Systems of second-order linear ODE's with constant coefficients and their symmetries II
The case of non-diagonal coefficient matrices

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Abstract

We complete the analysis of the symmetry algebra $\mathcal{L}$ for systems of n second-order linear ODEs with constant real coefficients, by studying the case of coefficient matrices having a non-diagonal Jordan canonical form. We also classify the Levi factor (maximal semisimple subalgebra) of $\mathcal{L}$, showing that it is completely determined by the Jordan form. A universal formula for the dimension of the symmetry algebra of such systems is given. As application, the case $n = 5$ is analyzed.

Keywords: Lie group method, point symmetry, Lie algebra, Levi factor, linearization
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1. Introduction

In contrast to the case of (scalar) ordinary differential equations, the analysis of the point symmetries and linearization criteria for systems of n ODEs remains still an unsolved problem for general n, although various results basing on different approaches have been developed in the literature [1, 2, 3]. A precise knowledge of the symmetries for linear systems constitutes a valuable tool for the study of non-linear systems, as it is known that systems of nonlinear ODEs can be locally mapped to a linear system of ODEs whenever they have the same structure of symmetry [4]. The case of systems consisting of order two differential equations is specially important, as it is related with many applications in Mechanics and dynamical systems, and the symmetry groups can be used to better comprehend the evolution and characteristics of such systems [5, 6]. The symmetries of linear second systems with $n \leq 3$ equations and constant coefficients have been recently studied in detail in [7, 8], while those with $n = 4$ equations were analyzed in [9]. The latter work also dealt with the general case of diagonal coefficient matrices and the structure of their symmetry Lie algebra $\mathcal{L}$.
The main objective of this work is to fill this gap in the literature, by finishing the study of symmetries of systems with constant coefficients, the coefficients matrices of which are non-diagonalizable. To this extent, we divide the task into various steps. We first consider the case of coefficient matrices J having only one real eigenvalue or two complex conjugated eigenvalues. The symmetry condition is explicitly integrated, and the dimension of the resulting symmetry algebra $\mathcal{L}$ given for arbitrary n. The argument bases heavily on the nilpotent part of the canonical Jordan forms [10]. We also prove that the Levi factor (i.e., the maximal semisimple subalgebra) of the symmetry algebra is determined by the Jordan form, and obtain a realization of this algebra. We also show that symmetries of systems $\ddot{\mathbf{x}} = J\mathbf{x}$ with coefficient matrices having more than one eigenvalue can be essentially reduced to the analysis of the cases with only one eigenvalue (or two complex conjugated eigenvalues). This allows to establish, for arbitrary n, a general formula for the

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dimension of the symmetry algebra. In particular, it will follow that the dimension of the symmetry algebra for a system of n equations and more than one eigenvalue can be described in terms of the dimension of the symmetry algebras of systems with one eigenvalue and $m < n$ equations. In similar form, the Levi subalgebra can be classified, resulting in a criterion to decide whether the symmetry algebra is solvable or not. As an illustration of the procedure, we tabulate the dimension and Levi factors for systems consisting of $n = 5$ ODEs.

1.1. Jordan forms of systems of ODEs and symmetries

Any second-order system of ODEs with constant coefficients $\ddot{\mathbf{x}} = M_1\dot{\mathbf{x}} + M_2\mathbf{x} + M_3(t)$, where $\mathbf{x}, M_3(t) \in \mathbb{R}^n$ and M_1, M_2 are real matrices, can be reduced, by means of canonical forms [7, 11], to a system of the form

$$\ddot{\mathbf{y}} = \left(M_2 - M_1^2\right)\mathbf{y}.$$

Since we are dealing with matrices with real coefficients, there always exists a (real) Jordan canonical form J such that $M_2 - M_1^2 = P^{-1}JP$ for some invertible matrix P [10]. Since the latter is related to changes of basis, a second change $\mathbf{y} = P\mathbf{z}$ reduces the system to the equivalent form $\ddot{\mathbf{z}} = J\,\mathbf{z}$. Therefore, the analysis of symmetries of systems with constant coefficients reduces to study the symmetries of the non-equivalent canonical forms.

To compute the symmetries, we maintain the approach by means of vector fields used in [9], and based on the equivalence of systems with PDEs of first order [12]. The system of ODEs $\ddot{x}_i = \omega_i(\dot{x}_j, x_j, t)$ is equivalent to the first order PDE

$$\mathbf{A}f = \left(\frac{\partial}{\partial t} + \dot{x}_i\frac{\partial}{\partial x_i} + \omega_i\frac{\partial}{\partial \dot{x}_i}\right)f = 0. \tag{1}$$

In this context, a vector field $X = \xi(t, x_1, \cdots, x_n)\frac{\partial}{\partial t} + \eta_i(t, x_1, \cdots, x_n)\frac{\partial}{\partial x_i}$ is a (point) symmetry if its prolongation $\dot{X} = X + \dot{\eta}_i(t, x_i, \dot{x}_i)\frac{\partial}{\partial \dot{x}_i}$ satisfies the commutator

$$\left[\dot{X}, \mathbf{A}\right] = -\frac{d\xi}{dt}\mathbf{A}, \tag{2}$$

where $\dot{\eta}_i = -\frac{d\xi}{dt}\dot{x}_i + \frac{d\eta_i}{dt}$. The n resulting equations of the components of the bracket are identities in t, x_i and $\dot{x}_i$. As for point symmetries the functions ξ and η_i of the symmetry X do not depend on the $\dot{x}_i$s, these n equations can be further split into more equations that express the symmetry condition analytically.

1.2. Non-diagonal canonical forms J

As commented, the case of systems with diagonal coefficient matrices was covered in [9], where the general dimension and the structure of the symmetries was obtained. To complete the study of systems with constants coefficients, it remains to see what happens for the non-diagonal canonical forms, corresponding to non-diagonalizable coefficient matrices. Certainly this case is more complicated, due to the high number of canonical forms and their structure. A generic non-diagonal canonical form J is of the shape

$$J = \begin{pmatrix} \lambda_1 & \nu_1 & & & & \\ & \lambda_2 & \nu_2 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \nu_{n-2} & \\ & & & & \lambda_{n-1} & \nu_{n-1} \\ & & & & & \lambda_n \end{pmatrix}, \tag{3}$$

where the ν_i are either zero or one, and $\nu_i = 0$ if $\lambda_i \neq \lambda_{i+1}$. For notational convenience we also define $\nu_n = 0$. Since J is not diagonalizable, at least one of the ν_i's must be nonzero. The eigenvalues λ_i of J are either real or complex conjugated [10]. If the eigenvalues are real, J is the canonical form to be analyzed, while for complex conjugated eigenvalues (3) must be further transformed to a real matrix involving rotations. For this reason it is convenient to analyze both possibilities separately, since the structure of the symmetry algebras will be essentially different.

2. Matrices with one real eigenvalue

Suppose that all eigenvalues λ_i of (3) are real. For such a matrix J, the symmetry condition (2) of the system $\ddot{\mathbf{x}} = J\mathbf{x}$ is expressed by means of the following system of PDEs:

$$\frac{\partial^2 \xi}{\partial x_i \partial x_j} = 0, \quad \frac{\partial^2 \eta_l}{\partial x_i \partial x_j} = 0, \ 1 \leq i, j \leq n, \ i, j \neq l \tag{4}$$

$$2\frac{\partial^2 \xi}{\partial t \partial x_j} - \frac{\partial^2 \eta_j}{\partial^2 x_j} = 0, \ 1 \leq j \leq n \tag{5}$$

$$\frac{\partial^2 \xi}{\partial t \partial x_j} - \frac{\partial^2 \eta_l}{\partial x_l \partial x_j} = 0, \ 1 \leq j, l \leq n, \ j \neq l \tag{6}$$

$$\left(\lambda_l x_l + \nu_l x_{l+1}\right) \frac{\partial \xi}{\partial x_j} - \frac{\partial^2 \eta_l}{\partial t \partial x_j} = 0, \ 1 \leq j, l \leq n, \ j \neq l \tag{7}$$

$$\sum_{i \neq l} \left(\lambda_i x_i + \nu_i x_{i+1}\right) \frac{\partial \xi}{\partial x_i} + 3\left(\lambda_l x_l + \nu_l x_{l+1}\right) \frac{\partial \xi}{\partial x_l} + \frac{\partial^2 \xi}{\partial t^2} - 2\frac{\partial^2 \eta_l}{\partial t \partial x_l} = 0, \ 1 \leq l \leq n \tag{8}$$

$$\lambda_l \eta_l + \nu_l \eta_{l+1} - \sum_{i=1}^{n} \left(\lambda_i x_i + \nu_i x_{i+1}\right) \frac{\partial \eta_l}{\partial x_i} - \frac{\partial^2 \eta_l}{\partial t^2} + 2\left(\lambda_l x_l + \nu_l x_{l+1}\right) \frac{\partial \xi}{\partial t} = 0, \ 1 \leq l \leq n \tag{9}$$

As a first simplification, we will suppose in this section that the matrix J of (3) has only one real eigenvalue, i.e., $\lambda = \lambda_i$ for $i = 1, \ldots, n$. In Section 4 we will prove that the general case of multiple (real) eigenvalues can be recovered from the analysis of the subsystems corresponding to the different eigenvalues.
The first step in the analysis of the symmetries is to obtain a generic form for the solution of the symmetry condition.

Theorem 1. *Let $X = \xi \frac{\partial}{\partial t} + \eta_i \frac{\partial}{\partial x_i}$ be a symmetry generator of the system $\ddot{\mathbf{x}} = J\mathbf{x}$. Then the component functions $\xi\left(t, x_1, \ldots, x_n\right)$ and $\eta_j\left(t, x_1, \ldots, x_n\right)$ $(j = 1, \ldots, n)$ have the following generic form:*

1. *If $\lambda \neq 0$:*

$$\xi\left(t, x_1, \ldots, x_n\right) = \alpha_0,$$
$$\eta_j\left(t, x_1, \ldots, x_n\right) = \sum_{l=1}^{n} \alpha_l^l x_j + \sigma_j(t).$$

2. *If $\lambda = 0$:*

$$\xi\left(t, x_1, \ldots, x_n\right) = \alpha_0 + \kappa_1 t,$$
$$\eta_j\left(t, x_1, \ldots, x_n\right) = \sum_{l=1}^{n} \alpha_l^l x_j + \sigma_j(t).$$

where $\alpha_0, \kappa_1, \alpha_j^l \in \mathbb{R}$.

From the set of equations (4) we immediately obtain that the component functions ξ and η_i can be written as

$$\xi\left(t, x_1, \ldots, x_n\right) = \varphi_0(t) + \sum_{i=1}^{n} \varphi_i(t)\, x_i,$$
$$\eta_l\left(t, x_1, \ldots, x_n\right) = \sum_{j \neq l} x_j \theta_j^l\left(t, x_l\right) + \rho_l\left(t, x_l\right).$$

By equation (5) we get the condition

$$2\dot{\varphi}_j(t) = \frac{\partial^2 \rho}{\partial x_j^2} + \sum_{l \neq j} x_l \frac{\partial^2 \theta_l^j}{\partial x_j^2},$$

which implies that $\frac{\partial^2 \theta_l^j}{\partial x_j^2} = 0$ for $l \neq j$ and $2\dot{\varphi}_j(t) = \frac{\partial^2 \rho}{\partial x_j^2}$. Now (6) further implies that $\frac{\partial^2 \theta_l^j}{\partial x_j \partial x_l} = \dot{\varphi}_l(t)$ for $l \neq j$. Thus the component functions $\eta_j(t, x_1, \cdots, x_n)$ can be rewritten as

$$\eta_j(t, x_1, \ldots, x_n) = \left(x_j^2 \dot{\varphi}_j(t) + x_j \zeta_j(t) + \sigma_j(t)\right) + \sum_{j \neq l} x_l \left(x_j \dot{\varphi}_l + \tau_j^l(t)\right), \ 1 \leq j \leq n. \tag{10}$$

For each j we now evaluate equation (7), which provides the conditions

$$(\lambda x_j + \nu_j x_{j+1}) \varphi_i(t) - x_j \ddot{\varphi}_i(t) - \dot{\tau}_j^l(t) = 0, \ i \neq j. \tag{11}$$

Since the matrix J is not diagonal, there exists at least one index $v_{j_0} \neq 0$. Without loss of generality we can suppose that $\nu_1 = 1$. From (11) we deduce that $\varphi_i(t) = 0$ for $i = 2, \cdots, n$.[1] The latter constraint implies that $\frac{\partial \xi}{\partial x_i} = 0$ for $i \geq 2$. On the other hand, integrating the condition $\dot{\tau}_j^l(t) = 0$ we get $\tau_j^l(t) = \alpha_j^l$. Taking into account this simplification, equation (8) with $j = 1$ reduces to

$$3(\lambda x_1 + x_2) \dot{\varphi}_1(t) + \ddot{\varphi}_0(t) + x_1 \ddot{\varphi}_1(t) - 4x_1 \ddot{\varphi}_1(t) - 2\dot{\zeta}_1(t) = 0. \tag{12}$$

From here we immediately see that $\varphi_1(t) = 0$, and further that the function $\zeta_1(t)$ satisfies the ODE

$$\ddot{\varphi}_0(t) - 2\dot{\zeta}_1(t) = 0.$$

The same condition is obtained for the remaining indices $j = 2, \cdots, n$, so that all functions $\zeta_j(t)$ are related to the second derivative of $\varphi_0(t)$. Straightforward integration gives $\zeta_j(t) = \frac{1}{2}\dot{\varphi}_0(t) + \alpha_j^j$, $j = 1, \cdots, n$. Up to this point, the general solution to the symmetry condition takes the shape

$$\xi(t, x_1, \ldots, x_n) = \varphi_0(t), \tag{13}$$

$$\eta_j(t, x_1, \ldots, x_n) = \frac{x_j}{2}\dot{\varphi}_0(t) + \sigma_j(t) + \sum_{l=1} \alpha_j^l x_l. \tag{14}$$

It remains to consider the last equation (9). As $\nu_1 = 1$, this means that can express $\eta_2(t, x_1, \ldots, x_n)$ in terms of $\eta_1(t, x_1, \ldots, x_n)$, $\varphi_0(t)$ and their (partial) derivatives:

$$\begin{aligned} \eta_2(t, x_1, \ldots, x_n) &= \sum_{i=1}^n (\lambda x_i + \nu_i x_{i+1}) \frac{\partial \eta_1}{\partial x_i} + \frac{\partial^2 \eta_1}{\partial t^2} - 2(\lambda x_1 + x_2) \dot{\varphi}_0(t) - \lambda \eta_1(t, x_1, \ldots, x_n) \\ &= \left(\frac{1}{2}\varphi_0^{(3)}(t) - 2\lambda \dot{\varphi}_0(t)\right) x_1 + \left(\alpha_1^1 - 2\dot{\varphi}_0(t)\right) x_2 + \sum_{l=3}^n \nu_{l-1} \alpha_1^{l-1} x_l + \left(\ddot{\sigma}_1(t) - \lambda \sigma_1(t)\right). \end{aligned} \tag{15}$$

Comparing the coefficients with those of (14) we deduce the identities

$$\begin{aligned} &\alpha_2^1 = \tfrac{1}{2}\varphi_0^{(3)}(t) - 2\lambda \dot{\varphi}_0(t); \quad &&\alpha_2^2 = \alpha_1^1 - 2\dot{\varphi}_0(t); \\ &\alpha_2^l = \nu_{l-1} \alpha_1^{l-1}, \ l \geq 3; &&\sigma_2(t) = \ddot{\sigma}_1(t) - \lambda \sigma_1(t). \end{aligned}$$

Now, if $\nu_2 \neq 0$, equation (9) allows us again to obtain $\eta_3(t, x_1, \ldots, x_n)$ expressed as a function of $\eta_1(t, x_1, \ldots, x_n)$ and $\varphi_0(t)$. Evaluating the identity and comparing the coefficients in both sides, we rewrite $\eta_3(t, x_1, \ldots, x_n)$ as

$$\eta_3(t, x_1, \ldots, x_n) = 2\left(\frac{1}{2}\varphi_0^{(3)}(t) - 2\lambda \dot{\varphi}_0(t)\right) x_2 + \left(\alpha_1^1 - 4\dot{\varphi}_0(t)\right) x_3 + \sum_{l=4}^n \nu_{l-1}\nu_{l-2} \alpha_1^{l-2} x_l + \left(\sigma_1^4(t) - 2\lambda \ddot{\sigma}_1(t) + \lambda^2 \sigma_1(t)\right).$$

[1] If there is another index $\nu_h \neq 0$ with $h \neq 1$, then we also have $\varphi_1(t) = 0$. However, in order to avoid a distinction of cases, we will not make further assumptions on the values of ν_h for $h \geq 2$.

The same pattern holds for successive nonzero indices ν_i. Therefore, if $\nu_1, \nu_2, \cdots, \nu_{l-1}$ are all nonzero, the component functions $\eta_j(t, x_1, \ldots, x_n)$ for $j = 2, \cdots, l$ are completely determined by $\eta_1(t, x_1, \ldots, x_n)$, $\varphi_0(t)$ and their derivatives. Applying recurrence, we arrive at the closed expression

$$\eta_l(t, x_1, \ldots, x_n) = (l-1)\left(\frac{1}{2}\varphi_0^{(3)}(t) - 2\lambda\dot{\varphi}_0(t)\right)x_{l-1} + \left(\alpha_1^1 - 2(l-1)\dot{\varphi}_0(t)\right)x_l + \sum_{m=l+1}^{n}\left(\prod_{s=1}^{l-1}\nu_{m-s}\right)\alpha_1^{m+1-l}x_m + \sigma_l(t), \tag{16}$$

where

$$\sigma_l(t) = \sum_{k=0}^{l-1}(-1)^k \binom{l-1}{k}\lambda^k \sigma_1^{(2l-2-2k)}(t). \tag{17}$$

Suppose that $\nu_l = 0$ is the first index that vanishes ($l \geq 2$).[2] In this case, equation (9) provides additional constraints on the coefficients and functions of $\eta_l(t, x_1, \ldots, x_n)$, and therefore on $\sigma_1(t)$, $\varphi_0(t)$ and the real constants α_1^j. Starting from the expression (14) for $\eta_l(t, x_1, \ldots, x_n)$ and applying (9), we obtain the identity

$$\left(2\lambda\dot{\varphi}_0(t) - \frac{1}{2}\varphi_0^{(3)}(t) - \alpha_l^{l-1}\right)x_l - \sum_{s=l+1}^{n-1}\nu_s\alpha_l^s x_{s+1} + \left(\ddot{\sigma}_l(t) - \lambda\sigma_l(t)\right) = 0. \tag{18}$$

Hence, inserting the values of the constants α_l^j from (16), we are led to the constraints

$$\begin{array}{l} l\left(2\lambda\dot{\varphi}_0(t) - \frac{1}{2}\varphi_0^{(3)}(t)\right) = 0 \\ \sum_{k=0}^{l}(-1)^k\binom{l}{k}\lambda^k\sigma_1^{(2l-2k)}(t) = 0 \\ \sum_{m=l+1}^{n}\left(\prod_{s=0}^{l-1}\nu_{m-s}\right)\alpha_1^{m+1-l}x_m = 0 \end{array} \tag{19}$$

The first of these equations must be analyzed in dependence on the value of the eigenvalue λ. Direct integration gives

$$\varphi_0(t) = \begin{cases} \alpha_0 + \kappa_1\exp\left(2\sqrt{\lambda}t\right) + \kappa_2\exp\left(-2\sqrt{\lambda}t\right), & \lambda > 0 \\ \alpha_0 + \kappa_1 t + \kappa_2 t^2, & \lambda = 0 \\ \alpha_0 + \kappa_1\sin\left(2\sqrt{-\lambda}t\right) + \kappa_2\cos\left(2\sqrt{-\lambda}t\right), & \lambda < 0 \end{cases}. \tag{20}$$

By (16), the coefficients α_i^i must be real constants, which further implies that

$$\alpha_l^l - 2(l-1)\dot{\varphi}_0(t) = \begin{cases} \alpha_l^l - 2(l-1)\left(2\sqrt{\lambda}\kappa_1\exp\left(2\sqrt{\lambda}t\right) - 2\sqrt{\lambda}\kappa_2\exp\left(-2\sqrt{\lambda}t\right)\right), & \lambda > 0 \\ \alpha_l^l - 2(l-1)(\kappa_1 + 2\kappa_2 t), & \lambda = 0 \\ \alpha_l^l - 2(l-1)\left(2\sqrt{-\lambda}\kappa_1\cos\left(2\sqrt{-\lambda}t\right) - 2\sqrt{-\lambda}\kappa_2\sin\left(2\sqrt{-\lambda}t\right)\right), & \lambda < 0 \end{cases}, \tag{21}$$

a condition that can only hold if $\kappa_1 = \kappa_2 = 0$ for $\lambda \neq 0$ and $\kappa_2 = 0$ for $\lambda = 0$, showing that $\dot{\varphi}_0(t)$ must be a constant. The components $\eta_j(t, x_1, \ldots, x_n)$ for $j \leq l$ are hence completely determined by equation (16). Observe further that for $j = l+1, \cdots, n$, equations (14) and (20) imply that the functions $\eta_j(t, x_1, \ldots, x_n)$ have the following generic form (where $\kappa_1 = 0$ if $\lambda \neq 0$):

$$\eta_j(t, x_1, \ldots, x_n) = \sum_{l=1}^{n}\left(\alpha_j^l + \frac{1}{2}\delta_j^l\kappa_1\right)x_l + \sigma_j(t). \tag{22}$$

This shows that the components functions $\xi(t, x_1, \ldots, x_n)$ and $\eta_j(t, x_1, \ldots, x_n)$ have the generic form claimed.

[2] I.e., $\nu_1 = \nu_2 = \cdots = \nu_{l-1} = 1$.

As follows from equation (16), it may seem difficult to count the exact number of independent integration constants α_i^l for a generic matrix J as given in (3), as the successive products of the indices ν_i are involved. In order to circumvent this difficulty and be able to determine the number of symmetries of the system, we consider the normal form J more closely. We can always rewrite it as

$$J = \begin{pmatrix} J_{m_1} & & & \\ & J_{m_2} & & \\ & & \ddots & \\ & & & J_{m_{p_0}} \end{pmatrix}, \tag{23}$$

where for each index m_i $(i = 1, \cdots, p_0)$ the submatrix J_{m_i} denotes the $(m_i + 1) \times (m_i + 1)$ −dimensional Jordan block

$$J_{m_i} = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}. \tag{24}$$

The scalar p_0 is nothing as the number of Jordan blocks of the matrix J. For convenience in the computations, we further order the blocks J_{m_i} after its size, i.e., taking $m_1 \geq m_2 \geq \cdots \geq m_{p_0} \geq 0$. It follows in particular that $\sum_{k=1}^{p_0} m_i + p_0 = n$.

Theorem 2. *Let $\mathcal{L}$ denote the Lie algebra of point symmetries of the system of n equations $\ddot{\mathbf{x}} = J\mathbf{x}$.*

1. *If $\lambda \neq 0$, then*

$$\dim \mathcal{L} = (2 + p_0)\, n + 1 - \sum_{i=1}^{p_0 - 1} \sum_{k=j+1}^{p_0} (m_j - m_k). \tag{25}$$

2. *If $\lambda = 0$,*

$$\dim \mathcal{L} = (2 + p_0)\, n + 2 - \sum_{i=1}^{p_0 - 1} \sum_{k=j+1}^{p_0} (m_j - m_k). \tag{26}$$

Proof. The proof of this assertion is enormously simplified if we use the nilpotent part of the Jordan form (3).
For each Jordan block J_{m_i}, the function $\eta_{\varsigma_i}(t, x_1, \cdots, x_n)$ determines the components $\eta_{\varsigma_i + j}(t, x_1, \cdots, x_n)$ for $1 \leq j \leq m_i$, where $\varsigma_1 = 1$, $\varsigma_i = \sum_{k=1}^{i-1} m_k + i$ for $(2 \leq i \leq p_0 - 1)$. Therefore we only need to consider the number of integration constants provided by the functions $\eta_{\varsigma_i}(t, x_1, \cdots, x_n)$. The generic form for the latter is

$$\eta_{\varsigma_i}(t, x_1, \cdots, x_n) = \sum_{k=1}^{n} \alpha_{\varsigma_i}^j x_j + \sigma_i(t), \; (1 \leq i \leq p_0).$$

For each m_i, the function $\sigma_i(t)$ satisfies the corresponding ODE

$$\sum_{k=0}^{m_i+1} (-1)^k \begin{pmatrix} m_i + 1 \\ k \end{pmatrix} \lambda^k \sigma_i^{(2m_i+2-2k)}(t) = 0, \; 1 \leq i \leq p_0. \tag{27}$$

If $\lambda \neq 0$, the general solution to such equation is given by

$$\sigma_i(t) = \left(\sum_{s=0}^{m_i} \beta_s^i t^s \right) \exp\left(\sqrt{\lambda} t\right) + \left(\sum_{s=0}^{m_i} \gamma_s^i t^s \right) \exp\left(-\sqrt{\lambda} t\right),$$

thus any function $\sigma_i(t)$ contributes with $2(m_i+1)$ integration constants and the part of the solution depending on the independent variable t has dimension $2\sum_{i=1}^{p_0}(m_i+1) = 2p_0 + 2\sum_{i=1}^{p_0} m_i$. For the case $\lambda = 0$, the equation (27) simply reduces to $\sigma_i^{(2m_i+2-2k)}(t) = 0$, with general solution

$$\sigma_i(t) = \sum_{k=0}^{2m_i+1} \frac{t^k}{k!}\beta_k.$$

Also in this case, we obtain $2m_i+2$ integration constants for each i, giving a total amount of $2p_0+2\sum_{i=1}^{p_0} m_i$. If we take into account that $n = p_0+\sum_{i=1}^{p_0} m_i$, the number of integration constants provided by the functions $\sigma_i(t)$ is simply $2n$, i.e., it depends only on the number of equations. Further, as follows from Theorem 1, the time component of a symmetry X is always $\xi(t, x_1, \cdots, x_n) = \alpha_0 + \kappa_1 t$, where $\kappa_1 \neq 0$ only if $\lambda = 0$.
In order to determine the dimension of $\mathcal{L}$, it remains to see how many integration constants arise from the linear part $\eta_{\varsigma_i}^{LT}(t, x_1, \cdots, x_n) := \sum_{k=1}^{n} \alpha_{\varsigma_i}^{j} x_j$ of the general solution to the symmetry condition.
In order to see how many of the constants α_i^l are independent, we use the nilpotent part of J as given in (23). It follows at once from (23) and (24) that

$$(J - \lambda \mathrm{Id}_n)^{m_1+1} = 0.$$

This condition implies that any for $i \geq 2$, the products $\prod_{k=1}^{m_1+1} \nu_{i-k}$ are identically zero. In particular, it follows that the third equation of (19) is identically zero taking $l = m_1+1$. As a consequence, the coefficients α_1^j are not subjected to any restriction, and the linear part of $\eta_1(t, x_1, \cdots, x_n)$ is simply $\sum_{k=1}^{n} \alpha_1^k x_k$ with n degrees of liberty.[3] This can be best seen rewriting the linear parts $\eta_j^{LT}(t, x_1, \cdots, x_n)$ of $\eta_j(t, x_1, \cdots, x_n)$ ($j = 2, \cdots, m_1+1$) using matrices: Let $\langle \cdot, \cdot \rangle$ denote the standard inner product of $\mathbb{R}^n$. By (15), the linear part of $\eta_2(t, x_1, \cdots, x_n)$ can be reformulated as

$$\eta_2^{LT}(t, x_1, \cdots, x_n) = \left\langle \begin{pmatrix} \alpha_1^1 & & & & \\ & \alpha_1^2 & & & \\ & & \ddots & & \\ & & & \alpha_1^{n-1} & \\ & & & & \alpha_1^n \end{pmatrix} \begin{pmatrix} 0 & \nu_1 & & & \\ & 0 & \nu_2 & & \\ & & 0 & \ddots & \\ & & & & \nu_{n-1} \\ & & & & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \right\rangle, \tag{28}$$

where $\nu_1 = \cdots = \nu_{m_1} = 1$, $\nu_{m_1+1} = 0$ etc by (23). Observe that the upper diagonal matrix depending on the ν_is is exactly $J - \lambda \mathrm{Id}_n$. Let us denote the diagonal matrix of (28) by A_1, and let $\mathbf{v} = (1, \cdots, 1)^T$. It follows from (16) that

$$\eta_j^{LT}(t, x_1, \cdots, x_n) = \left\langle A_1 (J - \lambda \mathrm{Id}_n)^{j-1} \mathbf{x}, \mathbf{v} \right\rangle, \; j = 2, \cdots, m_1+1. \tag{29}$$

Now the third condition of (19) implies that the identity

$$\left\langle A_1 (J - \lambda \mathrm{Id}_n)^{m_1+1} \mathbf{x}, \mathbf{v} \right\rangle = 0$$

must be satisfied. However, as $(J - \lambda \mathrm{Id}_n)^{m_1+1} = 0$, this means that there are no constraints on the scalars α_1^k, as seen above. In analogous manner we can treat the remaining $\eta_{\varsigma_i}^{LT}(t, x_1, \cdots, x_n) := \sum_{k=1}^{n} \alpha_{\varsigma_i}^{j} x_j$. Written in matrix form, these functions are expressed as

$$\eta_{\varsigma_i}^{LT}(t, x_1, \cdots, x_n) = \langle A_i (J - \lambda \mathrm{Id}_n) \mathbf{x}, \mathbf{v} \rangle \, ; \; i = 2, \cdots, p_0 - 1, \tag{30}$$

[3] If $\lambda = 0$, we have the additional term $\kappa_1 x_1$, which does however not add dimension, since $\xi(t, x_1, \cdots, x_n) = \alpha_0 + \kappa_1 t$ by Theorem 1.

where A_i denotes the diagonal matrix whose entries are $\left(\alpha_{\varsigma_i}^1, \cdots, \alpha_{\varsigma_i}^n\right)$. Again, the functions $\eta_{\varsigma_i+j}^{LT}(t, x_1, \cdots, x_n)$, where $j = 1, \cdots, m_i$, are given by successive powers of $(J - \lambda \mathrm{Id}_n)$:

$$\eta_{\varsigma_i+j}^{LT}(t, x_1, \cdots, x_n) = \left\langle A_i (J - \lambda \mathrm{Id}_n)^{j-1} \mathbf{x}, \mathbf{v} \right\rangle. \tag{31}$$

For the last component $\eta_{\varsigma_i+m_i}^{LT}(t, x_1, \cdots, x_n)$ of the block J_{m_i}, equation (9) implies the identity

$$\left\langle A_i (J - \lambda \mathrm{Id}_n)^{m_i+1} \mathbf{x}, \mathbf{v} \right\rangle = 0. \tag{32}$$

If $m_2 = m_1$, then (32) vanishes identically, and $\eta_{\varsigma_2}^{LT}(t, x_1, \cdots, x_n) = \sum_{k=1}^n \alpha_{\varsigma_2}^j x_j$ with n integration constants. Now, if $m_1 > m_2$, we get a nontrivial identity

$$\sum_{k=1}^{m_1-m_2} \alpha_{\varsigma_2}^k x_{k+m_2+1} = 0.$$

This implies that

$$\alpha_{\varsigma_2}^1 = \alpha_{\varsigma_2}^2 = \cdots = \alpha_{\varsigma_2}^{m_1-m_2} = 0,$$

and $\eta_{\varsigma_2}^{LT}(t, x_1, \cdots, x_n)$ only provides $n - (m_1 - m_2)$ new integration constants. For $\eta_{\varsigma_3}^{LT}(t, x_1, \cdots, x_n) = \sum_{k=1}^n \alpha_{\varsigma_3}^k x_k$ the situation is slightly more complicated. If $m_1 = m_2 = m_3$, all the $\alpha_{\varsigma_3}^k$ are independent and provide n integration constants. If $m_1 > m_2 = m_3$, (32) implies the constraint

$$\sum_{k=1}^{m_1-m_3} \alpha_{\varsigma_3}^k x_{k+(m_3+1)} = 0, \tag{33}$$

hence $\alpha_{\varsigma_3}^1 = \alpha_{\varsigma_3}^2 = \cdots = \alpha_{\varsigma_3}^{m_1-m_2} = 0$ and only $n - (m_1 - m_3)$ integration constants arise. For the last possibility, $m_1 > m_2 > m_3$, the constraint (32) involves two sums

$$\sum_{k=1}^{m_1-m_3} \alpha_{\varsigma_3}^k x_{k+(m_3+1)} + \sum_{k=1}^{m_2-m_3} \alpha_{\varsigma_3}^{k+\varsigma_2-1} x_{k+(\varsigma_2+1)} = 0, \tag{34}$$

and thus $\alpha_{\varsigma_3}^1 = \alpha_{\varsigma_3}^2 = \cdots = \alpha_{\varsigma_3}^{m_1-m_3} = \alpha_{\varsigma_3}^{\varsigma_2} = \cdots = \alpha_{\varsigma_3}^{m_2-m_3+\varsigma_2-1} = 0$. The total number of integration constants of $\eta_{\varsigma_3}^{LT}(t, x_1, \cdots, x_n)$ would be the given by

$$n - (m_1 - m_3) - (m_2 - m_3).$$

A recurrence argument shows that for m_i the condition (32) leads to the identity

$$\sum_{k=1}^{m_1-m_i} \alpha_{\varsigma_i}^k x_{k+(m_i+1)} + \sum_{k=1}^{m_2-m_i} \alpha_{\varsigma_3}^{k+\varsigma_2-1} x_{k+\varsigma_2} + \sum_{k=1}^{m_3-m_i} \alpha_{\varsigma_3}^{k+\varsigma_3-1} x_{k+\varsigma_3} + \cdots + \sum_{k=1}^{m_{i-1}-m_i} \alpha_{\varsigma_3}^{k+\varsigma_{i-1}-1} x_{k+\varsigma_{i-1}} = 0, \tag{35}$$

and therefore the linear function $\eta_{\varsigma_i}^{LT}(t, x_1, \cdots, x_n)$ provides exactly

$$n - (m_1 - m_i) - (m_2 - m_i) - \cdots (m_{i-1} - m_i)$$

integration constants. Summing together the integration constants for $m_1, \cdots, m_{p_0}$, the linear part of the solution has

$$p_0 n - \sum_{j=1}^{p_0-1} \sum_{k=j+1}^{p_0} (m_j - m_k) \tag{36}$$

degrees of liberty. To these we must add $2n$ for the functional part coming from the functions $\sigma_i(t)$, and either 1 or 2 from $\xi(t, x_1, \cdots, x_n)$, depending whether the eigenvalue is nonzero or not. Taken together, the dimension of the symmetry algebra $\mathcal{L}$ is

$$\dim \mathcal{L} = (2 + p_0)\, n - \sum_{j=1}^{p_0-1} \sum_{k=j+1}^{p_0} (m_j - m_k) + \left\{ \begin{array}{ll} 1 & \text{for } \lambda \neq 0 \\ 2 & \text{for } \lambda = 0 \end{array} \right. ,$$

as claimed. ■

Our subdivision of the functions $\eta_i(t, x_1, \cdots, x_n)$ into a linear and a "functional" part has an additional advantage to that of simplifying the expression of the solution to the symmetry condition. It allows us to explicitly construct the symmetries for each given matrix J, by simply taking into account equations (28)-(31). Moreover, we can make precise assertions on the existence and structure of the Levi subalgebra of $\mathcal{L}$.

Proposition 1. *Let $\mathcal{L}$ be the symmetry algebra of the system $\ddot{\mathbf{x}} = J\mathbf{x}$, where J is of the form (3). Then the Levi subalgebra $\mathfrak{s}$ of $\mathcal{L}$ is isomorphic to $\mathfrak{sl}(q, \mathbb{R})$, where q denotes the number of Jordan blocks of order one of J.*

Proof. As follows from the preceding result, the component functions $\xi(t, x_1, \cdots, x_n)$ and $\eta_j(t, x_1, \cdots, x_n)$ of a symmetry X have the shape

$$\xi(t, x_1, \cdots, x_n) = \alpha_0 + \kappa_1 t,$$
$$\eta_i(t, x_1, \cdots, x_n) = \sum_{k=1}^{n} a_i^k x_k + \kappa_1 x_i + \sigma_i(t),$$

where $\kappa_1 = 0$ if $\lambda \neq 0$ and the functions $\sigma_i(t)$ and the coefficients α_i^k are determined by equations (27) and (29) respectively. Now the symmetries of the form

$$\phi(t) \frac{\partial}{\partial x_i},\ i = 1, \cdots, n \tag{37}$$

will always commute, which means that the symmetries determined by the functions $\sigma_i(t)$ will generate a $2n$-dimensional Abelian Lie subalgebra of $\mathcal{L}$, as the functional part of the solution provided $2n$ integration constants. On the other hand, the symmetries $Y_1 = \frac{\partial}{\partial t}$ and $Y_2 = t\frac{\partial}{\partial t}$ (for $\lambda = 0$) will only provide nonzero commutators with the symmetries of type (37), in addition to the commutator $[Y_1, Y_2] = Y_1$. Thus the subalgebra spanned by Y_1, Y_2 and the symmetries (37) is solvable of dimension $2n + 2$. This means that if a semisimple subalgebra $\mathfrak{s}$ of $\mathcal{L}$ exists, it will come from the symmetries of type $a_i^k x_k \frac{\partial}{\partial x_i}$, that is, will be determined by what we have called the linear part of $\eta_i(t, x_1, \cdots, x_n)$.

Like before, we assume that J is written in the block form (23). Suppose that some index m_k vanishes,[4] and let $m_i = 0$ be the first of such vanishing indices. Then necessarily $m_{i+1} = \cdots = m_{p_0} = 0$ because of the order $m_1 \geq m_2 \geq \cdots m_{p_0} \geq 0$. It follows in particular that $\varsigma_{i+j} = \varsigma_i + j$ for $j = 1, \cdots, p_0$ and $\varsigma_{p_0} = n$. Further, using equation (29) it can be easily seen that the linear part of the function $\eta_{\varsigma_i+j}(t, x_1, \cdots, x_n)$ is

$$\eta_{\varsigma_i+j}^{LT}(t, x_1, \cdots, x_n) = \sum_{l=1}^{i-1} \alpha_{\varsigma_i+j}^{m_l+1} x_{m_l+1} + \sum_{l=\varsigma_i}^{n} \alpha_{\varsigma_i+j}^{l} x_l. \tag{38}$$

Taking into account the symmetries arising from the first summand, it follows at once that

$$\left[\alpha_{\varsigma_i+j}^{m_l+1} x_{m_l+1} \frac{\partial}{\partial x_{\varsigma_i+j}}, \alpha_{\varsigma_i+k}^{m_l+1} x_{m_l+1} \frac{\partial}{\partial x_{\varsigma_i+k}}\right] = 0,$$

[4] In our previous notation, this means that the block J_{m_a} reduces to $[\lambda]$.

since $\varsigma_i + j \neq m_l + 1$ for $l = 1, \cdots, i-1$. These $(i-1)(n+1-\varsigma_i)$ symmetries therefore generate an Abelian subalgebra of $\mathcal{L}$. Now we consider those symmetries arising from the second summand of (38) . We obtain the $(n+1-\varsigma_i)^2$ symmetries

$$X_{l,j} = x_{\varsigma_i+l}\frac{\partial}{\partial x_{\varsigma_i+j}},\ 0 \leq l, j \leq n-\varsigma_i. \tag{39}$$

It is not difficult to verify that the symmetries of (39) generate a Lie algebra isomorphic to $\mathfrak{sl}(q,\mathbb{R}) \oplus \mathbb{R}$, where $q = p_0 + 1 - i$. Since $m_j \geq 1$ for $j = 1, \cdots, i-1$, the scalar q is exactly the number of one dimensional Jordan blocks of the matrix J.[5]
It remains to see that from the symmetries determined by the components $\eta_l(t, x_1, \cdots, x_n)$ for $l = 1, \cdots, \varsigma_i - 1$ associated to the Jordan blocks J_{m_l} of order at least two we cannot extract a semisimple Lie algebra. By equation (30), the linear part of $\eta_{\varsigma_k}(t, x_1, \cdots, x_n)$ for $k = 1, \cdots, i-1$ is given by

$$\eta^{LT}_{\varsigma_k}(t, x_1, \cdots, x_n) = \sum_{s=m_1-m_k+1}^{m_1+1} \alpha^s_{\varsigma_k} x_s + \cdots + \sum_{s=m_{k-2}-m_k+1}^{m_{k-2}+1} \alpha^s_{\varsigma_k} x_s + \sum_{s=m_{k-1}-m_k+1}^{n} \alpha^s_{\varsigma_k} x_s, \tag{40}$$

while the remaining components $\eta^{LT}_{\varsigma_k+j}(t, x_1, \cdots, x_n)$ being determined by equation (31). For any fixed k, each integration constant $\alpha^s_{\varsigma_k}$ gives rise to a symmetry generator

$$X^{(\varsigma_k)}_s = x_s\frac{\partial}{\partial x_{\varsigma_k}} + x_{s+1}\frac{\partial}{\partial x_{\varsigma_k+1}} + \cdots + x_{s+\varepsilon(s)}\frac{\partial}{\partial x_{\varsigma_k+\varepsilon(s)}}, \tag{41}$$

where $\varepsilon(s) \leq m_k$ is completely determined by s as a consequence of equation (31). Among all these symmetries, there is a distinguished one, namely

$$H^{(\varsigma_k)}_{\varsigma_k} = x_{\varsigma_k}\frac{\partial}{\partial x_{\varsigma_k}} + \cdots + x_{\varsigma_{k+1}-1}\frac{\partial}{\partial x_{\varsigma_{k+1}-1}}, \tag{42}$$

which corresponds to the integration constant $\alpha^1_{\varsigma_k}$.[6] It is immediate to see that for $k \neq k'$ we have

$$\left[H^{(\varsigma_k)}_{\varsigma_k}, H^{(\varsigma_{k'})}_{\varsigma_{k'}}\right] = 0.$$

Let $\mathcal{L}_1$ be the Lie subalgebra of $\mathcal{L}$ generated by all the symmetries $X^{(\varsigma_k)}_s, H^{(\varsigma_k)}_{\varsigma_k}$ for $k = 1, \cdots, i-1$. If $\mathcal{L}_1$ contains a semisimple subalgebra $\mathfrak{s}$, then in particular it must contain three symmetries Y_1, Y_2, Y_3 with commutators

$$[Y_1, Y_2] = Y_2,\ [Y_1, Y_3] = -Y_3,\ [Y_2, Y_3] = Y_1 \tag{43}$$

that generate a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ [13]. For fixed k, we analyze the commutator of the symmetries (41). Now, any commutator $\left[X^{(\varsigma_k)}_s, X^{(\varsigma_k)}_{s'}\right]$ must be a linear combination of vector fields of the type (41), as there are no symmetries whose first component is $x_s\frac{\partial}{\partial x_l}$ with $\varsigma_k < l < \varsigma_{k+1}$ (this is a consequence of equations (30) and (31)). Consider now two indices $s, s' \neq \varsigma_k$. Without loss of generality we can suppose that $s < s'$. Let $X^{(\varsigma_k)}_s, X^{(\varsigma_k)}_{s'}$ be the corresponding symmetries. Computing the commutator formally we obtain

$$\begin{aligned}\left[X^{(\varsigma_k)}_s, X^{(\varsigma_k)}_{s'}\right] &= \left[x_s\frac{\partial}{\partial x_{\varsigma_k}} + \cdots + x_{s+\varepsilon(s)}\frac{\partial}{\partial x_{\varsigma_k+\varepsilon(s)}}, x_{s'}\frac{\partial}{\partial x_{\varsigma_k}} + \cdots + x_{s+\varepsilon(s')}\frac{\partial}{\partial x_{\varsigma_k+\varepsilon(s')}}\right] \\ &= x_{s+a}\frac{\partial x_{s'}}{\partial x_{\varsigma_k+a}}\frac{\partial}{\partial x_{\varsigma_k}} + \cdots - x_{s'+b}\frac{\partial x_s}{\partial x_{\varsigma_k+b}}\frac{\partial}{\partial x_{\varsigma_k}} - \cdots = \delta^{\varsigma_k+a}_{s'} X^{(\varsigma_k)}_{s+a} - \delta^{\varsigma_k+b}_{s} X^{(\varsigma_k)}_{s'+b}.\end{aligned}$$

[5] Obviously, for $p_0 = 2$ and $m_1 = n-1$ we get $q = 1$, in which case (39) only provides one symmetry. Nontrivial semisimple Lie algebras are hence obtained for $q \geq 2$.

[6] As $\alpha^1_{\varsigma_k}$ is directly related to the Jordan block J_{m_k}, it follows at once that that $\varepsilon(\varsigma_k) = m_k$ for $H^{(\varsigma_k)}_{\varsigma_k}$. It is also the only symmetry of length $m_k + 1$.

If the commutator is not zero, then there are $a \leq \varepsilon(s)$, $b \leq \varepsilon(s')$ such that $s' = \varsigma_k + a$, $s = \varsigma_k + b$, which further implies that $s, s' > \varsigma_k$ and $s + a \neq s'$. Observe that $s < s'$ implies that $s' + b > s$. Hence the only possibility to obtain a commutator of the shape

$$\left[X_s^{(\varsigma_k)}, X_{s'}^{(\varsigma_k)}\right] = h\, X_{s'}^{(\varsigma_k)} \tag{44}$$

with $h \in \mathbb{R}^*$ is that $s = \varsigma_k$, i.e., that $X_s^{(\varsigma_k)} = H_{\varsigma_k}^{(\varsigma_k)}$. The preceding computation also shows that $H_{\varsigma_k}^{(\varsigma_k)}$ does not appear as a commutator of two symmetries $X_s^{(\varsigma_k)}, X_{s'}^{(\varsigma_k)}$. By the classical structure theorems [13], it follows that the symmetries (41) for a fixed k generate a solvable Lie algebra. As a consequence, if $\mathcal{L}_1$ contains a subalgebra of type (43) , then we need to consider at least two different indices k, k'. Equation (44) shows that the only possible candidates for the symmetry Y_1 are the $H_{\varsigma_k}^{(\varsigma_k)}$ generators. Moreover, let $k \neq k'$ and $\varsigma_k < \varsigma_{k'}$. If $X_s^{(\varsigma_k)}, X_{s'}^{(\varsigma_{k'})}$ are the corresponding symmetries, it follows from (30)-(31) that $s' \neq \varsigma_k$. Now, if

$$\left[X_s^{(\varsigma_k)}, X_{s'}^{(\varsigma_{k'})}\right] = H_{\varsigma_{k''}}^{(\varsigma_{k''})},$$

then necessarily $k'' = k$ or $k'' = k'$ by the structure of these symmetries. Suppose that

$$\left[X_s^{(\varsigma_k)}, X_{s'}^{(\varsigma_{k'})}\right] = \delta_{s'}^{\varsigma_k + a} X_{s+a}^{(\varsigma_{k'})} - \delta_s^{\varsigma_{k'} + b} X_{s'+b}^{(\varsigma_k)} = H_{\varsigma_{k'}}^{(\varsigma_{k'})} \tag{45}$$

for some $k \neq k'$. Then $s' = \varsigma_k + a$ and $s + a = \varsigma_{k'}$ with $a > 0$.[7] By (42)

$$H_{\varsigma_{k'}}^{(\varsigma_{k'})} = x_{\varsigma_{k'}} \frac{\partial}{\partial x_{\varsigma_{k'}}} + \cdots + x_{\varsigma_{k'+1}-1} \frac{\partial}{\partial x_{\varsigma_{k'+1}-1}} = x_{s+a} \frac{\partial}{\partial x_{s+a}} + \cdots + x_{s+a+m_{k'}} \frac{\partial}{\partial x_{s+a+m_{k'}}}.$$

It is straightforward to verify that $\left[H_{\varsigma_{k'}}^{(\varsigma_{k'})}, X_{s'}^{(\varsigma_{k'})}\right] = -X_{s'}^{(\varsigma_{k'})}$. If these three symmetries generate a Lie algebra, then following constraint must be satisfied

$$\left[H_{\varsigma_{k'}}^{(\varsigma_{k'})}, X_s^{(\varsigma_k)}\right] = X_s^{(\varsigma_k)}, \tag{46}$$

otherwise the Jacobi condition is violated. Expanding the bracket we get

$$\left[H_{\varsigma_{k'}}^{(\varsigma_{k'})}, X_s^{(\varsigma_k)}\right] = \left[x_{s+a} \frac{\partial}{\partial x_{s+a}} + \cdots + x_{s+a+m_{k'}} \frac{\partial}{\partial x_{s+a+m_{k'}}}, x_s \frac{\partial}{\partial x_{\varsigma_k}} + \cdots + x_{s+\varepsilon(s)} \frac{\partial}{\partial x_{\varsigma_k+\varepsilon(s)}}\right]. \tag{47}$$

If (46) holds, then there must be some $0 \leq b \leq m_{k'}$ such that

$$\frac{\partial x_s}{\partial x_{s+a+b}} = 1,$$

which contradicts the fact that a, b must be positive. We conclude that no symmetries satisfying (45) can exist, which implies that $\mathcal{L}_1$ does not contain a semisimple Lie algebra. ■

Corollary 1. *If $m_i \geq 1$ for $i = 1, \cdots, p_0$, then the Lie algebra $\mathcal{L}$ of point symmetries is solvable.*

3. Matrices with complex conjugated eigenvalues

We now to consider matrices, the eigenvalues of which are non-real. In order to find the corresponding real form, we have to introduce rotation matrices. Although these matrices represent a practical difficulty to

[7] If $a = 0$, this would imply that $s' = \varsigma_k$, which cannot happen. Further, $\delta_s^{\varsigma_{k'}+b} X_{s'+b}^{(\varsigma_k)} = 0$ because $\varsigma_k = s' + b = \varsigma_k + a + b$ would imply that b is negative, which cannot happen by the structure of the symmetries (41).

solve explicitly the symmetry condition (2) for the corresponding system, formally the argument to count the dimension and see the generic shape of the symmetry generators is very similar to the case already seen, although computationally more complicated. In order to avoid repetition of the proofs, we only indicate the general procedure to solve this type of canonical forms.
Suppose that $n = 2p$ and that the coefficient matrix J has the form

$$J(\lambda,\mu) = \begin{pmatrix} J(\lambda+\mathrm{i}\mu) & 0 \\ 0 & J(\lambda-\mathrm{i}\mu) \end{pmatrix}, \tag{48}$$

where the block matrices are given by

$$J(\lambda+\mathrm{i}\mu) = \begin{pmatrix} \lambda+\mathrm{i}\mu & \nu_1 & & & \\ & \lambda+\mathrm{i}\mu & & & \\ & & \ddots & & \\ & & & \lambda+\mathrm{i}\mu & \nu_{p-1} \\ & & & & \lambda+\mathrm{i}\mu \end{pmatrix},\; J(\lambda-\mathrm{i}\mu) = \begin{pmatrix} \lambda-\mathrm{i}\mu & \nu_1 & & & \\ & \lambda-\mathrm{i}\mu & & & \\ & & \ddots & & \\ & & & \lambda-\mathrm{i}\mu & \nu_{p-1} \\ & & & & \lambda-\mathrm{i}\mu \end{pmatrix} \tag{49}$$

and $\mu \neq 0$, i.e, the eigenvalues are non-real. It follows from the general theory [10] that for the matrix (48) we can always find a real matrix S such that

$$S^{-1}J(\lambda,\mu)\,S = \begin{pmatrix} C(\lambda,\mu) & \nu_1\mathrm{Id}_2 & & & \\ & C(\lambda,\mu) & \nu_2\mathrm{Id}_2 & & \\ & & \ddots & & \\ & & & C(\lambda,\mu) & \nu_{p-1}\mathrm{Id}_2 \\ & & & & C(\lambda,\mu) \end{pmatrix}, \tag{50}$$

where $\nu_l = 0,1$ for $l = 1,\cdots,p-1$ and

$$C(\lambda,\mu) = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix},\; \nu_l\mathrm{Id}_2 = \begin{pmatrix} \nu_l & 0 \\ 0 & \nu_l \end{pmatrix}. \tag{51}$$

Moreover, we can suppose without loss of generality that the nonzero ν_is are all consecutive, as other orderings belong to the same similarity class of matrices (see e.g [10], page 151 for details).

Theorem 3. *For the system $\ddot{\mathbf{x}} = J(\lambda,\mu)\,\mathbf{x}$, the symmetry algebra $\mathcal{L}$ has dimension*

$$\dim\mathcal{L} = \frac{1}{2}n^2 + 2n + 1 + 2\left(k^2 + k - kn\right), \tag{52}$$

where k is the number of non-vanishing ν_is in (50).

As commented, the symmetry condition (2) for the above type matrices is very similar to equations (4)-(9). Actually, the first three equations remain exactly the same, while the others must be replaced by the following:

$$\omega_l(\mathbf{x})\frac{\partial\xi}{\partial x_j} - \frac{\partial^2\eta_l}{\partial t\partial x_j} = 0,\; j \neq l \tag{53}$$

$$\sum_{i\neq l}\omega_i(\mathbf{x})\frac{\partial\xi}{\partial x_i} + 3\omega_l(\mathbf{x})\frac{\partial\xi}{\partial x_l} + \frac{\partial^2\xi}{\partial t^2} - 2\frac{\partial^2\eta_l}{\partial t\partial x_l} = 0, \tag{54}$$

$$\lambda\eta_{2k+1} + \mu\eta_{2k+2} + \nu_{k+1}\eta_{2k+3} - \sum_{i=1}^{n}\omega_i(\mathbf{x})\frac{\partial\eta_{2k+1}}{\partial x_i} - \frac{\partial^2\eta_{2k+1}}{\partial t^2} + 2\left(\lambda x_{2k+1} + \mu x_{2k+2} + \nu_{k+1}x_{2k+3}\right)\frac{\partial\xi}{\partial t} = 0, \tag{55}$$

$$-\mu\eta_{2k+1} + \lambda\eta_{2k+2} + \nu_{k+1}\eta_{2k+4} - \sum_{i=1}^{n}\omega_i(\mathbf{x})\frac{\partial\eta_{2k+1}}{\partial x_i} - \frac{\partial^2\eta_{2k+1}}{\partial t^2} + 2\left(-\mu x_{2k+1} + \lambda x_{2k+2} + \nu_{k+1}x_{2k+4}\right)\frac{\partial\xi}{\partial t} = 0, \tag{56}$$

where $k = 0, \cdots, p-1$ and

$$\omega_i(\mathbf{x}) = \begin{cases} \lambda x_{2k+1} + \mu x_{2k+2} + \nu_{k+1} x_{2k+3}, & i = 2k+1 \\ -\mu x_{2k+1} + \lambda x_{2k+2} + \nu_{k+1} x_{2k+4}, & i = 2k+2 \end{cases}.$$

Analyzing stepwise this system, as done in the proof of Theorem 1, it can be shown in similar way that the solution to the symmetry condition is again of the shape

$$\xi(t, x_1, \ldots, x_n) = \alpha_0; \quad \eta_j(t, x_1, \ldots, x_n) = \sum_{l=1}^{n} \alpha_l^j x_j + \psi_j(t).$$

Now let $\nu_1 = \cdots = \nu_k = 1$, $\nu_j = 0$ for $j = k+1, \cdots, p-1$. Equations (55) and (56) imply that the functions $\eta_j(t, x_1, \cdots, x_n)$ with $j = 2, \cdots, 2k+2$ are completely determined by $\eta_1(t, x_1, \cdots, x_n)$, while for $j > 2k+2$ the function $\eta_{2k+2+2l}(t, x_1, \cdots, x_n)$ is determined by $\eta_{2k+1+2l}(t, x_1, \cdots, x_n)$. Like done before, we separate the functions $\eta_j(t, x_1, \ldots, x_n)$ into a linear and functional part:

$$\eta_j^{LT}(t, x_1, \ldots, x_n) := \sum_{l=1}^{n} \alpha_l^l x_j.$$

We can again rewrite these linear terms $\eta_j^{LT}(t, x_1, \ldots, x_n)$ in matrix form: Define the matrices

$$A = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}, \ P = \begin{pmatrix} 0 & \nu_1 \mathrm{Id}_2 & & & \\ & 0 & \nu_2 \mathrm{Id}_2 & & \\ & & \ddots & & \\ & & & 0 & \nu_{p-1} \mathrm{Id}_2 \\ & & & & 0 \end{pmatrix}. \tag{57}$$

and vectors

$$\mathbf{v} = \left(\alpha_1^1, \cdots, \alpha_1^n\right), \ \mathbf{x} = \left(x_1, \cdots, x_n\right)^T. \tag{58}$$

Then, for $j = 1, \cdots, 2k+2$ we have

$$\eta_{2j}^{LT}(t, x_1, \ldots, x_n) = \mathbf{v}\, A\, P^{j-1} \mathbf{x}, \quad 1 \le j \le k+1 \tag{59}$$

$$\eta_{2j-1}^{LT}(t, x_1, \ldots, x_n) = \mathbf{v}\, P^{j-1} \mathbf{x}, \quad 1 \le j \le k+1 \tag{60}$$

Observe that since $P^{k+1} = 0$, the coefficients $\alpha_1^1, \cdots, \alpha_1^n$ are not subjected to any condition, and therefore we get $\eta_1^{LT}(t, x_1, \ldots, x_n) := \sum_{l=1}^{n} \alpha_1^j x_j$, providing n constants of integration. For the linear parts $\eta_{2k+2+l}^{LT}(t, x_1, \cdots, x_n)$ the situation is much simpler. We have

$$\eta_{2k+1+2l}^{LT}(t, x_1, \cdots, x_n) = \sum_{l=1}^{n} \alpha_{2k+1+2l}^j x_j, \tag{61}$$

$$\eta_{2k+2+2l}^{LT}(t, x_1, \cdots, x_n) = \left(\alpha_{2k+1+2l}^1, \cdots, \alpha_{2k+1+2l}^n\right) A\, \mathbf{x}, \tag{62}$$

but since $v_j = 0$ for $j > k$, we get the constraint

$$\left(\alpha_{2k+1+2l}^1, \cdots, \alpha_{2k+1+2l}^n\right) P\, \mathbf{x} = 0.$$

Therefore only $n - 2k$ constants of $\eta_{2k+1+2l}^{LT}(t, x_1, \cdots, x_n)$ are nonzero. Taken together, the linear parts of $\eta_j(t, x_1, \cdots, x_n)$ provide

$$n + (n-2k)\left(\frac{n}{2} - k - 1\right) = \frac{n^2}{2} - 2kn + 2k^2 + 2k$$

integration constants. Now to the functional part of the $\eta_j(t, x_1, \cdots, x_n)$, given by the functions $\psi_j(t)$. Again, for $j = 2, \cdots, 2k+2$ the function $\psi_j(t)$ is determined by $\psi_1(t)$. By equations (55) and (56), they must satisfy the system

$$\begin{aligned} \psi_{2j+3}(t) + \mu\psi_{2+2j}(t) + \lambda\psi_{2j+1}(t) - \frac{d^2\psi_{2k+1+2l}(t)}{dt^2} &= 0, \\ \psi_{2j+4}(t) - \mu\psi_{2k+1+2l}(t) + \lambda\psi_{2k+2+2l}(t) - \frac{d^2\psi_{2k+2+2l}(t)}{dt^2} &= 0, \\ \mu\psi_{2k+2}(t) + \lambda\psi_{2k+1}(t) - \frac{d^2\psi_{2k+1}(t)}{dt^2} &= 0, \\ -\mu\psi_{2k+1}(t) + \lambda\psi_{2k+2}(t) - \frac{d^2\psi_{2k+2}(t)}{dt^2} &= 0, \end{aligned} \tag{63}$$

where $j = 0, \cdots, k-2$. Although not entirely trivial to integrate, it can be seen with some lengthly computations that $\psi_1(t)$ has $4k+4$ degrees of liberty.

For the remaning cases, it follows also from (55) and (56) that the functions $\{\psi_{2k+1+2l}(t), \psi_{2k+2+2l}(t)\}$ satisfy the equations

$$\begin{aligned} \mu\psi_{2k+2+2l}(t) + \lambda\psi_{2k+1+2l}(t) - \frac{d^2\psi_{2k+1+2l}(t)}{dt^2} &= 0, \\ -\mu\psi_{2k+1+2l}(t) + \lambda\psi_{2k+2+2l}(t) - \frac{d^2\psi_{2k+2+2l}(t)}{dt^2} &= 0, \end{aligned}$$

or in equivalent form, that $\psi_{2k+1+2l}(t)$ satisfies the fourth order equation

$$\frac{d^4\psi_{2k+1+2l}(t)}{dt^4} - 2\lambda\frac{d^2\psi_{2k+1+2l}(t)}{dt^2} + \left(\lambda^2 + \mu^2\right)\psi_{2k+1+2l}(t) = 0, \tag{64}$$

which provides four integration constants. Therefore the functional part of the symmetry condition provides exactly

$$4k + 4 + 4\left(\frac{n}{2} - k - 1\right) = 2n$$

integration constants. Bearing in mind that $\xi(t, x_1, \cdots, x_n) = \alpha_0$, this implies that the dimension of the symmetry algebra $\mathcal{L}$ of the system $\ddot{\mathbf{x}} = J(\lambda, \mu)\,\mathbf{x}$ is

$$\dim \mathcal{L} = 2n + 1 + \frac{n^2}{2} - 2kn + 2k^2 + 2k.$$

For this case, we can also determine the Levi subalgebra of $\mathcal{L}$ in dependence of the Jordan form. The proof is formally the same as that of proposition 1, although much more involved from the computational point of view. For this reason we only give the general outline.

Proposition 2. *Let J be coefficient matrix of type (50). Then the Levi subalgebra of $\mathcal{L}$ is isomorphic to real simple Lie algebra $\mathfrak{sl}(q, \mathbb{C})$, where*

$$q = p - 2\nu_1 - \sum_{i=2}^{p-1} \nu_i. \tag{65}$$

If the symmetry algebra $\mathcal{L}$ admits a semisimple subalgebra, this must be generated by the symmetries determined by the linear parts of the components functions $\eta_i(t, x_1, \cdots, x_n)$. If $\nu_1 = \cdots = \nu_k = 1$, $\nu_{k+1} = \cdots = \nu_{p-1} = 0$ (recall that we can chose the non-zero ν_is as consecutive) then using equations (59) and (60) it can be shown that the symmetries associated to the integration constants α_1^j $(j = 1, \cdots, n)$ of $\eta_1(t, x_1, \cdots, x_n)$

generate a solvable Lie algebra. For the remaining components $\eta_{2k+3}(t,x_1,\cdots,x_n),\cdots,\eta_n(t,x_1,\cdots,x_n)$ the linear parts are simply

$$\eta_{2k+2+j}(t,x_1,\cdots,x_n) = \alpha_{2k+2+j}^{2k+1}x_{2k+1} + \alpha_{2k+2+j}^{2k+2}x_{2k+2} + \cdots + \alpha_{2k+2+j}^{n}x_n, \tag{66}$$

$$\eta_{2k+3+j}(t,x_1,\cdots,x_n) = \alpha_{2k+2+j}^{2k+1}x_{2k+2} - \alpha_{2k+2+j}^{2k+2}x_{2k+1} + \cdots - \alpha_{2k+2+j}^{n}x_{n-1}, \tag{67}$$

where $j=1,3,5,\cdots,n-2k-3$. The integration constants $\alpha_{2k+2+j}^{2k+1},\alpha_{2k+2+j}^{2k+2}$ define symmetries

$$X_{2k+2+j}^{2k+1} = x_{2k+1}\frac{\partial}{\partial x_{2k+2+j}} + x_{2k+2}\frac{\partial}{\partial x_{2k+3+j}};\ X_{2k+2+j}^{2k+2} = x_{2k+2}\frac{\partial}{\partial x_{2k+2+j}} - x_{2k+1}\frac{\partial}{\partial x_{2k+3+j}}$$

that are easily seen to satisfy the identity $\left[X_{2k+2+j}^{2k+1},X_{2k+2+l}^{2k+1}\right]=\left[X_{2k+2+j}^{2k+1},X_{2k+2+l}^{2k+2}\right]=0$. Now, considering the integration constants $\alpha_{2k+2+j}^{2k+1+2l},\alpha_{2k+2+j}^{2k+2+2l}$ with $l\geq 1$ we obtain additional $2(p-k-1)^2$ symmetries

$$X_{2k+2+j}^{2k+1+2l} = x_{2k+1+2l}\frac{\partial}{\partial x_{2k+2+j}} + x_{2k+2+2l}\frac{\partial}{\partial x_{2k+3+j}};\ X_{2k+2+j}^{2k+2+2l} = x_{2k+2+2l}\frac{\partial}{\partial x_{2k+2+j}} - x_{2k+1+2l}\frac{\partial}{\partial x_{2k+3+j}}. \tag{68}$$

Taking arbitrary indices j,j',l,l' and computing the brackets leads to

$$\begin{aligned}
\left[X_{2k+2+j}^{2k+1+2l},X_{2k+2+j'}^{2k+1+2l'}\right] &= \delta_{1+j}^{2l'}X_{2k+2+j'}^{2k+1+2l} - \delta_{1+j'}^{2l}X_{2k+2+j}^{2k+1+2l'},\\
\left[X_{2k+2+j}^{2k+1+2l},X_{2k+2+j'}^{2k+2+2l'}\right] &= \delta_{1+j}^{2l'}X_{2k+2+j'}^{2k+2+2l} - \delta_{1+j'}^{2l}X_{2k+2+j}^{2k+2+2l'},\\
\left[X_{2k+2+j}^{2k+2+2l},X_{2k+2+j'}^{2k+2+2l'}\right] &= -\delta_{1+j}^{2l'}X_{2k+2+j'}^{2k+1+2l} + \delta_{1+j'}^{2l}X_{2k+2+j}^{2k+1+2l'}.
\end{aligned} \tag{69}$$

Let $\mathcal{L}_1$ be the Lie algebra generated by these symmetries. For convenience, we take the basis formed by the vector fields $\left\{H_0^{(1)},H_0^{(2)},H_j^{(1)},H_j^{(2)},X_{2k+2+j}^{2k+1+2l},X_{2k+2+j}^{2k+1+2l}\right\}_{l\neq\frac{1+j}{2}}$, where $H_0^{(1)}=\sum_{j=1}^{n-2k-3}X_{2k+2+j}^{2k+1+\frac{j+1}{2}}$, $H_0^{(2)}=\sum_{j=1}^{n-2k-3}X_{2k+2+j}^{2k+2+\frac{j+1}{2}}$, $H_j^{(1)}=X_{2k+2+j}^{2k+1+\frac{j+1}{2}}-X_{2k+4+j}^{2k+3+\frac{j+1}{2}}$ and $H_j^{(2)}=X_{2k+2+j}^{2k+2+\frac{j+1}{2}}-X_{2k+4+j}^{2k+4+\frac{j+1}{2}}$. In particular

$$\begin{aligned}
H_j^{(1)} &= x_{2k+2+j}\frac{\partial}{\partial x_{2k+2+j}} + x_{2k+3+j}\frac{\partial}{\partial x_{2k+3+j}} - x_{2k+4+j}\frac{\partial}{\partial x_{2k+4+j}} - x_{2k+5+j}\frac{\partial}{\partial x_{2k+5+j}},\\
H_j^{(2)} &= x_{2k+3+j}\frac{\partial}{\partial x_{2k+2+j}} - x_{2k+2+j}\frac{\partial}{\partial x_{2k+3+j}} - x_{2k+5+j}\frac{\partial}{\partial x_{2k+4+j}} + x_{2k+4+j}\frac{\partial}{\partial x_{2k+5+j}}.
\end{aligned}$$

Using (69) it can be easily verified that $H_j^{(1)}$ and $H_0^{(a)}$ all commute with each other, and further that

$$\left[H_0^{(a)},X_{2k+2+j'}^{2k+1+2l'}\right]=\left[H_0^{(a)},X_{2k+2+j'}^{2k+1+2l'}\right]=0,\ a=1,2.$$

This shows that $\mathcal{L}_1$ splits into the Abelian algebra generated by $\left\{H_0^{(1)},H_0^{(2)}\right\}$ and a Lie algebra $\mathcal{L}_2$ generated by the remaining vector fields $\left\{H_j^{(1)},H_j^{(2)},X_{2k+2+j}^{2k+1+2l},X_{2k+2+j}^{2k+1+2l}\right\}$. We claim that the latter is isomorphic to the simple algebra $\mathfrak{sl}(q,\mathbb{C})$, where $q=p-k-1$ (if $k=0$, we simply take $q=p$). This is best proved by induction on $n=2p$ and computing the spectrum of the Killing form κ associated to $\mathcal{L}_2$ [13]. For each generator $X\in\mathcal{L}_2$ define the linear operator

$$\alpha d(X)(Z):=[X,Z]$$

and the bilinear symmetric form

$$\kappa(X,Y)=Trace\left(ad(X)\cdot ad(Y)\right),\ X,Y\in\mathcal{L}_2 \tag{70}$$

Using the commutation relations (69) and diagonalizing the matrix κ resulting from (70), a cumbersome but routine computation shows that the eigenvalues are given by

$$Spec\,(\kappa) = \left\{ \pm n^2, (\pm 2n)^{q^2-p}, \pm n \right\}. \tag{71}$$

Now the signature of the Killing form is given by the difference of positive and negative eigenvalues, thus from (71) we get $\sigma(\mathcal{L}_2) = 0$. It is known that real simple Lie algebras are determined by the signature of the Killing form (see [13], chapter 14), it follows at once that $\mathcal{L}_2$ must be isomorphic to the real Lie algebra $\mathfrak{sl}(q, \mathbb{C})$.[8]

4. Matrices with more than one eigenvalue

Until now we have only considered the case with one eigenvalue and two complex conjugated eigenvalues. We next show that the general case corresponding to various different eigenvalues can be easily obtained in terms of the cases already studied.

Let J be a non-diagonalizable coefficient matrix, and let us rewrite it as

$$J = \begin{pmatrix} J(\lambda_1) & & & \\ & J(\lambda_2) & & \\ & & \ddots & \\ & & & J(\lambda_k) \end{pmatrix}, \tag{72}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_k$ $(k \leq n-1)$ are the different eigenvalues of J. If some eigenvalue λ_i is complex, we replace the pair $\left\{ J(\lambda_i), J(\overline{\lambda}_i) \right\}$ by the corresponding real matrix $S^{-1} J(\alpha_i, \beta_i)\; S$ of (50), where $\lambda_i = \alpha_i + \mathrm{i}\beta_i$. In some sense, we can interpret the system $\ddot{\mathbf{x}} = J\mathbf{x}$ as the different systems corresponding to the matrices $J(\lambda_i)$ glued together with respect to the component function $\xi(t, x_1, \cdots, x_n)$ (which is common to all different eigenvalues and therefore imposes some restrictions). It turns out that the equations of the system $\ddot{\mathbf{x}} = J\mathbf{x}$ are then of (at least) one of the following types:

1. $\ddot{\mathbf{x}}_i = \lambda_i \mathbf{x}_i + \nu_i \mathbf{x}_{i+1}$,
2. $\ddot{\mathbf{x}}_j = \alpha_j \mathbf{x}_j + \beta_j \mathbf{x}_{j+1} + \nu_j \mathbf{x}_{j+2}$, $\ddot{\mathbf{x}}_{j+1} = -\beta_j \mathbf{x}_j + \alpha_j \mathbf{x}_{j+1} + \nu_j \mathbf{x}_{j+3}$,

where $\lambda_i, \alpha_i, \beta_i \in \mathbb{R}$, $\beta_i \neq 0$ and $\nu_i = 0, 1$.

Proposition 3. *Let $X = \xi(t, x_1, \cdots, x_n) \frac{\partial}{\partial t} + \eta_j(t, x_1, \cdots, x_n) \frac{\partial}{\partial x_j}$ be a symmetry of the system $\ddot{\mathbf{x}} = J\mathbf{x}$. Suppose that at least one of the following conditions holds:*

1. $\ddot{\mathbf{x}}_i = \lambda_i \mathbf{x}_i + \nu_i \mathbf{x}_{i+1}$ *and* $\ddot{\mathbf{x}}_j = \lambda_j \mathbf{x}_j + \nu_j \mathbf{x}_{j+1}$ *with* $\lambda_i \neq \lambda_j$
2. $\ddot{\mathbf{x}}_i = \lambda_i \mathbf{x}_i + \nu_i \mathbf{x}_{i+1}$ *and* $\ddot{\mathbf{x}}_j = \alpha_j \mathbf{x}_j + \beta_j \mathbf{x}_{j+1} + \nu_j \mathbf{x}_{j+2}$
3. $\ddot{\mathbf{x}}_i = \alpha_i \mathbf{x}_i + \beta_i \mathbf{x}_{i+1} + \nu_i \mathbf{x}_{i+2}$ *and* $\ddot{\mathbf{x}}_j = \alpha_j \mathbf{x}_j + \beta_j \mathbf{x}_{j+1} + \nu_j \mathbf{x}_{j+2}$ *with* $\alpha_i \neq \alpha_j$

Then

$$\frac{\partial \eta_j(t, x_1, \cdots, x_n)}{\partial x_i} = \frac{\partial \eta_i(t, x_1, \cdots, x_n)}{\partial x_j} = 0.$$

Proof. For simplicity, we only made the explicit computations for the first case, the remaining ones being very similar. Further, it suffices to show the formula when J has only two eigenvalues λ_1, λ_2, as the

[8] By the structure of the symmetries (68), it is not surprising that the resulting symmetry algebra is $\mathfrak{sl}(q, \mathbb{C})$, as $\mathfrak{sl}(q, \mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} \simeq A_{q-1} \oplus A_{q-1}$. This is consistent with the fact that the complex Jordan form of J contains complex conjugated eigenvalues of multiplicity q.

general case follows by recurrence. Renumbering the dependent variables $\mathbf{x}_1, \cdots, \mathbf{x}_n$ we can suppose that the system is given by

$$\begin{array}{ll} \ddot{\mathbf{x}}_i = \lambda_1 \mathbf{x}_i + \nu_i \mathbf{x}_{i+1}, & 1 \leq i \leq s_0 - 1 \\ \ddot{\mathbf{x}}_{s_0} = \lambda_1 \mathbf{x}_{s_0} & \\ \ddot{\mathbf{x}}_{s_0+i} = \lambda_2 \mathbf{x}_{s_0+i} + \nu_{s_0+i} \mathbf{x}_{s_0+i+1}, & 1 \leq i \leq n - s_0 - 1 \\ \ddot{\mathbf{x}}_n = \lambda_2 \mathbf{x}_n & \end{array}$$

Considering separately the equations corresponding to the different eigenvalues, we know by Theorem 1 that the generic form of the component functions $\xi(t, x_1, \cdots, x_n)$ and $\eta_j(t, x_1, \cdots, x_n)$ is given by

$$\begin{aligned} \xi(t, x_1, \cdots, x_n) &= \alpha_0, \\ \eta_i(t, x_1, \cdots, x_n) &= \sum_{k=1}^{n} a_i^k x_k + \sigma_i(t). \end{aligned}$$

Observe that $\xi(t, x_1, \cdots, x_n)$ is necessarily constant, since either λ_1 or λ_2 is nonzero. We now consider equation (9) for $i = s_0$ $(\nu_{s_0} = 0)$:

$$\begin{array}{l} \lambda_1 \eta_{s_0}(t, x_1, \cdots, x_n) - \ddot{\sigma}_{s_0} - \sum_{l=1}^{s_0-1} (\lambda_1 x_l + \nu_l x_{l+1}) \alpha_{s_0}^l - \lambda_1 x_{s_0} \alpha_{s_0}^{s_0} \\ - \sum_{m=1}^{n-1-s_0} (\lambda_2 x_{s_0+m} + \nu_{s_0+m} x_{s_0+m-1}) \alpha_{s_0}^m - \lambda_2 x_n \alpha_{s_0}^n = 0. \end{array} \tag{73}$$

From this identity, we are only interested on the terms involving the dependent variables $\{x_{s_0+1}, \cdots, x_n\}$:

$$(\lambda_1 - \lambda_2) \left(\sum_{l=1}^{n-s_0} \alpha_l^{l+s_0} x_{s_0+l} \right) - \sum_{m=1}^{n-s_0-1} \nu_{s_0+m} a_m^{m+s_0} x_{s_0+m+1} = 0.$$

Reordering these terms we obtain the identity

$$(\lambda_1 - \lambda_2) \alpha_{s_0}^{1+s_0} x_{s_0+1} + \sum_{m=2}^{n-s_0} \left\{ (\lambda_1 - \lambda_2) \alpha_{s_0}^{m+s_0} - \nu_{s_0+m-1} \alpha_{s_0}^{m+s_0-1} \right\} x_{s_0+m} = 0. \tag{74}$$

Since $\lambda_1 \neq \lambda_2$, it follows immediately that $\alpha_{s_0}^{1+s_0} = 0$. This implies that the coefficient of x_{s_0+2} is $(\lambda_1 - \lambda_2) \alpha_{s_0}^{2+s_0}$, which must also vanish as the eigenvalues are different. We thus successively obtain that

$$\alpha_{s_0}^{l+s_0} = 0, \ 1 \leq l \leq n - s_0,$$

proving that

$$\frac{\partial \eta_{s_0}}{\partial x_i} = 0, \ i = s_0 + 1, \cdots, n. \tag{75}$$

We next consider the equation (9) for $i = s_0 - 1$. If $\nu_{s_0-1} = 0$, the same argument as before shows that $\frac{\partial \eta_{s_0-1}}{\partial x_i} = 0, \ i = s_0 + 1, \cdots, n$. In the case that $\nu_{s_0-1} = 1$, we get the expression

$$\eta_{s_0}(t, x_1, \cdots, x_n) = \sum_{i=1}^{s_0} (\lambda_1 x_i + \nu_i x_{i+1}) \frac{\partial \eta_{s_0-1}}{\partial x_i} + \sum_{i=s_0+1}^{n} (\lambda_2 x_i + \nu_i x_{i+1}) \frac{\partial \eta_{s_0-1}}{\partial x_i} - \frac{\partial^2 \eta_{s_0-1}}{\partial t^2} - \lambda_1 \eta_{s_0-1}(t, x_1, \cdots, x_n).$$

As before, we only consider the terms in the variables $\{x_{s_0+1}, \cdots, x_n\}$. By (75), these terms satisfy the identity

$$(\lambda_1 - \lambda_2) \alpha_{s_0-1}^{1+s_0} x_{s_0+1} + \sum_{m=2}^{n-s_0} \left\{ (\lambda_1 - \lambda_2) \alpha_{s_0-1}^{m+s_0} - \nu_{s_0+m-1} \alpha_{s_0-1}^{m+s_0-1} \right\} x_{s_0+m} = 0, \tag{76}$$

which is quite similar to (74), and we again conclude that $\frac{\partial \eta_{s_0-1}}{\partial x_i}=0,\ i=s_0+1,\cdots,n$. Repeating the process for the indices $i=s_0-2,\cdots,2,1$, we show recursively that

$$\frac{\partial \eta_i}{\partial x_j}=0,\ j=s_0+1,\cdots,n.$$

In analogous manner, analyzing the functions $\eta_i\left(t,x_1,\cdots,x_n\right)$ for $i=s_0+1,\cdots,n$ it is proved in straightforward manner that

$$\frac{\partial \eta_i}{\partial x_j}=0,\ \ j=1,\cdots,s_0.$$

■

This result allows us to establish the dimension formula for non-diagonalizable matrices J of type (72) with more than one eigenvalue.

Proposition 4. *For the system* $\ddot{\mathbf{x}}=J\mathbf{x}$*, where* J *has the form (72), the symmetry algebra* $\mathcal{L}$ *has dimension*

$$\dim\mathcal{L}=\sum_{l=1}^{k}\left(\dim\mathcal{L}\left(J\left(\lambda_l\right)\right)-\varepsilon_l\right)+1, \tag{77}$$

where $\varepsilon_l=1$ *if* $\lambda_l\neq 0$ *(or* $\lambda_l\in\mathbb{C}$*) and* $\varepsilon_l=2$ *if* $\lambda_l=0$*.*

The proof is a consequence of the Theorems 2 and 3. If J has the form (72), then we compute the symmetries of the equations $\ddot{\mathbf{x}}=J\left(\lambda_i\right)\mathbf{x}$ separately. For each λ_i we obtain $\dim\mathcal{L}\left(J\left(\lambda_i\right)\right)$ symmetries. Now, as $\xi\left(t,x_1,\cdots,x_n\right)$ is considered as many times as different eigenvalues the matrix J has, we must subtract to each $\dim\mathcal{L}\left(J\left(\lambda_i\right)\right)$ either one or two, depending whether λ_i is nonzero (or complex) or zero (observe further that zero appears at most once). As there is at least one eigenvalue that is nonzero, by Theorem 1 we conclude that $\xi\left(t,x_1,\cdots,x_n\right)$ must a constant. Counting the dimensions $\dim\mathcal{L}\left(J\left(\lambda_i\right)\right)-\varepsilon_i$ and adding the integration constant provided by $\xi\left(t,x_1,\cdots,x_n\right)$, we obtain formula (77).

The preceding results, combined with those obtained for diagonal coefficient matrices [9], allow to obtain the number of symmetries for an arbitrary (non-diagonalizable) matrix J. We first rewrite J as a block matrix, separating the non-diagonalizable part J_1 from the diagonal part J_2 :

$$J=\begin{pmatrix} J_1 & \\ & J_2\end{pmatrix}.$$

First of all, it is important to observe that none of the eigenvalues of the diagonal part J_2 can appear in the sub-matrix J_1, as this matrix corresponds to the non-diagonal Jordan blocks of J. Now, for the subsystem $\ddot{\mathbf{x}}=J_1\mathbf{x}$, the symmetries are determined by Theorems 2 and 3, thus their number is given by formula (77). For the diagonal part, however, some caution is necessary. We have that for any symmetry X of $\ddot{\mathbf{x}}=J\mathbf{x}$, the time component function $\xi\left(t,x_1,\cdots,x_n\right)$ is either a constant or a linear function of the independent variable t. This constraint implies that the symmetries of the subsystem $\ddot{\mathbf{x}}=J_2\mathbf{x}$ will have the generic form $X=\left(\alpha_0+\kappa_1 t\right)\frac{\partial}{\partial t}+\eta_i\frac{\partial}{\partial x_i}$, and therefore, even if $J_2=\rho\mathrm{Id}$, they do not coincide with the symmetries of the free particle system.[9] In fact, their number will be given by proposition 3 of [9]:

$$\dim\mathcal{L}\left(J_2\right)=3m+1+\sum_{i=1}^{r}k_i\left(k_i-1\right),$$

where m is the dimension of J_2 and $k_1,\cdots,k_r$ is the multiplicity of the different eigenvalues of J_2. Summarizing, the dimension of the symmetry algebra $\mathcal{L}$ of the system $\ddot{\mathbf{x}}=J\mathbf{x}$ is

$$\dim\mathcal{L}=\dim\mathcal{L}\left(J_1\right)+\dim\mathcal{L}\left(J_2\right)+1-\varepsilon_1-\varepsilon_2. \tag{78}$$

Using propositions 1 and 2, we can further determine the Levi factor of the symmetry Lie algebra L, by analyzing the sub-matrices corresponding to the different eigenvalues.

[9]This fact can be seen as a constrained symmetry of the free particle system, where only symmetry generators of certain type are allowed.

4.1. Symmetries for $n = 5$

As an application of the results obtained in this work, we determine the dimension and solvability of the symmetry algebra $\mathcal{L}$ of a system $\ddot{\mathbf{x}} = J\mathbf{x}$ with $n = 5$ equations. For this case, there are 36 types of real canonical forms, which can be comprised in the three following generic types:

$$J_1 = \begin{pmatrix} \lambda_1 & \nu_1 & & & \\ & \lambda_2 & \nu_2 & & \\ & & \lambda_3 & \nu_3 & \\ & & & \lambda_4 & \nu_4 \\ & & & & \lambda_5 \end{pmatrix}, \quad J_2 = \begin{pmatrix} \lambda_1 & \mu_1 & & & \\ -\mu_1 & \lambda_1 & & & \\ & & \lambda_2 & \nu_2 & \\ & & & \lambda_3 & \nu_3 \\ & & & & \lambda_4 \end{pmatrix}, \quad J_3 = \begin{pmatrix} \lambda_1 & \mu_1 & \alpha & & \\ -\mu_1 & \lambda_1 & & \alpha & \\ & & \lambda_2 & \mu_2 & \\ & & -\mu_2 & \lambda_2 & \\ & & & & \lambda_3 \end{pmatrix}. \tag{79}$$

To distinguish the different canonical forms, we denote the matrices of (79) by $J_1(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \nu_1, \nu_2, \nu_3, \nu_4)$, $J_2(\lambda_1, \mu_1, \lambda_2, \lambda_3, \lambda_4, \nu_3, \nu_4)$ and $J_3(\lambda_1, \mu_1, \lambda_2, \mu_2, \lambda_3, \alpha)$. We give the result in tabular form. If the symmetry algebra $\mathcal{L}$ does not contain a Levi factor, then it is solvable.

Table 1: Dimensions $d = \dim(\mathcal{L})$ and Levi factors of symmetry algebras $\mathcal{L}$ for systems with $n = 5$ equations.

Matrix	d	Levi factor	Matrix	d	Levi factor
$J_1(\lambda,\lambda,\lambda,\lambda,\lambda,0,0,0,0)$	48	$\mathfrak{sl}(7,\mathbb{R})$	$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_2,0,0,0,1)$	22	$\mathfrak{sl}(3,\mathbb{R})$
$J_1(\lambda,\lambda,\lambda,\lambda,\lambda,1,1,1,1)$	16	-	$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_2,1,0,0,1)$	18	-
$J_1(0,0,0,0,0,1,1,1,1)$	17	-	$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_2,1,0,0,0)$	20	$\mathfrak{sl}(2,\mathbb{R})$
$J_1(\lambda,\lambda,\lambda,\lambda,\lambda,1,1,1,0)$	18	-	$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_3,0,0,0,0)$	22	$\mathfrak{sl}(3,\mathbb{R})$
$J_1(0,0,0,0,0,1,1,1,0)$	19	-	$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_3,1,1,0,0)$	16	-
$J_1(\lambda,\lambda,\lambda,\lambda,\lambda,1,1,0,0)$	22	$\mathfrak{sl}(2,\mathbb{R})$	$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_3,1,0,0,0)$	18	-
$J_1(0,0,0,0,0,1,1,0,0)$	23	$\mathfrak{sl}(2,\mathbb{R})$	$J_1(\lambda_1,\lambda_1,\lambda_2,\lambda_2,\lambda_3,0,0,0,0)$	20	$2\mathfrak{sl}(2,\mathbb{R})^{\dagger}$
$J_1(\lambda,\lambda,\lambda,\lambda,\lambda,1,0,0,0)$	28	$\mathfrak{sl}(3,\mathbb{R})$	$J_1(\lambda_1,\lambda_1,\lambda_2,\lambda_2,\lambda_3,1,0,1,0)$	16	-
$J_1(0,0,0,0,0,1,0,0,0)$	29	$\mathfrak{sl}(3,\mathbb{R})$	$J_1(\lambda_1,\lambda_1,\lambda_2,\lambda_2,\lambda_3,1,0,0,0)$	18	$\mathfrak{sl}(2,\mathbb{R})$
$J_1(\lambda,\lambda,\lambda,\lambda,\lambda,1,1,0,1)$	20	-	$J_1(\lambda_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,0,0,0,0)$	18	$\mathfrak{sl}(2,\mathbb{R})$
$J_1(0,0,0,0,0,1,1,0,1)$	21	-	$J_1(\lambda_1,\lambda_1,\lambda_2,\lambda_3,\lambda_4,1,0,0,0)$	16	-
$J_1(\lambda,\lambda,\lambda,\lambda,\lambda,1,0,1,0)$	24	-	$J_1(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,0,0,0,0)$	16	-
$J_1(0,0,0,0,0,1,0,1,0)$	25	-	$J_2(\lambda_1,\mu_1,\lambda_2,\lambda_2,\lambda_2,0,0)$	22	$\mathfrak{sl}(3,\mathbb{R})$
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_1,\lambda_2,0,0,0,0)$	28	$\mathfrak{sl}(4,\mathbb{R})$	$J_2(\lambda_1,\mu_1,\lambda_2,\lambda_2,\lambda_2,1,0)$	18	-
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_1,\lambda_2,1,1,1,0)$	16	-	$J_2(\lambda_1,\mu_1,\lambda_2,\lambda_2,\lambda_2,1,1)$	16	-
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_1,\lambda_2,1,1,0,0)$	18	-	$J_2(\lambda_1,\mu_1,\lambda_2,\lambda_2,\lambda_3,0,0)$	18	$\mathfrak{sl}(2,\mathbb{R})$
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_1,\lambda_2,1,0,0,0)$	22	$\mathfrak{sl}(2,\mathbb{R})$	$J_2(\lambda_1,\mu_1,\lambda_2,\lambda_2,\lambda_3,1,0)$	16	-
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_1,\lambda_2,1,0,1,0)$	20	-	$J_2(\lambda_1,\mu_1,\lambda_2,\lambda_3,\lambda_4,0,0)$	16	-
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_2,0,0,0,0)$	24	$\mathfrak{sl}(3,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	$J_3(\lambda_1,\mu_1,\lambda_1,\mu_1,\lambda_3,0)$	20	$\mathfrak{sl}(2,\mathbb{C})$
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_2,1,1,0,1)$	16	-	$J_3(\lambda_1,\mu_1,\lambda_1,\mu_1,\lambda_3,1)$	16	-
$J_1(\lambda_1,\lambda_1,\lambda_1,\lambda_2,\lambda_2,1,1,0,0)$	18	$\mathfrak{sl}(2,\mathbb{R})$	$J_3(\lambda_1,\mu_1,\lambda_2,\mu_2,\lambda_3,0)$	16	-

† $2\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$

5. Conclusions

In this work we have completed the study of the possible dimensions and Levi subalgebras of the Lie algebra $\mathcal{L}$ of point symmetries of a linear system of n second order ODEs with constant real coefficients. While the symmetries for low values of n and the case of diagonal coefficient matrices were known [7, 8, 9], for the remaining types of matrices, corresponding to non-diagonalizable coefficient matrices, no general result was known beyond $n = 4$. This has been solved here for arbitrary values of n by means of a detailed analysis of the Jordan blocks of the coefficient matrices, and computing the symmetries stepwise.

By analyzing first the non-diagonalizable canonical forms J having only one eigenvalue or two complex conjugated eigenvalues, we have developed a constructive method to explicitly obtain the symmetry generators from the corresponding structure of the Jordan blocks of J. For such systems we have further classified the Levi factor of the symmetry algebra. Combining these results, we have proved that the general case with more than one eigenvalue can be essentially reduced to the analysis of the matrices corresponding to the different eigenvalues, allowing us to obtain the general formula for the dimension of $\mathcal{L}$, as well as a classification of the Levi factor. Therefore, for any given canonical form J, the dimension and the symmetries can be directly deduced from the various Jordan blocks corresponding to the different eigenvalues, and without being forced to integrate the symmetry condition (2). In particular, we can directly infer from the Jordan form whether the symmetry algebra $\mathcal{L}$ is solvable or not. As an application of the procedure, we have determined the dimension and the structure of the symmetry algebras for systems with $n = 5$ equations. Once the symmetry analysis of linear systems of second order ODEs with constant coefficients completed, it is natural to ask whether the procedure used can be enlarged to cover more general types of systems. The potential value of this case resides in its applicability to either systems with non-constant coefficient matrices or non-linear systems. Although a much more complicated problem from the formal point of view, a first approach in this direction could be to consider perturbations (i.e., by means of introducing a perturbation parameter or contraction of realizations [14]) of linear systems of the type analyzed, and classify the corresponding symmetries. Whether such an ansatz provides alternative criteria to simplify the analysis of general systems, or allows to decide on the solvability of the corresponding symmetry algebras, is still an open problem.

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