Counting function of the embedded eigenvalues for some manifold with cusps, and magnetic laplacian

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Abstract

We consider a non compact, complete manifold $\mathbf{M}$ of finite area with cuspidal ends. The generic cusp is isomorphic to $\mathbf{X}\times]1,+\infty[$ with metric $ds^2 = (h + dy^2)/y^{2\delta}$. $\mathbf{X}$ is a compact manifold equipped with the metric h. For a one-form A on $\mathbf{M}$ such that in each cusp A is a non exact one-form on the boundary at infinity, we prove that the magnetic Laplacian $-\Delta_A = (id + A)^\star(id + A)$ satisfies the Weyl asymptotic formula with sharp remainder. We deduce an upper bound for the counting function of the embedded eigenvalues of the Laplace-Beltrami operator $-\Delta = -\Delta_0$. [1]

1 Introduction

We consider a smooth, connected n-dimensional Riemannian manifold $(\mathbf{M}, \mathbf{g})$, $(n \geq 2)$, such that

$$\mathbf{M} \;=\; \bigcup_{j=0}^{J} \mathbf{M}_j \quad (J \geq 1)\;, \tag{1.1}$$

where the $\mathbf{M}_j$ are open sets of $\mathbf{M}$. We assume that the closure of $\mathbf{M}_0$ is compact and that the other $\mathbf{M}_j$ are cuspidal ends of $\mathbf{M}$.

[1] *Keywords* : spectral asymptotics, magnetic Laplacian, embedded eigenvalues , cuspidal manifold.

This means that $\mathbf{M}_j \cap \mathbf{M}_k = \emptyset$, if $1 \leq j < k$, and that there exists, for any $j,\ 1 \leq j \leq J$, a closed compact $(n-1)$-dimensional Riemannian manifold $(\mathbf{X}_j, \mathbf{h}_j)$ such that $\mathbf{M}_j$ is isometric to $\mathbf{X}_j \times]a_j^2, +\infty[$, $(a_j > 0)$ equipped with the metric

$$ds_j^2 \ = \ y^{-2\delta_j}(\mathbf{h}_j \ + \ dy^2\ ) \ ; \quad (1/n < \delta_j \leq 1). \tag{1.2}$$

So there exists a smooth real one-form $A_j \in T^\star(\mathbf{X}_j)$, non exact, such that

$$\begin{cases} i)\ dA_j \neq 0 \\ \text{or} \\ ii)\ dA_j = 0 \text{ and } [A_j] \text{ is not integer.} \end{cases} \tag{1.3}$$

In $ii)$ we mean that there exists a smooth closed curve γ in $\mathbf{X}_j$ such that

$$\int_\gamma A_j \ \notin \ 2\pi\mathbb{Z} \ .$$

Then one can always find a smooth real one-form $A \ \in \ T^\star(\mathbf{M})$ such that

$$\forall\ j,\ 1 \leq j \leq J, \quad A \ = \ A_j \quad \text{on} \ \ \mathbf{M}_j \ . \tag{1.4}$$

We define the magnetic Laplacian, the Bochner Laplacian

$$-\Delta_A \ = \ (i\ d + A)^\star(i\ d + A)\ , \tag{1.5}$$

$(i = \sqrt{-1}\ ,\ (i\ d + A)u = i\ du + uA\ ,\ \forall\ u \ \in \ C_0^\infty(\mathbf{M}; \mathbb{C})$, the upper star, *, stands for the adjoint in $L^2(\mathbf{M})$) .

As $\mathbf{M}$ is a complete metric space, by Hopf-Rinow theorem $\mathbf{M}$ is geodesically complete, so it is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(\mathbf{M})$, containing in its domain $C_0^\infty(\mathbf{M}; \mathbb{C})$, the space of smooth and compactly supported functions. The spectrum of $-\Delta_A$ is gauge invariant : for any $f \in C^1(\mathbf{M}; \mathbb{R})$, $-\Delta_A$ and $-\Delta_{A+df}$ are unitary equivalent, hence they have the same spectrum.

For a self-adjoint operator P on a Hilbert space H, $\mathrm{sp}(P)$, $\mathrm{sp}_{\mathrm{ess}}(P)$, $\mathrm{sp}_p(P)$ and $\mathrm{sp}_d(P)$ will denote respectively the spectrum, the essential spectrum, the point spectrum and the discrete spectrum of P. We recall that $\mathrm{sp}(P) = \mathrm{sp}_{\mathrm{ess}}(P) \cup \mathrm{sp}_d(P)$, $\mathrm{sp}_d(P) \subset \mathrm{sp}_p(P)$ and $\mathrm{sp}_{\mathrm{ess}}(P) \cap \mathrm{sp}_d(P) = \emptyset$.

Theorem 1.1 *Under the above assumptions on* $\mathbf{M}$*, the essential spectrum of the Laplace-Beltrami operator on* $\mathbf{M}$*,* $-\Delta = -\Delta_0$ *is given by*

$$\begin{cases} \mathrm{sp}_{\mathrm{ess}}(-\Delta) = [0, +\infty[, & \text{if} \quad 1/n < \delta < 1 \\ \mathrm{sp}_{\mathrm{ess}}(-\Delta) = [\frac{(n-1)^2}{4}, +\infty[, & \text{if} \quad \delta = 1 \end{cases} \ . \tag{1.6}$$

When (1.3) and (1.4) are satisfied, the magnetic Laplacian $-\Delta_A$ *has a compact resolvent. The spectrum* $\mathrm{sp}(-\Delta_A) = \mathrm{sp}_d(-\Delta_A)$ *is a sequence of non-decreasing eigenvalues* $(\lambda_j)_{j\in\mathbb{N}^\star}$, $\lambda_j \leq \lambda_{j+1}$, $\lim_{j\to+\infty} \lambda_j = +\infty$, *such that the sequence of normalized eigenfunctions* $(\varphi_j)_{j\in\mathbb{N}^\star}$ *is a Hilbert basis of* $L^2(\mathbf{M})$. *Moreover* $\lambda_0 > 0$.

For any self-adjoint operator P with compact resolvent, and for any real λ, $N(\lambda, P)$ will denote the number of eigenvalues, (repeated according to their multiplicity), of P less then λ,

$$N(\lambda, P) \ = \ \mathrm{trace}\ (\chi_{]-\infty,\lambda[}(P)) \ , \tag{1.7}$$

(for any $I \subset \mathbb{R}$, $\chi_I(x) = 1$ if $x \in I$ and $\chi_I(x) = 0$ if $x \in \mathbb{R} \setminus I$).

The asymptotic behavior of $N(\lambda, -\Delta_A)$ satisfies the Weyl formula with the following sharp remainder.

Theorem 1.2 *Under the above assumptions on* $\mathbf{M}$ *and on* A, *we have the Weyl formula with remainder as* $\lambda \to +\infty$,

$$N(\lambda, -\Delta_A) \ = \ |\mathbf{M}|\frac{\omega_n}{(2\pi)^n}\lambda^{n/2} \ + \ \mathbf{O}(\mathbf{r}(\lambda)) \ , \tag{1.8}$$

with

$$r(\lambda) \ = \ \begin{cases} \lambda^{(n-1)/2}\ln(\lambda), & \text{if} \quad 1/(n-1) \leq \delta \\ \lambda^{1/(2\delta)}, & \text{if} \quad 1/n < \delta < 1/(n-1) \end{cases} \ , \tag{1.9}$$

$\delta = \min_{1\leq j\leq J} \delta_j$, $|\mathbf{M}|$ *is the Riemannian measure of* $\mathbf{M}$ *and* ω_d *is the euclidian volume of the unit ball of* $\mathbb{R}^d$, $\omega_d = \dfrac{\pi^{d/2}}{\Gamma(1+\frac{d}{2})}$.

The asymptotic formula (1.8) without remainder is given in [Go-Mo], and with remainder but only for $n = 2$ (and $\delta_j = 1$ for any $1 \leq j \leq J$) in [Mo-Tr].

The Laplace-Beltrami operator $-\Delta = -\Delta_0$ may have embedded eigenvalues in its essential spectrum $\mathrm{sp}_{\mathrm{ess}}(-\Delta)$. Let $N_{\mathrm{ess}}(\lambda, -\Delta)$ denote the number of eigenvalues of $-\Delta$, (counted according to their multiplicity), less then λ.

Theorem 1.3 *There exists a constant* $C_\mathbf{M}$ *such that, for any* $\lambda >> 1$,

$$N_{\mathrm{ess}}(\lambda, -\Delta) \ \leq \ |\mathbf{M}|\frac{\omega_n}{(2\pi)^n}\lambda^{n/2} \ + \ C_\mathbf{M} r_0(\lambda) \ , \tag{1.10}$$

with $r_0(\lambda)$ *defined by*

$$r_0(\lambda) \;=\; \begin{cases} \lambda^{\frac{n-1}{2}}\ln(\lambda), & \text{if} \quad 2/n \le \delta \le 1 \\ \lambda^{\frac{n-(n\delta-1)}{2}}, & \text{if} \quad 1/n < \delta < 2/n \end{cases} \quad ; \tag{1.11}$$

δ *is the one defined in Theorem 1.2 .*

The above upper bound proves that any eigenvalue of $-\Delta$ has finite multiplicity.

The estimate (1.10) is sharp when $n = 2$. There exist hyperbolic surfaces $\mathbf{M}$ of finite area so that

$$N_{\text{ess}}(\lambda, -\Delta) \;=\; |\mathbf{M}|\frac{\omega_2}{(2\pi)^2}\lambda + \Gamma_{\mathbf{M}}\lambda^{1/2}\ln(\lambda) + \mathbf{O}(\lambda^{1/2}) \;,$$

for some constant $\Gamma_{\mathbf{M}}$. See [Mul] for such examples.

Still in the case of surfaces, a compact perturbation of the metric of non compact hyperbolic surface $\mathbf{M}$ of finite area can destroy all embedded eigenvalues, see [Col1].

2 Proof

2.1 Proof of Theorem 1.1

Since the essential spectrum of an elliptic operator on a manifold is invariant by compact perturbation of the manifold, (see for example [Do-Li], Proposition 2.1), we can write

$$\text{sp}_{\text{ess}}(-\Delta_A) \;=\; \bigcup_{j=1}^{J} \text{sp}_{\text{ess}}(-\Delta_A^{\mathbf{M}_j, D}) \;, \tag{2.1}$$

where $-\Delta_A^{\mathbf{M}_j, D}$ denotes the self-adjoint operator on $L^2(\mathbf{M}_j)$ associated to $-\Delta_A$ with Dirichlet boundary conditions on the boundary $\partial\mathbf{M}_j$ of $\mathbf{M}_j$.

Let us consider a cusp $\mathbf{M}_j \;=\; \mathbf{X}_j \times]a_j^2, +\infty[$ equipped with the metric (1.2). Then for any $u \in C^2(\mathbf{M}_j)$,

$$-\Delta_A u \;=\; -y^{2\delta_j}\Delta_{A_j}^{\mathbf{X}_j} u \;-\; y^{n\delta_j}\partial_y(y^{(2-n)\delta_j}\partial_y u) \;, \tag{2.2}$$

where $\Delta_{A_j}^{\mathbf{X}_j}$ is the magnetic Laplacian on $\mathbf{X}_j$: if for local coordinates $\mathbf{h}_j = \sum_{k,\ell} G_{k\ell}\ dx_k dx_\ell$ and $A_j = \sum_{k=1}^{n-1} a_{j,k}\ dx_k$, then

$$-\Delta_{A_j}^{\mathbf{X}_j} \ = \ \frac{1}{\sqrt{det(G)}} \sum_{k,\ell} (i\partial_{x_k} + a_{j,k}) \left(\sqrt{det(G)} G^{k\ell} (i\partial_{x_\ell} + a_{j,\ell}) \right) \ .$$

We perform the change of variables $y = e^t$, and define the unitary operator $U : L^2(\mathbf{X}_j \times]2\ln(a_j), +\infty[) \ \rightarrow \ L^2(\mathbf{M}_j)$, where $]2\ln(a_j), +\infty[$ is equipped with the standard euclidian metric dt^2, by $U(f) = y^{(n\delta_j - 1)/2} f$. Thus $L^2(\mathbf{M}_j)$ is unitary equivalent to $L^2(\mathbf{X}_j \times]2\ln(a_j), +\infty[)$, and

$$-U^\star \Delta_A U f = -e^{2\delta_j t} \Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n\delta_j - 1)[3 + \delta_j(n-4)]}{4} e^{2t(\delta_j - 1)} f - \partial_t (e^{2t(\delta_j - 1)} \partial_t f) \ . \tag{2.3}$$

Let us denote by $(\mu_\ell(j))_{\ell \in \mathbb{N}}$ the increasing sequence of eigenvalues of $-\Delta_{A_j}^{\mathbf{X}_j}$, each eigenvalue repeated according to its multiplicity. Then $-\Delta_A^{\mathbf{M}_j, D}$ is unitary equivalent to $\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D$,

$$\mathrm{sp}(-\Delta_A^{\mathbf{M}_j, D}) \ = \ \mathrm{sp}(\bigoplus_{\ell=0}^{+\infty} L_{j,\ell}^D) \ , \tag{2.4}$$

where $L_{j,\ell}^D$ is the Dirichlet operator on $L^2(]2\ln(a_j), +\infty[)$ associated to

$$L_{j,\ell} \ = \ e^{2\delta_j t} \mu_\ell(j) \ + \ \frac{(n\delta_j - 1)}{4} [3 + \delta_j(n-4)] e^{2t(\delta_j - 1)} \ - \ \partial_t (e^{2t(\delta_j - 1)} \partial_t) \ . \tag{2.5}$$

If $\mu_\ell(j) > 0$ then $\mathrm{sp}(L_{j,\ell}^D) = \mathrm{sp}_d(L_{j,\ell}^D) = \{\mu_{\ell,k}(j);\ k \in \mathbb{N}\}$, where $(\mu_{\ell,k}(j))_{k \in \mathbb{N}}$ is the increasing sequence of eigenvalues of $L_{j,\ell}^D$, $\lim_{k \to +\infty} \mu_{\ell,k}(j) = +\infty$.

If $\mu_\ell(j) = 0$ then $\mathrm{sp}(L_{j,\ell}^D) = \mathrm{sp}_{\mathrm{ess}}(L_{j,\ell}^D) = [\alpha_n, +\infty[$, with $\alpha_n = 0$ if $\delta_j < 1$, and $\alpha_n = (n-1)^2/4$ if $\delta_j = 1$.

Since we have $\mu_0(j) = 0$ when $A = 0$, we get that $\mathrm{sp}_{\mathrm{ess}}(-\Delta_0) = [\alpha_n, +\infty[$.

If A satisfies assumptions (1.3) and (1.4), then $0 < \mu_0(j) \leq \mu_\ell(j)$ for all j and ℓ, (see for example [Hel]), so $\mathrm{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \{\mu_{\ell,k}(j);\ (\ell, k) \in \mathbb{N}^2\}$. As $\lim_{\ell \to +\infty} \mu_{\ell,k}(j) = +\infty$, each $\mu_{\ell,k}(j)$ is an eigenvalue of $-\Delta_A^{\mathbf{M}_j, D}$ of finite multiplicity, so $\mathrm{sp}(-\Delta_A^{\mathbf{M}_j, D}) = \mathrm{sp}_d(-\Delta_A^{\mathbf{M}_j, D})$. Therefore, we get that $\mathrm{sp}_{\mathrm{ess}}(-\Delta_A) = \emptyset$ $\square$

2.2 Proof of Theorem 1.2

We proceed as in [Mo-Tr].

We begin by establishing formula (1.8) for $\mathbf{M}_j$, with $-\Delta_A^{\mathbf{M}_j,D}$ defined in (2.1), instead of $-\Delta_A$. When $\delta_j = 1$ we make the same change of variables and functions as in the proof of Theorem 1.1, but when $1/n < \delta_j < 1$, we set $y = [(1-\delta_j)t]^{1/(1-\delta_j)}$, and define the unitary operator

$U :\ L^2(\mathbf{X}_j \times]\frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, +\infty[) \ \to\ L^2(\mathbf{M}_j)$, by $\ U(f) = y^{(n-1)\delta_j/2} f.$

Then when $1/n < \delta_j < 1$,

$$-U^\star \Delta_A U f = -[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} \Delta_{A_j}^{\mathbf{X}_j} f + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2 t^2} f - \partial_t^2 f\ . \quad (2.6)$$

As a matter of fact,
$-U^\star y^{n\delta_j} \partial_y [y^{(2-n)\delta_j} \partial_y U(f)] = -y^{(n+1)\delta_j/2} \partial_y [y^{(3-n)\delta_j/2} \partial_y f] - \frac{(n-1)\delta_j}{2} y^{2\delta_j - 1} \partial_y f + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4} y^{-2(1-\delta_j)} f$,
then using that $y^{\delta_j}\partial_y = \partial_t$ and that $t^\rho \partial_t = \partial_t(t^\rho .) - \rho t^{\rho-1}$, we get easily (2.6).

Equality (2.4) still holds when $L_{j,\ell}^D$ is the Dirichlet operator on $L^2(]\frac{a^{2(1-\delta_j)}}{1-\delta_j}, +\infty[)$ associated to

$$L_{j,\ell} \ =\ \mu_\ell(j)[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} \ +\ \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2 t^2} \ -\ \partial_t^2\ . \quad (2.7)$$

From now on, any constant depending only on δ_j and on $\min_j \mu_0(j)$ will be invariably denoted by C .

As in [Mo-Tr], we will follow Titchmarsh's method. Using Theorem 7.4 in [Tit] page 146, we prove the following Lemma.

Lemma 2.1 *There exists $C > 1$ so that for any $\lambda >> 1$ and any $\ell \in K_\lambda = \{l \in \mathbb{N};\ \mu_\ell(j) \in [0, \lambda/\min_j a_j^2[\}$,*

$$|N(\lambda, L_{j,\ell}^D) \ -\ \frac{1}{\pi} w_{j,\ell}(\lambda)| \ \le\ C \ln(\lambda)\ , \quad (2.8)$$

with $\ w_{j,\ell}(\mu) \ =\ \int_{\alpha_j}^{+\infty} [\mu - V_{j,\ell}(t)]_+^{1/2}\, dt \ =\ \int_{\alpha_j}^{T_j(\mu)} [\mu - V_{j,\ell}(t)]_+^{1/2}\, dt.$

The potential $V_{j,\ell}$ is defined as following:

$$\begin{cases} \text{if } \delta_j = 1 & V_{j,\ell}(t) = \mu_\ell(j)e^{2t} + \frac{(n-1)^2}{4} \\ \text{if } 1/n < \delta_j < 1 & V_{j,\ell}(t) = \mu_\ell(j)[(1-\delta_j)t]^{\frac{2\delta_j}{1-\delta_j}} + \frac{(n-1)\delta_j[(n-3)\delta_j+2]}{4(1-\delta_j)^2}t^{-2} \end{cases}, \tag{2.9}$$

and

$$\begin{cases} \text{if } \delta_j = 1 & \alpha_j = 2\ln(a_j), \quad T_j(\mu) = \dfrac{1}{2}\ln\left(\mu/\mu_0(j)\right) \\ \text{if } 1/n < \delta_j < 1 & \alpha_j = \frac{a_j^{2(1-\delta_j)}}{1-\delta_j}, \quad T_j(\mu) = \dfrac{1}{1-\delta_j}\left(\dfrac{\mu}{\mu_0(j)}\right)^{\frac{1-\delta_j}{2\delta_j}} \end{cases}. \tag{2.10}$$

Proof of Lemma 2.1

When $1/n < \delta_j < 1$, by enlarging $\mathbf{M}_0$ and reducing $\mathbf{M}_j$, we can take α_j large enough so that $V_{j,\ell}(t)$ is an increasing function on $[\alpha_j, +\infty[$ and $\lambda/\mu_\ell(j) >> 1$ when $\ell \in K_\lambda$. Then, if $\alpha_j \leq Y < X(\lambda) = V_{j,\ell}^{-1}(\lambda)$, following the proof of Theorem 7.4 in [Tit] pages 146-147, we get that

$$|N(\lambda, L_{j,\ell}^D) - \frac{1}{\pi}w_{j,\ell}(\lambda)| \leq \tag{2.11}$$

$$C[\ln(\lambda - V_{j,\ell}(\alpha_j)) - \ln(\lambda - V_{j,\ell}(Y)) + (X(\lambda) - Y)(\lambda - V_{j,\ell}(Y)) + 1].$$

When $\delta_j = 1$, we choose $Y = X(\lambda) - \frac{\sqrt{\ln\lambda}}{\sqrt{\lambda}}$.

When $1/n < \delta_j < 1$, we choose $Y = X(\lambda) - \dfrac{\sqrt{\ln\lambda}}{\sqrt{\lambda}}\left(\dfrac{\lambda}{\mu_\ell(j)}\right)^{\frac{1-\delta_j}{4\delta_j}}$;

$(X(\lambda) \sim \frac{1}{1-\delta_j}\left(\frac{\lambda}{\mu_\ell(j)}\right)^{\frac{1-\delta_j}{2\delta_j}}$) $\square$

Let us recall the sharp asymptotic Weyl formula of L. Hörmander [Hor1] (see also [Hor2]).

Theorem 2.2 *There exists $C > 0$ so that for any $\mu >> 1$*

$$|N(\mu, -\Delta_{A_j}^{\mathbf{X}_j}) - \frac{\omega_{n-1}}{(2\pi)^{n-1}}|\mathbf{X}_j|\mu^{(n-1)/2}| \leq C\mu^{(n-2)/2}. \tag{2.12}$$

Lemma 2.3 *There exists $C > 0$ such that for any $\lambda >> 1$*

$$|N(\lambda, -\Delta_A^{\mathbf{M}_j,D}) - \frac{\omega_n}{(2\pi)^n}|\mathbf{M}_j|\lambda^{n/2}| \leq \tag{2.13}$$

$$C\begin{cases} \lambda^{(n-1)/2}\ln(\lambda), & \text{if } \; 1/(n-1) \leq \delta_j \leq 1 \\ \lambda^{1/(2\delta_j)}, & \text{if } \; 1/n < \delta_j < 1/(n-1) \end{cases}.$$

Proof of Lemma 2.3 By the formula (2.4),

$$N(\lambda, -\Delta_A^{\mathbf{M}_j,D}) \;=\; \sum_{\ell=0}^{+\infty} N(\lambda, L_{j,\ell}^D)\;. \tag{2.14}$$

As $N(\lambda, L_{j,\ell}^D) = 0$ when $\ell \notin K_\lambda$, (K_λ is defined in Lemma 2.1), the estimates (2.8), (2.12) and formula (2.14) prove that

$$|N(\lambda, -\Delta_A^{\mathbf{M}_j,D}) \;-\sum_{\ell=0}^{+\infty}\frac{1}{\pi}w_{j,\ell}(\lambda)| \;\leq\; C\lambda^{(n-1)/2}\ln(\lambda)\;. \tag{2.15}$$

Let us denote

$$\Theta_j(\lambda) \;=\; \sum_{\ell=0}^{+\infty}\frac{1}{\pi}w_{j,\ell}(\lambda) \quad \text{and} \quad R_j(\mu) \;=\; \sum_{\ell=0}^{+\infty}[\mu-\mu_\ell(j)]_+^{1/2}\;. \tag{2.16}$$

As $R_j(\mu) = \dfrac{1}{2}\displaystyle\int_0^{+\infty}[\mu-s]_+^{-1/2}N(s, -\Delta_{A_j}^{\mathbf{X}_j})ds,$
the Hörmander estimate (2.12) entails the following one.

There exists a constant $C>0$ such that, for any $\mu >> 1$,

$$|R_j(\mu) \;-\; \frac{\omega_{n-1}}{2(2\pi)^{n-1}}|\mathbf{X}_j|\int_0^{+\infty}[\mu-s]_+^{-1/2}s^{(n-1)/2}ds| \;\leq\; C\mu^{(n-1)/2}\;. \tag{2.17}$$

Writing in (2.9)

$$V_{j,\ell}(t) = \mu_\ell(j)V_j(t) \;+\; W_j(t)\;, \tag{2.18}$$

we get that $\Theta_j(\lambda) \;=\; \dfrac{1}{\pi}\displaystyle\int_{\alpha_j}^{T_j(\lambda)} V_j^{1/2}(t)R_j(\frac{\lambda-W_j(t)}{V_j(t)})dt\;.$
So according to (2.17)

$$|\Theta_j(\lambda) \;-\; \frac{\omega_{n-1}\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})}{(2\pi)^n\Gamma(1+\frac{n}{2})}|\mathbf{X}_j|\int_{\alpha_j}^{T_j(\lambda)}\frac{(\lambda-W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)}dt| \;\leq \tag{2.19}$$

$$C\int_{\alpha_j}^{T_j(\lambda)}\frac{(\lambda-W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)}dt\;.$$

From the definitions (2.9) and (2.18) we get that

$$|\int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{n/2}}{V_j^{(n-1)/2}(t)} dt \ - \ \lambda^{n/2} \frac{1}{(\delta_j n - 1) a_j^{2(\delta_j n-1)}}| \ \leq \ C\lambda^{(n-1)/2} \ , \quad (2.20)$$

and

$$\int_{\alpha_j}^{T_j(\lambda)} \frac{(\lambda - W_j(t))^{(n-1)/2}}{V_j^{(n-2)/2}(t)} dt \ \leq \quad (2.21)$$

$$C \left\{ \begin{array}{lll} \lambda^{(n-1)/2} & \text{if} & 1/(n-1) < \delta_j \leq 1 \\ \lambda^{(n-1)/2} \ln \lambda & \text{if} & 1/(n-1) = \delta_j \\ \lambda^{1/(2\delta_j)} & \text{if} & 1/n < \delta \leq 1/(n-1) \end{array} \right. .$$

As $|\mathbf{M}_j| = \dfrac{|\mathbf{X}_j|}{(\delta_j n - 1) a_j^{2(\delta_j n - 1)}}$, we get (2.13) from (2.15), (2.16) and (2.19)—(2.21) □

To achieve the proof of Theorem 1.2, we proceed as in [Mo-Tr].

We denote $\mathbf{M}_0^0 = \mathbf{M} \setminus (\bigcup_{j=1}^{J} \overline{\mathbf{M}_j})$, then

$$\mathbf{M} \ = \ \overline{\mathbf{M}_0^0} \bigcup \left(\bigcup_{j=1}^{J} \overline{\mathbf{M}_j} \right) \ . \quad (2.22)$$

Let us denote respectively by $-\Delta_A^{\Omega,D}$ and by $-\Delta_A^{\Omega,N}$ the Dirichlet operator and the Neumann-like operator on an open set Ω of $\mathbf{M}$ associated to $-\Delta_A$. $-\Delta_A^{\Omega,N}$ is the Friedrichs extension defined by the associated quadratic form $q_A^{\Omega}(u) \ = \ \int_{\Omega} |idu + Au|^2 d\mathbf{m}$, $u \in C^\infty(\overline{\Omega}; \mathbb{C})$, u with compact support in $\overline{\Omega}$. ($d\mathbf{m}$ is the n-form volume of $\mathbf{M}$).
The minimax principle and (2.22) imply that

$$N(\lambda, -\Delta_A^{\mathbf{M}_0^0,D}) + \sum_{1\leq j\leq J} N(\lambda, -\Delta_A^{\mathbf{M}_j,D}) \leq N(\lambda, -\Delta_A) \quad (2.23)$$

$$\leq N(\lambda, -\Delta_A^{\mathbf{M}_0^0,N}) + \sum_{1\leq j\leq J} N(\lambda, -\Delta_A^{\mathbf{M}_j,N})$$

The Weyl formula with remainder, (see [Hor2] for Dirichlet boundary condition and [Sa-Va] p. 9 for Neumann-like boundary condition), gives that

$$N(\lambda, -\Delta_A^{\mathbf{M}_0^0,Z}) = \frac{\omega_n}{(2\pi)^n}|\mathbf{M}_0^0|\lambda^{n/2}+\mathbf{O}(\lambda^{(n-1)/2}) \ ; \quad (\text{for } Z = D \text{ and for } Z = N) \ . \tag{2.24}$$

For $1 \leq j \leq J$, the asymptotic formula for $N(\lambda, -\Delta_A^{\mathbf{M}_j,N})$,

$$N(\lambda, -\Delta_A^{\mathbf{M}_j,N}) \ = \ \frac{\omega_n}{(2\pi)^n}|\mathbf{M}_j|\lambda^{n/2} + \mathbf{O}(r(\lambda)) \ , \tag{2.25}$$

is obtained as for the Dirichlet case (2.13) by noticing that $N(\lambda, L_{j,\ell}^D) \leq N(\lambda, L_{j,\ell}^N) \leq N(\lambda, L_{j,\ell}^D) + 1$, where $L_{j,\ell}^D$ and $L_{j,\ell}^N$ are Dirichlet and Neumann-like operators on a half-line $I =]\alpha_j, +\infty[$, associated to the same differential Schrödinger operator $L_{j,\ell}$ defined by (2.5) when $\delta_j = 1$, and by (2.7) otherwise. (The Neumann-like boundary condition is of the form $\partial_t u(\alpha_j) + \beta_j u(\alpha_j) = 0$ because of the change of functions performed by $U^\star$).

We get (1.8) from (2.13) and (2.23)— (2.25) $\square$

2.3 Proof of Theorem 1.3

Lemma 2.4 *For any $j \in \{1, \ldots, J\}$, there exists a one-form A_j satisfying (1.3) and the following property.*

There exists $\tau_0 = \tau_0(A_j) > 0$ and $C = C(A_j) > 0$ such that for any $\lambda >> 1$, if $e(\tau, j) = \inf_{u \in C^\infty(\mathbf{X}_j),\ \|u\|_{L^2(\mathbf{X}_j)}=1} \|idu + \tau u A_j\|^2_{L^2(\mathbf{X}_j)}$ denotes the first eigenvalue of $-\Delta_{\tau A_j}^{\mathbf{X}_j}$, then

$$e(\tau, j) \ \geq \ C\tau^2 \ ; \quad \forall \ \tau \in]0, \tau_0] \ . \tag{2.26}$$

Proof of Lemma 2.4. When $n = 2$, we can take $A_j = \omega_j d\mathbf{x}_j$, ($d\mathbf{x}_j$ is the $(n-1)$-form volume of $\mathbf{X}_j$), for some constant $\omega_j \in \mathbb{R} \setminus \frac{2\pi}{|\mathbf{X}_j|}\mathbb{Z}$, then $e(\tau, j) = \tau^2\omega_j^2$ for small $|\tau|$.

When $n \geq 3$, we have $e(0, j) = 0$, $\partial_\tau e(0, j) = 0$ and

$$\partial_\tau^2 e(0, j) \ = \ \int_{\mathbf{X}_j} \Big[\ |A_j|^2 - (-\Delta_0^{\mathbf{X}_j})^{-1}(d^\star A_j).(d^\star A_j) \ \Big] \ d\mathbf{x}_j \ .$$

($d^\star$ is the adjoint of d defined on functions, and $(-\Delta_0^{\mathbf{X}_j})^{-1}$ is the inverse of the Laplace-Beltrami operator on functions, which is well-defined on the space $\{f \in L^2(\mathbf{X}_j);\ \int_{\mathbf{X}_j} f d\mathbf{x}_j = 0\}$).

To the non-negative quadratic form $A_j \to \partial_\tau^2 e(0,j)$, we associate a self-adjoint operator P on $T^\star(\mathbf{X}_j)$, which is a pseudodifferential operator of order 0 with principal symbol, the square matrix $p_0(x,\xi) = (p_0^{ik}(x,\xi))_{1\le i,k\le n-1}$ defined as follows. In local coordinates, if $\mathbf{h}_j = G_{ik}(x)dx_i dx_k$, then

$$p_0^{ik}(x,\xi) \ = \ G^{ik}(x) \ - \ \sum_{\ell,m} G^{im}(x)G^{\ell k}(x)\frac{\xi_m}{|\xi|}\frac{\xi_\ell}{|\xi|} \ ; \quad (|\xi|^2 = \sum_{\ell,m} G^{m\ell}(x)\xi_m\xi_\ell) \ .$$

As the non-negative symmetric matrix $p_0(x,\xi)$ is not the zero matrix, there exists A_j such that $P(A_j) \neq 0$ and by the positivity of P, $\partial_\tau^2 e(0,j) = \int_{\mathbf{X}_j} \langle P(A_j)|A_j\rangle d\mathbf{x}_j \ > 0$ □

Lemma 2.5 *For a one-form A satisfying (1.4), there exists a constant $C_A > 0$ such that, if u is a function in $L^2(\mathbf{M})$ such that $du \in L^2(\mathbf{M})$ and*

$$\forall j = 1,\ldots,J, \quad \int_{\mathbf{X}_j} u(x_j,y)d\mathbf{x}_j = 0 \ , \quad \forall y \in]a_j^2, +\infty[\ , \tag{2.27}$$

then $\forall \tau \in]0,1]$,

$$\|idu + \tau uA\|^2_{L^2(\mathbf{M})} \ \le \ (1+\tau C_A)\|idu\|^2_{L^2(\mathbf{M})} + C_A\|u\|^2_{L^2(\mathbf{M})} \ . \tag{2.28}$$

Proof of Lemma 2.5. First we remark that the inequality

$$|idu + \tau uA|^2 \le (1+\rho)|du|^2 + (1+\rho^{-1})|\tau uA|^2 \tag{2.29}$$

is satisfied for any $\rho > 0$.

For $\rho = \tau$ we get that there exists a constant $C_A^0 > 0$, depending only on $A/\mathbf{M}_0$, such that

$$\|idu + \tau uA\|^2_{L^2(\mathbf{M}_0)} \ \le \ (1+\tau)\|idu\|^2_{L^2(\mathbf{M}_0)} + \tau C_A^0\|u\|^2_{L^2(\mathbf{M}_0)} \ . \tag{2.30}$$

We get also for $\rho = \tau$ that for any $j \in \{1,\ldots,J\}$,

$$\int_{a_j^2}^{+\infty} \|idu + \tau uA\|^2_{L^2(\mathbf{X}_j)} y^{(2-n)\delta_j} dy \ \le \tag{2.31}$$

$$\int_{a_j^2}^{+\infty} \Big((1+\tau)\|idu\|^2_{L^2(\mathbf{X}_j)} + \tau C_A^j \|u\|^2_{L^2(\mathbf{X}_j)}\Big)\, y^{(2-n)\delta_j} dy \ ,$$

for some constant C_A^j depending only on A/X_j.

But (2.27) implies that

$$\|u\|^2_{L^2(\mathbf{X}_j)} \leq \frac{1}{\mu_1(j)} \|idu\|^2_{L^2(\mathbf{X}_j)} \ , \tag{2.32}$$

with $(\mu_\ell(j))_{\ell\in\mathbb{N}}$ the sequence of eigenvalues of Laplace-Beltrami operator on $\mathbf{X}_j,\ \mu_0(j) = 0 < \mu_1(j) \leq \mu_2(j) \leq \ldots$. So if (2.27) is satisfied then (2.31) and (2.32) imply that

$$\|idu + \tau u A\|^2_{L^2(\mathbf{M}_j)} \ \leq\ (1 + \tau c_A^j)\|idu\|^2_{L^2(\mathbf{M}_j)} \ , \tag{2.33}$$

for some constant c_A^j depending only on A/X_j.

The existence of a constant $C_A > 0$ satisfying the inequality (2.28) follows from (2.30) and (2.33) for $j = 1, \ldots J$ $\square$

Lemma 2.6 *When A satisfies (1.3), (1.4) and Lemma 2.4 , then as $\lambda \to +\infty$, the following Weyl formula is satisfied.*

$$N(\lambda, -\Delta_{(\lambda^{-\rho}A)}) \ =\ |\mathbf{M}| \frac{\omega_n}{(2\pi)^n} \lambda^{n/2} \ +\ \mathbf{O}(\mathbf{r_0}(\lambda)) \ , \tag{2.34}$$

with

$$\rho \ =\ \begin{cases} 1/2, & \text{if} \quad 2/n \leq \delta \leq 1 \\ (n\delta - 1)/2, & \text{if} \quad 1/n < \delta < 2/n \end{cases} \ , \tag{2.35}$$

δ and ω_d are as in Theorem 1.2, and the function $r_0(\lambda)$ is the one defined by (1.11) .

Proof of Lemma 2.6. Since A satisfies Lemma 2.4, we have

$$C/\lambda^{2\rho} \leq \mu_0(j) \quad \text{and} \quad C \leq \mu_1(j) \ ,$$

where $(\mu_\ell(j))_{\ell\in\mathbb{N}}$ denotes now the increasing sequence of eigenvalues of $-\Delta^{\mathbf{X}_j}_{\lambda^{-\rho}A_j}$. Hence we can mimick the proof of Theorem 1.2. More precisely Lemma 2.1 holds for any $\ell \in K_\lambda, \ell \neq 0$, and to get the result it only remains to prove that we have, for $L_{j,0}$ defined by (2.5) if $\delta_j = 1$, and by (2.7) otherwise,

$$N(\lambda, L^D_{j,0}) \ =\ \mathbf{O}(r_0(\lambda)) \ .$$

This can easily be done as follows.

When $\delta_j = 1,\ (\rho = 1/2)$, it is easy to see that

$$N(\lambda, L^D_{j,0}) \ \le\ N(\lambda + C, L^{D,\lambda}) \ \le\ C\lambda^{1/2}\ln(\lambda)\ ,$$

where $L^{D,\lambda}$ is the Dirichlet operator on $]0,+\infty[$ associated to $\dfrac{C}{\lambda}e^{2t} - \partial_t^2$.

When $0 < \delta_j < 1$, by scaling we have that

$$N(\lambda, L^D_{j,0}) \ \le\ N((\lambda + C)^{1+2\rho(1-\delta_j)}, L^D) \ \le\ C\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)}\ ,$$

where L^D is the Dirichlet operator on $]0,+\infty[$ associated to $\dfrac{1}{C^2}t^{\frac{2\delta_j}{1-\delta_j}} - \partial_t^2$.
When $2/n \le \delta < 1$, as $2/n \le \delta \le \delta_j$, then
$\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} = \lambda^{(2-\delta_j)/(2\delta_j)} \le \lambda^{(2-\delta)/(2\delta)} \le \lambda^{(n-1)/2} = \mathbf{O}(r_0(\lambda))$.
When $1/n < \delta < 2/n$, as $\delta \le \delta_j$, then
$\lambda^{(1+2\rho(1-\delta_j))/(2\delta_j)} \le \lambda^{(1+2\rho(1-\delta))/(2\delta)} = \lambda^{(n-(n\delta-1))/2} = \mathbf{O}(r_0(\lambda))$ $\square$

To achieve the proof of Theorem 1.3, we take a one-form A satisfying the assumptions of Lemma 2.6.

We remark that any eigenfunction u of the Laplace-Beltrami operator $-\Delta$ on $\mathbf{M}$ associated to an eigenvalue in $]\inf \mathrm{sp}_{\mathrm{ess}}(-\Delta), +\infty[$, satisfies (2.27). So if H_λ is the subspace of $L^2(\mathbf{M})$ spanned by eigenfunctions of $-\Delta$ associated to eigenvalues in $]0,+\infty[$, then, by (2.28) of Lemma 2.5 with $\tau = 1/\lambda^\rho$, with ρ defined by (2.35), we have

$$\forall u \in H_\lambda, \quad \|idu + \frac{1}{\lambda^\rho}uA\|^2_{L^2(\mathbf{M})} \ \le\ (1 + \frac{C_A}{\lambda^\rho})\|du\|^2_{L^2(\mathbf{M})} + C_A\|u\|^2_{L^2(\mathbf{M})} \quad (2.36)$$

So

$$dim(H_\lambda) \ \le\ N((1 + \frac{C_A}{\lambda^\rho})\lambda + C_A, -\Delta_{(\lambda^{-\rho}A)})\ . \qquad (2.37)$$

The estimates (2.34) and (2.37) prove (1.10), by noticing that $\lambda^{n/2}/\lambda^\rho = \mathbf{O}(r_0(\lambda))$ $\square$

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