Application of Core Identification to Starshape Constant Conductivity Inclusion

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ABSTRACT

The problem of determining the interface separating a constant conductivity inclusion with star-shaped support from boundary measurement data of a solution of the corresponding PDEs is considered. An equivalent statement as a nonlinear integral equation is obtained. The problem is analyzed and implemented numerically using truncated Fourier series expansion. Numerical experiments based on simplified iteratively regularized Gauss-Newton method (sIRGNM) are presented.

Keywords: impedance tomography, inverse source problem, nonlinear integral equation, Fourier series expansion, simplified regularized Gauss-Newton.

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1 Introduction

Consider the following boundary value problem

$$\Delta u = S(x), \quad \text{in } \Omega_1 \subset \mathbf{R}^2; \quad u = 0 \quad \text{on} \quad \partial\Omega_1, \tag{1.1}$$

where Ω_1 is a unit disc with boundary $\partial\Omega_1$, and $\Omega_S \subset \Omega_1$ is the support of S, i.e. $\Omega_S = \text{supp}(S)$. The inverse problem consist of identifying the shape of Ω_S given the Neumann data $\frac{\partial u}{\partial n}$ of the solution on $\partial\Omega_1$. Therefore, we define F as the operator mapping q to $\frac{\partial u}{\partial n}$, where q is the parameterization of the shape boundary of Ω_S. Ring (Ring, 1995) studied $S(x) = \chi_{\Omega_S}$, where Ω_S to be a star-shaped with respect to the origin and χ_{Ω_S} denoted the characteristic function of Ω_S. He named the inverse problem as *core identification*. Hohage (Hohage, 2001) mentioned briefly the shape identification of the inverse source problem (1.1) in general domain and for general $S(x)$ without details, complement the work of (Hettlich and Rundell, 1996), which is more general than the study of (Ring, 1995) with different approach.
In this work, we are particularly interested in the case of $S(x)$ of the form

$$S(x) = -\nabla a \nabla u_0, \qquad 0 < a \quad \text{a constant}, \tag{1.2}$$

where the support of a, supp$(a) \subset\subset \Omega_1$, and u_0 is the solution of the following boundary value problem

$$\Delta u_0 = 0, \quad u_0 = g \quad \text{on} \quad \partial\Omega_1. \tag{1.3}$$

This work extends the work of Ring (Ring, 1995), and studying in more details of a particular case of Hohage (Hohage, 2001) work. Furthermore, this particular problem arises from identification of conductivity inclusion shape from boundary measurement.

This work is organized into some sections and an appendix as follows. In the next section, we state the inclusion problem in Electrical Impedance Tomography (EIT) as inverse source problem particularly as core identification. The third section, we recall the results from core identification and recast it to the problem at our hand. We owe much of these results in this section to Ring (Ring, 1995). In this section, we also verify the singular system of the Fréchet derivative at a particular situation, which demonstrate the degree of ill-posedness of the problem. Before closing by the last section, a section solely devoted to some numerical results, to reconstruct the star-shaped support of conductivity inclusion using simpified Iteratively Regularized Gauss-Newton Method (sIRGNM). For the sake of conveniencies, we provide in the appendix some nomenclatures and facts used in this work.

2 Object inclusion in Electrical Impedance Tomography and inverse source problem of Poisson equation.

Consider the inverse boundary value problem of electric impedance tomography (EIT) in a bounded simply connected domain $\Omega_1 \subset \mathrm{R}^2$:

Determine the conductivity κ : $\Omega_1 \to \mathrm{R}$, with $0 < \kappa_m = \kappa(x) = \kappa_M < \infty$, in the elliptic equation

$$\nabla \bullet (\kappa \nabla w) = 0 \text{ in } \Omega_1, w = g \text{ on } \partial\Omega_1, \tag{2.1}$$

using all possible boundary measurement pairs $(g, \kappa \frac{\partial w}{\partial \nu}|_{\partial\Omega_1})$. Here, w is the electrostatic potential, and g is the prescribed potential at the boundary. Due to the severe ill-posedness of the problem, it is almost futile effort to build a reliable method for reconstructing general conductivities. This is particularly true if only few of Cauchy data pairs $(g, \kappa \frac{\partial w}{\partial \nu}|_{\partial\Omega_1})$ are available, which is often the case in practice.

One specific application of EIT which has practical importance is locating inhomogeneities inside objects with known background conductivities. For example, finding tissue anomalies from normal healthy background fall into this category of problem. Assuming the background conductivity to be constant, say 1, our problem setting can be expressed as follow:

Assuming that

$$a := (\kappa - 1) \qquad \text{has compact support within} \quad \Omega_1,$$

i.e., κ is 1 within the vicinity of the boundary of Ω_1, find information of *supp*(a) from single (or a few) Dirichlet-Neumann pair(s) of boundary data for problem (2.1).

Assuming that the conductivity perturbation a is small, let us linearize the equations with respect to the conductivity perturbation. The linearized equation for small a is an equation for the additional potential $u = w - u_0$, and can be obtained by solving equations

$$\nabla(1 + a)\nabla w = 0 \qquad \text{in } \Omega_1 \qquad w = g \quad \text{on } \partial\Omega_1 \tag{2.2}$$

and

$$\Delta u_0 = 0 \quad \text{in } \Omega_1 \quad u_0 = g \quad \text{on } \partial\Omega_1 \ . \tag{2.3}$$

We assume that *supp*$(a) = \tilde{\Omega}_1$ for some subdomain $\tilde{\Omega}_1 \subset\subset \Omega_1$, and $a(x) = 0$ for $x \in \Omega_1 \backslash \tilde{\Omega}_1$, and suppose that $a \in L^2(\tilde{\Omega}_1)$. Then from (2.2) and (2.3) we have equivalent system of elliptic equations

$$\begin{aligned} \Delta u_0 &= 0 && \text{in } \Omega_1 \quad u_0 = g \quad \text{on } \partial\Omega_1, \\ \Delta u &= -\nabla a \nabla u_0 && \text{in } \Omega_1 \quad u = 0 \quad \text{on } \partial\Omega_1. \end{aligned} \tag{2.4}$$

Here g is applied boundary voltage, u_0 is a potential of the electrical field in the model background problem, or a harmonic reference potential, and u is fluctuation of the potential due to the presence of the inclusion.
From (2.4), by generating the corresponding harmonic reference potential u_0, we can reduce our problem to finding the support of the source in the following Poisson equation

$$\Delta u = S(x) \text{ in } \Omega_1, u = 0 \text{ on } \partial\Omega_1, \tag{2.5}$$

where the source is expressed formally as $S(x) = - \nabla \cdot a \nabla u_0$.
Note that this notation has to be properly interpreted in a weak sense if κ lacks the appropriate smoothness, but nonetheless, S is always supported within the support of a. Also note that each pair of Dirichlet-to-Neumann data for problem (2.1) leads to a different source, and thus, in principle, to additional information about the support of a. Furthermore, the smallness of a is not necessarily to be of low-contrast conductivity but also of high-contrast conductivity as well such as arises in the problem in geophysics, see for example (Cherkaeva and Tripp, 1996).
The inverse linearized problem is to find a conductivity distribution a which fits the measured currents on the surface , $\frac{\partial u}{\partial \nu}|_{\partial\Omega_1}$, of potential difference u satisfying (2.4)for the applied voltages g.

Starshape Object Inclusion problem as core identification.

From now on, in this work, we consider Ω_1 the unit disc. It is well known that the unique solution to the problem (1.1) is given by

$$u(x) = \int_{\Omega_1} G(x,\xi) S(\xi) d\xi, \tag{2.6}$$

where

$$G(x,\xi) := \frac{1}{2\pi} \ln|x-\xi| - \frac{1}{2\pi} \ln\left(|\xi|\,|x - \frac{\xi}{|\xi|^2}| \right),$$

is the Green's function for Ω_1. We assume that Ω_S to be a star-shaped with respect to the origin. We consider the particular case that $a = \chi_{\Omega_S}$ and $\bar{\Omega}_S \subset\subset \Omega_1$, hence $a|_{\partial\Omega_1} = 0$. Then $\partial\Omega_S = \{q(t)(\cos t, \sin t) : t \in [0, 2\pi]\}$ for q which is a positive function and $2\pi-$periodic.
Let $\mathbb{Z}_0 = \mathbb{Z}\backslash\{0\}$, the set of non-zero integers, and let the Poisson Kernel

$$P(r,s) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} (r)^{|n|} e^{i\,ns}, \tag{2.7}$$

which is the Green's function of Laplace equation in the unit disc Ω_1 in polar coordinates (r, s).
We restrict ourselves by choosing *the harmonic reference potential* u_0 of the form

$$u_0 := U_k(r, s) = \frac{r^{|k|}}{|k|} e^{i\,ks}, k \in \mathbb{Z}_0,$$

which is the solution of boundary value problem (1.3) under scaled trigonometric function

$$g = \frac{e^{i\,ks}}{|k|}, k \in \mathbb{Z}_0,$$

as a Dirichlet boundary value at $\partial\Omega_1$.
Forward Map F. The defined operator F mapping q to $\frac{\partial u}{\partial n}$ is by taking the normal derivative of (2.6)

$$\begin{aligned} F(q)(t) &= \frac{\partial u}{\partial \nu_x} = \frac{\partial}{\partial \nu_x} \int_{\Omega_1} G(x, \xi) S(\xi) d\xi, \\ &= \int_{\Omega_1} \frac{\partial}{\partial \nu_x} G(x, \xi) \nabla_\xi a \nabla_\xi U_k(\xi) d\xi, \\ &\quad \text{by applying Green's second formula} \\ &\quad \text{and use the fact that} \quad a|_{\partial\Omega_1} = 0 \\ &= \int_{\Omega_S} a \nabla_\xi \frac{\partial}{\partial \nu_x} G(x, \xi) \nabla_\xi U_k(\xi) d\xi, \\ &\quad \text{due to change of variables from cartesian to polar} \\ &\quad \text{and the use of Poisson Kernel for Green's function} \\ &\quad \text{also inserting } a = \chi_{\Omega_S} \\ &= \int_0^{2\pi} \int_0^{q(s)} \nabla_\xi P(r, t-s) \cdot \nabla_\xi U_k(r, s) r dr ds; \quad \xi(r, s) \\ &\quad \text{after inserting an explicit expression of the Poisson Kernel} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{q(s)} \sum_{n \neq 0, n \in \mathrm{Z}} |n| e^{i\,n(t-s)} e^{i\,ks} r^{|n|+|k|-2} r dr ds \end{aligned}$$

We arrive at the forward map F mapping q to $\frac{\partial u}{\partial n}$,

$$F(q)(t) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n \neq 0, n \in \mathrm{Z}} \frac{|n|}{|n| + |k|} q(s)^{|n|+|k|} e^{i\,ks} e^{i\,n(t-s)} ds. \tag{2.8}$$

In the next section we address some propeties of the forward operator (2.8) as we derived above. We address the smoothness property of $F(q)(t)$ and derive the Fréchet derivatives of F.

3 Smoothness and differentiability

Denote, $\mathbb{T} = [0, 2\pi]$ and $\mathbb{Z}_o = \mathbb{Z} \backslash \{0\}$, then (2.8) is a well-defined map

$$\begin{aligned} F &: H^s(\mathbb{T}) \supset D(F) \to H_\diamond^{-1/2}(\mathbb{T}), \\ &: q \mapsto F(q)(t). \end{aligned}$$

The domain for F is defined as the set

$$\text{dom}F = \{q \in H^1(\mathbb{T}) : 0 \leq q(t) \leq 1 \text{ for all } t \in \mathbb{T}\}, \tag{3.1}$$

and frequently we shall consider the interior $(\text{dom}F)^\circ$ of dom$F$ in $H^1(\mathbb{T})$, given by

$$(\text{dom}F)^\circ = \{q \in H^1(\mathbb{T}) : 0 < q(t) < 1 \text{ for all } t \in \mathbb{T}\}. \tag{3.2}$$

Proposition 3.1. Let F be as defined in (2.8) and suppose that dom$F$ and $(\text{dom}F)^\circ$ are given as in (3.1) and (3.2), respectively.

1. Then
$$F(q) \in H^l(\mathbb{T})$$
for every $q \in \text{dom}F$ and $l < \frac{1}{2}$.

2. If $q \in (\text{dom}F)^\circ$, then
$$F(q) \in \mathbb{C}^\infty(\mathbb{T}).$$

3. For every $q \in (\text{dom}F)^\circ$, there exists a neighbourhood $U(q)$ of q in $H^1(\mathbb{T})$ such that
$$F : U(q) \subset H^1(\mathbb{T}) \to H^l(\mathbb{T})$$
is Lipschitz continuous for every $l \in \mathrm{R}$, with Lipschitz constant given by $\tilde{c}$, where $\tilde{c}$ is depending on $U(q)$ and l.

Proof. We follow in parallel to the proof of Proposition 4.1. in Ring (Ring, 1995).

1. Observe that,
$$\begin{aligned} \|F(q)\|^2_{H^l(\mathbb{T})} &= (\frac{1}{2\pi})^2 \sum_{n\in \mathrm{Z}_0} (1+n^2)^l \Big| \int_0^{2\pi} \frac{|n| q(s)^{|n|+|k|}}{|n|+|k|} e^{-ins} ds \Big|^2 \\ &\leq \sum_{n\in \mathrm{Z}_0} \frac{(1+n^2)^{l+1}}{|n|+|k|} . \end{aligned}$$
Since $(1+n^2)^{l+1}/(|n|+|k|) \leq (1+n^2)^{l+1}/(|n|+1) \leq 2^{l+2}|n|^{2l-2}$ for all $l \geq 0, |k| \geq 1$, and $|n| \geq 1$, the series converges if $2l-2 < -1$,i.e., if $l < \frac{1}{2}$.

   Note that we also have a tighter bound due to the following inequality $(1+n^2)^{l+1}/(|n|+1) \leq 2^{l+2}|n|^{2l+1}$, but we don't use this tighter bound, as the current scale will be of the similar result as in (Ring, 1995).

2. As argued in (Ring, 1995), suppose that $q \in (\text{dom}F)^\circ$. The n-th terms of $F(q)$ under Fourier Transform is
$$|n||(F(q))\hat{}(n)| = |n|^k \frac{1}{2\pi} \Big| \int_0^{2\pi} \frac{|n| q(s)^{|n|+|k|}}{|n|+|k|} e^{-in\, s} ds \Big|$$
Since $H^1(\mathbb{T}) \subset \mathcal{C}(\mathbb{T})$, clearly $\sup_{s\in\mathbb{T}} |q(s)| = \|q\|_\infty < 1$, then we have
$$|n|^k \frac{1}{2\pi} \Big| \int_0^{2\pi} \frac{|n| q(s)^{|n|+|k|}}{|n|+|k|} e^{-in\, s} ds \Big| \leq \frac{|n|^{k+1}}{|n|+|k|} \|q(s)^{|n|+|k|}\|_\infty \quad \to 0,$$
as $|n| \to \infty$ since $\|q\|_\infty < 1$.

3. Since the embedding $H^1(\mathbb{T}) \hookrightarrow \mathcal{C}(\mathbb{T})$ is bounded, we can find a neighbourhood $U(q)$ of $q \in (\text{dom}F)^\circ$ such that $m := \{\|\tilde{q}\|_\infty : \tilde{q} \in U(q)\} < 1$. We will take a look at

$$F : U(\tilde{q}) \subset H^1(\mathbb{T}) \to \mathcal{C}^\infty(\mathbb{T}) \subset H^l(\mathbb{T})$$

with $l \in \mathrm{R}$. Therefore

$$\begin{aligned}
\|F(q_1) - F(q_2)\|_{H^l(\mathbb{T})} &= \left(\sum_{n \in \mathrm{Z}_0} (1+n^2)^l \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{|n|(q_1^{|n|+|k|} - q_2^{|n|+|k|})}{|n|+|k|} e^{-ins} ds \right|^2 \right)^{1/2} \\
&\leq \frac{1}{2\pi} \left(\sum_{n \in \mathrm{Z}_0} \frac{(1+n^2)^l |n|^2}{(|n|+|k|)^2} \left| \int_0^{2\pi} (q_1^{|n|+|k|} - q_2^{|n|+|k|}) e^{-ins} ds \right|^2 \right)^{1/2} \\
&\leq \frac{1}{2\pi} \left(\sum_{n \in \mathrm{Z}_0} \frac{(1+n^2)^{l+1}}{(|n|+|k|)^2} \times \right. \\
&\qquad \left. \left(\int_0^{2\pi} |q_1 - q_2| \left| \sum_j^{|n|+|k|} q_1^j q_2^{|n|+|k|-j} \right| ds \right)^2 \right)^{1/2} \\
&\leq \left(\sum_{n \in \mathrm{Z}_0} (1+n^2)^{l+1} m^{2(|n|+|k|)} \right)^{1/2} \|q_1 - q_2\|_\infty \qquad (3.3) \\
&\leq L \|q_1 - q_2\|_{H^l(\mathbb{T})}
\end{aligned}$$

where

$$L := \left(\sum_{n \in \mathrm{Z}_0} (1+n^2)^{l+1} m^{2(|n|+|k|)} \right)^{1/2} c_1,$$

with c_1 such that $\|f\|_\infty \leq c_1 \|f\|_{H^l(\mathbb{T})}$. The series in (3.3) is convergent via the quotient criterion. Hence we have demonstrated that for every $l \in \mathrm{R}$,

$$F : U(\tilde{q}) \subset H^1(\mathbb{T}) \to H^l(\mathbb{T}),$$

is locally Lipschitz continuous. □

Observe that in the above proposition, we also proved Lipschitz continuity of F with respect to $C(\mathbb{T})-$norm on $(\text{dom}F)^\circ$.

Derivative of F. Now we turn to deriving Fréchet derivative of F.

Proposition 3.2. The forward map $F(q) : (\text{dom}F)^\circ \subset H^1(\mathbb{T}) \to H^l(\mathbb{T})$ is Fréchet differentiable at every element $q \in (\text{dom}F)^\circ$ with derivative given by

$$(DF[q]h)(t) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{n \in \mathbb{Z}_0} |n| q(s)^{|n|+|k|} e^{iks} h(s) e^{in(t-s)} ds, \qquad (3.4)$$

for all $h \in H^1(\mathbb{T})$. The Fourier transform of $DF[q]h$ is given by

$$\widehat{(DF[q]h)}(n) = \frac{1}{2\pi} \int_0^{2\pi} |n| q(s)^{|n|+|k|} e^{iks} h(s) e^{-ins} ds, \qquad (3.5)$$

for all $h \in H^1(\mathbb{T})$ and $n \in \mathbb{N}$.

Proof. Before we proceed to show differentiability of F, it is necessary first to show that (3.5) is the Fourier Transform of (3.4). Observe that for $q \in (\text{dom}F)^\circ$

$$|\sum_{n\in\mathbb{Z}_0} |n|q(s)^{|n|+|k|}e^{i\,k\,s}e^{i\,n(t-s)}| \leq \|q\|_\infty^{|k|}\frac{\|q\|_\infty}{(1-\|q\|_\infty)^2} < \infty,$$

we interchange the summation and integration in (3.4) and obtain

$$(DF[q]h)(t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}_0}\int_0^{2\pi} |n|q(s)^{|n|+|k|}e^{i\,k\,s}h(s)e^{i\,n(-s)}ds\; e^{i\,nt},$$

which confirms (3.5) to be correct.

To prove Fréchet differentiability of F, it is sufficient to show the following

(1.) $DF[q] : H^1(\mathbb{T}) \to H^l(\mathbb{T})$ is a bounded linear operator.

(2.) DF is a locally Lipschitz continuous dependent on q, i.e., there exists $L(q) > 0$, such that

$$\|F(q) - F(\tilde{q})\|_{\mathcal{L}(H^l(\mathbb{T}),H^l(\mathbb{T}))} \leq L\|q-\tilde{q}\|_{H^l(\mathbb{T})}$$

(3.) $DF[q]$ is Gateaux derivative of F at q, i.e.

$$\lim_{\tau\to 0}\|F(q+\tau h) - F(q) - \tau DF[q]h\|_{H^l(\mathbb{T})} = 0,$$

where $\tau \in [0,\infty)$ for every $h \in H^l(\mathbb{T})$.

Let us show first that $DF[q]$ as defined in (3.4) is a bounded linear operator from $H^1(\mathbb{T})$ into $H^l(\mathbb{T})$ for some arbitrary but fixed $l \in \mathrm{R}$. Using

$$\begin{aligned}
\|DF[q]h\|_{H^l(\mathbb{T})} &= \left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^l|\frac{1}{2\pi}\int_0^{2\pi}|n|q(s)^{|n|+|k|}e^{i\,k\,s}h(s)e^{i\,n(t-s)}ds|^2\right)^{1/2},\\
&\leq \left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^l|n|^2\|q\|_\infty^{2(|n|+|k|)}|\frac{1}{2\pi}\int_0^{2\pi}h(s)e^{i\,n(-s)}ds|^2\right)^{1/2},\\
&\leq \left((\sum_{n\in\mathbb{Z}_0}(1+n^2)^l(1+n^2)\|q\|_\infty^{2(|n|+|k|)}|\hat{h}(n)|^2)\right)^{1/2},\\
&\leq \left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^{(l+1)}\|q\|_\infty^{2(|n|+|k|)}|\hat{h}(n)|^2\right)^{1/2},\\
&\leq c\left(\sum_{n\in\mathbb{Z}}|\hat{h}(n)|^2\right)^{1/2} < c\|h\|_{L_2(\mathbb{T})},
\end{aligned}$$

where we chose c sufficiently large such that $(1+n^2)^{(l+1)/2}\|q\|_\infty^{(|n|+|k|)} < c$ for all $n \in \mathbb{Z}$. (Note that $\|q\|_\infty < 1$ since $q \in (\text{dom}F)^\circ$). Hence $DF[q] : H^1(\mathbb{T}) \subset L_2(\mathbb{T}) \to H^l(\mathbb{T})$ is a bounded linear operator. Moreover the results show a stronger result than we require and also $DF[q]$ can be uniquely extended to a bounded linear operator from $L_2(\mathbb{T})$ into $H^1(\mathbb{T})$.

For the proof of (2.) we pick a neighbourhood $U(q)$ of q in $H^1(\mathbb{T}) \subset L^2(\mathbb{T})$ such that $\|\tilde{q}\|_\infty \leq m < 1$, for all $\tilde{q}$. Then we have

$$\begin{aligned}
\|DF[q]h - DF[\tilde{q}]h\|_{H^l(\mathbb{T})} &= \Bigg(\sum_{n\in\mathbb{Z}_0}(1+n^2)^l||n| \times \\
&\quad \frac{1}{2\pi}\int_0^{2\pi}(q-\tilde{q})[\sum_{j=0}^{|n|+|k|} q^j\tilde{q}^{(|n|+|k|-j}]h(s)e^{-ns}ds|^2\Bigg)^{1/2} \\
&\leq \left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^l|n|^2(|n|+|k|+1)^2m^{2(|n|+|k|)}|\hat{h}(n)|^2\right)^{1/2} \times \\
&\quad \|q-\tilde{q}\|_\infty \\
&\leq \left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^{l+1}(|n|+|k|+1)^2m^{2(|n|+|k|)}|\hat{h}(n)|^2\right)^{1/2} \times \\
&\quad \|q-\tilde{q}\|_\infty
\end{aligned}$$

We choose c_1 and c_2 two constants such that $\|q-\tilde{q}\|_\infty \leq c_1\|q-\tilde{q}\|_{H^1(\mathbb{T})}$ for all $\tilde{q}$ elements in $U(q)$, the neighbourhood of q, and

$$(1+n^2)^{l+1}(|n|+|k|+1)^2m^{2(|n|+|k|)} < c_2^2, \text{for all } n \in \mathbb{Z}.$$

Then we have

$$\|DF[q]h - DF[\tilde{q}]h\|_{H^l(\mathbb{T})} \leq L\|q-\tilde{q}\|_{H^l(\mathbb{T})}\|h\|_{H^l(\mathbb{T})},$$

with $L = c_1c_2$, and consequently

$$\|DF[q]h - DF[\tilde{q}]h\|_{\mathcal{L}(H^1(\mathbb{T}),H^l(\mathbb{T}))} \leq L\|q-\tilde{q}\|_{H^l(\mathbb{T})},$$

for all $\tilde{q} \in U(q)$.
Now, it remains to show that DF is the Gateaux derivative of F. We find that

$$\|F(q+\tau h) - F(q) - \tau DF[q]h\|_{H^l(\mathbb{T})} =$$

$$\begin{aligned}
&= \left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^l|\frac{1}{2\pi}\int_0^{2\pi}(\frac{|n|((q+\tau h)^{|n|+|k|}-q^{|n|+|k|})}{|n|+|k|} - \tau q^{|n|+|k|}|n|h(s))e^{-ns}ds|^2\right)^{1/2} \\
&= \frac{1}{2\pi}\left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^l|n|^2|\int_0^{2\pi}(\frac{((q+\tau h)^{|n|+|k|}-q^{|n|+|k|})}{|n|+|k|} - \tau q^{|n|+|k|}h(s))e^{-ns}ds|^2\right)^{1/2} \\
&\leq \frac{1}{2\pi}\left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^{(l+1)}|\int_0^{2\pi}\frac{1}{|n|+|k|}[\sum_{j=2}^{|n|+|k|}\binom{|n|+|k|}{j}\tau^j h(s)^j q^{|n|+|k|-j}]e^{-ns}ds|^2\right)^{1/2} \\
&\leq \frac{1}{2\pi}\left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^{(l+1)}|\int_0^{2\pi}\frac{1}{|n|+|k|}[\sum_{j=2}^{|n|+|k|}\binom{|n|+|k|}{j}\tau^j h(s)^j q^{|n|+|k|-j}]e^{-ns}ds|^2\right)^{1/2} \\
&\leq \tau\left(\sum_{n\in\mathbb{Z}_0}(1+n^2)^{(l+1)}(\frac{1}{|n|+|k|}\|q+\tau|h|\|_\infty^{|n|+|k|})^2\right)^{1/2}
\end{aligned}$$

where we used $\tau^{k-1} \leq \tau^{k/2}$ for all $k \geq 2$ and $\tau \in (0,1)$. The last series converges if $\|q + \tau|h|\|_\infty < \infty$ by the quotient criterion, hence the whole expression goes to 0 as $\tau \to 0$.

□

The following theorem is in parallel to the result of Theorem 4.5 in Ring (Ring, 1995).

Theorem 3.3. *The operator $DF[q] : H^1(\mathbb{T}) \to H^l_\diamond(\mathbb{T})$ with $l \in \mathrm{R}$ is injective for all $q \in (\text{dom}F)^\circ$.*

Proof. Let $h(s) \in \text{ker}DF[q] \subset H^1(\mathbb{T}), then$

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}_0} |n| q(s)^{|n|+|k|} e^{i\,k\,s} h(s) e^{i\,n\,s} ds$$

after exchanging integral and sum, then

$$0 = \frac{1}{2\pi}\int_0^{2\pi} q(s)^{|n|+|k|} e^{i\,k\,s} h(s) e^{i\,n\,s} ds, \quad \text{for all } n \neq 0, n \in \mathrm{Z}.$$

Let $\tilde{h}(s) = q(s)^{|k|-1} e^{i\,k\,s} h(s)$, this equivalently with

$$0 = \frac{1}{2\pi}\int_0^{2\pi} q(s)^{|n|+1} \tilde{h}(s) e^{i\,n\,s} ds, \quad \text{for all } n \neq 0, n \in \mathrm{Z}.$$

It was shown in (Ring, 1995)[Theorem 4.5], that the functions $\{q(s)^{|n|} e^{i\,n\,s}; n \in \mathrm{Z}\}$ are dense in $H^1(\mathbb{T})$, thus we may conclude that $q(s)\tilde{h}(s) = 0$. However, since $q(s)^{|k|-1} e^{i\,k\,s} \neq 0, s \in \mathbb{T}\backslash\{0\}$ and $q(s) > 0$, then leads to $h(s) = 0$, or $h \in \text{ker}DF[q]$. □

Theorem 3.4. *The evaluation of adjoint operator $DF[q]^*$ of (3.4) is given by:*

$$(DF[q]^* g)(s) = q(s)^{|k|} e^{i\,ks} \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}_0} |n| q(s)^{|n|} e^{i\,n(t-s)} g(t) dt, \tag{3.6}$$

for $g(t) \in H^l_\diamond(\mathbb{T})$.

Proof. We follow in parallel to the idea of the proof of Theorem 5.1 and utilize some results of the appendix in the works of Ring (Ring, 1995).
(The following geometric series: $r/(1-r)^2 = \sum_{n\in\mathbb{N}_0} nr^n, |r| < 1,$ is needed in the proof.)

$$\begin{aligned}
\langle DF[q]h, g\rangle_{H^{-1/2}(\mathbb{T}),H^{1/2}(\mathbb{T})} &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}_0} |n| q(s)^{|n|+|k|} e^{i\,k\,s} h(s) e^{i\,n(t-s)} ds \right) g(t)\, dt, \\
&= \frac{1}{2\pi}\int_0^{2\pi} q(s)^{|k|} e^{i\,k\,s} h(s) \\
&\quad \left(\frac{1}{2\pi}\int_0^{2\pi} \sum_{n\in\mathbb{Z}_0} |n| q(s)^{|n|} e^{i\,n(t-s)} g(t)\, dt \right) ds.
\end{aligned} \tag{3.7}$$

The order of integration can be reversed since we have

$$|\sum_{n\in\mathbb{Z}_0} |n| q(s)^{|n|+|k|} e^{i\,k\,s} h(s) e^{i\,n(t-s)} g(t)| \leq \frac{\|q\|_\infty^{|k|+1} \|h\|_\infty}{(1 - \|q\|_\infty)^2} |g(t)|$$

for all $(s,t) \in \mathbb{T} \times \mathbb{T}$ and $g \in H^{1/2} \subset L_2(\mathbb{T})$. We put

$$g^*(s) = q(s)^{|k|} e^{i\,k\,s} \left(\frac{1}{2\pi} \int_0^{2\pi} \sum_{n \in \mathbb{Z}_0} |n| q(s)^{|n|} e^{i\,n(t-s)} g(t)\; dt \right). \tag{3.8}$$

From

$$| \sum_{n \in \mathbb{Z}_0} |n| q(s)^{|n|+|k|-1} e^{i\,k\,s} e^{i\,n(t-s)} g(t)| \leq \frac{\|q\|_\infty^{|k|}}{(1-\|q\|_\infty)^2} |g(t)|,$$

thus allows us to reorder the summation and integration in (3.8), so we obtain

$$\begin{aligned} g^*(s) &= \sum_{n \in \mathbb{Z}_0} |n| q(s)^{|n|} q(s)^{|k|} e^{i\,k\,s} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{i\,n(t)} g(t)\; dt \right) e^{i\,n(-s)}, \\ &= \sum_{n \in \mathbb{Z}_0} |n| q(s)^{|n|} q(s)^{|k|} e^{i\,k\,s} \hat{g}(n) e^{i\,n(-s)}. \end{aligned} \tag{3.9}$$

The following estimates

$$\begin{aligned} \sum_{n \in \mathbb{Z}_0} |\hat{g}(n)|^2 |n| \|q\|_\infty^{|n|} &\leq \sum_{n \in \mathrm{Z}} |\hat{g}(n)|^2 |n| \|q\|_\infty^{|n|}, \\ &\leq (\sum_{n \in \mathrm{Z}} (1+n^2)^{1/2} |\hat{g}(n)|^2)^{1/2} (\sum_{n \in \mathrm{Z}} (1+n^2)^{-1/2} |n|^2 \|q\|_\infty^{2|n|})^{1/2}, \\ &\leq (\sum_{n \in \mathrm{Z}} (1+n^2)^{1/2} |\hat{g}(n)|^2)^{1/2} (\sum_{n \in \mathrm{Z}} (1+n^2)^{1-1/2} \|q\|_\infty^{2|n|})^{1/2}, \\ &\leq (\|g\|^2_{H^{1/2}(\mathbb{T})}) (\sum_{n \in \mathrm{Z}} (1+n^2)^{1/2} \|q\|_\infty^{2|n|})^{1/2}, \\ &< \infty, \end{aligned}$$

imply the uniform convergence of the series (3.9) on $\mathbb{T}$. Therefore we conclude that $g^* \in \mathcal{C}(\mathbb{T}) \subset H^{-1}(\mathbb{T})$. Moreover (3.7) implies that

$$\langle DF[q]h, g \rangle_{H^{-1/2}(\mathbb{T}), H^{1/2}(\mathbb{T})} = \langle h, g^* \rangle_{H^1(\mathbb{T}), H^{-1}(\mathbb{T})}$$

and consequently

$$(DF[q]^* g)(s) = q(s)^{|k|} e^{i\,ks} \frac{1}{2\pi} \int_0^{2\pi} \sum_{n \in \mathbb{Z}_0} |n| q(s)^{|n|} e^{i\,n(t-s)} g(t) dt. \tag{3.10}$$

□

Singular system of $DF[q_0]$ for circular support inclusion.

Let $q_0, 0 < q_0 < 1$, be a constant radius of support inclusion.
Denote $\epsilon_j(s) = e^{i\,js}$, we have

$$(DF[q_0]\epsilon_j)(t) = \sum_{n \in \mathbb{Z}_0} \frac{|n|}{2\pi 1^{|n|+|k|+1}} q_0^{|n|+|k|} e^{i\,nt} \int_0^{2\pi} e^{i\,js} e^{i\,(k-n)s} ds \tag{3.11}$$

Under complex unit monomial basis in $\mathbb{T}$ the entry of $DF[q_0]$ can be expressed as

$$\begin{aligned} (DF[q_0])_{jn} &= \langle (DF[q_0]\epsilon_j)(t), \epsilon_n(t) \rangle_{H^{-1/2}(\mathbb{T}), H^{1/2}(\mathbb{T})} \\ &= |n| q_0^{|n|+|k|} \delta_n^{j+k} \end{aligned}$$

for $n \neq 0, n, j \in \mathbb{Z}$, and δ denoting delta Dirac notation. We observe here we have an infinite matrix representation of $DF[q_0]$, which is a non-zero upper-lower k-diagonal or k-banded.
Let us make some notes related to the infinite matrix representation of $DF[q_0]$ induced by harmonic reference potential u_0.

Note 3.5. Observe that under 'harmonic reference potential' u_0 of the form of a finite linear combination of *elementary harmonic reference potential*

$$u_0 := \sum_{k \in K} \alpha_k U_k(r,s) = \sum_{k \in K} \alpha_k (r)^k e^{i\,ks}, K = \{k_i \in \mathbb{Z}_0, i = 1, \cdots, n\},$$

which is the solution of boundary value problem (1.3) under boundary value

$$g = \sum_{k \in K} \alpha_k e^{i\,ks}, K = \{k_i \in \mathbb{Z}_0, i = 1, \cdots, n\},$$

the infinite matrix $DF[q_0]$ is a non-zero banded matrix with bandwidth $|\min\{k_i; k_i \in K\}| + \max\{k_i; k_i \in K\}$.
In the case of 'harmonic reference potential' is $u_0 := U_k^c(r,s) = (r^k \cos ks)/k, k \in \mathbb{Z}_0$, which is the solution of boundary value problem (1.3) under cosine boundary value $g = (\cos ks)/k, k \in \mathbb{Z}_0$, we observe that the infinite matrix $DF[q_0]$ under trigonometrical polynomial basis is a non-zero symmetric k-bidiagonal.
While in the case of 'harmonic reference potential' is $u_0 := U_k^s(r,s) = (r^k \sin ks)/k, k \in \mathbb{Z}_0$, which is the solution of boundary value problem (1.3) under boundary value $g = (\sin ks)/k, k \in \mathbb{Z}_0$, we obtain that the infinite matrix $DF[q_0]$ under trigonometrical polynomial basis is a non-zero skew-symmetric k-bidiagonal.

Singular system of $DF[q_0]$.
Let $\epsilon_j^s(t) = (1+j^2)^{-s/2} e^{i\,jt}$, then

$$\begin{aligned}(DF[q_0]\epsilon_j^s)(t) &= (1+j^2)^{-s/2}(DF[q_0]\epsilon_j)(t), \\ &= (1+j^2)^{-s/2} \sum_{n \in \mathbb{Z}_0} |n| q_0^{|n|+|k|} \delta_n^{j+k} \epsilon_n(t), \\ &= (1+j^2)^{-s/2} |j| q_0^{|j|+|k|} \epsilon_{j+k}(t).\end{aligned}$$

Denote

$$\sigma_j = q_0^{|k|+|j|}\, |j|(1+j^2)^{-s/2},$$

then from (3.6) and Theorem (3.3), we can show :

$$DF[q_0]^* \epsilon_{j+k}(t) = \sigma_j\, \epsilon_j^s(t).$$

Then we demonstrate that the triple $\{\epsilon_j^s(t), \epsilon_{j+k}(t), \sigma_j\}$ is singular system of $DF[q_0]$, the Fréchet derivative of F at positive constant q_0, since $\{\epsilon_j^s(t), j \in \mathbb{Z}\}$ and $\{e^{i\,jt}, j \in \mathbb{Z}_0\}$ are complete orthonormal basis of $H^s(\mathbb{T})$ and $L^2_\diamond(\mathbb{T})$, respectively. Also it shows that $DF[q_0]$ is a compact linear operator.

Remark 3.1. From the knowledge of the singular system of $DF[q_0]$, we could obtain the information of *modified source condition* needed for the class of iterative method using fixed Fréchet derivative such as simplified IRGNM, the method that we use in the numerical section later. This method studied in details by Mahale & Nair (Mahale and Nair, 2009) and Jin (Jin, 2010), and by following the argument given by (Hohage, 2001) in his discussion on source condition for the inverse (constant) source problem, we might highlights the modified source condition for simplified IRGNM. This issue will be addressed in the future works.

The following theorem highlights the degree of ill-posedness of $DF[q_0]$.

Theorem 3.6. *Assume that* $0 < q_0(t) = q_0 < 1$, *a positive constant. Then the operator*

$$f_p(DF[q_0]^*DF[q_0]) : H^s(\mathbb{T}) \to H^{s+p}(\mathbb{T})$$

is bounded and boundedly invertible.

Proof. By the help of the singular system of $DF[q_0]$, we have

$$f_p(DF[q_0]^*DF[q_0])\epsilon_j^s = f_p(\sigma_j^2)\epsilon_j^s.$$

Choosing $\lambda_0 = \|DF[q_0]\|$ in the definition of f_p, we obtain

$$f_p(\sigma_j^2) = (|j| + |k|)\ln R + c_R + s\ln(1+j^2) - \ln|j|, \qquad R := 1/q_0.$$

Since

$$(|j| + |k|)\ln R + c_R + s\ln(1+j^2) - \frac{1}{2}\ln(1+j^2)| < (|j| + |k|)\ln R + c_R + s\ln(1+j^2) - \ln|j|,$$

then there exists $c > 0$ a constant, and for all $j \in \mathbb{Z}$

$$c(\sqrt{1+j^2}) \leq (|j| + |k|)\ln R + c_R + s\ln(1+j^2) - \ln|j|,$$

therefore

$$f_p(\sigma_j^2) \leq c^{-p}(1+j^2)^{-p/2}.$$

This leads to

$$\|f_p(DF[q_0]^*DF[q_0])\|_{H^s(\mathbb{T})\to H^{s+p}(\mathbb{T})} \leq c^{-p}.$$

Using the fact $-\ln|j| < 0$ for all $j \in \mathbb{Z}_0$, and use of the following inequality:

$$(|j| + |k|)\ln R + c_R + s\ln(1+j^2) - \ln|j| < (|j| + |k|)\ln R + c_R + s\ln(1+j^2),$$

then there exists $C > 0$ a constant, and for all $j \in \mathbb{Z}_0$

$$(|j| + |k|)\ln R + c_R + s\ln(1+j^2) \leq C(\sqrt{1+j^2}),$$

therefore

$$C^{-p}(1+j^2)^{-p/2} \leq f_p(\sigma_j^2).$$

So we conclude

$$\|f_p(DF[q_0]^*DF[q_0])^{-1}\|_{H^{s+p}(\mathbb{T})\to H^s(\mathbb{T})} \leq C^{-p}.$$

Hence $f_p(DF[q_0]^*DF[q_0])$ is boundedly invertible. ☐

Remark 3.2. The above result, highlights the degree of ill-posedness of the problem. Let $q^\sharp$ be the exact value, the condition $q_0 - q^\sharp = f_p(DF[q_0]^* DF[q_0])w$ for some $w \in H^s(\mathbb{T})$ is equivalent to the fact that $q_0 - q^\sharp \in H^{s+p}(\mathbb{T})$. Moreover, there are constants c, C such that $c\|w\|_{H^s(\mathbb{T})} \leq \|q_0 - q^\sharp\|_{H^{s+p}(\mathbb{T})} \leq C\|w\|_{H^s(\mathbb{T})}$

Remark 3.3. In the case inclusion of the form $a(r)$, that is a known radial function, the forward map (2.8) will be of the form

$$F(q)(t) = \sum_{n\neq 0, n\in \mathbb{Z}} \frac{|n|}{2\pi} \int_0^{2\pi} e^{i\,n(t-s)} e^{i\,ks} \left(\int_0^{q(s)} r^{|n|+|k|-2} a(r) r dr \right) ds.$$

Observe that in our case for star-shaped inclusion, it is necessary that radially supp$(a(r))$ is an interval $[0, b] \subseteq [0, 1], 0 < b \leq 1$. This condition rules out the following studies on non-uniqueness result :

1. Counter example of constant radial object by Kang & Seo (Kang and Seo, 2001).
2. Identification of radial function by El Badia & Ha-Duong (Badia and Ha-Duong, 1998).

Assuming that $a(r)$ is bounded over its support and with finite jump discontinuity, the results for star-shaped constant conductivity support still valid without essential change in the proof.We don't pose the general radial case result in this work, as to maintain the simplicity of exposition in mind.

4 Numerical Implementation.

We may follow the implementation of either (Ring, 1995) or (Hohage, 2001) for star-shaped support of inverse source problem, for numerical implementation to reconstructs the shape of conductivity inclusion. Rather than working in complex arithmetic as in (Ring, 1995), in this work we follow the numerical implementation of (Hohage, 2001), hence we need to recast the expression of the forward operator and the Fréchet derivative in terms of trigonometrical polynomials. To ease the work, we consider three simple cases of general form.

Cosine boundary input. The first case is the case of $u_k^c(t) = \frac{r^k}{k}\cos(kt)$, or a harmonic reference potential under boundary input of the form $g_k^c = \frac{cos(kt)}{k}, k \neq 0$. We have the following expression on the forward map F and its Fréchet derivative.

- Forward map from $q(t)$ to $\frac{\partial u}{\partial n}$:

$$\begin{aligned} F_k^c(q)(t) &= \frac{1}{\pi} \sum_{n\in\mathbb{N}_0} \frac{n}{(n+k)} \Bigg((\int_0^{2\pi} q(s)^{n+k} \cos{(k-n)s}\, ds)\ \cos nt - \\ &\quad (\int_0^{2\pi} q(s)^{n+k} \sin{(k-n)s}\, ds)\ \sin nt \Bigg). \end{aligned} \tag{4.1}$$

- Derivative :

$$\begin{aligned} (DF_k^c[q]h)(t) &= \frac{1}{\pi} \sum_{n\in\mathbb{N}_0} n \Bigg((\int_0^{2\pi} q(s)^{n+k-1} h(s) \cos{(k-n)s}\, ds)\ \cos nt - \\ &\quad (\int_0^{2\pi} q(s)^{n+k-1} h(s) \sin{(k-n)s}\, ds)\ \sin nt \Bigg). \end{aligned} \tag{4.2}$$

Sine boundary input. The second case is $u_k^s(t) = \frac{r^k}{k}\sin(kt)$, or a harmonic reference potential under scaled sine boundary input of the form $g_k^s = \frac{\sin(kt)}{k}, k \neq 0$. The corresponding forward map F and its Fréchet derivative for this case are the following,

- Forward map from $q(t)$ to $\frac{\partial u}{\partial n}$:

$$F_k^s(q)(t) = \frac{1}{\pi}\sum_{n\in\mathbb{N}_0}\frac{n}{(n+k)}\Bigg(\Big(\int_0^{2\pi} q(s)^{n+k}\sin(k-n)s\,ds\Big)\cos nt + \Big(\int_0^{2\pi} q(s)^{n+k}\cos(k-n)s ds\Big)\sin nt\Bigg). \tag{4.3}$$

- Derivative :

$$(DF_k^s[q]h)(t) = \frac{1}{\pi}\sum_{n\in\mathbb{N}_0} n\Bigg(\Big(\int_0^{2\pi} q(s)^{n+k-1}h(s)\sin(k-n)s\,ds\Big)\cos nt + \Big(\int_0^{2\pi} q(s)^{n+k-1}h(s)\cos(k-n)s\,ds\Big)\sin nt\Bigg). \tag{4.4}$$

Finite linear combination of Cosine & Sine boundary input case. And lastly, a case of $u_0(t) = \sum_{k\in K\subset\mathbb{N}_0} r^k(g_k^c\cos(kt) + g_k^s\sin(kt))/k, K = \{k_1,\cdots,k_K\}$ or a harmonic reference potential under finite linear combination of scaled cosine and sine boundary input of the form $g = \sum_{k\in K\subset\mathbb{N}_0}(g_k^c\cos(kt) + g_k^s\sin(kt))/k$, the corresponding forwad map and it's Fréchet derivative provided as follows.

- Forward map from $q(t)$ to $\frac{\partial u}{\partial n}$:

$$\begin{aligned} F(q)(t) &= \sum_{k\in K\subset\mathbb{N}_0}\frac{1}{\pi}\sum_{n\in\mathbb{N}_0}\frac{n}{(n+k)} \\ &\Bigg(\Big[\int_0^{2\pi} q(s)^{n+k}(g_k^c\cos(k-n)s + g_k^s\sin(k-n)s)\,ds\,\Big]\cos nt + \\ &\Big[\int_0^{2\pi} q(s)^{n+k}(g_k^s\cos(k-n)s - g_k^c\sin(k-n)s)\,ds\,\Big]\sin nt\Bigg).\end{aligned}$$

- Derivative :

$$\begin{aligned} (DF[q]h)(t) &= \sum_{k\in K\subset\mathbb{N}_0}(g_k^c(DF_k^c[q]h)(t) + g_k^s(DF_k^s[q]h)(t)). \\ &= \sum_{k\in K\subset\mathbb{N}_0}\frac{1}{\pi}\sum_{n\in\mathbb{N}_0}\frac{n}{(n+k)} \\ &\Bigg(\Big[\int_0^{2\pi} q(s)^{n+k-1}h(s)(g_k^c\cos(k-n)s + g_k^s\sin(k-n)s)\,ds\,\Big]\cos nt + \\ &\Big[\int_0^{2\pi} q(s)^{n+k-1}h(s)(g_k^s\cos(k-n)s - g_k^c\sin(k-n)s)\,ds\,\Big]\sin nt\Bigg).\end{aligned}$$

For general case of a harmonic reference potential, we may take the last case as an example, and extend it to suit the case.

Numerical Implementation (Discrete approximation).

To solve the inverse problem finding q from data $\frac{\partial u}{\partial n} = g^\delta$, we use an iterative method called the simplified iteratively regularized Gauss-Newton method (sIRGNM), defined by the iteration process

$$q^\delta_{(k+1)} := q^\delta_k + (F'[q_0]^* F'[q_0] + \alpha_k I)^{-1} (F'[q_0]^* (y^\delta - F(q^\delta_k)) + \alpha_k (q^\delta_k - q_0)), \ k \in \mathbb{N}_0, \tag{4.5}$$

where $F'[q_0]$ is the Fréchet derivative of F at initial guess q_0, and α_k is regularization parameter. This method is a simplified version of iteratively regularized Gauss-Newton method (IRGNM), which falls within the class of regularized Gauss-Newton method, and studied recently in details by Mahale & Nair (Mahale and Nair, 2009), Jin (Jin, 2010), and George (George, 2010). The reader may consult the monograph by (Kaltenbacher, Neubauer and Scherzer, 2008) or (Bakushinsky, Kokurin and Smirnova, 2011) on the subject of regularized Gauss-Newton method for solving nonlinear inverse problem.

To arrive at a finite dimensional system, we introduce the space T_M of trigonometric polynomials of degree $\leq M, M$ is a positive integer, and solve the following minimization problem

$$\|P_N(F'[q_0]h_k + F(q_k) - g^\delta)\|^2_{L^2} + \alpha_k \|h_k + q_k - q_0\|^2_{H^s,K} = \text{min!} \tag{4.6}$$

over $h_k \in T_M$ at each step. Here K and N are positive integers, P_N is the orthogonal projection onto T_N in $L_2([0, 2\pi])$ and $\|f\|^2_{H^s,K}, s \in \mathrm{N}$, is the approximation of $\|f\|^2_{H^s}$ using the trapezoidal rule with K equidistant grid points, i.e.

$$\|f\|^2_{H^s,K} := \frac{1}{2\pi K} \sum_{k=0}^{K-1} \left(|f(\frac{k}{2\pi K})|^2 + |f^{(s)}(\frac{k}{2\pi K})|^2 \right).$$

It turns out that (4.6) is equivalent to a linear least squares problem for the Fourier coeffficients of h_k.

$P_N(F'[q_0]h_k + F(q_k) - g^\delta)$ is easily evaluated by truncating the series in (4.1) and (4.2). We approximate the integrals in these formulas using the trapezoidal rule with N_t grid points. Since the L_2-norm of a function $f(t) = a_0 + \sum_{j=1}^N a_j \cos(jt) + b_j \sin(jt)$ provided by

$$\|f\|^2_{L_2} := a_0^2 + \frac{1}{2} \sum_{j=1}^N a_j^2 + b_j^2,$$

we obtain $(2N + 1)$ linear equations for the $2M + 1$ Fourier coefficients of h_k from the term $\|P_N(F'[q_0]h_k + F(q_k) - g^\delta)\|^2_{L_2}$ in (4.6). The term $\|h_k + q_k - q_0\|^2_{H^s,K}$ yields another $2K$ equations. Hence in total we obtain a linear least squares problem with $(2(N + K) + 1)$ equations for the $(2M + 1)$ Fourier coefficients of h_k.

Test Cases

Two cases of conductivity inclusion shape have been tested, a rose petal and a dented circular shapes. The first case, a rose petal shaped inclusion, a domain described parametrically by the function

$$q_1(t) := (0.5 + 0.1 \cos(5 * t)).$$

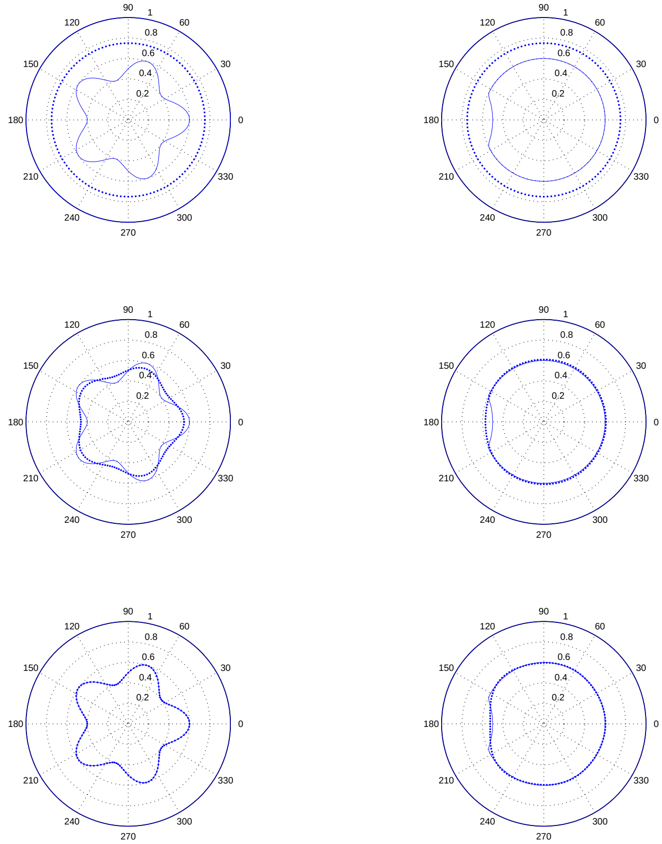

Figure 1: We show the result of two test cases of conductivity star shape support and its domain reconstruction for exact data. On the left column, we show the first case where the shape support is $q^\sharp(t) := (0.5 + 0.1\cos(5 * t))$. While on the right column, we show the similar result for the second test case where $q^\sharp(t)$ is a circle with inward dent. From top to bottom we show the domain shape reconstruction at initial step, fifth step, and the fiftieth step.

The harmonic reference potential used in this case induced by boundary source $g = cos(t)$. The second case we chose a dented circular shaped domain described parametrically by the function

$$q_2(t) := \begin{cases} 0.5, & t \in [0, \pi - 1/2] \cup [\pi + 1/2, 2\pi], \\ 0.5 - 0.1 * \exp(-(\frac{1}{1-4(t-\pi)^2})), & t \in [\pi - 1/2, \pi + 1/2]. \end{cases}$$

The harmonic reference potential used in this second case induced by boundary source $g = cos(3 * t)$.

In both cases, the numerical experiment we perform using forward map F given in (4.1) and its Fréchet derivative (4.2). The synthetic data generated using (4.1) over finer grid points than the number of grid used in sIRGNM, to avoid an obvious inverse crime. The reconstruction and the initial guess of both domain are shown in figure 1.On both numerical experiments, we use $M = 5, N = 32, N_t = 128$, while the data generated using forward operator, using $n = 256$ grid points to avoid inverse crime.

5 Concluding Remarks

We have presented a result to identify a star-shaped conductivity inclusion from boundary measurement as a core identification problem. Using the Newton method requires the knowledge of the derivative of the measurements with respect to the shape perturbation. This derivative has been computed along the lines of the works of Ring for the core identification (Ring, 1995). Based on this result we developed a numerical method for reconstructing the shape of the conductivity inclusion.

The numerical implementation has made use of the simplified iteratively regularized Gauss-Newton method in order to solve the inverse problem, the forward problem being solved by the Fourier expansion method. Our method works satisfactorily to identify conductivity inclusion with star-shaped boundary.

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Appendix A: Nomenclature and facts

Let $H^1(\Omega_1)$ is the sobolev space of all functions $u \in L^2(\Omega_1)$ for which $\frac{\partial u}{\partial x_i} \in L^2(\Omega_1)$ for $i = 1, 2$, endowed with the inner product

$$\langle u, v\rangle_{H^1(\Omega_1)} = \langle u, v\rangle_{L^2(\Omega_1)} + \langle \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_1}\rangle_{L^2(\Omega_1)} + \langle \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_2}\rangle_{L^2(\Omega_1)}.$$

Moreover we define the Hilbert space

$$H(\Delta, \Omega_1) = \{v \in H^1(\Omega_1) : \Delta v \in L^2(\Omega_1)\}, \tag{5.1}$$

with inner product

$$\langle u, v\rangle_{H(\Delta,\Omega_1)} = \langle u, v\rangle_{H^1(\Omega_1)} + \langle \Delta u, \Delta v\rangle_{L^2(\Omega_1)}. \tag{5.2}$$

Points on the boundary of Ω_1 are identified with their corresponding angle in polar coordinates, i.e. we set $\partial\Omega_1 = \{t + 2\pi\mathbb{Z} : t \in \mathbb{R}\} = \mathbb{R}/2\pi\mathbb{Z} =: \mathbb{T}$. For $l \geq 0$, the Sobolev space $H^l(\mathbb{T})$ is defined by

$$H^l(\mathbb{T}) = \{f \in L^2(\mathbb{T}) \ \sum_{n\in\mathbb{Z}} (1 + n^2)^l |\tilde{f}(n)|^2 < \infty\}, \tag{5.3}$$

where

$$\tilde{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(s)e^{-ins} ds, \tag{5.4}$$

denotes the Fourier transform of f. $H^l(\mathbb{T})$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle_{H^l(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (1 + n^2)^l \tilde{f}(n)\widehat{\tilde{g}(n)}. \tag{5.5}$$

The Sobolev space $H^l_\diamond(\mathbb{T})(l \geq 0)$ is defined by

$$H^l(\mathbb{T}) = \{f \in L^2(\mathbb{T}) \ \sum_{n\in\mathbb{Z}_0} (1 + n^2)^l |\tilde{f}(n)|^2 < \infty\}. \tag{5.6}$$

By $H^{-l}(\mathbb{T})(l \geq 0)$ we denote the dual of $H^l(\mathbb{T})$. With the Fourier transform defined on $H^{-l}(\mathbb{T})$ by $\hat{f}(n) := \langle f, e^{-int}\rangle_{H^{-l}(\mathbb{T}),H^l(\mathbb{T})}$ we find

$$\langle f, e^{-int}\rangle_{H^{-l}(\mathbb{T}),H^l(\mathbb{T})} = \sum_{n\in\mathbb{Z}} \tilde{f}(n)\widehat{\tilde{g}(n)} \tag{5.7}$$

for the duality pairing $\langle .,. \rangle_{H^{-l}(\mathbb{T}),H^l(\mathbb{T})}$ and for $f \in H^{-l}(\mathbb{T})$ and $g \in H^l(\mathbb{T})$. Moreover, $H^{-l}(\mathbb{T})$ is characterized by (5.3) and the inner product on $H^{-l}(\mathbb{T})$ is given by (5.7), in both cases with l replaced by $-l$. We have

$$f = \sum_{n\in\mathbb{Z}} \tilde{f}(n)e^{int} \tag{5.8}$$

for $f \in H^l(\mathbb{T}), l \in \mathbb{R}$, where the series (5.8) converges in $H^l(\mathbb{T})$. We define the differential operator $D : H^l(\mathbb{T}) \to H^{l-1}(\mathbb{T})$ by

$$Df = \sum_{n\in\mathbb{Z}} in\tilde{f}(n)e^{int}. \tag{5.9}$$

It is a bounded linear operator with $\ker(D)$ given by the constant functions on $\mathbb{T}$. The space $\mathcal{C}^\infty(\mathbb{T})$ of all infinitely differentiable functions on $\mathbb{T}$ is characterized by

$$\mathcal{C}^\infty(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}\ |n|^k|\tilde{f}(n) \to 0 \text{as} n \to \infty \text{for all } k \in \mathbb{N}\}. \tag{5.10}$$

Those facts follows as a special case of theorems on Sobolev spaces on smooth compact manifolds as given for example in Wloka (Wloka, 1987).