TRANSITION MAP AND SHADOWING LEMMA FOR NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

AMADEU DELSHAMS† , MARIAN GIDEA‡ AND PABLO ROLDÁN †

Abstract. For a given a normally hyperbolic invariant manifold, whose stable and unstable manifolds intersect transversally, we consider several tools and techniques to detect orbits with prescribed trajectories: the scattering map, the transition map, the method of correctly aligned windows, and the shadowing lemma. We provide an user's guide on how to apply these tools and techniques to detect unstable orbits in Hamiltonian systems.

1. Introduction

Consider a normally hyperbolic invariant manifold for a flow or a map, and assume that the stable and unstable manifolds of the normally hyperbolic invariant manifold have a transverse intersection along a homoclinic manifold. One can distinguish an inner dynamics, associated to the restriction of the flow or of the map to the normally hyperbolic invariant manifold, and an outer dynamics, associated to the homoclinic orbits. There exist pseudo-orbits obtained by alternatively following the inner dynamics and the outer dynamics for some finite periods of time. An important question in dynamics is whether there exist true orbits with similar behavior. In this paper, we develop a toolkit of instruments and techniques to detect true orbits near a normally hyperbolic invariant manifold, that alternatively follow the inner dynamics and the outer dynamics, for all time. Some of the tools discussed below have already been used in other works. The aim of this paper is to provide a general recipe on how to make a systematic use of these tools in general situations.

The first tool is the scattering map, introduced in [8], and further investigated in [10]. The scattering map is defined on the normally hyperbolic invariant manifold and it assigns to the foot of an unstable fiber passing through a point in the homoclinic manifold, the foot of the corresponding stable fiber that passes through the same point in the homoclinic manifold. The scattering map can be defined both in the flow case and in the map case. In Section 3 we describe the relationship between the scattering map for a flow and the scattering map for the return map to a surface of section. We note that the scattering map is defined in terms of the geometric structure, however it is not dynamically defined – there is no actual orbit that is given by the scattering map.

The second tool that we discuss is the transition map, that actually follows the homoclinic orbits for a prescribed time. The transition map can be computed in

---

1991 *Mathematics Subject Classification.* Primary, 37J40; 37C50; 37C29; Secondary, 37B30.

† Research of A.D. and P.R. was partially supported by MICINN-FEDER grant MTM2009-06973 and CUR-DIUE grant 2009SGR859.

‡Research of M.G. was partially supported by NSF grant: DMS 0601016.

terms of the scattering map. Again, we will have a transition map for the flow and one for the return map, and we will describe the relationships between them. The transition map is presented in Section 4.

The third tool is the topological method of correctly aligned windows (see [22]), which is used to detect orbits with prescribed itineraries in a dynamical system. A window is a homeomorphic copy of a multi-dimensional rectangle, with a choice of an exit direction and of an entry direction. A window is correctly aligned with another window if the image of the first window crosses the second window all the way through and across its exit set. This method is reviewed briefly in Section 5.

The fourth tool is a shadowing lemma type of result for a normally hyperbolic invariant manifold, presented in Section 6. The assumption is that a bi-infinite sequence of windows lying in the normally hyperbolic invariant manifold is given, with the consecutive pairs of windows being correctly aligned, alternately, under the transition map (outer map), and under some power of the inner map. The role of the windows is to approximate the location of orbits. Then there exists a true orbit that follows closely these windows, in the prescribed order. To apply this lemma for a normally hyperbolic invariant manifold for a map, one needs to reduce the dynamics from the continuous case to the discrete case by considering the return map to a surface of section, and construct the sequence of correctly aligned windows for the resulting normally hyperbolic invariant manifold for the return map. For this situation, the relationships between the scattering map for the flow and the scattering map for the return map, and between the transition map for the flow and the transition map for the return map, explored in Section 3 and Section 4, are useful.

A remarkable feature of these tools is that they can be used for both analytic arguments and rigorous numerical verifications. The scattering map and the transition map can be computed explicitly in concrete systems. They can also be used to reduce the dimensionality of the problem: from the phase space of a flow to a normally hyperbolic invariant manifold for the flow, and further to the normally hyperbolic invariant manifold for the return map to a surface of section. The shadowing lemma also plays a key role in reducing the dimensionality of the problem: it requires the verification of topological conditions in the normally hyperbolic invariant manifold for the return map to conclude the existence of trajectories in the phase space of the flow. In numerical applications, reducing the number of dimensions of the objects computed is very crucial. The potency of these tools in numerical application is illustrated in [7].

A main motivation for developing these tools resides with the instability problem for Hamiltonian systems. We now describe two models where the above techniques can be applied to show the existence of unstable orbits.

The first model is emblematic to the Arnold diffusion problem for Hamiltonian systems [1]. This problem conjectures that generic Hamiltonian systems that are close to integrable possess trajectories the move ‘wildly’ and ‘arbitrarily far’.

The model consists of a rotator and a pendulum with a small, periodic coupling. This model is described by a time-dependent Hamiltonian. By considering the return map to the surface of section given by the period of the perturbation, the problem is reduced to the discrete case. The phase space of the rotator can be described in action-angle coordinates, and its dynamics satisfies a twist condition.

When the rotator and the pendulum are decoupled the system is integrable. The phase space of the rotator is a normally hyperbolic invariant manifold for the return map, and is foliated by invariant tori. The separatrix of the pendulum determines hyperbolic stable and unstable manifolds of the normally hyperbolic invariant manifold; these stable and unstable manifolds coincide. All trajectories of the system are stable, in the sense that they experience no change in their action variable.

The situation changes dramatically when a small, generic coupling is added to the system. The phase space of the rotator is survived by a normally hyperbolic invariant manifold. Its foliation by invariant tori is destroyed, however, the KAM theory yields a Cantor family of invariant tori that survives the small coupling. The surviving tori are slight deformations of some tori from the integrable case, and they are referred as primary tori. The stable and unstable manifolds of the normally hyperbolic invariant manifold no longer coincide, but they intersect transversally along a homoclinic manifold. The unstable manifold of an invariant torus intersects transversally the stable manifolds of all sufficiently close invariant tori. Thus, by following the unstable manifold of a torus and then the stable manifold of a different torus, one can obtain trajectories that exhibit a change in the action variable. The scattering map associated to the homoclinic manifold can be explicitly computed in terms of the coupling, so one can estimate the change in the action variable along a homoclinic excursion. The actual homoclinic excursion is described by the transition map. The Arnold instability problem requires to show that there exist trajectories along which the action variable changes by some arbitrary quantity that is independent of the size of the coupling. One difficulty is that the coupling creates large gaps between the invariant tori, of size larger than the change in the action variable achieved by the scattering map.

A geometric method to cross the large gaps, used in [9, 11], is to consider secondary invariant tori that are formed inside the large gaps by resonances. The estimates on the scattering map show that there exist homoclinic orbits that connect invariant primary tori outside the large gaps with invariant secondary tori inside the gaps. The invariant tori (primary and secondary) and their heteroclinic orbits form a 'transition chain'.

Another geometric method to cross the large gaps, used in [16, 17] is to treat the large gaps as Birkhoff Zones of Instability (regions between two invariant primary tori that contain no other invariant primary tori in their interior) and to use connecting orbits inside the gaps that travel from one boundary of the Birkhoff Zones of Instability to the other boundary. In this case, one obtains a sequence of transition chains of invariant primary tori that are interspersed with Birkhoff Zones of Instability.

Both methods, the one based on transition chain of primary and secondary tori, and the one based on transition chains of invariant tori interspersed with Birkhoff Zones of Instability, yield pseudo-orbits that follow the corresponding geometric structures. To show the existence of true orbits that follow the geometric structures, one can use the method of correctly aligned windows, as in [14, 16, 17]. The windows used in these papers are full dimensional, and consecutive pairs of windows are correctly aligned under suitable powers of the first return map. Alternatively, one can construct windows contained in the phase space of the rotator, such that the consecutive pairs of windows are correctly aligned, alternately, under the transition map, and under some power of the inner map. The shadowing lemma stated in Theorem 6.1 implies that there is an orbit that follows closely these windows, in the prescribed order. By choosing the initial and the final windows far from one another, one obtains unstable orbits as claimed by the Arnold diffusion problem.

The second model is the spatial circular restricted three-body problem, in the case of the Sun-Earth system. We follow [7]. This problem considers the spatial motion of an infinitesimal body under the gravitational influence of Sun and Earth that are assumed to move on circular orbits about their center of mass. When the equations of motion of the infinitesimal body are described relative to a co-rotating frame, then the dynamics is given by a Hamiltonian system. The system has five equilibrium points; one of them, denoted L_1, lies between the two primaries. We will focus our attention on the dynamics near L_1. This problem is not close to integrable, and so the methods from perturbation theory do not apply. The approach below is numerical.

Fixing an energy manifold close to that of L_1, inside the energy manifold one computes a 3-dimensional normally hyperbolic invariant manifold for the Hamiltonian flow. One shows that the stable and unstable manifolds of the normally hyperbolic invariant manifold intersect transversally. Fixing a homoclinic intersection, one computes the corresponding scattering map for the the normally hyperbolic invariant manifold of the flow [3, 12]. By making a choice on how closely to follow the homoclinic orbits to the normally hyperbolic invariant manifold, one also computes the corresponding transition map. Inside the normally hyperbolic invariant manifold, one identifies a family of 2-dimensional invariant tori with the property that the stable and unstable manifolds of nearby tori intersect transversally. Such invariant tori can be enchained in a transition chain. In order to detect orbits that follow the transition chain, one can use the method of correctly aligned windows. Since the energy manifold is 5-dimensional, it is convenient to reduce the problem to a lower dimension by considering the return map relative to a suitable local surface of section to the Hamiltonian flow. The surface of section is 4-dimensional, the normally hyperbolic invariant manifold relative to the return map is 2-dimensional, and the corresponding tori are 1-dimensional. It is possible to endow the 2-dimensional normally hyperbolic invariant manifold with a system of action-angle coordinates, where the action represents the out-of-plane amplitude of the motion of the infinitesimal body. The scattering map and the transition map corresponding to the return map can be computed as in Section 3 and in Section 4. Then one constructs 2-dimensional windows contained inside the 2-dimensional normally hyperbolic invariant manifold, with the property that the consecutive pairs of windows are correctly aligned, alternately, under the transition map, and under some power of the inner map. The shadowing lemma stated in Theorem 6.1 implies that there exist orbits that follow closely these windows, in the prescribed order. These orbits increase the out-of-plane amplitude of the motion of the infinitesimal body. The argument is entirely numerical, however, due to the robustness of the correct alignment of windows given by Proposition 5.4, it can be translated into a rigorous verification with the aid of the computer, using the methods from [4, 5].

It seems possible that the above argument can be proved analytically, using the same methodology, in the case of the spatial circular restricted three-body problem when the relative mass of the smaller primary is sufficiently small, and the energy level is close to the energy of L_1.

As a practical conclusion, we propose a possible recipe for finding trajectories with prescribed itineraries for a normally hyperbolic invariant manifold with the property that its stable and unstable manifolds have a transverse intersection along a homoclinic manifold:

- Compute the scattering map associated to the homoclinic manifold.
- For some prescribed forward and backwards integration times, compute the corresponding transition map.
- If necessary, reduce the dynamics from a flow to the return map relative via some surface of section. Determine the normally hyperbolic invariant manifold relative to the surface of section, and compute the inner map – the restriction of the return map relative to the normally hyperbolic invariant manifold.
- Compute the scattering map and the transition map for the return map.
- Construct windows within the normally hyperbolic invariant manifold relative to the surface of section, with the the property that the consecutive pairs of windows are correctly aligned, alternately, under the transition map and under some power of the inner map.
- Apply the shadowing lemma stated in Theorem 6.1 to conclude that there exist orbits that follow closely these windows.

2. Preliminaries

In this section we review the concepts of normal hyperbolicity for flows and maps, normally hyperbolic invariant manifold for the return map to a surface of section, and we state a version of the Lambda Lemma that will be used in the subsequent sections.

2.1. **Normally hyperbolic invariant manifolds.** In this section we recall the concept of a normally hyperbolic invariant manifolds for a map and for a flow, following [13, 18].

Let M be a C^r-smooth, m-dimensional manifold, with $r \geq 1$, and $\Phi : M\times\mathbb{R} \to M$ a C^r-smooth flow on M.

Definition 2.1. A submanifold Λ of M is said to be a normally hyperbolic invariant manifold for Φ if Λ is invariant under Φ, there exists a splitting of the tangent bundle of TM into sub-bundles

$$TM = E^u \oplus E^s \oplus T\Lambda,$$

that are invariant under $d\Phi^t$ for all $t \in \mathbb{R}$, and there exist a constant $C > 0$ and rates $0 < \beta < \alpha$, such that for all $x \in \Lambda$ we have

$$\begin{aligned} v \in E^s_x &\Leftrightarrow \|D\Phi^t(x)(v)\| \leq Ce^{-\alpha t}\|v\| \text{ for all } t \geq 0,\\ v \in E^u_x &\Leftrightarrow \|D\Phi^t(x)(v)\| \leq Ce^{\alpha t}\|v\| \text{ for all } t \leq 0,\\ v \in T_x\Lambda &\Leftrightarrow \|D\Phi^t(x)(v)\| \leq Ce^{\beta|t|}\|v\| \text{ for all } t \in \mathbb{R}. \end{aligned}$$

It follows that there exist stable and unstable manifolds of Λ, as well as stable and unstable manifolds of each point $x \in \Lambda$, which are defined by

$$\begin{aligned} W^s(\Lambda) =& \{y \in M \mid d(\Phi^t(y), \Lambda) \leq C_y e^{-\alpha t} \text{ for all } t \geq 0\}, \\ W^u(\Lambda) =& \{y \in M \mid d(\Phi^t(y), \Lambda) \leq C_y e^{\alpha t} \text{ for all } t \leq 0\}, \\ W^s(x) =& \{y \in M \mid d(\Phi^t(y), \Phi^t(y)(x)) \leq C_{x,y} e^{-\alpha t} \text{ for all } t \geq 0\}, \\ W^u(x) =& \{y \in M \mid d(\Phi^t(y), \Phi^t(y)(x)) \leq C_{x,y} e^{\alpha t} \text{ for all } t \leq 0\}, \end{aligned}$$

for some constants $C_y, C_{x,y} > 0$.

The stable and unstable manifolds of Λ are foliated by stable and unstable manifolds of points, respectively, i.e., $W^s(\Lambda) = \bigcup_{x \in \Lambda} W^s(x)$ and $W^u(\Lambda) = \bigcup_{x \in \Lambda} W^u(x)$.

In the sequel we will assume that Λ is a compact and connected manifold. With no other assumptions, E^s_x and E^u_x depend continuously (but non-smoothly) on $x \in M$; thus the dimensions of E^s_x and E^u_x are independent of x. Below we only consider the case when the dimensions of the stable and unstable bundles are equal. We denote $n = \dim(E^s_x) = \dim(E^u_x)$, $l = \dim(T_x\Lambda)$, where $2n + l = m$.

The smoothness of the invariant objects defined by the normally hyperbolic structure depends on the rates α and β. Let ℓ be a positive integer satisfying $1 \leq \ell < \min\{r, \alpha/\beta\}$. The manifold Λ is C^ℓ-smooth. The stable and unstable manifolds $W^s(\Lambda)$ and $W^u(\Lambda)$ are $C^{\ell-1}$-smooth. The splittings E^s_x and E^u_x depend $C^{\ell-1}$-smoothly on x. The stable and unstable fibers $W^s(x)$ and $W^u(x)$ are C^r-smooth. The stable and unstable fibers $W^s(x)$ and $W^u(x)$ depend $C^{\ell-1-j}$-smoothly on x when $W^s(x), W^u(x)$ are endowed with the C^j-topology. In the sequel we will assume that the rates are such that there exists such an integer $\ell \geq 2$.

The notion of normal hyperbolicity for maps is very similar. Let $F : M \to M$ be a C^r-smooth map on M.

Definition 2.2. A submanifold Λ of M is said to be a normally hyperbolic invariant manifold for F if Λ is invariant under F, there exists a splitting of the tangent bundle of TM into sub-bundles

$$TM = E^u \oplus E^s \oplus T\Lambda,$$

that are invariant under dF, and there exist a constant $C > 0$ and rates $0 < \lambda < \mu^{-1} < 1$, such that for all $x \in \Lambda$ we have

$$\begin{aligned} v \in E^s_x &\Leftrightarrow \|DF^k_x(v)\| \leq C\lambda^k \|v\| \text{ for all } k \geq 0, \\ v \in E^u_x &\Leftrightarrow \|DF^k_x(v)\| \leq C\lambda^{-k} \|v\| \text{ for all } k \leq 0, \\ v \in T_x\Lambda &\Leftrightarrow \|DF^k_x(v)\| \leq C\mu^{|k|} \|v\| \text{ for all } k \in \mathbb{Z}. \end{aligned}$$

There exist stable and unstable manifolds of Λ, as well as the stable and unstable manifolds of each point $x \in \Lambda$, that are defined similarly as in the flow case, and they carry analogous properties. The smoothness properties of the invariant objects defined by the normally hyperbolic structure for a map are analogous of those for a flow, if we set $1 \leq \ell < \min\{r, (\log \lambda^{-1})(\log \mu)^{-1}\}$.

2.2. **Normal hyperbolicity relative to the return map.** Let $\Phi : M \times \mathbb{R} \to M$ be a C^r-smooth flow defined on an m-dimensional manifold M. Denote by X the vector field associated to Φ, where $X(x) = \frac{\partial}{\partial t}\Phi(x,t)_{|t=0}$. As before, assume that $\Lambda \subseteq M$ is an l-dimensional normally hyperbolic invariant manifold for Φ. The dimensions of $T\Lambda$, E^u and E^s are l, n, n, respectively, with $l + 2n = m$.

Let Σ be an $(m-1)$-dimensional local surface of section, i.e., Σ is a C^1 submanifold of M such that $X(x) \notin T_x\Sigma$ for all $x \in \Sigma$. Let $\Lambda_\Sigma = \Lambda \cap \Sigma$. Then Λ_Σ is a $(l-1)$-dimensional submanifold in Σ, assuming that the intersection is non-empty.

Assume that each forward and backward orbit through a point in Λ_Σ intersects again Λ_Σ. Since $X(x) \notin T_x\Sigma$ for all $x \in \Sigma$, then the intersection of the forward and backward orbits with Σ are transverse. Also, $X(x) \notin T_x\Lambda_\Sigma$ for all $x \in \Lambda_\Sigma$. Additionally, assume that the function

$$\tau : \Lambda_\Sigma \to (0, \infty), \text{ given by } \tau(x) = \inf\{t > 0 \,|\, \Phi(x, \tau(x)) \in \Lambda_\Sigma\},$$

is a continuous function. Following [13], we will refer to Λ_Σ with these properties as a thin surface of section.

By the Implicit Function Theorem, τ can be extended to a C^1-smooth function in a neighborhood U_Σ of Λ_Σ in Σ such that $\Phi(x, \tau(x)) \in \Sigma$ for all $x \in U_\Sigma$. The Poincaré first return map to Σ is the map $F : U_\Sigma \to \Sigma$ given by $F(x) = \Phi^{\tau(x)}(x)$.

Let $\Lambda^X_\Sigma \subseteq \Lambda$ be the union of the orbits of the flow through points in Λ_Σ. Since Λ^X_Σ is a C^1-submanifold of Λ, and is invariant under Φ, then is a normally hyperbolic invariant manifold for the flow Φ. The theorem below [13] implies that the manifold Λ_Σ is normally hyperbolic for the return map F.

Theorem 2.3. *Let Λ_Σ be a thin surface of section for the vector field X on M. Then Λ_Σ is normally hyperbolic with respect to F if and only if Λ^X_Σ is a normally hyperbolic invariant manifold with respect to Φ.*

The invariant sub-bundles $T\Lambda$, E^u, E^s associated to the normal hyperbolic structure on Λ correspond to sub-bundles $T\Lambda_\Sigma$, E^u_Σ, E^s_Σ in the following way. Let $\pi : TM = \text{span}(X) \oplus T\Sigma \to T\Sigma$ be the projection onto $T\Sigma$. Then $T\Lambda_\Sigma = \pi(T_\Lambda)$, $E^u_\Sigma = \pi(E^u)$, and $E^s_\Sigma = \pi(E^s)$.

2.3. **Lambda Lemma.** We describe a Lambda Lemma type of result for normally hyperbolic invariant manifolds that appears in J.-P. Marco [19].

We consider a normally hyperbolic invariant manifold Λ for a diffeomorphism $F : M \to M$; also $\dim(M) = l+2n$, $\dim(\Lambda) = l$, and $\dim(W^s(\Lambda)) = \dim(W^u(\Lambda)) = l + n$. We fix an integer $2 \leq k \leq \ell$ so that all the manifolds and maps considered below are C^k-smooth. By a normal form in a neighborhood V of Λ in M we mean a C^k-smooth coordinate system (c, s, u) on V such that V is diffeomorphic through (c, s, u) with a product $\Lambda \times \mathbb{R}^n \times \mathbb{R}^n$, where $\Lambda = \{(c, s, u) \,|\, c \in \Lambda,\ u = s = 0\}$, and $W^u(x) = \{(c, s, u) \,|\, c = c(x), s = 0\}$, $W^s(x) = \{(c, s, u) \,|\, c = c(x), u = 0\}$ for each $x \in \Lambda$ of coordinates $(c(x), 0, 0)$.

Theorem 2.4 (Lambda Lemma)**.** *Suppose that Λ is a normally hyperbolic invariant manifold for F and (c, s, u) is a normal form in a neighborhood of Λ. Consider a submanifold Δ of M of dimension n which intersects the stable manifold $W^s(\Lambda)$ transversely at some point $z = (c, s, 0)$. Set $F^N(z) = z_N = (c_N, s_N, 0)$ for $N \in \mathbb{N}$. Then there exists $\delta > 0$ and $N_0 > 0$ such that for each $N \geq N_0$ the connected component Δ_N of $F^N(\Delta)$ in the δ-neighborhood $V(\delta) = \Lambda \times B^s_\delta(0) \times B^u_\delta(0)$ of Λ in M admits a graph parametrization of the form*

$$\Delta_N := \{(C_N(u), S_N(u), u) \,|\, u \in B^u_\delta(0)\}$$

such that

$$\|C_N - c_N\|_{C^1(B^u_\delta(0))} \to 0, \text{ and } \|S_N\|_{C^1(B^u_\delta(0))} \to 0 \text{ as } N \to \infty.$$

3. Scattering map

In this section we review the scattering map associated to a normally hyperbolic invariant manifold for a flow or for a map, and discuss the relationship between the scattering map for a flow and the scattering map for the corresponding return map to some surface of section.

3.1. **Scattering map for continuous and discrete dynamical systems.** Consider a flow $\Phi : M \times \mathbb{R} \to M$ defined on a manifold M that possesses a normally hyperbolic invariant manifold $\Lambda \subseteq M$.

As the stable and unstable manifolds of Λ are foliated by stable and unstable manifolds of points, respectively, for each $x \in W^u(\Lambda)$ there exists a unique $x_- \in \Lambda$ such that $x \in W^u(x_-)$, and for each $x \in W^s(\Lambda)$ there exists a unique $x_+ \in \Lambda$ such that $x \in W^s(x_+)$. We define the wave maps $\Omega_+ : W^s(\Lambda) \to \Lambda$ by $\Omega_+(x) = x_+$, and $\Omega_- : W^u(\Lambda) \to \Lambda$ by $\Omega_-(x) = x_-$. The maps Ω_+ and Ω_- are C^ℓ-smooth.

We now describe the scattering map, following [10]. Assume that $W^u(\Lambda)$ has a transverse intersection with $W^s(\Lambda)$ along a l-dimensional homoclinic manifold Γ. The manifold Γ consists of a $(l-1)$-dimensional family of trajectories asymptotic to Λ in both forward and backwards time. The transverse intersection of the hyperbolic invariant manifolds along Γ means that $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$ and, for each $x \in \Gamma$, we have

$$\begin{aligned} T_xM &= T_xW^u(\Lambda) + T_xW^s(\Lambda),\\ T_x\Gamma &= T_xW^u(\Lambda) \cap T_xW^s(\Lambda). \end{aligned} \tag{3.1}$$

Let us assume the additional condition that for each $x \in \Gamma$ we have

$$\begin{aligned} T_xW^s(\Lambda) &= T_xW^s(x_+) \oplus T_x(\Gamma),\\ T_xW^u(\Lambda) &= T_xW^u(x_-) \oplus T_x(\Gamma), \end{aligned} \tag{3.2}$$

where x_-, x_+ are the uniquely defined points in Λ corresponding to x.

The restrictions $\Omega_+^\Gamma, \Omega_-^\Gamma$ of the wave maps Ω_+, Ω_- to Γ are local $C^{\ell-1}$-diffeomorphisms. By restricting Γ even further if necessary, we can ensure that $\Omega_+^\Gamma, \Omega_-^\Gamma$ are $C^{\ell-1}$-diffeomorphisms. A homoclinic manifold Γ for which the corresponding restrictions of the wave maps are $C^{\ell-1}$-diffeomorphisms will be referred as a homoclinic channel.

Definition 3.1. Given a homoclinic channel Γ, the scattering map associated to Γ is the $C^{\ell-1}$-diffeomorphism $S^\Gamma = \Omega_+^\Gamma \circ (\Omega_-^\Gamma)^{-1}$ defined on the open subset $U_- := \Omega_-^\Gamma(\Gamma)$ in Λ to the open subset $U_+ := \Omega_+^\Gamma(\Gamma)$ in Λ.

In the sequel we will regard S as a partially defined map, so the image of a set A by S means the set $S(A \cap U_-)$.

If we flow Γ backwards and forward in time we obtain the manifolds $\Phi^{-t_u}(\Gamma)$ and $\Phi^{t_s}(\Gamma)$ that are also homoclinic channels, where $t_u, t_s > 0$. The associated wave maps are $\Omega_+^{\Phi^{-t_u}(\Gamma)}, \Omega_-^{\Phi^{-t_u}(\Gamma)}$, and $\Omega_+^{\Phi^{t_s}(\Gamma)}, \Omega_-^{\Phi^{t_s}(\Gamma)}$, respectively. The scattering map can be expressed with respect to these wave maps as

$$S^\Gamma = \Phi^{-t_s} \circ (\Omega_+^{\Phi^{t_s}(\Gamma)}) \circ \Phi^{t_s+t_u} \circ (\Omega_-^{\Phi^{-t_u}(\Gamma)})^{-1} \circ \Phi^{-t_u}. \tag{3.3}$$

We recall below some remarkable properties of the scattering map.

Proposition 3.2. *Assume that* $\dim M = 2n + l$ *is even (i.e.,* l *is even) and* M *is endowed with a symplectic (respectively exact symplectic) form* ω *and that* $\omega_{|\Lambda}$

is also symplectic. Assume that Φ^t is symplectic (respectively exact symplectic). Then, the scattering map S^Γ is symplectic (respectively exact symplectic).

Proposition 3.3. *Assume that T_1 and T_2 are two invariant submanifolds of complementary dimensions in Λ. Then $W^u(T_1)$ has a transverse intersection with $W^s(T_2)$ inside Γ if and only if $S(T_1)$ has a transverse intersection with T_2 in Λ.*

In the case of a discrete dynamical system consisting of a diffeomorphism $F : M \to M$ defined on a manifold M, the scattering map is defined in a similar way. We assume that F has a normally hyperbolic invariant manifold $\Lambda \subseteq M$. The wave maps are defined by $\Omega_+ : W^s(\Lambda) \to \Lambda$ with $\Omega_+(x) = x_+$, and $\Omega_- : W^u(\Lambda) \to \Lambda$ with $\Omega_-(x) = x_-$.

Assume that $W^u(\Lambda)$ and $W^s(\Lambda)$ have a differentiably transverse intersection along a homoclinic l-dimensional $C^{\ell-1}$-smooth manifold Γ. We also assume the transverse foliation condition (3.2).

A homoclinic manifold Γ for which the corresponding restrictions of the wave maps are $C^{\ell-1}$-diffeomorphisms is referred as a homoclinic channel.

Definition 3.4. Given a homoclinic channel Γ, the scattering map associated to Γ is the $C^{\ell-1}$-diffeomorphism $S^\Gamma = \Omega_+^\Gamma \circ (\Omega_-^\Gamma)^{-1}$ defined on the open subset $U_- := \Omega_-^\Gamma(\Gamma)$ in Λ to the open subset $U_+ := \Omega_+^\Gamma(\Gamma)$ in Λ.

Note that for $M, N > 0$, the manifolds $F^{-M}(\Gamma)$ and $F^N(\Gamma)$ are also homoclinic channels. The associated wave maps are $\Omega_-^{F^{-M}(\Gamma)}, \Omega_+^{F^{-M}(\Gamma)}$, and $\Omega_-^{F^N(\Gamma)}, \Omega_+^{F^N(\Gamma)}$. The scattering map can be expressed with respect to these wave map as

$$S^\Gamma = F^{-N} \circ (\Omega_+^{F^N(\Gamma)}) \circ F^{M+N} \circ (\Omega_-^{F^{-M}(\Gamma)})^{-1} \circ F^{-M}. \tag{3.4}$$

The scattering map for the discrete case satisfies symplectic and transversality properties similar to those in Proposition 3.2 and Proposition 3.3 for the continuous case.

3.2. **Scattering map for the return map.** Let $\Phi : M \times \mathbb{R} \to M$ be a C^r-smooth flow defined on an m-dimensional manifold M, and X be the vector field associated to Φ. Let $\Lambda \subseteq M$ be an l-dimensional normally hyperbolic invariant manifold for Φ. Assume that Σ is a local surface of section and $\Lambda_\Sigma = \Lambda \cap \Sigma$ satisfies the conditions in Subsection 2.2.

Consider Γ a homoclinic channel for Φ. First, we assume that Γ has a non-empty intersection with Σ. Note that Γ is a $(l-1)$-parameter family of orbits; we further assume that each trajectory intersects Σ transversally. Since Γ is a homoclinic channel, each orbit intersects Σ exactly once. Let $\Gamma_\Sigma = \Gamma \cap \Sigma$. It is easy to see that Γ_Σ is a homoclinic channel for F. Thus, we have a scattering map S^Γ for Γ associated to the flow Φ, and we also have a scattering map S^{Γ_Σ} for Γ_Σ associated to the map F.

We want to understand the relationship between S^Γ and S^{Γ_Σ}. Associated to the homoclinic channels Γ and Γ_Σ there exist wave maps $\Omega_\pm^\Gamma : \Gamma \to \Lambda$ and $\Omega_\pm^{\Gamma_\Sigma} : \Gamma_\Sigma \to \Lambda_\Sigma$, respectively. These maps are diffeomorphisms. Let $x \in \Gamma_\Sigma$, and let $x_- = \Omega_-^\Gamma(x)$, $x_+ = \Omega_+^\Gamma(x)$, and $\hat{x}_- = \Omega_-^{\Gamma_\Sigma}(x)$, $\hat{x}_+ = \Omega_+^{\Gamma_\Sigma}(x)$. We have $S^\Gamma(x_-) = x_+$ and $S^{\Gamma_\Sigma}(\hat{x}_-) = \hat{x}_+$.

We want to relate x_- with $\hat{x}_-$, and x_+ with $\hat{x}_+$. These points are all in Λ. It is clear that $\hat{x}_- = \Omega_-^{\Gamma_\Sigma} \circ (\Omega_-^\Gamma)^{-1}(x_-)$, and $\hat{x}_+ = \Omega_+^{\Gamma_\Sigma} \circ (\Omega_+^\Gamma)^{-1}(x_+)$. Denote by $P_-^\Gamma : \Omega_-^\Gamma(\Gamma) \to \Omega_-^{\Gamma_\Sigma}$ the map given by $P_-^\Gamma = \Omega_-^{\Gamma_\Sigma} \circ (\Omega_-^\Gamma)^{-1}$, and denote by

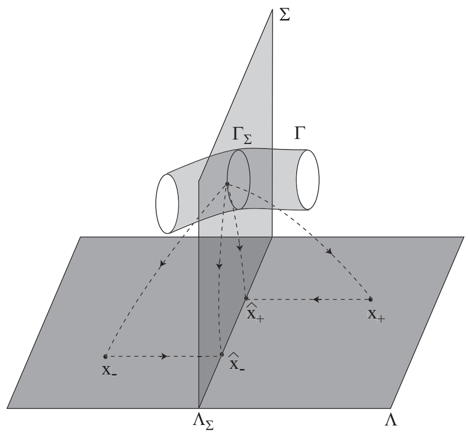

FIGURE 1. Scattering map for the return map.

$P_+^\Gamma : \Omega_+^\Gamma(\Gamma) \to \Omega_+^{\Gamma_\Sigma}$ the map given by $P_+^\Gamma = \Omega_+^{\Gamma_\Sigma} \circ (\Omega_+^\Gamma)^{-1}$. We want to express these maps in terms of the dynamics restricted to Λ.

Let V be a flow box at $\hat{x}_-$ (for definition see [21]). This means each trajectory through a point $y \in V$ intersects Σ exactly once. Then there exists a differentiable function $\hat{\tau} : V \to \mathbb{R}$ defined by $\tau(z) = 0$ if $z \in \Sigma$ and $\Phi^{\hat{\tau}(y)}(y) \in \Sigma$ for each $y \in V$. The function $\hat{\tau}$ can be extended in a unique way on each trajectory passing though V. Due to the relationship between the invariant bundles for the flow and the invariant bundles for the map described in Subsection 2.2, the fiber $E^u_\Sigma(\hat{x}_-)$ is the projection onto $T\Sigma$ of the image of the fiber $E^u(x_-)$ under $D\Phi_{x_-}^{\hat{\tau}(x_-)}$. This means that $\Phi^{\hat{\tau}(x_-)}(x_-) = \hat{x}_-$. In other words, $\hat{x}_-$ is at the intersection of the trajectory through x_- with Σ. Thus, the projection P_-^Γ that takes x_- to $\hat{x}_-$ is given by $P_-^\Gamma(x_-) = \Phi^{\hat{\tau}(x_-)}(x_-)$. This projection map is invertible. If $\hat{y}_-$ is a point in $\Omega_-^{\Gamma_\Sigma}$, there exists a unique point $y_- \in \Omega_-^\Gamma(\Gamma)$ such that $\Phi^{\hat{\tau}(y_-)}(y_-) = \hat{y}_-$. If there exist two such points, y_- and y'_-, to them they correspond two points y, y' in Γ_Σ such that $y \in W_F^u(y_-)$ and $y' \in W_F^u(y'_-)$. The points y, y' should belong to the same unstable fiber $W_F^u(\hat{y}_-)$. Then it means that y, y' are on the same trajectory. As they are also in Γ and Γ is a homoclinic channel, than $y = y'$ and $y_- = y'_-$. In summary, the projection map $P_-^\Gamma : \Omega_-^\Gamma(\Gamma) \to \Omega_-^{\Gamma_\Sigma}$ is given by $P_-^\Gamma(x_-) = \Phi^{\tau(x_-)}(x_-)$. Similarly, the projection map $P_+^\Gamma : \Omega_+^\Gamma(\Gamma) \to \Omega_+^{\Gamma_\Sigma}$ is given by $P_+^\Gamma(x_+) = \Phi^{\tau(x_+)}(x_+)$. See Figure 1.

Now we can formulate the relationship between the scattering map S^Γ associated to the flow Φ, and the scattering map S^{Γ_Σ} associated to the map F.

Proposition 3.5. *Assume that Γ is a homoclinic channel for the flow Φ, and $\Gamma_\Sigma = \Gamma \cap \Sigma$ is the corresponding homoclinic channel for the map F. Let S^Γ be the scattering map corresponding to Γ, and let S^{Γ_Σ} be the scattering map corresponding to Γ_Σ. Then:*

$$S^{\Gamma_\Sigma} = P_+^\Gamma \circ S^\Gamma \circ (P_-^\Gamma)^{-1}. \tag{3.5}$$

Proof. We have that $S^\Gamma(x_-) = x_+$, $S^{\Gamma_\Sigma}(\hat{x}_-) = \hat{x}_+$, $P_-^\Gamma(x_-) = \hat{x}_-$, and $P_-^\Gamma(x_+) = \hat{x}_+$. Thus $S^{\Gamma_\Sigma}(\hat{x}_-) = P_+^\Gamma(x_+) = P_+^\Gamma \circ S^\Gamma(x_-) = P_+^\Gamma \circ S^\Gamma \circ (P_-^\Gamma)^{-1}(\hat{x}_-)$. □

4. Transition map

The scattering map for a flow Φ is geometrically defined: $S^\Gamma(x_-) = x_+$ means that $W^u(x_-)$ intersects $W^s(x_+)$ at a unique point $x \in \Gamma$, with $W^u(x_-)$ and $W^s(x_+)$ being n-dimensional manifolds. However, there is no trajectory of the system that goes from near x_- to near x_+. Instead, the trajectory of x approaches asymptotically the backwards orbit of x_- in negative time, and approaches asymptotically the forward orbit of x_+ in positive time. For applications we need a dynamical version of the scattering map. That is, we need a map that takes some backwards image of x_- into some forward image of x_+. We will call this map a transition map. The transition map depends on the amounts of times we want to flow in the past and in the future. The transition map carries the same geometric information as the scattering map. Since in perturbation problems the scattering map can be computed explicitly, the transition map is also computable. The notion of transition map below is similar to the transition map defined in [6], however, their version is not related to the scattering map.

4.1. **Transition map for continuous and discrete dynamical systems.** Consider a flow $\Phi : M \times \mathbb{R} \to M$ defined on a manifold M that possesses a normally hyperbolic invariant manifold $\Lambda \subseteq M$. Assume that $W^u(\Lambda)$ and $W^s(\Lambda)$ have a transverse intersection, and that there exists a homoclinic channel Γ. Given $t_u, t_s > 0$, the time-map $\Phi^{t_s+t_u}$ is a diffeomorphism from $\Phi^{-t_u}(\Gamma)$ to $\Phi^{t_s}(\Gamma)$. Using (3.3) we can express the restriction of $\Phi^{t_s+t_u}$ to $\Phi^{-t_u}(\Gamma)$ in terms of the scattering map as

$$\Phi^{t_s+t_u}_{|\Phi^{-t_u}(\Gamma)} : (\Omega_-^{\Phi^{-t_u}(\Gamma)})^{-1}(\Phi^{-t_u}(U_-)) \to (\Omega_+^{\Phi^{t_s}(\Gamma)})^{-1}(\Phi^{t_s}(U_+)),$$

given by

$$\Phi^{t_s+t_u}_{|\Phi^{-t_u}(\Gamma)} = (\Omega_+^{\Phi^{t_s}(\Gamma)})^{-1} \circ \Phi^{t_s} \circ S^\Gamma \circ \Phi^{t_u} \circ (\Omega_-^{\Phi^{-t_u}(\Gamma)}), \tag{4.1}$$

where $S^\Gamma : U_- \to U_+$ is the scattering map associated to the homoclinic channel Γ. We use this to define the transition map as an an approximation of $\Phi^{t_s+t_u}_{|\Phi^{-t_u}(\Gamma)}$ provided that t_u, t_s are sufficiently large.

Definition 4.1. Let Γ be a homoclinic channel for Φ. Let $t_u, t_s > 0$ fixed. The transition map $S^\Gamma_{t_u,t_s}$ is a diffeomorphism

$$S^\Gamma_{t_u,t_s} : \Phi^{-t_u}(U_-) \to \Phi^{t_s}(U_+)$$

given by

$$S^\Gamma_{t_u,t_s} = \Phi^{t_s} \circ S^\Gamma \circ \Phi^{t_u},$$

where $S^\Gamma : U_- \to U_+$ is the scattering map associated to the homoclinic channel Γ.

Alternatively, we can express the transition map as

$$S^\Gamma_{t_u,t_s} = \Omega_+^{\Phi^{t_s}(\Gamma)} \circ \Phi^{t_u+t_s} \circ (\Omega_-^{\Phi^{-t_u}(\Gamma)})^{-1}$$

The symplectic property and the transversality property of the scattering map lend themselves to similar properties of the transition map.

In the case of a dynamical system given by a map $F : M \to M$, the transition map can be defined in a similar manner to the flow case, and enjoys similar properties. As before, we assume that $\Lambda \subseteq M$ is a normally hyperbolic invariant manifold for F.

Definition 4.2. Let Γ be a homoclinic channel for F. Let $N_u, N_s > 0$ fixed. The transition map $S^{\Gamma}_{N_u,N_s}$ is a diffeomorphism

$$S^{\Gamma}_{N_u,N_s} : F^{-N_u}(U_-) \to F^{N_s}(U_+)$$

given by

$$S^{\Gamma}_{N_u,N_s} = F^{N_s} \circ S^{\Gamma} \circ F^{N_u},$$

where $S^{\Gamma} : U_- \to U_+$ is the scattering map associated to the homoclinic channel Γ.

4.2. **Transition map for the return map.** We will consider the reduction of the transition map to a local surface of section. Let Σ be a local surface of section and $\Lambda_\Sigma = \Lambda \cap \Sigma$. By Theorem 2.3, Λ_Σ is normally hyperbolic with respect to the first return map to Σ. Assume that Γ intersects Σ as in Subsection 2.2, and let $\Gamma_\Sigma = \Gamma \cap \Sigma$.

Let x be a point in Γ_Σ. Then $\Phi^{-t_u}(x)$ lies on $W^u(\Phi^{-t_u}(x_-))$, approaches asymptotically Λ as $t_u \to \infty$, and intersects Σ infinitely many times. Similarly, $\Phi^{t_s}(x)$ lies on $W^u(\Phi^{t_s}(x_+))$, approaches asymptotically Λ as $t_s \to \infty$, and intersects Σ infinitely many times.

We want to choose and fix some times t_u, t_s, depending on $x \in \Gamma$, such that $\Phi^{-t_u}(x), \Phi^{t_s}(x)$ are both in Σ, and moreover, $\Phi^{-t_u}(x), \Phi^{t_s}(x)$ are sufficiently close to $\Phi^{-t_u}(x_-), \Phi^{t_s}(x_+)$, respectively.

Let $\upsilon > 0$ be a small positive number. We define $t_u = t_u(x)$ to be the smallest time such that $\Phi^{-t_u(x)}(x) \in \Sigma$, and the distance between $\Phi^{-t_u}(x)$ and $\Phi^{-t_u}(x_-)$, measured along the unstable fiber $W^u(\Phi^{-t_u}(x_-))$, is less than υ. Let $N_u > 0$ be such that $\Phi^{-t_u}(x) = F^{-N_u}(x)$. Similarly, we define $t_s = t_s(x)$ to be the smallest time such that $\Phi^{t_s}(x) \in \Sigma$, and the distance between $\Phi^{t_s}(x)$ and $\Phi^{t_s}(x_+)$, measured along the stable fiber $W^s(\Phi^{t_s}(x_+))$, is less than υ. Let $N_s > 0$ be such that $\Phi^{t_s}(x) = F^{N_S}(x)$.

At this point, we have a transition map $S^{\Gamma}_{t_u,t_s}$ associated to the flow Φ and to the homoclinic channel Γ for the flow, and a transition map $S^{\Gamma_\Sigma}_{N_u,N_s}$ associated to the map F and to the homoclinic channel Γ_Σ for the map.

We have that $\Phi^{-t_u}(\Gamma)$ and $\Phi^{t_s}(\Gamma)$ are both homoclinic channels for the flow Φ, and $F^{-N_u}(\Gamma)$ and $F^{N_s}(\Gamma)$ are both homoclinic channels for the map F. Let us consider the projection mappings $P_-^{F^{-N_u}}(\Gamma), P_+^{F^{-N_u}}(\Gamma)$ associated to the homoclinic channel $F^{-N_u}(\Gamma)$, and the projection mappings $P_-^{F^{N_s}}(\Gamma), P_+^{F^{N_s}}(\Gamma)$ associated to the homoclinic channel $F^{N_s}(\Gamma)$. These projections mappings are defined as in Subsection 2.2.

The relationship between the transition map for the flow Φ and the transition map for the return map F is given by the following:

Proposition 4.3. *Assume that Γ is a homoclinic channel for the flow Φ, and $\Gamma_\Sigma = \Gamma \cap \Sigma$ is the corresponding homoclinic channel for the map F. Let t_u, t_s, N_u, $N_s > 0$ be fixed. Let $S^{\Gamma}_{N_u,N_s}$ be the transition map corresponding to Γ for the flow Φ, and let $S^{\Gamma_\Sigma}_{N_u,N_s}$ be the transition map corresponding to Γ_Σ for the return map F. Then*

$$S^{\Gamma_\Sigma}_{N_u,N_s} = P_+^{F^{N_s}(\Gamma)} \circ S^{\Gamma}_{t_u,t_s} \circ (P_-^{F^{-N_u}(\Gamma)})^{-1}.$$

Proof. We have that $S^{\Gamma_\Sigma}(\hat{x}_-) = \hat{x}_+$. Note that $\hat{x}_- = F^{N_u} \circ P_-^{F^{-N_u}(\Gamma)} \circ \Phi^{-t_u}(x_-)$ and $\hat{x}_+ = F^{-N_s} \circ P_+^{F^{N_s}(\Gamma)} \circ \Phi^{t_s}(x_+)$.

Thus

$$\begin{aligned} S^{\Gamma\Sigma}(\hat{x}_-) &= \hat{x}_+ \\ &= F^{-N_s} \circ P_+^{F^{N_s}(\Gamma)} \circ \Phi^{t_s}(x_+) \\ &= F^{-N_s} \circ P_+^{F^{N_s}(\Gamma)} \circ \Phi^{t_s} \circ S(x_-) \\ &= F^{-N_s} \circ P_+^{F^{N_s}(\Gamma)} \circ \Phi^{t_s} \circ S \circ \Phi^{t_u} \circ (P_-^{F^{-N_u}(\Gamma)})^{-1} \circ F^{-N_u}(\hat{x}_-). \end{aligned}$$

Hence

$$F^{N_s} \circ S^{\Gamma\Sigma} \circ F^{N_u} = P_+^{F^{N_s}(\Gamma)} \circ \Phi^{t_s} \circ S^{\Gamma} \circ \Phi^{t_u} \circ (P_-^{F^{-N_u}(\Gamma)})^{-1}.$$

The conclusion of the proposition now follows from the definition of the transition map in the flow case and the definition of the transition map in the map case. □

5. Topological method of correctly aligned windows

We review briefly the topological method of correctly aligned windows. We follow [22]. See also [15, 14].

Definition 5.1. An (m_1, m_2)-window in an m-dimensional manifold M, where $m_1 + m_2 = m$, is a compact subset R of M together with a C^0-parametrization given by a homeomorphism χ from some open neighborhood of $[0,1]^{m_1} \times [0,1]^{m_2}$ in $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ to an open subset of M, with $R = \chi([0,1]^{m_1} \times [0,1]^{m_2})$, and with a choice of an 'exit set'

$$R^{\text{exit}} = \chi\left(\partial[0,1]^{m_1} \times [0,1]^{m_2}\right)$$

and of an 'entry set'

$$R^{\text{entry}} = \chi\left([0,1]^{m_1} \times \partial[0,1]^{m_2}\right).$$

We adopt the following notation: $R_\chi = \chi^{-1}(R)$, $(R^{\text{exit}})_\chi = \chi^{-1}(R^{\text{exit}})$, and $(R^{\text{entry}})_\chi = \chi^{-1}(R^{\text{entry}})$. (Note that $R_\chi = [0,1]^{m_1} \times [0,1]^{m_2}$, $(R^{\text{exit}})_\chi = \partial[0,1]^{m_1} \times [0,1]^{m_2}$, and $(R^{\text{entry}})_\chi = [0,1]^{m_1} \times \partial[0,1]^{m_2}$.) When the local parametrization χ is evident from context, we suppress the subscript χ from the notation.

Definition 5.2. Let R_1 and R_2 be (m_1, m_2)-windows, and let χ_1 and χ_2 be the corresponding local parametrizations. Let F be a continuous map on M with $F(\text{im}(\chi_1)) \subseteq \text{im}(\chi_2)$. We say that R_1 is correctly aligned with R_2 under F if the following conditions are satisfied:

(i) There exists a continuous homotopy $h : [0,1] \times (R_1)\chi_1 \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$, such that the following conditions hold true

$$\begin{aligned} h_0 &= F_\chi, \\ h([0,1], (R_1^{\text{exit}})_{\chi_1}) \cap (R_2)_{\chi_2} &= \emptyset, \\ h([0,1], (R_1)_{\chi_1}) \cap (R_2^{\text{entry}})_{\chi_2} &= \emptyset, \end{aligned}$$

(ii) the map $A_{y_0} : \mathbb{R}^{m_1} \to \mathbb{R}^{m_1}$ defined by $A_{y_0}(x) = \pi_{m_1}(h_1(x, y_0))$ satisfies

$$\begin{aligned} A_{y_0}\left(\partial[0,1]^{m_1}\right) &\subseteq \mathbb{R}^{m_1} \setminus [0,1]^{m_1}, \\ \deg(A_{y_0}, 0) &\neq 0, \end{aligned}$$

where $\pi_{m_1} : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}$ is the orthogonal projection onto the first component, and deg is the Brouwer degree of the map A_{y_0} at 0.

The following result allows the detection of orbits with prescribed itineraries.

Theorem 5.3. *Let R_i be a collection of (m_1, m_2)-windows in M, where $i \in \mathbb{Z}$ or $i \in \{0, \dots, d-1\}$, with $d > 0$ (in the latter case, for convenience, we let $R_i = R_{(i \bmod d)}$ for all $i \in \mathbb{Z}$). Let F_i be a collection of continuous maps on M. If R_i is correctly aligned with R_{i+1}, for all i, then there exists a point $p \in R_0$ such that*

$$(F_i \circ \dots \circ F_0)(p) \in R_{i+1},$$

Moreover, if $R_{i+k} = R_i$ for some $k > 0$ and all i, then the point p can be chosen periodic in the sense

$$(F_{k-1} \circ \dots \circ F_0)(p) = p.$$

Often, the maps F_i represent different powers of the return map associated to a certain surface of section.

The correct alignment of windows is robust, in the sense that if two windows are correctly aligned under a map, then they remain correctly aligned under a sufficiently small perturbation of the map.

Proposition 5.4. *Assume R_1, R_2 are (m_1, m_2)-windows in M. Let G be a continuous maps on M. Assume that R_1 is correctly aligned with R_2 under G. Then there exists $\epsilon > 0$, depending on the windows R_1, R_2 and G, such that, for every continuous map F on M with $\|F(x) - G(x)\| < \epsilon$ for all $x \in R_1$, we have that R_1 is correctly aligned with R_2 under F.*

Also, the correct alignment satisfies a natural product property. Given two windows and a map, if each window can be written as a product of window components, and if the components of the first window are correctly aligned with the corresponding components of the second window under the appropriate components of the map, then the first window is correctly aligned with the second window under the given map. For example, if we consider a pair of windows in a neighborhood of a normally invariant normally hyperbolic invariant manifold, if the center components of the windows are correctly aligned and the hyperbolic components of the windows are also correctly aligned, then the windows are correctly aligned. Although the product property is quite intuitive, its rigorous statement is rather technical, so we will omit it here. The details can be found in [14].

6. A shadowing lemma for normally hyperbolic invariant manifolds

In this section we present a shadowing lemma-type of result saying that, given a sequence of windows in a normally hyperbolic invariant manifold, if each pair of successive windows is correctly aligned under some appropriate mappings, then there exists a true orbit in the full space dynamics that follows these windows. In the sequence, the pairs of windows are correctly aligned under the transition map, alternating with pairs of windows that are correctly aligned under the inner map.

The result below provides a method to reduce the problem of the existence of orbits in the full dimensional phase space to a lower dimensional problem of the existence of pseudo-orbits in the normally hyperbolic invariant manifold.

Theorem 6.1. *Let $\varepsilon > 0$. Let $\{D_i^+, D_i^-\}_{i \in \mathbb{Z}}$ be a bi-infinite sequence of l-dimensional windows contained in a compact subset of Λ. Assume that for any integers $n_1^0, n_1^-, n_1^+ > 0$ there exist integers $n_2^0 > n_1^0$, $n_2^- > n_1^-$, $n_2^+ > n_1^+$ and sequences of integers $\{N_i^0, N_i^-, N_i^+,\}_{i \in \mathbb{Z}}$ with $n_1^0 < N_i^0 < n_2^0$, $n_1^- < N_i^- < n_2^-$, $n_1^+ < N_i^+ < n_2^+$ such that the following properties hold for all $i \in \mathbb{Z}$:*

(i) $F^{-N_i^+}(D_i^+) \subseteq U_+$ *and* $F^{N_i^-}(D_i^-) \subseteq U_-$.

(ii) D_i^- *is correctly aligned with* D_{i+1}^+ *under the transition map* $S^\Gamma_{N_i^-, N_{i+1}^+} = F^{N_{i+1}^+} \circ S \circ F^{N_i^-}$.

(iii) D_i^+ *is correctly aligned with* D_i^- *under the iterate* $F^{N_i^0}$ *of* $F_{|\Lambda}$.

Then there exist an orbit $F^N(z)$ *of* F *for some* $z \in M$ *and an increasing sequence of integers* $\{N_i\}_{i\in\mathbb{Z}}$ *with* $N_{i+1} = N_i + N_{i+1}^+ + N_{i+1}^0 + N_{i+1}^-$ *such that, for all* i*:*

$$d(F^{N_i}(z), \Gamma) < \varepsilon,$$
$$d(F^{N_i - N_i^-}(z), D_i^-) < \varepsilon,$$
$$d(F^{N_i + N_{i+1}^+}(z), D_{i+1}^+) < \varepsilon.$$

Proof. The idea of this proof is to 'thicken' the windows D_i^+, D_i^- in Λ to full-dimensional windows R_i^-, R_i^+ in M, so that the successive windows in the sequence $\{R_i^-, R_i^+\}_i$ are correctly aligned under some appropriate iterations of the map F. The argument is done in several steps. In the first three steps, we only specify the relative sizes of the windows involved in each step. In the fourth step, we explain how to make the choices of the sizes of the windows uniform.

Step 1. Note that conditions (i) and (ii) imply that $\hat{D}_i^- := F^{N_i^-}(D_i^-) \subseteq U_- \subseteq \Lambda$ is correctly aligned with $\hat{D}_{i+1}^+ := F^{-N_{i+1}^+}(D_{i+1}^+) \subseteq U_+ \subseteq \Lambda$ under the scattering map S. Let $\bar{D}_i^- = (\Omega_-^\Gamma)^{-1}(\hat{D}_i^-)$ and $\bar{D}_{i+1}^+ = (\Omega_+^\Gamma)^{-1}(\hat{D}_{i+1}^+)$ be the copies of $\hat{D}_i^-$ and $\hat{D}_{i+1}^+$, respectively, in the homoclinic channel Γ. By making some arbitrarily small changes in the sizes of their exit and entry directions, we can alter the windows $\hat{D}_i^-$ and $\hat{D}_{i+1}^+$ such that $\hat{D}_i^-$ is correctly aligned with $\bar{D}_i^-$ under $(\Omega_\Gamma^-)^{-1}$, $\bar{D}_i^-$ is correctly aligned with $\bar{D}_{i+1}^+$ under the identity mapping, and $\bar{D}_{i+1}^+$ is correctly aligned with $\hat{D}_{i+1}^+$ under Ω_Γ^+.

We 'thicken' the l-dimensional windows $\bar{D}_i^-$ and $\bar{D}_{i+1}^+$ in Γ, which are correctly aligned under the identity mapping, to $(l+2n)$-dimensional windows that are correctly aligned under the identity map. We now explain the 'thickening' procedure.

First, we describe how to thicken $\bar{D}_i^-$ to a full dimensional window $\bar{R}_i^-$. We choose some $0 < \bar{\delta}_i^- < \varepsilon$ and $0 < \bar{\eta}_i^- < \varepsilon$. At each point $x \in \bar{D}_i^-$ we choose an n-dimensional closed ball $\bar{B}^-_{\bar{\delta}_i^-}(x)$ of radius $\bar{\delta}_i^-$ centered at x and contained in $W^u(x_-)$, where $x_- = \Omega_-^\Gamma(x)$. We take the union $\bar{\Delta}_i^- := \bigcup_{x\in\bar{D}_i^-} \bar{B}^u_{\bar{\delta}_i^-}(x)$. Note that $\bar{\Delta}_i^-$ is contained in $W^u(\Lambda)$ and is homeomorphic to an $(l+n)$-dimensional rectangle. We define the exit set and the entry set of this rectangle as follows:

$$(\bar{\Delta}_i^-)^{\text{exit}} := \bigcup_{x\in(\bar{D}_i^-)^{\text{exit}}} \bar{B}^u_{\bar{\delta}_i^-}(x) \cup \bigcup_{x\in\bar{D}_i^-} \partial\bar{B}^u_{\bar{\delta}_i^-}(x),$$
$$(\bar{\Delta}_i^-)^{\text{entry}} := \bigcup_{x\in(\bar{D}_i^-)^{\text{entry}}} \bar{B}^u_{\bar{\delta}_i^-}(x).$$

We consider the normal bundle N^+ to $W^u(\Lambda)$. At each point $y \in \bar{\Delta}_i^-$, we choose an n-dimensional closed ball $\bar{B}^+_{\bar{\eta}_i^-}(y)$ centered at y and contained in the image of $N_y^+ \subseteq T_yM$ under the exponential map $\exp_y : N_y^+ \to M$. We let $\bar{R}_i^- := \bigcup_{y\in\bar{\Delta}_i^-} \bar{B}^s_{\bar{\eta}_i^-}(y)$. By the Tubular Neighborhood Theorem (see, for example [2]), we have that for $\bar\eta^-_i > 0$ sufficiently small, the set $\bar R^-_i$ is a homeomorphic copy of an $(l+2n)$-rectangle. We now define the exit set and the entry set of $\bar R^-_i$ as follows:

$$(\bar R^-_i)^{\mathrm{exit}} := \bigcup_{y\in(\bar\Delta^-_i)^{\mathrm{exit}}} \bar B^s_{\bar\eta^-_i}(y),$$

$$(\bar R^-_i)^{\mathrm{entry}} := \bigcup_{y\in(\bar\Delta^-_i)^{\mathrm{entry}}} \bar B^s_{\bar\eta^-_i}(y) \cup \bigcup_{y\in(\bar\Delta^-_i)} \partial\bar B^s_{\bar\eta^-_i}(y).$$

Second, we describe in a similar fashion how to thicken $\bar D^+_{i+1}$ to a full dimensional window $\bar R^+_{i+1}$. We choose $0<\bar\delta^+_{i+1}<\varepsilon$ and $0<\bar\eta^+_{i+1}<\varepsilon$. We consider the $(l+n)$-dimensional rectangle $\bar\Delta^+_{i+1} := \bigcup_{x\in\bar D^+_{i+1}} \bar B^s_{\bar\eta^+_{i+1}}(x) \subseteq W^s(\Lambda)$, where $\bar B^+_{\bar\eta^+_{i+1}}(x)$ is the n-dimensional closed ball of radius $\bar\eta^+_{i+1}$ centered at x and contained in $W^s(x_+)$, with $x_+ = \Omega^\Gamma_+(x)$. The exit set and entry set of this window are defined as follows:

$$(\bar\Delta^+_{i+1})^{\mathrm{exit}} := \bigcup_{x\in(\bar D^+_{i+1})^{\mathrm{exit}}} \bar B^s_{\bar\eta^+_{i+1}}(x),$$

$$(\bar\Delta^+_{i+1})^{\mathrm{entry}} := \bigcup_{x\in(\bar D^+_{i+1})^{\mathrm{entry}}} \bar B^s_{\bar\eta^+_{i+1}}(x) \cup \bigcup_{x\in(\bar D^+_{i+1})} \partial\bar B^s_{\bar\eta^+_{i+1}}(x).$$

We let $\bar R^+_{i+1} := \bigcup_{y\in\bar\Delta^+_{i+1}} \bar B^u_{\bar\delta^+_{i+1}}(y)$, where $\bar B^-_{\bar\delta^+_{i+1}}(y)$ is the n-dimensional closed ball centered at y and contained in the image of $N^-_y \subseteq T_yM$ under the exponential map $\exp_y : N^-_y \to M$, and N^- is the normal bundle to $W^s(\Lambda)$. The Tubular Neighborhood Theorem implies that for $\bar\delta^+_{i+1}>0$ sufficiently small the set $\bar R^+_{i+1}$ is a homeomorphic copy of a $(l+2n)$-rectangle. The exit set and the entry set of $\bar R^+_{i+1}$ are defined by:

$$(\bar R^+_{i+1})^{\mathrm{exit}} := \bigcup_{y\in(\bar\Delta^+_{i+1})^{\mathrm{exit}}} \bar B^u_{\bar\delta^+_{i+1}}(y) \cup \bigcup_{y\in(\bar\Delta^+_{i+1}} \partial\bar B^u_{\bar\delta^+_{i+1}}(y),$$

$$(R^+_{i+1})^{\mathrm{entry}} := \bigcup_{y\in(\bar\Delta^+_{i+1})^{\mathrm{entry}}} \bar B^u_{\bar\delta^+_{i+1}}(y).$$

This completes the description of the thickening of the l-dimensional window $\bar D^-_i$ into an $(l+2n)$-dimensional window $\bar R^-_i$, and of the thickening of the l-dimensional window $\bar D^+_{i+1}$ into an $(l+2n)$-dimensional window $\bar R^+_{i+1}$. Note that, by construction, $\bar R^-_i$ and $\bar R^+_{i+1}$ are both contained in an ε-neighborhood of Γ.

Now we want to make $\bar R^-_i$ correctly aligned with $\bar R^+_{i+1}$ under the identity map. This is achieved by choosing $\bar\delta^+_{i+1}$ sufficiently small relative to $\bar\delta^-_i$, and by choosing $\bar\eta^-_i$ sufficiently small relative to $\bar\eta^+_{i+1}$. Thus, we have $\bar\delta^-_i > \bar\delta^+_{i+1}$ and $\bar\eta^-_i < \bar\eta^+_{i+1}$ (we stress that these inequalities alone may not suffice for the correct alignment). Choosing $\bar\delta^+_{i+1}$ and $\bar\eta^-_i$ small enough agrees with the constraints imposed by the Tubular Neighborhood Theorem.

Step 2. We take a negative iterate $F^{-M}(\bar R^-_i)$ of $\bar R^-_i$, where $M>0$. We have that $F^{-M}(\Gamma)$ is ε-close to Λ on a neighborhood in the C^1-topology, for all M sufficiently large. The vectors tangent to the fibers $W^u(x_-)$ in $\bar R^-_i$ are contracted, and the vectors transverse to $W^u(\Lambda)$ along $\bar R^-_i \cap W^u(\Lambda)$ are expanded by the derivative of F^{-M}. We choose and fix $M = N^-_i$ sufficiently large. We obtain that, in particular, $F^{-N^-_i}(\bar R^-_i)$ is ε-close to $D^-_i = F^{-N^-_i}(\hat D^-_i)$.

We now construct a window R_i^- about D_i^- that is correctly aligned with $F^{-N_i^-}(\bar{R}_i^-)$ under the identity. Note that each closed ball $\bar{B}^u_{\delta_i^-}(x)$, which is a part of $\bar{\Delta}_i^-$, gets exponentially contracted as it is mapped into $W^u(F^{-N_i^-}(x_-))$ by $F^{-N^-{}_i}$. By the Lambda Lemma (Proposition 2.4), each closed ball $\bar{B}^s_{\eta_i^-}(y)$ with $y \in \bar{\Delta}_i^-$, which is a part of $\bar{R}_i^-$, C^1-approaches a subset of $W^s(F^{-M}(y_-))$ under F^{-M}, as $M \to \infty$. For N_i^- sufficiently large, we may assume that $F^{-N_i^-}(\bar{B}^s_{\eta_i^-}(y))$ is ε-close to a subset of $W^s(F^{-N_i^-}(y^-))$ in the C^1-topology, for all $y \in \bar{\Delta}_i^-$. As $\hat{D}_i^-$ is correctly aligned with $\bar{D}_i^-$ under $(\Omega_-^\Gamma)^{-1}$, we have that $D_i^- = F^{-N_i^-}(\hat{D}_i^-)$ is correctly aligned with $F^{-N_i^-}(\bar{D}_i^-)$ under $(\Omega_-^{F^{-N_i^-}(\Gamma)})^{-1}$. In other words, D_i^- is correctly aligned under the identity mapping with the projection of $F^{-N_i^-}(\bar{D}_i^-)$ onto Λ along the unstable fibres. Let us consider $0 < \delta_i^- < \varepsilon$ and $0 < \eta_i^- < \varepsilon$.

To define the window R_i^- we use a local linearization of the normally hyperbolic invariant manifold. By Theorem 1 in [20], there exists a homeomorphisms h from an open neighborhood of $T\Lambda \times \{0\} \times \{0\}$ in $T\Lambda \oplus E^s \oplus E^u$ to a neighborhood of Λ in M such that $h \circ DF = F \circ h$. At each point $x \in D_i^-$ we consider a a rectangle $H_i^-(x)$ of the type $h(\{\tilde{x}\} \times \bar{B}^u_{\delta_i^-}(0) \times \bar{B}^s_{\eta_i^-}(0))$, where $\tilde{x} \in T\Lambda$ is such that $h(\{\tilde{x}\} \times \{0\} \times \{0\}) = x$, $\bar{B}^u_{\delta_i^-}(0)$ is the closed ball centered at 0 of radius δ_i^- in the unstable bundle E^u, and $\bar{B}^s_{\eta_i^-}$ is the closed ball centered at 0 of radius η_i^- in the stable bundle E^s. We set the exit and entry sets of $H_i^-(x)$ as $(H_i^-(x))^{\text{exit}} = h(\{\tilde{x}\} \times \partial \bar{B}^u_{\delta_i^-}(0) \times \bar{B}^s_{\eta_i^-}(0))$ and $(H_i^-(x))^{\text{entry}} = h(\{\tilde{x}\} \times \bar{B}^u_{\delta_i^-}(0) \times \partial \bar{B}^s_{\eta_i^-}(0))$.

Then we define the window R_i^- as follows:

$$\begin{aligned} R_i^- &= \bigcup_{x \in D_i^-} H_i^-(x), \\ (R_i^-)^{\text{exit}} &= \bigcup_{x \in (D_i^-)^{\text{exit}}} H_i^-(x) \cup \bigcup_{x \in D_i^-} (H_i^-(x))^{\text{exit}}, \\ (R_i^-)^{\text{entry}} &= \bigcup_{x \in (D_i^-)^{\text{entry}}} H_i^-(x) \cup \bigcup_{x \in D_i^-} (H_i^-(x))^{\text{entry}}. \end{aligned}$$

In order to ensure the correct alignment of R_i^- with $F^{-N_i^-}(\bar{R}_i^-)$ under the identity map, it is sufficient to choose δ_i^-, η_i^- such that $\bigcup_{x \in D_i^-} h(\{\tilde{x}\} \times \bar{B}^u_{\delta_i^-}(0) \times \{0\})$ is correctly aligned with $F^{-N_i^-}(\bar{\Delta}_i^-)$ under the identity map (the exit sets of both windows being in the unstable directions), and that each closed ball $F^{-N_i^-}(\bar{B}^s_{\eta_i^-})$ intersects R_i in a closed ball that is contained in the interior of $F^{-N_i^-}(\bar{B}^s_{\eta_i^-})$. The existence of suitable δ_i^-, η_i^- follows from the exponential contraction of $\bar{\Delta}_i^-$ under negative iteration, and from the Lambda Lemma applied to $\bar{B}^s_{\eta_i^-}(y)$ under negative iteration.

In a similar fashion, we construct a window R_{i+1}^+ contained in an ε-neighborhood of Λ such that $\bar{R}_{i+1}^+$ is correctly aligned with R_{i+1}^+ under $F^{N_{i+1}^+}$. The window R_{i+1}^+, and its entry and exit sets, are defined by:

$$R^+_{i+1} = \bigcup_{x\in D^+_{i+1}} H^+_{i+1}(x),$$
$$(R^+_{i+1})^{\rm exit} = \bigcup_{x\in (D^+_{i+1})^{\rm exit}} H^+_{i+1}(x) \cup \bigcup_{x\in D^+_{i+1}} (H^+_{i+1}(x))^{\rm exit},$$
$$(R^+_{i+1})^{\rm entry} = \bigcup_{x\in (D^+_{i+1})^{\rm entry}} H^+_{i+1}(x) \cup \bigcup_{x\in D^+_{i+1}} (H^+_{i+1}(x))^{\rm entry},$$

where $H^+_{i+1}(x) = h(\{\tilde{x}\} \times \bar{B}^u_{\delta^+_{i+1}}(0) \times \bar{B}^s_{\eta^+_{i+1}}(0))$, $(H^+_{i+1}(x))^{\rm exit}$, and $(H^+_{i+1}(x))^{\rm entry}$ are defined as before for some appropriate choices of radii $\delta^+_{i+1}, \eta^+_{i+1} > 0$.

Step 3. Suppose that we have constructed the window R^+_{i+1} about the l-dimensional rectangle $D^+_{i+1} \subseteq \Lambda$ and the window R^-_{i+1} about the l-dimensional rectangle $R^-_{i+1} \subseteq \Lambda$. Under positive iterations, the rectangle $\bar{B}^u_{\delta^+_{i+1}}(0) \times \bar{B}^s_{\eta^+_{i+1}}(0) \subseteq E^u \oplus E^s$ gets exponentially expanded in the unstable direction and exponentially contracted in the stable direction by DF. Thus $\bar{B}^u_{\delta^+_{i+1}}(0) \times \bar{B}^s_{\eta^+_{i+1}}(0)$ is correctly aligned with $\bar{B}^u_{\delta^-_{i+1}}(0) \times \bar{B}^s_{\eta^-_{i+1}}(0)$ under the power $DF^{N^0_{i+1}}$ of DF, provided N^0_{i+1} is sufficiently large. This implies that $F^{N^0_{i+1}}(h(\{\tilde{x}\} \times \bar{B}^u_{\delta^+_{i+1}}(0) \times \bar{B}^s_{\delta^+_{i+1}}(0)))$ is correctly aligned with $h(F^{N^0_{i+1}}(\tilde{x}) \times \bar{B}^u_{\delta^-_{i+1}}(0) \times \bar{B}^s_{\delta^-_{i+1}}(0))$ under the identity map (both rectangles are contained in $h(F^{N^0_{i+1}}(\tilde{x}) \times E^u \times E^s)$).

Since D^+_{i+1} is correctly aligned with D^-_{i+1} under $F^{N^0_i}$, the product property of correctly aligned windows implies that R^+_{i+1} is correctly aligned with R^-_{i+1} under $F^{N^0_{i+1}}$, provided that N^0_{i+1} is sufficiently large.

Step 4. At this step we will use the previous steps to construct a bi-infinite sequences of windows $\{R^\pm_i, \bar{R}^\pm_i\}_{i\in\mathbb{Z}}$ such that, for each i, the windows $\{R^\pm_i\}$ are obtained by thickening the rectangles $\{D^\pm_i\} \subseteq \Lambda$, the windows $\{\bar{R}^\pm_{i+1}\}$ are obtained by thickening some rectangles $\{\bar{D}^\pm_i\} \subseteq \Gamma$, and, moreover, R^-_i is correctly aligned with $\bar{R}^-_i$ under $F^{N^-_i}$, $\bar{R}^-_i$ is correctly aligned with $\bar{R}^+_{i+1}$ under the identity map, $\bar{R}^+_{i+1}$ is correctly aligned with R^+_{i+1} under $F^{N^+_{i+1}}$, and R^+_{i+1} is correctly aligned with R^-_{i+1} under $F^{N^0_i}$.

We can assume without loss of generality that Λ and Γ are compact. We fix an ε-neighborhood V of Λ. Using the compactness of Λ and Γ and the uniform boundedness of the iterates N^-_i, N^+_i, N^0_i, we now show how to choose the sizes of the stable and unstable components of the windows $\{R^\pm_i, \bar{R}^\pm_i\}_{i\in\mathbb{Z}}$ constructed in the previous steps in a uniformly bounded manner.

For each point x in Λ we consider a $(2n)$-dimensional window $h(\{\tilde{x}\} \times \bar{B}^u_\delta \times \bar{B}^s_\eta)$, for some $0 < \delta, \eta < \varepsilon$, where h is the local conjugacy between F and DF near Λ, and $h(\tilde{x}) = x$. Then $F^{N^0_i}(h(\{\tilde{x}\} \times \bar{B}^u_\delta \times \bar{B}^s_\eta)$ is correctly aligned with $h(\{F^{N^0_i}(\tilde{x})\} \times \bar{B}^u_\delta(0) \times \bar{B}^s_\eta(0))$, for all $n^0_1 \leq N^0_i \leq n^0_2$, provided that n^0_1 is chosen sufficiently large. For each i, we thicken D^+_i and D^-_i into full dimensional windows R^+_i and of R^-_i respectively, as described in Step 2, where for the sizes of the components of these windows we choose $\delta^\pm_i = \delta$ and $\eta^\pm_i = \eta$ for all i. Since D^+_i is correctly aligned with

D_i^- under $F^{N_i^0}$, then, as in Step 3, it follows that R_i^+ is correctly aligned with R_i^- under $F^{N_i^0}$.

We also define the set

$$\Upsilon^0 = \bigcup_{x\in\Lambda} h(\{\tilde{x}\} \times \bar{B}^s_\eta(0) \times \bar{B}^s_\delta(0)).$$

This set cannot be realized as a window since it does not have exit/entry directions associated to the Λ components. However, for each $x \in \Lambda$, the set $h(\{\tilde{x}\} \times \bar{B}^s_\eta(0) \times \bar{B}^u_\delta(0))$ is a well defined window, with the exit given by the hyperbolic unstable directions. Note that $\Upsilon^0(x) \subseteq h(\{\tilde{x}\} \times \bar{W}^u(x) \times \bar{W}^s(x))$ for each $x \in \Gamma$.

We let $\bar{\Delta}^- = \bigcup_{x\in\Gamma} \bar{B}^u_{\bar{\delta}^-}(x)$, with $\bar{B}^u_{\bar{\delta}^-}(x)$ being the closed ball centered at x of radius $\bar{\delta}^-$ in $W^u(x_-)$. For each point $y \in \bar{\Delta}^-$ we consider the closed ball $\bar{B}^s_{\bar{\eta}^-}(y)$ centered at y of radius $\bar{\eta}^-$ in the image under $\exp_y$ of the normal subspace N_y to $W^u(\Lambda)$ at y. Similarly, we let $\bar{\Delta}^+ = \bigcup_{x\in\Gamma} \bar{B}^s_{\bar{\eta}^+}(x)$, where $\bar{B}^u_{\bar{\eta}^+}(x) \subseteq W^s(x_+)$, and for each $y \in \bar{\Delta}^+$ we consider the closed ball $\bar{B}^u_{\bar{\delta}^+}(y)$ in the image under $\exp_y$ of the normal subspace N_y to $W^s(\Lambda)$ at y. We define the sets

$$\Upsilon^- = \bigcup_{y\in\bar{\Delta}^-} \bar{B}^s_{\bar{\eta}^-}(y),\ \Upsilon^+ = \bigcup_{y\in\bar{\Delta}^+} \bar{B}^u_{\bar{\delta}^+}(y).$$

These sets cannot be realized as windows as there are no well defined exit/entry directions associated to their Γ components. However, for each $x \in \Gamma$, the set $\Upsilon^-(x) = \bigcup_{y\in B^u_{\bar{\delta}^-}(x)} \bar{B}^s_{\bar{\eta}^-}(y)$ is a well defined $(2n)$-dimensional window, with the exit given by the hyperbolic unstable directions. Note that $\Upsilon^-(x) \subseteq \bigcup_{y\in W^u(x_-)} \exp_y(N_y)$. The intersection of $\bigcup_{y\in W^u(x_-)} \exp_y(N_y)$ with Υ^+ defines a window $\Upsilon^+(x)$ with the exit given by the hyperbolic unstable directions. Due to the compactness of Γ, there exist $\delta^\pm, \eta^\pm$ such that $\Upsilon^-(x)$ is correctly aligned with $\Upsilon^+(x)$ for all $x \in \Gamma$. We choose and fix such $\delta^\pm, \eta^\pm$. We define the windows $\bar{R}_i^\pm$ at Step 1 with the choices of $\delta_i^\pm = \delta^\pm$, and $\eta_i^\pm = \eta^\pm$, for all i. It follows that $\bar{R}_i^-$ is correctly aligned with $\bar{R}_i^+$ under the identity map for all i.

Due to the compactness of Γ and the uniform expansion and contraction of the hyperbolic directions, there exist n_1^-, n_1^+ such that, for all $N_i^- > n_1^-$, $N_i^+ > n_2^+$, we have $F^{-N_i^-}(\Gamma) \subseteq V$ and $F^{N_i^+}(\Gamma) \subseteq V$ for all $i \in \mathbb{Z}$, where V is the neighborhood of Λ where the local linearization is defined. For any such n_1^-, n_1^+, the assumptions of Lemma 6.1 provide us with some $n_2^- > n_i^-$, $n^+ > n_i^-$. Moreover, we choose n_1^-, n_1^+ such that for all $N_i^- > n_1^-$, $N_i^+ > n_2^+$ we have

(i) $\Upsilon^0(F^{-N_i^-}(x_-))$ is correctly aligned with $F^{-N_i^-}(\Upsilon^-) \cap h(\{\tilde{F}^{-N_i^-}(x_-)\} \times \bar{W}^u(F^{-N_i^-}(x_-)) \times \bar{W}^s(F^{-N_i^-}(x_-)))]$ under the identity map,

(ii) $F^{N_i^+}(\Upsilon^+(x))$ is correctly aligned with $\Upsilon^0 \cap h(\{\tilde{F}^{N_i^+}(x_+)\} \times \bar{W}^u(F^{N_i^+}(x_+)) \times \bar{W}^s(F^{N_i^+}(x_+)))]$ under the identity map.

From these choices, it follows that the windows R_i^-, R_i^+ constructed in Step 2 satisfy that R_i^- is correctly aligned with $\bar{R}_i^-$ under $F^{N_i^-}$, and $\bar{R}_i^+$ is correctly aligned with R_i^+ under $F^{N_i^+}$.

This concludes the construction of windows $\{R_i^\pm, \bar{R}_i^\pm\}_{i\in\mathbb{Z}}$ of uniform sizes, such that R_i^- is correctly aligned with $\bar{R}_i^-$ under $F^{N_i^-}$, $\bar{R}_i^-$ is correctly aligned with $\bar{R}_{i+1}^+$ under the identity map, $\bar{R}_{i+1}^+$ is correctly aligned with R_{i+1}^+ under $F^{N_{i+1}^+}$, and

R^+_{i+1} is correctly aligned with R^-_{i+1} under $F^{N^0_i}$. The windows $R^\pm_i$ are contained in ε-neighborhoods of the given rectangles $D^\pm_i$, respectively, and the windows $R^\pm_i$ are contained in ε-neighborhoods of some rectangles $\bar{D}^\pm_i$, respectively.

By Theorem 5.3, there exits an orbit $F^N(z)$ that visits the windows $\{R^\pm_i, \bar{R}^\pm_i\}_{i\in\mathbb{Z}}$ in the prescribed order. More precisely, if $F^{N_i}(z)$ is the corresponding point in $\bar{R}^-_i \cap R^+_{i+1}$, then $F^{N_i+N^+_{i+1}}(z)$ is in R^+_{i+1}, $F^{N_i+N^+_{i+1}+N^0_{i+1}}(z)$ is in R^-_{i+1}, and $F^{N_i+N^+_{i+1}+N^0_{i+1}+N^-_{i+1}}(z)$ is in $\bar{R}^-_{i+1} \cap \bar{R}^+_{i+2}$, for all i. This means that $N_{i+1} = N_i + N^+_{i+1} + N^0_{i+1} + N^-_{i+1}$ for all i. The existence of the shadowing orbit concludes the proof. □

Acknowledgement

Part of this work has been done while M.G. was visiting the Centre de Recerca Matemàtica, Barcelona, Spain, for whose hospitality he is very grateful.

References

[1] V.I. Arnold. Instability of dynamical systems with several degrees of freedom. Sov. Math. Doklady, 5:581–585, 1964.
[2] Keith Burns and Marian Gidea. Differential geometry and topology. With a view to dynamical systems. Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 2005.
[3] E. Canalias, A. Delshams, J. J. Masdemont, and P. Roldán. The scattering map in the planar restricted three body problem. Celestial Mech. Dynam. Astronom., 95(1-4):155–171, 2006.
[4] Maciej J. Capiński. Covering relations and the existence of topologically normally hyperbolic invariant sets. Discrete Contin. Dyn. Syst., 23(3):705–725, 2009.
[5] Maciej J. Capiński and Piotr Zgliczyński. Transition tori in the planar restricted elliptic three body problem, preprint, 2011.
[6] Jacky Cresson and Christophe Guillet. Hyperbolicity versus partial-hyperbolicity and the transversality-torsion phenomenon. J. Differential Equations, 244(9):2123–2132, 2008.
[7] A. Delshams, M. Gidea, and P. Roldan. Arnold's mechanism of diffusion in the spatial circular restricted three-body problem: A semi-numerical argument, 2010.
[8] Amadeu Delshams, Rafael de la Llave, and Tere M. Seara. A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbf{T}^2$. Comm. Math. Phys., 209(2):353–392, 2000.
[9] Amadeu Delshams, Rafael de la Llave, and Tere M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. Mem. Amer. Math. Soc., 179(844):viii+141, 2006.
[10] Amadeu Delshams, Rafael de la Llave, and Tere M. Seara. Geometric properties of the scattering map of a normally hyperbolic invariant manifold. Adv. Math., 217(3):1096–1153, 2008.
[11] Amadeu Delshams and Gemma Huguet. Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems. Nonlinearity, 22(8):1997–2077, 2009.
[12] Amadeu Delshams, Josep Masdemont, and Pablo Roldán. Computing the scattering map in the spatial Hill's problem. Discrete Contin. Dyn. Syst. Ser. B, 10(2-3):455–483, 2008.
[13] N. Fenichel. Asymptotic stability with rate conditions. Indiana Univ. Math. J., 23:1109–1137, 1973/74.
[14] Marian Gidea and Rafael de la Llave. Topological methods in the instability problem of Hamiltonian systems. Discrete Contin. Dyn. Syst., 14(2):295–328, 2006.
[15] Marian Gidea and Clark Robinson. Topologically crossing heteroclinic connections to invariant tori. J. Differential Equations, 193(1):49–74, 2003.
[16] Marian Gidea and Clark Robinson. Shadowing orbits for transition chains of invariant tori alternating with Birkhoff zones of instability. Nonlinearity, 20(5):1115–1143, 2007.
[17] Marian Gidea and Clark Robinson. Obstruction argument for transition chains of tori interspersed with gaps. Discrete Contin. Dyn. Syst. Ser. S, 2(2):393–416, 2009.
[18] M.W. Hirsch, C.C. Pugh, and M. Shub. Invariant manifolds, volume 583 of Lecture Notes in Math. Springer-Verlag, Berlin, 1977.

[19] Jean-Pierre Marco. A normally hyperbolic lambda lemma with applications to diffusion. Preprint, 2008.
[20] Charles Pugh and Michael Shub. Linearization of normally hyperbolic diffeomorphisms and flows. Invent. Math., 10:187–198, 1970.
[21] Clark Robinson. Dynamical systems. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition, 1999. Stability, symbolic dynamics, and chaos.
[22] Piotr Zgliczyński and Marian Gidea. Covering relations for multidimensional dynamical systems. J. Differential Equations, 202(1):32–58, 2004.

Departament de Matemática Aplicada I, ETSEIB-UPC, 08028 Barcelona, Spain
E-mail address: Amadeu.Delshams@upc.edu

Department of Mathematics, Northeastern Illinois University, Chicago, IL 60625, USA
E-mail address: mgidea@neiu.edu

Departament de Matemática Aplicada I, ETSEIB-UPC, 08028 Barcelona, Spain
E-mail address: Pablo.Roldan@upc.edu