 1, Def. 1 and Ch. IX, §1, No. 5, Th. 2).

Topological groups were in the air 'at the creation': A. Haar had recently proved (1933) the existence, on every locally compact group having a countable base for open sets, a measure invariant under, say, left-translation—a result to whose generalization to arbitrary locally compact groups, along with the proof of uniqueness up to a scalar multiple, J. von Neumann, A. Weil and H. Cartan contributed. If, in addition, one imposes commutativity on the group, one arrives at a platform that supports abstract harmonic analysis [W2].

Whence the author's objective of ascertaining the fundamental structures (of which the "core" high-lights order structure, algebraic structure, topological structure and uniform structure), and demonstrating the possibility, in principle, of reconstructing 'all of mathematics' by a judicious combination of appropriate structures.

4. In particular, to bring 'abstract harmonic analysis' as conceived in [W2] (at the 'frontier' in 1934) within range of an advanced graduate course.

5. *To set a standard for mathematical textbook exposition.*

6. *To set a standard for a basic set of symbols.*

7. *To set a standard for mathematical typesetting.* Before the $\TeX$ era, formulas were a chronic headache for author and typesetter. The last page-and-a-half of Weil's review of Chevalley's AMS 'Survey' [W3] is an essay on when and where a formula can be broken and when it must be displayed (regardless of its 'importance'); and publishers are cautioned that the tendency to cram (Weil's word) as much as possible onto a page has a cost in intelligibility.

8. *To learn from each other.* An objective implicit in Chevalley's description of Bourbaki's working style ([DG], p. 20, col. 1): "Strong bonds

of friendship existed between us, and when the problem of recruiting new members was raised we were all in agreement that this should be as much for their social manner as their mathematical ability. This allowed our work to submit to a rule of unanimity: anyone had the right to impose a veto. As a general rule, unanimity over a text only appeared at the end of seven or eight successive drafts. When a draft was rejected, there was a procedure foreseen for its improvement ... The general lines in which the new draft should go was indicated in such a way that the new author would know what he had to do. It was always someone else who was charged with the next draft. There was never an example of a first draft being accepted."

9. *The reader in* Y + 2K. We are the Y + 2K readers of Euclid's "Elements". Bourbaki has played with this idea from the beginning.

The first paragraph of the *Introduction* to Book I: "Ever since the time of the Greeks, mathematics has involved proof; and it is even doubted by some whether proof, in the precise and rigorous sense which the Greeks gave to this word, is to be found outside mathematics. We may fairly say that this sense has not changed, because what constituted a proof for Euclid is still a proof for us; and in times when the concept has been in danger of oblivion, and consequently mathematics itself has been threatened, it is to the Greeks that men have turned again for models of proof. But this venerable bequest has been enlarged during the past hundred years by important acquisitions."

The early *Summary of results* fascicles featured, as frontispiece, a photograph of ruins of an ancient sculpture, tactfully omitted from the final version in the bound edition of Book I.

Dieudonné, when asked why the word "Elements" was chosen for the title of the author's series, replied "Because of Euclid, obviously!"; while Weil, noting that unusually little is known of Euclid, acknowledged having always suspected that Euclid was a fictitious person in the genre of Bourbaki ([PP], Part 1).

Cartier, in an ironic vein: "Bourbaki was to be the New Euclid, he would write a textbook for the next 2000 years." ([PC], p. 27, col. 2)

Pierre Samuel (in an 'insider's review' [MR 43#2] of the *édn. reliée* of A-I): "... in a time in which the indiscriminate use of science and technology threatens the future of the human race, or at least the future of what we now call 'civilization', it is surely essential that a well integrated report about our mathematical endeavors be written and kept for the use of a latter day 'Renaissance'. As Thucydides said ..."

Can there remain any doubt?

Were the author's objectives attained?

l. *Initial objective.* A remark by Cartier suggests that FVR might be regarded as partially fulfilling the original objective described by Chevalley: " ... there is [in the "core"] an elementary calculus text, a very good book, that was the influence of Jean Delsarte" ([PC], p. 25, col. 3). However, the scope of FVR is far from that of the classical treatises on analysis (absent, for example, are complex function theory and partial differential equations).

Dieudonné's monumental *Éléments d'analyse* [D2] attains the objective single-handedly. From the Introduction to Volume 1 (I do not have the English translation): "*J'ai donc finalement été amené à tenter d'écrire l'équivalent, pour les mathématiciens de 1970, de ce qu'avaient été pour les étudiants des années 1880–1920 les* Traités d'analyse *de Jordan, Picard et Goursat.*" Renouncing the 'utmost generality' of Bourbaki, topological spaces are generally assumed to be, if not metrizable, then at least 'uniformizable' (definable by a family of pseudometrics). Metrizable spaces are taken to be separable when possible. Compactness is defined only for metrizable spaces ([D2], Vol. 1, Ch. III, §16), in sequential terms—obviating the need for a detour through axiomatic set theory—and one has a sequential proof of the compactness of a countable product of metrizable compact spaces ([D2], Vol. 2, Ch. XII, 12.5.9). The construction of Haar measure is restricted to separable, metrizable, locally compact groups (*loc. cit.*, Ch. XIV, §1).

2. *Revised objective.* According to Cartier (in [PP], Part 2), the 'objective as originally contemplated' had been accomplished by 1975. The publication dates of the bound volumes (listed later in this review) are consistent with viewing the "core" as the 'revised objective'.

3. *Applications of the method.* FRV, TVS and INT already provide examples of the method of 'structures'. Books subsequently derived from the "core" by the author's method: AC (*Algèbre commutative*), VAR (*Variétés différentielles et analytiques*), LIE (*Groupes et algèbres de Lie*) and TS (*Théories spectrales*). The objective was sufficiently accomplished to raise the question as to whether there was any point in continuing (Serre and Schwartz in [PP], Cartier in [PP] and in [PC], p. 26, col. 3). The last segment in [PP] is the recitation of a formal invitation to the funeral ceremony for Nicolas Bourbaki, deceased November 11, 1968 at his residence in Nancago. Closing remarks by André Revuz, that Bourbaki is not dead, or rather, that if he is then he lives on in at least the subconscious of every mathematician.

4. *Abstract harmonic analysis.* Accomplished in Ch. II of *Théories spectrales.*

5. *A standard for mathematical textbook exposition.*

From the Introduction of a recent paper in a distinguished journal: "The editor ... has asked me to survey the recent work of .... Space is limited and I have been advised to use 'Bourbaki style', and so this is an account of the essentials of the theory and a few of its applications, with complete proofs as far as possible."

Q.E.D.

6. *A standard for a basic set of symbols.* The success of the author's choice may be seen in its evident influence on mathematical typesetting.

As measure of the stability of the Index of Notations in the Summary of Results fascicle for the Theory of Sets, a complete list of the changes of symbols in the 2nd edition (1951) of the fascicle to the final version in E, R (1970) is as follows:

the line for $\complement\mathrm{A}$ becomes $\complement\mathrm{A}$, $\mathrm{X} - \mathrm{A}$ (an alternative notation for the complement),

$x \to f(x)$ becomes $x \mapsto f(x)$,

f_{A} becomes $f|\mathrm{A}$ (the restriction of f to A),

c_1, c_2, pr_1, pr_2 becomes pr_1, pr_2 (the alternative c_i for the i'th coordinate projection is dropped).

The symbols sup and inf are introduced in the 3rd edition [B1] (in the 2nd edition the concepts of supremum and infimum are defined, but the notations are not) as are $\varinjlim$ and $\varprojlim$ (in the 2nd edition, direct and inverse limits are not discussed).

The door is not shut for new notations in later Books (for example, Supp for the support of a numerical function, defined in Book III).

7. *Mathematical typesetting.* The stability and the widespread acceptance of Bourbaki's notations surely helped prepare the ground for the development and adoption of typesetting programs such as TEX, assured in advance of widespread applicability.

In particular, apart from the title pages, the volumes under review were typeset entirely in Leo (v. 3.4), a TEX-based WYSIWYG typesetting program running in DOS, from ABK Software. I welcome this opportunity to express my debt and my gratitude to the family (Arlan, Karen and Bruce) that created this superb tool.

8. *To learn from each other.* Chevalley, in connection with the inclusion of the axiomatic set theory treated in Book I, to which there was initial resistance but eventual accord: "Bourbaki had a great advantage: one always accepted the possibility of a sharp change of opinion ... No one in Bourbaki had the impression of talking to a wall. In this sense it was a very remarkable phenomenon of collaboration." ([DG], p. 20)

The success of the strategy is clearly visible in the great advances that were made in successive editions (of, for example, *Espaces vectoriels topologiques* and *Intégration*).

The testimony of Dieudonné ([D1], pp. 143–144): " . . . a Bourbaki member is supposed to take an interest in everything he hears . . . There is no question of asking everyone to be a universal mathematician; this is reserved for a small number of geniuses. But still, one should take an interest in everything, and be able, when the time comes, to write a chapter of the treatise, even if it is not in one's specialty. This is something which has happened to practically every member, and I think most of them have found it extremely beneficial. In any case, in my personal experience, I believe that if I had not been submitted to this obligation to draft questions I did not know a thing about, and to manage to pull through, I should never have done a quarter or even a tenth of the mathematics I have done."

9. *The reader in* $\mathrm{Y} + 2\mathrm{K}$. Stay tuned . . . .

Miscellaneous comments on the Books of the "core"

As expert reviews of the constituent fascicles are already available in reviewing journals, the present comments are more in the nature of supplementing the remarks in *To the reader*. For Books II–V, a few results cited in INT are indicated so as to give an idea of the demands of INT on the earlier Books.

For Book I, the comments pertain to the "core" as a whole. I know very little of what is in Book I. My attempt to read the Book from the beginning bogged down early in Chapter I (page I.36), with the feeling that I was not learning anything I wanted to know and disbelief that I would ever finish the task.

On occasion, I felt the need to study several topics in Chapter III: well-ordered sets (§2), cardinality (§3), combinatorial analysis (§5, No. 8), inverse limits and direct limits (§7).

Book I: Theory of sets

Why is it here?

1. In a word, *Zorn.* In GT, convergence in a topological space is defined in terms of the concept of 'filter', and compactness is characterized in terms of an 'ultrafilter'; for the concept to be useful, it is essential to know that every filter can be enlarged to an ultrafilter, the proof of which entails . . . Zorn's Lemma. Also, the existence of a basis for a vector space is proved using Zorn's Lemma (A, II, §7, No. 1, Th. 1). And Zorn's Lemma figures in the proof of the Hahn-Banach theorem (TVS, Ch. II, §3, No. 2, Th. 1).

In Book I, Zorn's Lemma (*théorème de Zorn*) is derived as a consequence of the well-ordering theorem (*théorème de Zermelo*), which is proved on the basis of the Axiom of Choice (*axiome de Zermelo*). As the "core" intends to be self-contained, and proofs to be complete, a unit on the theory of sets is imposed.

A necessary prelude to the preceding paragraph is the definition of ordered (totally ordered, well-ordered) set (S, III, §§1, 2), in effect, the concept of a set equipped with an 'order structure'. But the concept of 'structure' is the subject of Ch. IV (see item 3 below), so the appearance of the term 'order structure' must await S, IV, §1, No. 4, *Example* 1. It is a curious anomaly that the term 'order structure' can be found in the index of S, R but not in the index of S.

2. *Cardinality.* Book I includes a restrained treatment of cardinal number (S, III, §3). The concept figures in the dimension of a vector space (A, II, §7, No. 2, Th. 3), in distinguishing between finite and infinite sets (S, III, §4, No. 1), and between countable and uncountable sets (S, III, §6, No. 4). Cardinal arithmetic is extensively developed, the cardinality of the denumerably infinite and the cardinality of the continuum are singled out, but the $\aleph$ notations are not defined. Well-ordered sets and Zermelo's theorem enter early (S, III, §2), and every set of cardinal numbers is shown to be well-ordered for the natural ordering (S, III, §3, No. 2, Th. 1), but ordinal numbers are not defined in the text (they are introduced and developed in the exercises). The Continuum Hypothesis and the contributions of Kurt Gödel and Paul Cohen are mentioned briefly in the text (E, p. III.50; S, p. 189) and discussed more fully (for experts in axiomatic set theory) in the Historical Note.

In S, R, countability is discussed in the text, and cardinal numbers are mentioned in a footnote that refers the reader to Ch. III, but ordinal numbers are not mentioned.

3. *Mathematical structure.* An informal definition of a mathematical 'structure' is given in §8, of S, R, which, in practice, the reader need not know. Chapter IV is devoted to a full-dress treatment of the subject. I do not understand what is going on on page IV.1.

From my readings in Books II–VI, I have acquired the naïve impression that a 'species of structure' (for example, 'topological group') is something like a category (a term not defined in the "core"), and that Chapter IV is akin to a recipe of how to construct a category from sets, with examples of some of the things one considers within a species of structure—such as morphisms, ordering (finer, coarser), initial and final structures, induced structures, product structures, quotient structures, universal mappings.

The option of recasting the Elements in terms of categories was considered but rejected (cf. P. Cartier in [PC], p. 26 and in [PP]).

4. *The reader in* Y + 2K . Think of S, R as the Rosetta Stone.

How to avoid reading Book I. The concept is not heresy. In [PP], Dieudonné describes Book I as an insupportable burden on the reader, that no one reads, and with cause; while Serre and Schwartz echo the sentiment, and suggest that the Summary of Results would have sufficed.

An alternate strategy is to read the set theory part (the first 66 pages) of Irving Kaplansky's *Set theory and metric spaces* (Allyn & Bacon, Boston, 1972; 2nd edn., Chelsea, New York, 1977), which includes a thorough treatment of cardinal and ordinal numbers, and rely thereafter on S, R.

The particular relevance of the book is that it grew out of a course at the University of Chicago ("Theory of sets", Math. 261) in the early 1950's of the 'Stone Age'; with Weil and supportive colleagues (Saunders MacLane, Marshall Stone, ...) on the faculty, and Dieudonné on the faculty of nearby Northwestern University, Chicago could fairly be called a hotbed of Bourbakism, earning its half (along with the University of Nancy) of the mythical 'University of Nancago'.

Book II: Algebra

Relatively few of the results in A are needed in Books III–VI, the lion's share being targeted for Algebra itself; for example, Galois theory, rings with minimum condition and the deeper results on multilinear algebra, while not needed in other books of the "core", impose themselves.

Beyond generalities on groups, rings, modules and vector spaces, as found in Chs. I and II of A, the demands of INT on A include the following:

The 'decomposition theorem' for lattice-ordered abelian groups (A, VI, §1, No. 10, Th. 1) is cited in the chapter on Riesz spaces (INT, II, §1, No. 1).

Some intricate tensor product identifications (A, II, §7, No. 7) are applied to function spaces (INT, III, §1, No. 2, Prop. 5), and in INT, III, §4, No. 2, the product of two measures (*loc. cit.*, No. 1, Def. 1) is shown to be an extension of their algebraic tensor product as linear forms (A, II, §3, No. 2).

In INT, VII, §3, No. 3 the 'classical groups' (general linear, affine, unimodular, triangular) provide examples for calculating Haar measure (cf. A, II, §10, No. 7; A, II, §9, No. 4; A, III, §8, No. 9; A, III, §8, No. 6, formula (31)).

The structure theorem for finitely generated abelian groups (A, VII, §4, No. 7, Th. 3) is cited in a structure theorem on connected topological groups (INT, VII, §3, No. 2, Prop. 5).

Book III: General topology

TG is largely driven by the needs of EVT and INT (and, to a lesser degree, the needs of FVR). When I started to read *Intégration*, I had read

Chs. I, II and X of TG; the citation of TG is so extensive that, by the end, I had found it 'time-efficient' to read TG from cover-to-cover (excluding the exercises).

It is important to note that GT *is not a translation of* TG; Volumes I and II of GT (both published in 1966) were translated from the most recent editions of the fascicles of *Topologie générale* available at the time, whereas the corresponding volumes of TG were not published until 1971 and 1974, respectively.

In particular, Chapter IX of TG is at least 2 editions later than Chapter IX of GT, incorporating changes that are critical for Chapter IX of INT; namely:

(i) in GT, Souslin spaces and Lusin spaces are required to be *metrizable*, whereas in TG they need only be *Hausdorff* (Ch. IX, §6, Nos. 2 and 4);

(ii) to Ch. IX, §6, TG adds a new No. 8 (on the Souslin graph theorem), so that the former Nos. 8, 9 become Nos. 9, 10;

(iii) to Ch. IX, TG also adds a new appendix (on Lindelöf spaces);

(iv) the definition of a 'capacity' in GT (Ch. IX, §6, No. 9, Def. 8) is changed in TG (Ch. IX, §6, No. 10, Def. 9), leading to a generalization of GT, *loc. cit.*, Th. 5 to TG, *loc. cit.*, Th. 6.

I have not systematically compared GT and TG, but I can report that to the four subsections of GT, Ch. VIII, §4, TG adds *No. 5. Continuité des racines d'un polynôme*.

Book IV: Functions of a real variable

The demands of INT on FRV are minimal: convex functions (Ch. I, §4), regulated functions (Ch. II, §§ 1, 2), the logarithm function (Ch. III, §1) and the gamma function (Ch. VII, §1). Explicitly:

A characterization of convexity for a twice-differentiable numerical function (FRV, I, §4, No. 4, Cor. of Prop. 8) is cited in proving Hölder's inequality (INT, I, No. 2, Prop. 2), with some help from the Hahn-Banach theorem (TVS, II, §5).

The natural logarithm ln (FRV, III, §1, No. 1) figures in a proposition (INT, I, No. 3, Prop. 5) destined for application in the study of the set of values of p for which a given measurable function belongs to $\mathscr{L}^p$ (INT, IV, §6, No. 5, Prop. 4).

The theory of regulated functions (FRV, II, §1) assures that when 'measure' is defined, 'Lebesgue measure' immediately qualifies (INT, III, §1, No. 3, Example II).

The gamma function (FRV, Ch. VII) figures in the formula for the measure of the unit ball in Euclidean space (INT, V, §8, No. 7) and in the calculation of Gaussian integrals (INT, IX, §6, No. 4).

Book V: Topological vector spaces

1. The destinies of Books V and VI are intertwined, as a 'measure' is defined in Ch. III of INT as a continuous linear form on a suitable topological vector space. Citations of TVS in INT range from the definition of a pointed convex cone (TVS, II, §2, No. 4) on page I.1 of INT, to the self-duality of a real Hilbert space (TVS, V, §1, No. 7, Th. 3) on page IX.97 of INT.

Before undertaking INT, it is advisable to read Chapters I and II of TVS, then study the more specialized topics in Chapters III–V as the need arises, and count on ending up having read TVS from cover-to-cover.

2. Assigned a major role in the definition of a 'measure' on a locally compact space is the concept of 'locally convex direct limit' (or 'inductive limit') of locally convex spaces (the term will be explained below), cited on page III.2 of INT. The purpose of the following remarks is to help the reader navigate—and even circumnavigate—this thorny topic.

To set the stage, let F be a topological vector space (over $\mathbf{C}$ or over $\mathbf{R}$), let E be a vector space over the same field, and let $u : \mathrm{F} \to \mathrm{E}$ be a linear mapping of F into E. One is interested in compatible topologies on E for which u is continuous. The trivial, or coarsest topology ($\varnothing$ and E the only open sets) is such a topology (cf. GT, III, §1, No. 1, *Example* 1, and TVS, I, §1, No. 1, Def. 1). It can happen that there is no other. For example, if F is a topological vector space (say over $\mathbf{R}$) whose only continuous linear form is identically zero, and if u is a nonzero linear form on F, then the linear mapping $u : \mathrm{F} \to \mathbf{R}$ (with $\mathbf{R}$ regarded as a 1-dimensional vector space over $\mathbf{R}$) is rendered continuous only by the coarsest topology, as the only other candidate—the usual topology—is ruled out by the assumption on F.

{For, suppose $\mathbf{R}$ has a compatible topology other than the coarsest topology. Let A be a closed subset other than $\varnothing$ and $\mathbf{R}$, and let $x \in \complement \mathrm{A}$. Then $\{x\}$ is not dense in $\mathbf{R}$, therefore its translate $\{0\}$ is not dense in $\mathbf{R}$, whence $\{0\}$ is closed in $\mathbf{R}$ (TVS, I, §2, No. 1, Cor. of Prop. 1), and so $\mathbf{R}$ is Hausdorff for the topology (GT, III, §1, No. 2, Prop. 2), consequently this topology on $\mathbf{R}$ is the usual one (TVS, I, §2, No. 2, Prop. 2).}

An example of such a space F is the classical space denoted (S) by Banach, derived from the vector space of Lebesgue-measurable functions $f : [0,1] \to \mathbf{R}$ and the pseudo-metric $(f,g) \mapsto \int \frac{|f-g|}{1+|f-g|}$ $\big(\int$ the Lebesgue integral$\big)$. {The details are thrashed out in [FA], p. 63, Theorem 15.10. See also Exercise 4 of TVS, Ch. I, §2, which avoids measure theory but entails modifying the absolute value function of the coefficient field $\mathbf{R}$.}

To get to the point: when we are assured that there exists a finest locally convex topology rendering u continuous (TVS, II, §4, No. 4, Prop. 5), we take the message with a grain of salt: it may be the coarsest topology (cf. [JH], p. 157, first paragraph) .

In preparation for the application in the definition of a measure in Ch. III of INT, if (F_α) is any family of topological vector spaces (over **C** or over **R**), E is any vector space over the same field, and $g_\alpha : F_\alpha \to E$ is any family of linear mappings, there exists a finest locally convex topology $\mathfrak{T}$ on E that renders continuous every g_α (TVS, *loc. cit.*, Prop. 5). It is called the *locally convex final topology* for the family of linear mappings (g_α). The essence of the proof is a clever description of a fundamental system of neighborhoods of 0 for such a topology $\mathfrak{T}$ (cf. the next-to-last paragraph of TVS, II, §8, No. 2 for the relation between the real and complex cases).

CAUTION: There may be a compatible topology strictly finer than $\mathfrak{T}$ that renders the g_α continuous (TVS, II, §4, Exer. 15)—but not a locally convex one.

In particular, if $(E_\alpha, f_{\beta\alpha})$ is a direct system (or 'inductive system') of locally convex spaces (indexed by an ordered set directed to the right), where, for $\alpha \leqslant \beta$, $f_{\beta\alpha} : E_\alpha \to E_\beta$ is a continuous linear mapping, $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$ when $\alpha \leqslant \beta \leqslant \gamma$, and $f_{\alpha\alpha}$ is the identity mapping on E_α for all α, and if $E = \varinjlim E_\alpha$ is the vector space direct limit of the system (A, Ch. II, §6, No. 2), the locally convex final topology for the family of canonical mappings $E_\alpha \to E$ is called the *locally convex direct limit* of the family (TVS, II, §4, No. 4, Example II). With the above CAUTION in mind, it is good terminological hygiene to not omit the 'locally convex' when using this term. Note that when all the $f_{\beta\alpha}$ are injective, then so are the f_α (S, R, §6, No. 13).

3. There is a situation in which the locally convex final topology for a family of mappings is assured of being nontrivial. Suppose (F_α) is a family of locally convex topological vector spaces, E is a vector space, and, for every index α, $g_\alpha : F_\alpha \to E$ is a linear mapping. Suppose, in addition, that there already exists a nontrivial (i.e., not the coarsest topology) locally convex topology $\mathfrak{S}$ on E such that every g_α is continuous, that is, $\overset{-1}{g}_\alpha(\mathfrak{S})$ is contained in the given topology $\mathfrak{S}_\alpha$ on F_α. Then the finest locally convex topology $\mathfrak{T}$ on E that renders continuous the g_α is finer than $\mathfrak{S}$, hence is also nontrivial. Moreover, if $\mathfrak{S}$ is Hausdorff, then so is the finer topology $\mathfrak{T}$. The property that characterizes $\mathfrak{T}$ uniquely is the following: *A linear mapping* $f : E \to G$ *of* E *into a locally convex space* G *is continuous if and only if the composition* $f \circ g_\alpha : F_\alpha \to G$ *is continuous for every* α (TVS, II, §4, No. 4, Prop. 5).

4. Continuing in the foregoing situation, the continuity of g_α for $\mathfrak{T}$ means that $\overset{-1}{g}_\alpha(\mathfrak{T}) \subset \mathfrak{S}_\alpha$. If, moreover, $\mathfrak{S}_\alpha = \overset{-1}{g}_\alpha(\mathfrak{S})$, that is, $\mathfrak{S}_\alpha$ is the

initial topology for g_α (GT, I, §2, No. 3, Example 1), then $\overset{-1}{g}_\alpha(\mathfrak{S}) = \mathfrak{S}_\alpha \supset \overset{-1}{g}_\alpha(\mathfrak{T})$; but the reverse inclusion results from $\mathfrak{S} \subset \mathfrak{T}$, and so

$$(*) \qquad \overset{-1}{g}_\alpha(\mathfrak{T}) = \overset{-1}{g}_\alpha(\mathfrak{S}) = \mathfrak{S}_\alpha \quad \text{for all } \alpha .$$

Thus $\mathfrak{S}_\alpha$ is also the initial topology for g_α when E is equipped with $\mathfrak{T}$.

5. A special case of importance for INT: If E is a locally convex topological vector space, with topology $\mathfrak{S}$, and if (F_α) is a family of linear subspaces of E, each equipped with the topology $\mathfrak{S}_\alpha = \mathfrak{S} \cap \mathrm{F}_\alpha$ induced by $\mathfrak{S}$ (i.e., the initial topology for the canonical injection $g_\alpha : \mathrm{F}_\alpha \to \mathrm{E}$) (GT, I, §3, No. 1, Def. 1), then the locally convex final topology $\mathfrak{T}$ for the family satisfies

$$(*) \qquad \mathfrak{T} \cap \mathrm{F}_\alpha = \mathfrak{S} \cap \mathrm{F}_\alpha \qquad \text{for all } \alpha ,$$

that is, $\mathfrak{S}_\alpha$ is also the topology induced by $\mathfrak{T}$ on F_α. If now $f : \mathrm{E} \to \mathrm{G}$ is a linear mapping of E into a locally convex space G, then $f \circ g_\alpha = f \big| \mathrm{F}_\alpha$, therefore f is continuous with respect to $\mathfrak{T}$ if and only if its restriction to F_α is continuous with respect to $\mathfrak{S}_\alpha$ for every α; thus, one has a test for the continuity of f with respect to $\mathfrak{T}$ that can be applied without reference to $\mathfrak{T}$ itself.

Book VI: Integration

In the first 8 chapters, all 'measures' are defined for a locally compact space, whereas the 9'th and final chapter presents a theory of measure for Hausdorff spaces, with an eye on applications in probability theory.

Why are locally compact spaces a natural setting for the theory? The question is addressed in the *Introduction* to the Book. To the answers given there, one may add the following. The existence of an invariant measure on a locally compact topological group is a major preoccupation of the theory, and raises the question of whether local compactness is essential. It is shown in an appendix to Weil's monograph [W2] that a group with no topology, but with a left-invariant measure defined on a tribe (σ-algebra) closed under left-translation, satisfying two innocent-looking measure-theoretic conditions, can be regarded as a dense subgroup of a locally compact group.

INT is largely driven by the needs of *Théories spectrales*, perhaps the first 'direct descendant' of the "core". It would not be frivolous to regard the first 8 chapters of INT as a 'Lemma' to Chapter II of TS (*Groupes localement compacts commutatifs*). {The 9th chapter, published in 1969, is not cited in TS (published in 1967), hence can be omitted by the reader headed for abstract harmonic analysis.}

Publisher and translator each made an embarrassing mistake, in Volumes I and II, respectively:

The blank page after the Ch. I *Historical Note* and the resulting incorrect running heads was the result of a mechanical production error—the printout was correct but the physical pages were incorrectly assembled. A reprinting should remedy the error.

The translator's blunder was in the referencing in Chapter IX, where some of the references to GT should be changed to TG; more about this in the notes for Chapter IX below.

Chapter I: *Inequalities of convexity*

The inequality proved in No. 1, Prop. 1 serves as the basis for the Hölder and Minkowski inequalities in No. 2, as well as the semi-norms N_p $(1 \leqslant p < +\infty)$ that are the work-horses of the development of the theory of integration generated by a 'measure' (the term is defined in Ch. III). Its proof begins with a citation of the Hahn-Banach theorem (TVS, II, §5).

Chapter II: *Riesz spaces*

A *Riesz space* (also known as a *vector lattice*) is a real vector space whose additive group is a lattice-ordered group (A, VI, §1, No. 9) in which a positive scalar multiple of a positive element is positive, so that the set of positive elements is a convex cone). A Riesz space is said to be *fully lattice-ordered* if every nonempty subset that is bounded above has a least upper bound.

The theory developed in Ch. II is the basis for showing that every 'real measure' on a locally compact space X is the difference of two positive measures, and that the vector space $\mathscr{M}(\mathrm{X};\mathbf{R})$ of real measures is a fully lattice-ordered Riesz space (Ch. III, §1, No. 5). This leads to the positive measure $|\mu|$ associated with a 'complex measure' μ (*loc. cit.*, No. 6), and eventually to the 'outer measure' $|\mu|^*$ from which flows the theory of integration with respect to μ (Ch. IV, §3, No. 4).

A linear subspace B of a fully lattice-ordered Riesz space E is called a *band* if (1) for every element x of B, the elements y of E such that $|y| \leqslant |x|$ also belong to B, and (2) for every nonempty subset C of B that is bounded above, the supremum of C in E also belongs to B. The concept resurfaces in the author's formulation of the Radon-Nikodym theorem (Ch. V, §5, No. 5, Cor. 2 of Th. 2).

Chapter III: *Measures on locally compact spaces*

The daunting topological preliminaries in §1, No. 1 of this chapter are sufficiently general to accommodate the definition of a measure with values in a locally convex topological vector space (Ch. VI, §2, No. 1, Def. 1), far more general than what is required for the definition of a complex measure.

A substantial simplification is available, that permits the reader primarily interested in complex measures to save a lot of time and effort, as follows.

A (complex) *measure* on the locally compact space X is defined to be a continuous linear form on the vector space $\mathscr{K}(\mathrm{X};\mathbf{C})$ of continuous complex-valued functions on X with compact support, equipped with an ingenious locally convex topology (described below). The proposed simplification is in the definition of that topology; the idea is to dispense with the theory of direct limits and arrive at the topology by a more direct path.

For each compact subset K of X, $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ denotes the linear subspace of $\mathscr{K}(\mathrm{X};\mathbf{C})$ formed by the functions $f \in \mathscr{K}(\mathrm{X};\mathbf{C})$ with support contained in K (i.e., such that $f = 0$ on $\mathrm{X} - \mathrm{K}$). Equipped with the norm $\|f\| = \sup_{x\in\mathrm{K}} |f(x)|$, $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is a Banach space (not to be confused with $\mathscr{C}(\mathrm{K};\mathbf{C})$). On the other hand, $\mathscr{K}(\mathrm{X};\mathbf{C})$ is itself a normed space for the sup-norm, which defines the topology $\mathfrak{T}_u$ of uniform convergence in X. The topology of $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ coincides with the topology induced by $\mathfrak{T}_u$, in other words, the initial topology for the canonical injection $\iota_{\mathrm{K}} : \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}) \to \mathscr{K}(\mathrm{X};\mathbf{C})$. Thus $\mathfrak{T}_u$ is a locally convex topology on $\mathscr{K}(\mathrm{X};\mathbf{C})$ that renders continuous every ι_{K}.

The topology on $\mathscr{K}(\mathrm{X};\mathbf{C})$ we are interested in is the *locally convex final topology* $\mathfrak{T}$ for the family of linear mappings $\iota_{\mathrm{K}} : \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}) \to \mathscr{K}(\mathrm{X};\mathbf{C})$ as described above in the remarks on Book V. Thus, $\mathfrak{T}$ is the finest locally convex topology on $\mathscr{K}(\mathrm{X};\mathbf{C})$ that renders every ι_{K} continuous. In particular, $\mathfrak{T} \supset \mathfrak{T}_u$, so $\mathfrak{T}$ is nontrivial (i.e., is not the coarsest topology); and since $\mathfrak{T}_u$ is Hausdorff, so is $\mathfrak{T}$. We know that $\mathfrak{T}$ is uniquely determined by the property that a linear mapping of $\mathscr{K}(\mathrm{X};\mathbf{C})$ into a locally convex space is continuous if and only if its composition with every ι_{K} is continuous. {The fact that $\mathfrak{T}$ can be regarded as the *locally convex direct limit topology* for $\mathscr{K}(\mathrm{X};\mathbf{C})$ regarded as the union of the increasing directed family of subspaces $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$, with the canonical mappings $\iota_{\mathrm{LK}} : \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}) \to \mathscr{K}(\mathrm{X},\mathrm{L};\mathbf{C})$ for $\mathrm{K} \subset \mathrm{L}$, is interesting but inessential for the definition of measure.}

A closed set in a topological space remains closed for a finer topology; in particular, every $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is closed for $\mathfrak{T}$. Since ι_{K} is continuous for $\mathfrak{T}$, $\overset{-1}{\iota}_{\mathrm{K}}(\mathfrak{T}) \subset \mathfrak{T}_u \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ (the topology induced by $\mathfrak{T}_u$ on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$), that is, $\mathfrak{T} \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}) \subset \mathfrak{T}_u \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$; the reverse inclusion follows from $\mathfrak{T} \supset \mathfrak{T}_u$, thus

$$\mathfrak{T} \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}) = \mathfrak{T}_u \cap \mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C}),$$

so that $\mathfrak{T}$ also induces on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ its original topology.

To summarize: A linear form $\mu : \mathscr{K}(\mathrm{X};\mathbf{C}) \to \mathbf{C}$ is called a *measure* (or *complex measure*) if it is continuous for the topology $\mathfrak{T}$ on $\mathscr{K}(\mathrm{X};\mathbf{C})$.

Thus the set $\mathscr{M}(\mathrm{X};\mathbf{C})$ of measures on X is the dual of the space $\mathscr{K}(\mathrm{X};\mathbf{C})$ equipped with the locally convex topology $\mathfrak{T}$, bringing to bear the resources of TVS to the theory of integration. {Extensive use is made of several topologies on $\mathscr{M}(\mathrm{X};\mathbf{C})$ (cf. Ch. III, §1, No. 10; §3, No. 4, Prop. 9; §4, No. 3; Ch. V, §3); for a critique of the author's treatment from the perspective of category theory, see [CH], [FK].}

What has the 'shortcut' saved? The raw materials for the direct limit are the family $\big(\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})\big)_{\mathrm{K}\in\mathfrak{K}}$ of linear subspaces of $\mathscr{K}(\mathrm{X};\mathbf{C})$, indexed by the set $\mathfrak{K}$ of all compact subsets of X, directed to the right by the inclusion relation $\subset$, along with the canonical injections $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})\to\mathscr{K}(\mathrm{X},\mathrm{L};\mathbf{C})$ when $\mathrm{K}\subset\mathrm{L}$. We have dispensed with the need to

(i) construct the vector space direct limit $\varinjlim_{\mathrm{K}\in\mathfrak{K}}\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$, already provided by $\mathscr{K}(\mathrm{X};\mathbf{C})$;

(ii) construct the associated canonical injections, already provided by the inclusion mappings $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})\to\mathscr{K}(\mathrm{X};\mathbf{C})$;

(iii) equip $\mathscr{K}(\mathrm{X};\mathbf{C})$ with a locally convex topology to assure the non-triviality of the locally convex direct limit topology, a role already played by the topology $\mathfrak{T}_u$.

Moreover, the fact that the norm topology on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is induced by $\mathfrak{T}_u$ leads to simple proofs that it is also induced by $\mathfrak{T}$, and that $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is also closed in $\mathscr{K}(\mathrm{X};\mathbf{C})$ for $\mathfrak{T}$. Since $\mathfrak{T}$ is a locally convex final topology, a linear mapping $g:\mathscr{K}(\mathrm{X};\mathbf{C})\to\mathrm{G}$ of $\mathscr{K}(\mathrm{X};\mathbf{C})$ into a locally convex space G is continuous for $\mathfrak{T}$ if and only if $g\circ\iota_{\mathrm{K}}$ is continuous for every K. To say that $g\circ\iota_{\mathrm{K}}$ is continuous means that $g\big|\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is continuous for the topology on $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ induced by that of $\mathscr{K}(\mathrm{X};\mathbf{C})$; this induced topology is the same whether it is induced by $\mathfrak{T}$ or by $\mathfrak{T}_u$, namely, it is the topology of uniform convergence. Thus g is continuous for $\mathfrak{T}$ if and only if $g\big|\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is continuous for the sup-norm topology. It follows that a linear form $\mu:\mathscr{K}(\mathrm{X};\mathbf{C})\to\mathbf{C}$ is a measure if and only if its restriction to $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{C})$ is continuous for the norm topology for every K.

That is, *the definition of a measure need not refer to direct limits at all.* This was, indeed, the procedure in the first edition of Chs. I–IV (published in 1952). The possibility of characterizing measures as linear forms continuous with respect to a suitable locally convex topology was noted (for real-valued functions) in Ch. III, §2, No. 2, and sketched in an Exercise. But a (real) measure on a locally compact space X was officially defined to be a linear form on the vector space $\mathscr{K}(\mathrm{X};\mathbf{R})$ of continuous real-valued functions on X with compact support, such that for every compact subset K of X the restriction of the form to the linear subspace $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{R})$ is continuous when $\mathscr{K}(\mathrm{X},\mathrm{K};\mathbf{R})$ is equipped with the topology of uniform convergence. That such measures can be regarded as the continuous linear forms on $\mathscr{K}(\mathrm{X};\mathbf{R})$

equipped with the locally convex direct limit topology is recognized in the first edition (1953) of *Esp. vect. top.* (Ch. II, §2, No. 4, *Exemple* 3), which cites the above-mentioned Exercise, and it is this reference that is cited in the Introduction to the first edition (1959) of Ch. VI of *Intégr.*, where, finally, a real measure on X is recognized as a continuous linear form on the real locally convex space $\mathscr{K}(\mathrm{X};\mathbf{R})$ (see p. VI.1 of INT, where the references have been up-dated). Before the second edition of Chs. I–IV of *Intégr.* was published (1965), one had to wait until Ch. VI, §2, No. 8 of *Intégr.* for a definition of a complex measure m—analyzed in terms of the vector-valued measure $m\big|\mathscr{K}(\mathrm{X};\mathbf{R})$ $\big($not to be confused with the 'real part' $\frac{1}{2}(m+\overline{m})$ of $m\big)$ with values in $\mathbf{C}$ regarded as a 2-dimensional Banach space over $\mathbf{R}$.

Chevalley records that the choice of definition between measure as a linear form (favored by Weil) and measure as a set function (favored by De Possel) was long and vigorously debated before settling on linear forms ([DG], p. 20, col. 2; [PP], Part 1). The principle of unanimous decisions must have been severely tested.

Chapter IV: *Extension of a measure.* L^p *spaces.*

The second edition of the French fascicle for Chs. I–IV, on which the translation is based, was reviewed by W. A. J. Luxemburg (MR 36#2763); the first edition was reviewed thoroughly by Leopoldo Nachbin (MR 14,960h).

In the 2nd edition, the theory of extremal points of compact convex sets in Hausdorff locally convex spaces is substantially expanded and sharpened (Ch. IV, §7), featuring work of Gustave Choquet (Choquet boundary) and Errett Bishop (peak points of function algebras).

In the 1st edition, the topic of measure as a linear form and as a set function, in the context of a set without topology, is treated 'in parallel' as the last subsection (*mesures abstraites*) of §§1–5 of Ch. IV. In the 2nd edition, measure in the context of a set without topology is abandoned, and measure as a set function is concentrated in Nos. 9–11 of Ch. IV, §4 and is restricted to locally compact spaces, culminating in a theorem relating numerical measures in the sense of Bourbaki with regular Borel measures on locally compact spaces in the sense of Halmos (INT, IV, §4, No. 11, Cor. of Th. 5; [H1], Sec. 56, Ths. D and E, pp. 247–248).

Chapter V: *Integration of measures*

The 2nd edition of the fascicle for this chapter, on which the translation is based, was reviewed by Bertram Walsh (MR 35#322), the first edition having been reviewed by J. C. Oxtoby (MR 18,881c).

From the wealth of ideas in this chapter, the treatment of two familiar theorems gives some insight into the author's approach.

1. *Fubini's theorem on multiple integrals.* Each point t of a locally compact space T defines the 'point measure' $\varepsilon_t : f \to f(t)$ $\big(f \in \mathscr{K}(\mathrm{T};\mathbf{R})\big)$.

If μ is a positive measure on T, the notation

$$\mu(f) = \int f(t)\, d\mu(t) = \int \varepsilon_t(f)\, d\mu(t)$$

invites the notation $\mu = \int \varepsilon_t\, d\mu(t)$. The 'integration of measures' alluded to in the chapter title is a theory of integration, with respect to μ, of positive-measure-valued functions $\mathrm{T} \to \mathscr{M}_+(\mathrm{X}; \mathbf{R})$, where X is another locally compact space, giving meaning to notations such as $\int \lambda_t\, d\mu(t)$ for functions $t \mapsto \lambda_t \in \mathscr{M}_+(\mathrm{X}; \mathbf{R})$.

Among the numerous applications is Fubini's theorem expressing an integral with respect to a product measure $\mu \otimes \mu'$ (μ, μ' measures on T, T') as an iterated integral (Ch. V, §8, No. 4, Th. 1); the idea is to regard $\mu \otimes \mu'$ as the integral $\int \lambda_t'\, d\mu(t)$ of the function $t \mapsto \lambda_t' = \varepsilon_t \otimes \mu'$.

A special class of functions $\Lambda : t \mapsto \lambda_t$, called 'diffusions', is singled out for development in §3, Nos. 5, 6; an impressive calculus for them is worked out, including a treatment of the composition of diffusions, for which an associative law is proved. From the text, one example and two exercises devoted to diffusions, I drew no insights into the applications for which they are destined, but the reader who knows will appreciate the exposition.

2. *Radon-Nikodym theorem.* In the author's formulation (Ch. V, §5, No. 5, Th. 2), if λ is a positive measure on X, the following conditions on a measure $\mu \in \mathscr{M}(\mathrm{X}; \mathbf{R})$ are equivalent:

a) every 'locally λ-negligible set is locally μ-negligible';

b) $\mu = f \cdot \lambda$ for some 'locally λ-integrable' function f (*loc. cit.*, No. 2, Def. 2);

c) μ belongs to the band generated by λ in the Riesz space $\mathscr{M}(\mathrm{X}; \mathbf{R})$ (*loc. cit.*, No. 5, Cor. 2 of Th. 2).

{In the terminology of [H1], the measures μ in the band generated by λ are the 'signed measures' that are 'absolutely continuous' with respect to λ ([H1], p. 97, Th. H and pp. 128–129, Th. B).}

A consequence of the foregoing is the possibility of defining a measure $f(\mu_1, \ldots, \mu_n)$ for real measures $\mu_1, \ldots, \mu_n$ and certain functions $f : \mathbf{R}^n \to \mathbf{R}$, leading to a startling formula: if θ is a complex measure, and if μ and ν are the real measures such that $\theta = \mu + i\nu$ (Ch. III, §1, No. 5), then $|\theta| = \sqrt{\mu^2 + \nu^2}$ (Ch. V, §5, No. 9; here, the right member means $f(\mu, \nu)$ for the function $f(s,t) = \sqrt{s^2 + t^2}$ on $\mathbf{R}^2$).

Chapter VI: *Vectorial integration*

The original French fascicle for this chapter was reviewed by Nicolae Dinculeanu (MR 23 #A2033). Published in 1959, it is 'unaware' of the second editions of Chs. I–IV and V. (The same is true of Chs. VII and VIII, but the revisions are less significant for them than for Ch. VI.)

Treated in this chapter are vector-valued integrals, resulting from :

(i) a vector-valued function and a numerical (real or complex) measure (§1, No. 1, Def. 1),

(ii) a vector-valued measure and a numerical function (§2, No. 2, Def. 2), and

(iii) a function with values in a Banach space F, a measure with values in a Banach space G, and a continuous bilinear mapping of $F \times G$ into H, leading to an integral in H by a procedure too complex to digest in a sentence (§2, No. 7, Prop. 11).

A key concept for Ch. VI is that of a *μ-adequate* mapping $t \mapsto \lambda_t \in \mathscr{M}_+(X)$ $(t \in T)$, where X and T are locally compact spaces, a term that was introduced in the 1st edition of Ch. V. In the 2nd edition of Ch. V, the term was redefined to be a weaker (more general) concept (INT, V, §3, No. 1, Def. 1), putting in question the validity of assertions involving it in Ch. VI. Not to worry. Anticipating this problem, the definition in the 2nd edition of Ch. V is followed by a theorem that explains the exact relation between the two definitions (INT, *loc. cit.*, Prop. 2b) and shows that they are equivalent when X has a countable base for open sets (INT, *loc. cit.*, Prop. 2c), thereby making it possible for assertions in the text of Ch. VI to remain valid for the new definition, as explained in the footnote on page VI.2. (But the status of the Exercises will have to be decided on a case-by-case basis; fortunately, proofs in the text are not permitted to depend on the exercises.)

The reader should also be alerted to two changes in No. 4 of §3: the definition of *μ-measurable equivalence relation* (*loc. cit.*, Def. 3), and the next-to-last sentence in the paragraph that precedes the definition, have been revised.

Chapter VII: *Haar measure*

Chapter VIII: *Convolution and representations*

The reviews by Edwin Hewitt of the original French fascicle for Chs. VII and VIII (MR 31# 3539), and of the fascicle for Ch. IX are models of technical mastery and expository skill.

Chapters VII and VIII are primarily devoted to general (not necessarily abelian) locally compact groups, the abelian case being the subject of Ch. II of TS; I found especially rewarding the calculation of Haar measures in examples of classical groups (INT, VII, §3, No. 3).

Chapter IX: *Measures on Hausdorff topological spaces*

The original French fascicle for Ch. IX was reviewed by Edwin Hewitt (MR 43#2183).

The chapter depends heavily on the 3rd edition of Ch. IX of *Top. gén.*, to which I did not have access; in the translation, I juggled references between GT (most convenient for readers) and TG (the gold standard) ... and I made some mistakes.

The differences between GT and TG signaled in items (i)–(iv) of the notes for Book III resulted in some incorrect translations of references (but the surrounding text is correct). I was well aware of the differences when I studied TG in 1975, but when I translated Ch. IX of INT in 2002, I seem to have forgotten item (i), i.e., that in GT, Souslin and Lusin spaces were required to be metrizable, rendering the results there inapplicable in the Hausdorff environment of Ch. IX of INT.

{In hindsight, I can infer that the above-mentioned 3rd edition was aware of items (i), (iii) and (iv), but not of (ii) and its effect on the numbering of subsequent subsections and theorems.}

The needed repairs to references and footnotes are as follows:

IX.10. In the footnote (1), change the reference to TG, IX, §6, No. 10, Th. 6.

IX.31, ℓ. 9. Change GT to TG.

IX.63, ℓ. $-3, -2$. Change the reference to GT, *loc. cit.*, Cor. 1 of Th. 1.

IX.64. ℓ. 14. Change GT to TG.

IX.64. In $\ell\ell$. 18 and 19, change the (repeated) reference to TG, *loc. cit.*, No. 4, Prop. 11.

IX.93, ℓ. 6. In the reference to TVS, replace "No. 2, Th. 1" by "No. 1, *Scholium*".

IX.97. In the reference to TVS in line 4 of the *Remark*, replace "No. 2, Th. 1" by "No. 1, *Scholium*".

IX.18. Revise footnote 1 as follows:

[1] The cited appendix on Lindelöf spaces does not appear in GT. Lindelöf spaces are defined in GT in Ch. I, §9, Exer. 14. Souslin spaces (and Lusin spaces) are defined in TG for Hausdorff spaces (TG, IX, §6, No. 2, Def. 2 and No. 4, Def. 7); in GT they are required to be metrizable (GT, IX, §6, No. 2, Def. 2 and No. 4, Def. 6).

IX.31. Revise footnote (1) to the following:

(1) A capacity f on T is said to be right-continuous if, for every compact set K in T, $f(\mathrm{K}) = \inf_{\mathrm{U}} f(\mathrm{U})$ as U runs over the open sets $\mathrm{U} \supset \mathrm{K}$. In GT, a "capacity" is defined by three axioms (GT, IX, §6, No. 9, Def. 8). In TG, a function satisfying only the first two is called a capacity, but a right-continuous capacity also satisfies the third (TG, *loc. cit.*, *Remarque*).

IX.40. Revise footnote (2) as follows:

(2) Cf. the footnote to *Remark* 1 of §1, No. 9.

IX.48. Revise footnote (2) to the following:

(2) In GT, every Souslin space has a countable base for open sets (GT, IX, §6, No. 2, Prop. 4), hence is Lindelöf (GT, I, §9, Exer. 14); but see the footnote on p. IX.18.

IX.48. In the line following the first display in the proof of Prop. 3, change the reference to TG, IX, §6, No. 10 and attach to it the following footnote (3):

(3) See footnote (1) on p. IX.31.

IX.49. Delete footnote (4) and revise footnote (3) to be the following new footnote (4):

(4) In GT, every Borel set in a Souslin space is a Souslin set (GT, IX, §6, No. 3, Prop. 11); but see the footnotes on pp. IX.18 and IX.31.

IX.64. Add the following footnote (1) to the statement of Cor. 2:

(1) Cf. the footnote on p. IX.18.

IX.87. Revise footnote (2) as follows:

(2) The term *espace mesuré* was used in the first edition of Ch. III (§2, No. 2, p. 52) for a space (locally compact, there) equipped with a measure.

The uneven history of the bound editions and translations.

The relation between the French and English versions is complicated. Books II, IV and V were translated from the definitive 'bound edition' (*édition reliée*). Books I and III were translated from fascicles of the chapters available prior to the publication of the French 'bound edition', hence do not reflect the author's 'last word'. Book VI never received an *édition reliée*; it was translated from the most recent fascicles of the constituent chapters, of which Chs. 1–5 received 2nd editions and Chs. 6–9 did not. Here they are, with dates of publication:

French	Date	English	Date
E	1970	S	1968
A–I	1970	A–I	1974
A–II	1981	A–II	1990
TG–I	1971	GT–I	1966
TG–II	1974	GT–II	1966
FVR	1976	FRV	2004
EVT	1981	TVS	1987
Intégration	1965 (Chs. 1–4)	INT–I	2004
	1967 (Ch. 5)		
	1959 (Ch. 6)		
	1963 (Chs. 7, 8)	INT–II	2004
	1969 (Ch. 9)		

Mathematical Reviews' treatment of 'bound editions' has been equally erratic. No translation volume has ever received more than an "editors' review" consisting of references to reviews of the French fascicles that preceded the French *édition reliée*. Of the French *éditions reliées*, only A–I, A–II and EVT received actual reviews. The situation in detail:

(S) The book was translated from fascicles and published 2 years before E; it received an editors' review (MR 38#5631) listing the fascicle reviews. When E was published, it received an editors' review (MR 43#1849) citing the 'review' of S. The translator is not identified.

(A–I) The French *édition reliée* was reviewed by Pierre Samuel (MR 43#2), presumably one of its authors. Apparently Bourbaki's rule of secrecy was still staunch, as self-reviews were contrary to MR policy, but we are grateful for the reviewer's privileged insight into the nature of the revisions incorporated in the edition. The translation, published 4 years later, received the predictable editors' review (MR 50#6689). The translator is not identified.

(A–II) The French *édition reliée* was reviewed by Robert Gilmer (MR 84d:00002). The translation (by P. M. Cohn and J. Howie) was published 9 years later; it is listed in MR on-line as MR1994218 with the comment "There will be no review of this item." As I do not have access to a printed version of MR, I can only guess from the 'unformatted' MR number that the listing was never printed.

(GT–I) Translated from French fascicles, its review (MR 34#5044a) identifies the edition numbers of the fascicles and gives their review numbers; TG–I, published 5 years later, received an editors' review (MR 50#11111) citing the 'review' of GT–I.

(GT–II) Translated from French fascicles, its review (MR 34#5044b) identifies the edition numbers of the fascicles and gives their review numbers. TG–II, published 8 years later, seems not to be listed at all in MR; this is regrettable, as there were changes to Chapter IX that were critical for Ch. IX of *Intégration*, rendering GT–II an inadequate reference (see the comments on Book VI, Chapter IX above).

The translations GT are excellent, the translator (unidentified) occasionally declining the author's terminology so as to conform with established usage in English (e.g., translating *espace séparé* as Hausdorff space, and *espace dénombrable à l'infini* as σ-compact).

(FRV) The first edition of *Fonctions d'une variable réelle* was published in two fascicles, Chs. I–III and Chs. IV–VII, in 1949 and 1951, respectively, and duly reviewed (MR 11, 86h and MR 13, 631a). A second edition of the first fascicle received an editor's review (MR 27#1340), referring the reader to the review of the first edition. When the *édition reliée* was published in 1976, it received an editors' review (MR 58 #28327), "There will be no review of this item", with no reference to the fascicle reviews. Presumably FVR corrected the errata (more than 75, a good many of them significant) accumulated in published lists (*Feuilles d'errata*). The English translation (by Philip Spain), published in 2004, is listed on-line as MR2013000, with the statement "There will be no review of this item"; a missed opportunity to signal a half-century of evolution of Book IV.

(TVS) The review of EVT by Eberhard Gerlach (MR 83k:46003) lists the table of contents of the 5 chapters by their section (§) titles. The editors' review of the translation by H. G. Eggleston and S. Madan (MR 88g:46002) adds no further information.

The first edition consisted of two fascicles, for Chs. I, II and Chs. III–V. They received detailed reviews by J. L. Kelley (MR 14,880b and MR 17,1109e); the heading of the on-line version of the latter review appears to be corrupted, labeling it as the review of the 39-page *Fascicule de résultats*.

The first fascicle received a thoroughly revised and expanded 2nd edition; its editors' review consists of the author's one-sentence Preface (MR 34#3277). Chs. I, II of EVT appear to be essentially the same as their 2nd edition (but the Appendix on fixed-point theorems at the end of Ch. II has been expanded and moved to the end of Ch. IV).

(INT) There was no *édition reliée* of *Intégration*. The editors' reviews of the two volumes (MR 2004i:28001 and MR 2005f:28001) consist of a listing of the reviews of the French fascicles from which the translation was made.

Although the 2nd editions of Chs. I–V created some terminology inconsistencies with Ch. VI, a result in Chapter V (§3, No. 1, Prop. 2) makes it possible for the chapters to co-exist without having to re-write Ch. VI, as explained in a footnote on page VI.2 (and in the notes above for Chapter VI).

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