Articles for an Encyclopædia on Mathematical Physics

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This is a collection of articles (on KAM theory and related topics) I was asked to write for an encyclopædia on mathematical physics at the end of 2006. By April 19, 2007, the articles were prepared, but publication of the encyclopædia is put off till the indefinite future. So I have decided to archive the articles lest they be lost. No changes or updates of the articles have been made.

KAM theory (Kolmogorov–Arnold–Moser theory) is the theory of **conditionally periodic motions** in nonintegrable dynamical systems. The name of the theory (proposed by F.M.Izrailev and B.V.Chirikov in 1968) stems from the first letters of the names of its founders A.N.Kolmogorov [K1, K2], V.I.Arnold [A2, A3], and J.Moser [M1, M3]. The theory started with A.N.Kolmogorov's note [K1] of 1954.

The prototype of most of the results in KAM theory is the theorem on the persistence of invariant tori under small perturbations of integrable Hamiltonian systems. Consider a **completely integrable Hamiltonian system** with $n \geq 2$ degrees of freedom and Hamilton function $H_0(I)$:

$$\dot{\varphi} = \omega(I) = \partial H_0(I)/\partial I, \qquad \dot{I} = 0, \tag{1}$$

where (I, φ) are the **action-angle variables** (I ranges in a certain finite domain G of the n-dimensional Euclidean space $\mathbb{R}^n$). The whole phase space of this system is smoothly foliated into **invariant n-tori** $I = \text{const}$, the motion on each torus being conditionally periodic with frequencies $\omega_1(I) = \partial H_0(I)/\partial I_1, \ldots, \omega_n(I) = \partial H_0(I)/\partial I_n$. Let us now subject system (1) to a small Hamiltonian perturbation, i.e., consider a Hamiltonian system

$$\dot{\varphi} = \omega(I) + \varepsilon \partial H_1(I, \varphi, \varepsilon)/\partial I, \qquad \dot{I} = -\varepsilon \partial H_1(I, \varphi, \varepsilon)/\partial \varphi \tag{2}$$

with Hamilton function $H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon)$, where $0 < \varepsilon \ll 1$ is the perturbation parameter.

Theorem. Suppose that the unperturbed system (1) is *Kolmogorov nondegenerate*, i.e., the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are functionally independent at each point of the domain G:

$$\det \frac{\partial \omega}{\partial I} = \det \frac{\partial^2 H_0}{\partial I^2} \neq 0. \tag{3}$$

Then for ε sufficiently small, most of the invariant n-tori $I = \text{const}$ of the unperturbed system (1) do not disappear but are only slightly deformed—so that in the phase space of the perturbed system (2), there also exist many invariant n-tori. The motion on each perturbed torus is still conditionally periodic and, moreover, with the same collection of frequencies $\omega_1, \omega_2, \ldots, \omega_n$ as those on the corresponding unperturbed torus. The perturbed tori generically form a nowhere dense set, but the Lebesgue measure of the complement to this set is $O\big(\sqrt{\varepsilon}\,\big)$.

On the other hand, assume that the unperturbed system (1) is *isoenergetically nondegenerate*, i.e.,

$$\det \begin{pmatrix} \partial\omega/\partial I & \omega \\ \omega & 0 \end{pmatrix} = \det \begin{pmatrix} \partial^2 H_0/\partial I^2 & \partial H_0/\partial I \\ \partial H_0/\partial I & 0 \end{pmatrix} \neq 0 \tag{4}$$

at each point of the domain G (for $\omega_1 \neq 0$, this inequality expresses the functional independence of the frequency ratios $\omega_2/\omega_1, \ldots, \omega_n/\omega_1$ on every energy level $H_0 = \text{const}$). Then for ε sufficiently small, most of the invariant n-tori $I = \text{const}$ of the unperturbed system (1) lying on any fixed energy level $H_0 = h$ do not disappear but are only slightly deformed—so that on the energy level $H = h$ of the perturbed system (2), there also exist many invariant n-tori.

The motion on each perturbed torus is still conditionally periodic and, moreover, with the same frequency ratios $\omega_1 : \omega_2 : \ldots : \omega_n$ as those for the corresponding unperturbed torus. The perturbed tori on the hypersurface $H = h$ generically form a nowhere dense set, but the measure of the complement to this set is $O\big(\sqrt{\varepsilon}\,\big)$.

In both the cases, there persist for sure the unperturbed invariant tori $I = I^*$ on which the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ of the conditionally periodic motion satisfy the so-called *strong incommensurability*, or *Diophantine* condition. This condition consists in the existence of positive constants τ and γ such that for any integers $k_1, k_2, \ldots, k_n$ that do not vanish simultaneously, the inequality

$$|k_1\omega_1 + k_2\omega_2 + \ldots + k_n\omega_n| > \gamma\big(|k_1| + |k_2| + \ldots + |k_n|\big)^{-\tau} \tag{5}$$

holds. Let the exponent $\tau > n - 1$ be fixed. Then there persist all the unperturbed tori on which the frequencies of the motion satisfy inequalities (5) with a coefficient γ no less than a certain quantity of the order of $\sqrt{\varepsilon}$.

In the phase space of the perturbed system (2), the perturbed invariant n-tori whose existence is guaranteed by this theorem are called the *Kolmogorov tori*, while their union is called the *Kolmogorov set*. The symplectic 2-form $dI \wedge d\varphi$ vanishes on each Kolmogorov torus. Taking into account that the dimension of the Kolmogorov tori is half the phase space dimension, one concludes that all the Kolmogorov tori are **Lagrangian submanifolds** of the phase space (like the unperturbed tori $I = \mathrm{const}$). The first part of the theorem (where the Kolmogorov nondegeneracy is considered) is called the *Kolmogorov theorem* [K1], or the *basic theorem of KAM theory*. The second part (where the isoenergetic nondegeneracy is treated) is due to V.I.Arnold [A2, A3]. The determinant in the left-hand side of inequality (4) is called sometimes the *Arnold determinant*. The estimate $O\big(\sqrt{\varepsilon}\,\big)$ of the measure of the complement to the Kolmogorov set was obtained by V.F.Lazutkin, N.V.Svanidze, A.I.Neishtadt, and J.Pöschel (see e.g. [N, P]).

It is important to emphasize that in the case of Kolmogorov nondegeneracy, an unperturbed invariant torus with Diophantine frequencies and the corresponding perturbed torus are characterized, generally speaking, by different energy values $H_0 = h$ and $H = h' \neq h$. As to the energy level $H = h$, it is possible that there will be no perturbed torus on this hypersurface with the given frequencies and even with the given frequency ratios. In the case of isoenergetic nondegeneracy, on the other hand, an unperturbed torus and the corresponding perturbed torus are characterized by the same energy value $H_0 = h$ and $H = h$, but in return only the frequency ratios (rather than the frequencies themselves) of these tori coincide. As to a perturbed torus with the given frequencies, it is possible that there will be no such torus on the energy level $H = h$ and even in the whole phase space. The conditions of Kolmogorov nondegeneracy (3) and isoenergetic nondegeneracy (4) are independent. For instance, consider system (1) with the Hamilton function $H_0(I) = a_1 \ln I_1 + a_2 \ln I_2 + \ldots + a_n \ln I_n$, where $a_1, a_2, \ldots, a_n$ are nonzero constants whose sum vanishes. This system is Kolmogorov nondegenerate in the domain $I_j > 0$, $1 \le j \le n$, but isoenergetically degenerate everywhere. On the other hand, system (1) with the Hamilton function $H_0(I) = I_1 + \frac{1}{2}I_2^2 + \ldots + \frac{1}{2}I_n^2$ is isoenergetically nondegenerate but Kolmogorov degenerate everywhere. System (1) with the Hamilton function $H_0(I) = \frac{1}{2}I_1^2 + \frac{1}{2}I_2^2 + \ldots + \frac{1}{2}I_n^2$ is both Kolmogorov nondegenerate and isoenergetically nondegenerate.

The gaps between Kolmogorov tori (the so-called *resonant zones*) contain: a) complicated infinite hierarchical structures of invariant tori of various dimensions from 1 to n; b) asymptotic surfaces constituted by the phase trajectories that approach these tori as $t \to +\infty$ or $t \to -\infty$; c) stochastic layers with chaotic behavior of the trajectories.

The theorem above is valid if both the unperturbed Hamilton function H_0 and the perturbed one H are of smoothness class C^l with $l > 2n$. The higher the smoothness of H_0 and H, the higher is the smoothness of the Kolmogorov tori. If H_0 and H are real analytic, so are the Kolmogorov tori. If H_0 and H are infinitely differentiable, then the Kolmogorov tori are of smoothness class C^∞ as well. If H_0 and H are of finite smoothness C^l with $l > 2n$, system (1) is Kolmogorov nondegenerate, and the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ of conditionally periodic motion on a perturbed torus satisfy inequalities (5) with $n - 1 < \tau < \frac{1}{2}(l-2)$, then the torus in question is of smoothness class $C^{l-\tau-1-\delta}$ for any $\delta > 0$. The initial results by A.N.Kolmogorov and V.I.Arnold [A2, A3, K1] on the persistence of conditionally periodic motions pertained to analytic Hamiltonian systems only. The case of finitely smooth systems was first examined by J.Moser in 1961–62 [M1].

Although the Kolmogorov set is nowhere dense, it is organized in some sense very regularly. The first results in this direction were due to V.F.Lazutkin (1972–74) [L]. In the sequel, a major contribution to studies of the structure of the Kolmogorov set has been made by J.Pöschel [P].

There are an enormous number of quite diverse modifications, refinements, and (sometimes very far) generalizations of the theorem above. It is these results that constitute, in the aggregate, KAM theory. Below, we list several most important achievements.

1. The nondegeneracy conditions in the theorem we formulated can be relaxed considerably. For instance, if the unperturbed Hamilton function H_0 is analytic and the domain G is connected, then *Rüssmann nondegeneracy* [R] of the unperturbed system (1) is sufficient for the existence of invariant n-tori filling in the most part of the phase space of a perturbed system (2). The Rüssmann nondegeneracy condition consists in that the image of the mapping $\omega\colon G \to \mathbb{R}^n$ does not lie in any hyperplane (of the frequency space) passing through the origin. This condition is optimal: if it fails then one can find an arbitrarily small perturbation εH_1 for which the system (2) will admit no invariant torus at all. Another example is provided by so-called *proper degeneracy* where the unperturbed Hamilton function H_0 is independent of some of the action variables: $H_0 = H_0(I_1, I_2, \ldots, I_s)$ with $s < n$, i.e., $\omega_{s+1}(I) = \ldots = \omega_n(I) \equiv 0$ (this rules Rüssmann nondegeneracy out), but the perturbation is of the special form $H_1 = H_{01}(I) + \varepsilon H_{11}(I, \varphi, \varepsilon)$.

2. There exists the local KAM theory that explores invariant tori near a critical element—an equilibrium or a closed trajectory (to be more precise, a one-parameter family of closed trajectories). In this situation (sometimes called *limit degeneracy*), the role of the perturbation parameter ε is played by the distance ρ to the critical element. In the **Arnold theorem on conditionally periodic motions** [A3] (the central result in the applications of KAM theory to Celestial Mechanics), one considers a combination of proper and limit degeneracies.

3. Recall that a submanifold L of a symplectic manifold is said to be **isotropic** if the symplectic 2-form vanishes on L, and is said to be **coisotropic** if the tangent space T_pL contains its skew-orthogonal complement (in the sense of the symplectic 2-form) at each point $p \in L$. By now, very many results have been obtained on isotropic invariant tori whose dimension m is less than the number n of degrees of freedom of the Hamiltonian system in question (such tori are said to be *lower dimensional*). For instance, consider again a completely integrable Hamiltonian system (1) and an invariant n-torus $I = I^*$ of this system. Let the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ of conditionally periodic motion on the torus $I = I^*$ be commensurable (such tori are said to be *resonant*). Suppose that these frequencies satisfy d independent resonance relations $k_{1i}\omega_1(I^*) + k_{2i}\omega_2(I^*) + \ldots + k_{ni}\omega_n(I^*) = 0$, $1 \leq i \leq d$, so that the unperturbed torus $I = I^*$ is smoothly foliated into invariant $(n-d)$-tori. Then a small Hamiltonian perturbation of system (1) leads, under some additional genericity conditions, to a break-up of the torus $I = I^*$ into a finite collection of invariant $(n-d)$-tori of the perturbed system. Other theorems treat invariant m-tori ($2 \leq m \leq n-1$) in a neighborhood of an equilibrium or a one-parameter family of closed trajectories. Besides, there often occurs the situation where the unperturbed system with n degrees of freedom is not completely integrable but possesses an invariant $2r$-dimensional surface S smoothly foliated into invariant r-tori, the motions on these tori being conditionally periodic ($2 \leq r \leq n-1$). Then one can establish, under certain conditions, the existence (near the surface S) of invariant r-tori of perturbed systems, as well as of invariant tori of dimensions m from $r+1$ to n in the unperturbed system and all its perturbations. The Arnold theorem on conditionally periodic motions mentioned above describes a particular case of this situation where for arbitrary r and n, one looks for tori of dimension $m = n$. Lower dimensional invariant tori of Hamiltonian systems were first considered in the context of KAM theory by V.K.Melnikov in 1965 [Mel]. The first results on break-up of resonant unperturbed tori into invariant tori of smaller dimension $n - d \geq 2$ are due to D.V.Treshchëv [T] (1989).

4. A number of results in KAM theory obtained mainly by I.O.Parasyuk (starting in 1984 [Par]) and partially also by M.R.Herman and other authors relate to coisotropic invariant tori whose dimension M is greater than the number n of degrees of freedom of the Hamiltonian system. In this situation, the phase space of the unperturbed system is smoothly foliated into such tori and the motions on these tori are conditionally periodic, while the most part of the phase space of a perturbed system is also filled in by invariant M-tori ($n+1 \leq M \leq 2n-1$). The symplectic 2-form is not exact in this set-up. In the case where $M = 2n-1$, each invariant

M-torus is a connected component of an energy level.

5. On all the invariant tori whose existence is guaranteed by various results in KAM theory, there take place *quasi-periodic motions*, i.e., conditionally periodic motions with incommensurable frequencies (these frequencies are, as a rule, Diophantine, but sometimes one succeeds in establishing the existence of invariant tori with less restrictive conditions on the incommensurability "degree" of the frequencies). On the other hand, invariant sets of nonintegrable systems are known that are analogues of the resonant (or nearly resonant) invariant tori of integrable systems. Such sets have, as a rule, a rather complicated structure and are called *cantori* or *Aubry–Mather sets*. The conventional Kolmogorov tori also turn to Aubry–Mather sets as the perturbation increases. The Kolmogorov tori as well as the Aubry–Mather sets satisfy a certain variational principle.

6. In some theorems of KAM theory, one treats perturbations dependent on time (periodically or even quasi-periodically).

7. In early works on KAM theory, only Hamiltonian systems were considered. In 1965–67, J.Moser [M2, M3] and independently Yu.N.Bibikov and V.A.Pliss [BP] obtained first results on quasi-periodic motions in nonintegrable **reversible** dynamical **systems** that are not Hamiltonian. By now, the KAM theory for various classes of non-Hamiltonian dynamical systems (first of all for reversible systems, but also for volume-preserving systems and general systems) has been developed to almost the same extent as the Hamiltonian KAM theory has. Invariant m-tori of typical systems of general form are isolated in the phase space, and the motions on such tori for $m \geq 2$ are not conditionally periodic. Therefore, in the KAM theory for general systems, one considers systems dependent on one or several external parameters, and one studies a quasi-periodic motion on an isolated invariant torus of such systems for special values of the parameters. Some results in KAM theory pertain to Hamiltonian, reversible, or volume-preserving systems also depending on external parameters.

8. Along with the KAM theory for dynamical systems with continuous time (i.e., for systems of ordinary differential equations), there exists the parallel KAM theory for dynamical systems with discrete time, i.e., for mappings (symplectic mappings, reversible ones, volume-preserving ones, or general mappings).

9. There is the quasi-periodic bifurcation theory which is a combination of KAM theory and Singularity Theory. In this direction of research, one studies, e.g., multi-dimensional quasi-periodic analogues of such well-known metamorphoses of closed trajectories as saddle-node, period doubling, and Poincaré–Andronov (also known as Hopf) bifurcations.

10. As a special branch of KAM theory, one may mention the theory of quasi-periodic or almost periodic motions in nonintegrable infinite dimensional systems, i.e., of quasi-periodic or almost periodic (in time) solutions of partial differential equations.

KAM theory implies failure of the **ergodic** and **quasi-ergodic hypotheses** in Hamiltonian Mechanics.

An overwhelming majority of the applications of KAM theory in Mathematics, Physics, and Astronomy relate to problems of stability of motion. For instance, consider a Hamiltonian system (2) with n degrees of freedom sufficiently close to the completely integrable system (1). If system (1) is nondegenerate (in the sense of Kolmogorov, isoenergetically, or at least in the sense of Rüssmann), then the trajectories of system (2) for most of the initial conditions (in the sense of Lebesgue measure) lie on Kolmogorov tori close to the unperturbed tori $I = \text{const}$. Hence, the values of the action variables $I_1(t), I_2(t), \ldots, I_n(t)$ for such trajectories will remain forever close to their initial values $I_1(0), I_2(0), \ldots, I_n(0)$, respectively. If $n = 2$ and system (1) is isoenergetically nondegenerate, then such an absence of evolution of the action variables along the trajectories of system (2) holds not only for a majority but for all the initial conditions. The reason is that in this case, the two-dimensional Kolmogorov tori of the perturbed system (2) divide the three-dimensional energy level $H = h$ into narrow resonant zones which do not overlap each other. Any trajectory that does not lie on one of the tori finds itself trapped inside one of such zones. Within the framework of KAM theory, one succeeds in proving the stability of equilibria and closed trajectories of Hamiltonian systems with two degrees of freedom in the so-called general elliptic case. One more example is applications of KAM theory to the problem of perpetual conservation of **adiabatic invariants**.

Less traditional applications of KAM theory pertain to Quantum Mechanics and are connected with the calculation of the short-wave approximation for the eigenvalues and eigenfunctions of the Schrödinger, Laplace, and Beltrami–Laplace operators.

What is a characteristic feature of KAM theory is an extreme complicacy of the proofs of almost all the results. The main difficulty here is connected with the appearance of the so-called **small divisors**, or **small denominators**, in any calculation scheme of constructing invariant tori carrying quasi-periodic motions. Small divisors are linear combinations of the motion frequencies (and sometimes of other auxiliary quantities) with integer coefficients. These combinations one has to divide by often lead to the divergence of the whole procedure. The method of an infinite sequence of coordinate transformations with a quadratic growth of the orders of "discrepancies" (this method was introduced by A.N.Kolmogorov in 1954 [K1]) was for many years (and, to a large extent, has been until now) the main tool of the proofs in KAM theory. Kolmogorov's method generalizes Newton's method of tangents for numerically solving equations $f(x) = 0$. After the N-th change of variables in the perturbed system (2), the dependence on the phases φ remains only in terms of order ε^{2^N}. Such "superconvergence" of Kolmogorov's procedure enables one to "paralyze" the influence of small divisors for strongly incommensurable frequencies. The first detailed exposition of Kolmogorov's method was given by V.I.Arnold in the article [A1] of 1961 devoted to mappings of a circle onto itself. There are also known other methods for finding quasi-periodic motions in nonintegrable systems. For instance, in 1974–76, E.Zehnder [Z] proposed a method based on rather general Implicit Function Theorems. Lately, the so-called direct proofs of the convergence of Lindstedt-type series for invariant tori in KAM theory have become widely used. These proofs exploit a complicated technique of regrouping the terms in the Lindstedt series. The technique was introduced by L.H.Eliasson in 1986–88 [E], it goes back to C.L.Siegel's paper [S] of 1942 on the linearization of analytic mappings in neighborhoods of fixed points.

See also **Arnold diffusion**, **Nekhoroshev theorem** on an exponential estimate of the diffusion.

For fundamentals of KAM theory for finite dimensional systems, basic theorems, their proofs, and a general review, the reader is referred to the monographs [BHS, Bru, Laz], manuals [Arn, AKN, SM, Tre], tutorials [Chi, CLHB, dlL, Pos], memoirs [BBH, BHT], and surveys [Bos, Sev1, Sev2, Sev3]. The works [AKN, BHS, dlL, Sev3] contain an extensive bibliography. The infinite dimensional KAM theory is presented in detail in the books [KP, Kuk1, Kuk2] and review [Kuk3].

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Integrability of a dynamical system is an extreme regularity of the behavior of its trajectories. This regularity manifests itself in, first, the existence of many independent first integrals of the system and, second, simplicity of the motions on the common level surfaces of these integrals. One distinguishes complete integrability (integrability in the whole phase space) and partial integrability (integrability on a separate invariant surface). Integrable systems are often encountered in studying various physical phenomena as the first approximation or as a result of averaging over angular variables. There are many nonequivalent precise definitions of integrability. The most explored class of integrable systems is constituted by integrable Hamiltonian systems. In the theory of such systems, the following statement is the central one.

Liouville–Arnold theorem [A]. Suppose that an autonomous Hamiltonian system with n degrees of freedom and Hamilton function H possesses n smooth integrals $F_1 = H, F_2, \ldots, F_n$ that are pairwise in involution. Let M be a connected component of one of the common level surfaces $\{F_i = c_i,\ 1 \le i \le n\}$ of these integrals, and let the differentials of the functions $F_1, F_2, \ldots, F_n$ be linearly independent at each point of the set M. Moreover, assume that whenever a trajectory of any of the Hamiltonian systems with Hamilton functions $F_1, F_2, \ldots, F_n$ lies on M, it is defined for all $t \in \mathbb{R}$. Then:

a) The surface M is diffeomorphic to the product of the k-torus $\mathbb{T}^k$ and the $(n-k)$-dimensional Euclidean space $\mathbb{R}^{n-k}$ for a certain k in the range $0 \le k \le n$ ($M \approx \mathbb{T}^k \times \mathbb{R}^{n-k}$).

b) In $\mathbb{T}^k \times \mathbb{R}^{n-k}$, one can introduce coordinates $\varphi = (\varphi_1, \ldots, \varphi_k) \in \mathbb{T}^k$, $x = (x_1, \ldots, x_{n-k}) \in \mathbb{R}^{n-k}$ in which the Hamilton equations with Hamilton functions F_i $(1 \le i \le n)$ on M take the form

$$\dot\varphi_\mu = \omega_{\mu i}, \qquad \dot x_\nu = a_{\nu i}$$

with constant $\omega_{\mu i}$, $a_{\nu i}$ $(1 \le \mu \le k,\ 1 \le \nu \le n-k)$.

c) The Hamilton equations with Hamilton functions $F_1, F_2, \ldots, F_n$ can be integrated by quadratures.

d) Suppose additionally that the manifold M is compact, i.e., $k = n$ (in this case, the condition of infinite extendibility of the trajectories on M is fulfilled automatically). Then some small neighborhood of the surface M in the phase space is diffeomorphic to the product $D \times \mathbb{T}^n$ of a domain D in the n-dimensional Euclidean space $\mathbb{R}^n$ and the n-torus $\mathbb{T}^n$ and, moreover, there are coordinates $I = (I_1, \ldots, I_n) \in D$, $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$ in $D \times \mathbb{T}^n$ with the following properties:

i) the torus M is given by the equation $I = I^*$ for a certain $I^* \in D$;

ii) the functions $F_1, F_2, \ldots, F_n$ in the variables (I, φ) depend on I only;

iii) the symplectic 2-form is

$$dI \wedge d\varphi = dI_1 \wedge d\varphi_1 + dI_2 \wedge d\varphi_2 + \ldots + dI_n \wedge d\varphi_n.$$

In particular, properties ii) and iii) imply that in the coordinates (I, φ), the Hamilton equations with Hamilton functions F_i $(1 \le i \le n)$ in a neighborhood of the manifold M have the form

$$\dot{\varphi} = \partial F_i(I)/\partial I, \qquad \dot{I} = 0.$$

A Hamiltonian system with Hamilton function H satisfying the hypotheses of this theorem is said to be *completely integrable* (in a neighborhood of the manifold M). The coordinates (I, φ) one speaks of in item d) of the theorem are called the **action-angle variables**. Under the hypotheses of item d), a neighborhood of the manifold M is foliated into **invariant n-tori** $I =$ const of the completely integrable system (M is one of these tori), the **motions** on the tori being **conditionally periodic**.

H.Poincaré called studying the motions in Hamiltonian systems close to completely integrable ones "the principal problem of dynamics" [P]. An essential progress in solving this problem has been achieved within the framework of **KAM theory**.

Some classical examples of completely integrable Hamiltonian systems with two degrees of freedom.

A. The motion of a particle of mass μ on the (x, y)-plane in the central force field with potential $U(r)$, where $r = (x^2 + y^2)^{1/2}$. The equations of motion read $\mu\ddot{x} = -xU'(r)/r$, $\mu\ddot{y} = -yU'(r)/r$. This system with two degrees of freedom always possesses two independent integrals in involution: the energy integral $h = \mu(\dot{x}^2+\dot{y}^2)/2+U(r)$, i.e., the Hamilton function, and the area integral $\sigma = x\dot{y} - y\dot{x}$, and is therefore integrable for any potential $U(r)$.

In the case of the gravitational potential $U(r) = -\gamma/r$ with $\gamma > 0$ (**Kepler's problem**), all the bounded orbits are closed (they are ellipses with a focus at the origin) due to the presence of an additional integral. As such an integral, one can take any of the two *Laplace integrals*

$$f_1 = x(\dot{x}^2 + \dot{y}^2) - \frac{\gamma x}{\mu r} - r\dot{r}\dot{x} \equiv \dot{y}(x\dot{y} - y\dot{x}) - \frac{\gamma x}{\mu r},$$
$$f_2 = y(\dot{x}^2 + \dot{y}^2) - \frac{\gamma y}{\mu r} - r\dot{r}\dot{y} \equiv \dot{x}(y\dot{x} - x\dot{y}) - \frac{\gamma y}{\mu r}.$$

Note that these integrals are connected by the relation $\mu^2(f_1^2 + f_2^2) \equiv \gamma^2 + 2\mu h\sigma^2$.

In the case where $U(r) = \gamma r^2$ with $\gamma > 0$ (oscillations obeying Hooke's law), all the orbits are also closed (they are ellipses with a center at the origin) due to the presence of an additional integral. As such an integral, one can take any of the two partial energy integrals

$$h_1 = \mu\dot{x}^2/2 + \gamma x^2, \qquad h_2 = \mu\dot{y}^2/2 + \gamma y^2.$$

The sum $h_1 + h_2$ of these integrals is equal to h.

The potentials const $- \gamma/r$ and const $+ \gamma r^2$ constitute all the analytic potentials $U(r)$ for which there is a stable circular orbit and all the orbits sufficiently close to this stable circular one are closed (the *Bertrand theorem*).

B. The planar motion in the gravitational field of two fixed centers is an integrable Hamiltonian system with two degrees of freedom (L.Euler, 1760). The same holds for the planar motion under the gravitational attraction of a fixed center and an additional constant force (J.L.Lagrange, 1766).

C. The motion of a heavy rigid body with a fixed (fastened) point is described by the Euler–Poisson equations $A\dot{\boldsymbol{\omega}} = A\boldsymbol{\omega} \times \boldsymbol{\omega} + P\mathbf{e} \times \mathbf{r}$, $\dot{\mathbf{e}} = \mathbf{e} \times \boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the angular velocity vector of the body with respect to the coordinate frame attached to the body, $\mathbf{e}$ is the unit vertical vector in this coordinate frame, $\mathbf{r}$ is the radius vector of the center-of-mass of the body with respect to the fixed point in this coordinate frame (so that $\dot{\mathbf{r}} = 0$), A is the inertia operator with respect to the fixed point, and P is the weight of the body. On the four-dimensional invariant manifolds N_C given by the relation $\langle A\boldsymbol{\omega}, \mathbf{e}\rangle = C = \text{const}$, the Euler–Poisson equations define Hamiltonian systems with two degrees of freedom. Here $\langle \mathbf{a}, \mathbf{b}\rangle$ and $\mathbf{a} \times \mathbf{b}$ denote respectively the scalar (inner) and vector (outer) products of vectors $\mathbf{a}$ and $\mathbf{b}$. Let A_1, A_2, A_3 be the eigenvalues of the inertia operator A and let r_1, r_2, r_3 be the coordinates of the center-of-mass of the body with respect to the principal axes of this operator. We list the known cases of integrability of the Euler–Poisson equations on N_C.

a) The Euler case (1750; also known as the Euler–Poinsot case): $r_1 = r_2 = r_3 = 0$, C is arbitrary.

b) The Lagrange case (1788; also known as the Lagrange–Poisson case): $A_1 = A_2$, $r_1 = r_2 = 0$, C is arbitrary.

c) The Kovalevskaya case (1889): $A_1 = A_2 = 2A_3$, $r_3 = 0$, C is arbitrary.

d) The Goryachev–Chaplygin case (1900): $A_1 = A_2 = 4A_3$, $r_3 = 0$, $C = 0$.

D. The geodesic flow on an n-dimensional Riemannian manifold is a Hamiltonian system with n degrees of freedom. There are the following three classical examples of integrable geodesic flows on two-dimensional surfaces ($n = 2$):

a) the geodesic flow on an ellipsoid in the three-dimensional Euclidean space $\mathbb{R}^3$ (in fact, the geodesic flow on an ellipsoid of any dimension n in $\mathbb{R}^{n+1}$ is integrable—this is the so-called Jacobi theorem),

b) the geodesic flow on a surface of revolution in the three-dimensional Euclidean space $\mathbb{R}^3$ (the statement that such a flow is integrable is called the Clairaut theorem),

c) the geodesic flow of a Liouville metric, i.e., a metric that has the form $ds^2 = [f(x) + g(y)](dx^2 + dy^2)$ with arbitrary smooth positive-valued functions f and g in some local coordinates (x, y) on the surface. The metrics on the surfaces of revolution constitute a particular class of Liouville metrics.

Many Hamiltonian systems with n degrees of freedom encountered in applications possess more than n independent first integrals (not all of these integrals are in involution). This situation is exemplified by the planar motion in a central force field with potential $-\gamma/r$ or γr^2. The Liouville–Arnold theorem is carried over to the case of more than n integrals as follows.

Nekhoroshev theorem (1969–72) [N1]. Suppose that an autonomous Hamiltonian system with n degrees of freedom possesses $n + m$ smooth integrals $F_1, \ldots, F_{n+m}$ ($0 \le m \le n$), each of the first $n-m$ functions $F_1,\dots,F_{n-m}$ being in involution with all the $n+m$ functions $F_1,\dots,F_{n+m}$. Let M be a compact connected component of one of the common level surfaces $\{F_i = c_i,\ 1 \le i \le n+m\}$, and let the differentials of the integrals $F_1,\dots,F_{n+m}$ be linearly independent at each point of the set M. Then the surface M is diffeomorphic to the $(n-m)$-torus. Moreover, in a neighborhood of the manifold M, there are coordinates $I = (I_1,\dots,I_{n-m})$, $\varphi = (\varphi_1,\dots,\varphi_{n-m}) \in \mathbb{T}^{n-m}$, $p = (p_1,\dots,p_m)$, $q = (q_1,\dots,q_m)$ such that

$$\begin{aligned} I_j &= I_j(F_1,\dots,F_{n-m}), && 1 \le j \le n-m,\\ p_s &= p_s(F_1,\dots,F_{n+m}), && 1 \le s \le m,\\ q_s &= q_s(F_1,\dots,F_{n+m}), && 1 \le s \le m, \end{aligned}$$

while the symplectic 2-form is

$$\sum_{j=1}^{n-m} dI_j \wedge d\varphi_j + \sum_{s=1}^{m} dp_s \wedge dq_s.$$

Finally, a neighborhood of the surface M is foliated into invariant $(n-m)$-tori $I = \text{const}$, $p = \text{const}$, $q = \text{const}$ of the system in question (M is one of these tori). The motions on the tori are conditionally periodic with frequencies dependent on $F_1,\dots,F_{n-m}$ only.

The coordinates (I,φ,p,q) here are called *generalized action-angle variables.*

There exist various methods of searching for first integrals of Hamiltonian systems: the method of separation of variables (or the Jacobi–Hamilton method), the method of L–A pairs (or the Lax method, this method is a particular case of the inverse scattering problem method), and others.

Typical Hamiltonian systems with $n \ge 2$ degrees of freedom are not integrable. The first rigorous results on non-integrability of Hamiltonian systems were due to H.Poincaré (1890). The essence of Poincaré's approach is that a complicated behavior (to be revealed) of the trajectories of the system in question is incompatible with the existence of many first integrals. As an obstruction to integrability of Hamiltonian systems close to integrable ones, one can encounter, for instance, the birth of isolated (on an energy level hypersurface) periodic trajectories or the so-called splitting of asymptotic surfaces. Poincaré's ideas have been strongly developed in V.V.Kozlov's works. The problems whose non-integrability follows from a complicated structure of the set of long-periodic solutions are exemplified by the problem of the planar motion in the gravitational field of two bodies that revolve around their common center-of-mass in circular orbits (H.Poincaré, 1890). Other methods of proving non-integrability are based on estimates from below for the coefficients of the power series for the so-called formal integrals in a neighborhood of an equilibrium of the Hamiltonian system (the Siegel method) or on examining the branching of solutions in the plane of complex time. There are also known a number of obstructions to integrability that are connected with the geometry and topology of the configuration space. For instance, obstructions of this kind are used in the proof of non-integrability of the problem of the planar motion in the gravitational field of $N \geq 3$ fixed centers (S.V.Bolotin, 1984).

The concept of integrability can be generalized to infinite dimensional systems. There are known many examples of completely integrable infinite dimensional nonlinear Hamiltonian systems: the Korteweg–de Vries (KdV) equation, the sine-Gordon equation, and others.

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Small divisors (small denominators) are expressions of the form

$$k_1\lambda_1 + k_2\lambda_2 + \ldots + k_N\lambda_N$$

which appear in the denominators (or enable one to estimate the denominators) of the terms of series in the Perturbation Theory or the Normal Form Theory for differential equations or differentiable mappings. Here $\lambda_1, \lambda_2, \ldots, \lambda_N$ are fixed real or complex quantities while $k_1, k_2, \ldots, k_N$ are integer coefficients (it is assumed that not all of $k_1, k_2, \ldots, k_N$ vanish). For $N \geq 2$ in the real case and for $N \geq 3$ in the complex case, among these expressions there are numbers arbitrarily close to zero or even (for resonant values of $\lambda_1, \lambda_2, \ldots, \lambda_N$) equal to zero. For instance, the mean angular velocities of the motions of Jupiter and Saturn around the Sun are equal respectively to $\omega_{\mathrm{J}} = 0.5297\ \mathrm{year}^{-1}$ and $\omega_{\mathrm{S}} = 0.2133\ \mathrm{year}^{-1}$. The ratio $\omega_{\mathrm{S}}/\omega_{\mathrm{J}} = 0.4027$ is very close to $2/5$, and the small divisor $\Omega = 5\omega_{\mathrm{S}} - 2\omega_{\mathrm{J}} \approx 0.007\ \mathrm{year}^{-1}$ is much smaller than any of the frequencies ω_{J} and ω_{S} ($\omega_{\mathrm{J}}/\Omega \approx 75$, $\omega_{\mathrm{S}}/\Omega \approx 30$). This leads to a large long-periodic perturbation in the motions of these planets (the so-called "great inequality"). The presence of small divisors often results in the divergence of the corresponding series—Taylor ones, Fourier ones, Poisson ones (combinations of Taylor and Fourier series), and so on—even when exact resonances are absent. In the cases where the series converge, the differential properties of the series sum (the smoothness class and estimates for the derivatives) depend in an essential way on the arithmetical properties of the collection of quantities $\lambda_1, \lambda_2, \ldots, \lambda_N$.

One can easily illustrate the appearance of small divisors by the following two model examples.

Example 1. The equation

$$\sum_{j=1}^{n} \omega_j \frac{\partial f(x)}{\partial x_j} = F(x), \tag{1}$$

where F is a given function in n variables $x_1, x_2, \ldots, x_n$ while f is an unknown function, ω_j being certain real constants ("frequencies"). Both the functions f and F are assumed to be 2π-periodic in each argument and to have zero mean values. Expand the function F in a Fourier series:

$$F(x) = \sum_k F_k e^{i\langle k,x\rangle},$$

where $k = (k_1, k_2, \ldots, k_n)$ is a collection of integers not equal to zero simultaneously and $\langle k, x\rangle = k_1x_1 + k_2x_2 + \ldots + k_nx_n$. The solution of equation (1) is given by the formula

$$f(x) = -i\sum_k \frac{F_k}{\langle k, \omega\rangle} e^{i\langle k,x\rangle}, \tag{2}$$

where $\langle k, \omega\rangle = k_1\omega_1 + k_2\omega_2 + \ldots + k_n\omega_n$ are small divisors.

Example 2. The equation

$$f(x + \omega) - f(x) = F(x) \tag{3}$$

in the same notation and under the same conditions on F and f as in example 1. The solution of equation (3) is given by the formula

$$f(x) = \sum_k \frac{F_k}{e^{i\langle k,\omega\rangle} - 1} e^{i\langle k,x\rangle}. \tag{4}$$

One can easily estimate the absolute values of the differences $e^{i\langle k,\omega\rangle} - 1$ in terms of the small divisors $\langle k,\omega\rangle - 2k_0\pi$ where k_0 is an arbitrary integer:

$$\frac{2}{\pi}\min_{k_0}\bigl|\langle k,\omega\rangle - 2k_0\pi\bigr| \le \bigl|e^{i\langle k,\omega\rangle} - 1\bigr| \le \min_{k_0}\bigl|\langle k,\omega\rangle - 2k_0\pi\bigr|,$$

since $2|\varphi|/\pi \le \bigl|e^{i\varphi} - 1\bigr| \le |\varphi|$ for $-\pi \le \varphi \le \pi$.

If the frequencies $\omega_1, \omega_2, \ldots, \omega_n$ are commensurable, i.e. if $\langle k,\omega\rangle = 0$ for some $k_1, k_2, \ldots, k_n$ not equal to zero simultaneously, then series (2) is not well-defined (only a finite segment of the series is defined). If the frequencies ω_j are incommensurable but anomalously well approximable by commensurable frequencies, then series (2) is generally divergent. In both the situations, there exists no solution f of equation (1). On the other hand, if the frequencies ω_j are incommensurable and, moreover, not too well approximable by commensurable frequencies, then series (2) converges. The same is valid for equation (3), but in the case of equation (3), one should consider commensurability of the $n+1$ numbers $\omega_1, \omega_2, \ldots, \omega_n, \pi$ instead of commensurability of the n numbers $\omega_1, \omega_2, \ldots, \omega_n$.

The differential properties of the solution f depend on those of the given function F and on the arithmetical properties of the frequency collection.

The expression "the frequencies ω_j are incommensurable and not too well approximable by commensurable frequencies" is usually understood as *strong incommensurability*, or *Diophantine* property of the frequency collection $\omega_1, \omega_2, \ldots, \omega_n$. The Diophantine condition consists in the existence of positive constants τ and γ such that for any integers $k_1, k_2, \ldots, k_n$ that do not vanish simultaneously, the inequality

$$\bigl|\langle k,\omega\rangle\bigr| > \gamma|k|^{-\tau} \tag{5}$$

holds, where $|k| = |k_1| + |k_2| + \ldots + |k_n|$. The Lebesgue measure of the set of non-Diophantine frequency collections is zero. Suppose that the frequency collection $(\omega_1, \omega_2, \ldots, \omega_n)$ ranges in a certain finite domain of the n-dimensional Euclidean space while the exponent $\tau > n-1$ is fixed. Then the Lebesgue measure of the set of the frequency collections that do not satisfy at least one of the inequalities (5) is (for $n \ge 2$) of the order of γ.

In the context of equation (3), the Diophantine property of the frequency collection consists in the existence of positive constants τ and γ such that for any integers $k_0, k_1, k_2, \ldots, k_n$ subject to the condition $|k| = |k_1| + |k_2| + \ldots + |k_n| > 0$, the inequality

$$\bigl|\langle k,\omega\rangle - 2k_0\pi\bigr| > \gamma|k|^{-\tau} \tag{6}$$

holds. The Lebesgue measure of the set of frequency collections non-Diophantine in this sense is also zero. If the frequency collection $(\omega_1, \omega_2, \ldots, \omega_n)$ ranges in a certain finite domain of the n-dimensional Euclidean space and the exponent $\tau > n$ is fixed, then the Lebesgue measure of the set of the frequency collections that do not satisfy at least one of the inequalities (6) is of the order of γ.

Problem (1) is often encountered in studies of **quasi-periodic motions** in the case of ordinary differential equations and problem (3), in the case of mappings.

Various methods of struggling with the influence of small divisors have been developed in the framework of **KAM theory**.

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See also the bibliography to the article "**KAM theory**".

Arnold theorem on conditionally periodic motions is one of the theorems in **KAM theory**. This theorem plays the central role in applications of KAM theory to problems of Celestial Mechanics. The Arnold theorem implies the existence, under certain assumptions, of numerous quasi-periodic solutions in the many-body problem, the Lebesgue measure of the union of **invariant tori** filled in by the trajectories of these solutions being positive. The theorem was announced by V.I.Arnold in 1962 [A1], a detailed proof was published in 1963 [A2]. Below we give a simplified formulation of the Arnold theorem.

Consider a Hamiltonian system with $n+m$ degrees of freedom and a real analytic Hamilton function of the form

$$H(I,\varphi,p,q,\mu) = H_0(I) + \mu H_1(I,\varphi,p,q) + \mu^2 H_2(I,\varphi,p,q,\mu), \tag{1}$$

where $I = (I_1,\dots,I_n)$ ranges in a finite domain G of the n-dimensional Euclidean space $\mathbb{R}^n$, $\varphi = (\varphi_1,\dots,\varphi_n) \in \mathbb{T}^n$ are angular variables ($\mathbb{T}^n$ denotes the n-torus), the variables $p = (p_1,\dots,p_m) \in \mathbb{R}^m$ and $q = (q_1,\dots,q_m) \in \mathbb{R}^m$ range in a neighborhood of the origin of the m-dimensional Euclidean space $\mathbb{R}^m$, and μ is a small parameter. The symplectic 2-form is

$$\sum_{l=1}^{n} dI_l \wedge d\varphi_l + \sum_{i=1}^{m} dp_i \wedge dq_i.$$

It is supposed that the function H_1 has the form

$$H_1(I,\varphi,p,q) = F_0(I,\tau) + F_1(I,p,q) + F_2(I,\varphi,p,q),$$
$$F_0(I,\tau) = \lambda_0(I) + \sum_{i=1}^{m}\lambda_i(I)\tau_i + \sum_{i,j=1}^{m}\lambda_{ij}(I)\tau_i\tau_j + \sum_{i,j,k=1}^{m}\lambda_{ijk}(I)\tau_i\tau_j\tau_k,$$

where $\tau_i = \frac{1}{2}(p_i^2+q_i^2)$ for $i = 1,\dots,m$, $\lambda_{ij}(I) \equiv \lambda_{ji}(I)$, the function F_1 satisfies the inequality

$$\bigl|F_1(I,p,q)\bigr| < C\sum_{i=1}^{m}\bigl(|p_i|^7 + |q_i|^7\bigr)$$

with a certain constant $C > 0$, while the mean value of the function F_2 over the variables φ vanishes:

$$\int_{\mathbb{T}^n} F_2(I,\varphi,p,q)\,d\varphi_1\dots d\varphi_n \equiv 0.$$

Theorem. Assume that in the domain G, there hold the inequalities

$$\det\frac{\partial^2 H_0(I)}{\partial I^2} \neq 0, \qquad \det\bigl(\lambda_{ij}(I)\bigr) \neq 0.$$

Then for any $\delta > 0$, there exists $\varepsilon_0 > 0$ with the following properties. For $\varepsilon > 0$, let the domain A_ε of the phase space of the system with Hamilton function (1) be defined by the inequalities $0 < \tau_i < \varepsilon$ for all $i = 1,\dots,m$. Then for any ε and μ such that $0 < \varepsilon < \varepsilon_0$ and $0 < \mu < \varepsilon^4$, the domain A_ε contains invariant analytic $(n+m)$-tori $T_{\omega,\theta}$ of the system in question that are given parametrically by the equations

$$\begin{aligned} I &= I_\omega + f_{\omega,\theta}(\Omega,\Theta), \\ \varphi &= \Omega + g_{\omega,\theta}(\Omega,\Theta), \\ p &= \left[2\big(\tau_{\omega,\theta} + \alpha_{\omega,\theta}(\Omega,\Theta)\big)\right]^{1/2} \cos\big(\Theta + \beta_{\omega,\theta}(\Omega,\Theta)\big), \\ q &= \left[2\big(\tau_{\omega,\theta} + \alpha_{\omega,\theta}(\Omega,\Theta)\big)\right]^{1/2} \sin\big(\Theta + \beta_{\omega,\theta}(\Omega,\Theta)\big). \end{aligned}$$

Here $\Omega = (\Omega_1,\dots,\Omega_n) \in \mathbb{T}^n$ and $\Theta = (\Theta_1,\dots,\Theta_m) \in \mathbb{T}^m$ are angular parameters, while $I_\omega \in G$ and $\tau_{\omega,\theta} \in \mathbb{R}^m$ are constants dependent on the label $(\omega,\theta) \in \mathbb{R}^{n+m}$ of the torus. Moreover,

$$\omega = \partial H_0(I_\omega)/\partial I, \qquad \theta = \mu \partial F_0(I_\omega, \tau_{\omega,\theta})/\partial\tau.$$

The Lebesgue measure mes of the complement to the union $B_{\varepsilon,\mu} = \bigcup_{\omega,\theta} T_{\omega,\theta}$ of these tori is small:

$$\operatorname{mes}(A_\varepsilon \setminus B_{\varepsilon,\mu}) < \delta \operatorname{mes} A_\varepsilon.$$

The invariant tori $T_{\omega,\theta}$ differs but slightly from the tori $I = I_\omega$, $\tau = \tau_{\omega,\theta}$, namely,

$$|f_{\omega,\theta}| < \delta\varepsilon, \qquad |g_{\omega,\theta}| < \delta\varepsilon, \qquad |\alpha_{\omega,\theta}| < \delta\varepsilon, \qquad |\beta_{\omega,\theta}| < \delta\varepsilon$$

(here the absolute value of a vector-valued function is defined as the sum of the absolute values of its components). Finally, each torus $T_{\omega,\theta}$ carries **quasi-periodic motions** with frequencies $(\omega_1,\dots,\omega_n,\theta_1,\dots,\theta_m)$:

$$\dot\Omega = \omega, \qquad \dot\Theta = \theta.$$

Later on, it was found that under the hypotheses of the theorem just formulated, the system with Hamilton function (1) admits in the domain A_ε (under some additional assumptions) not only invariant $(n+m)$-tori but also invariant analytic tori of smaller dimensions $n, n+1,\dots,n+m-1$. These latter tori carry quasi-periodic motions as well. The invariant d-tori are exponentially "condensing" as they approach a torus of smaller dimension $d' < d$ [JV].

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Nekhoroshev theorem on an exponential estimate of the diffusion is the theorem stating that the rate of **Arnold diffusion** (to be more precise, the average rate of evolution of the action variables in analytic Hamiltonian systems close to **completely integrable** ones) is exponentially small with respect to the perturbation parameter, with the exception of very degenerate cases. Consider a Hamiltonian system with $n \geq 2$ degrees of freedom and real analytic Hamilton function $H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon)$:

$$\dot{\varphi} = \omega(I) + \varepsilon \partial H_1(I, \varphi, \varepsilon)/\partial I, \qquad \dot{I} = -\varepsilon \partial H_1(I, \varphi, \varepsilon)/\partial \varphi, \tag{1}$$

where (I, φ) are the **action-angle variables** for the unperturbed system with Hamilton function $H_0(I)$ and $\omega(I) = \partial H_0(I)/\partial I$ is the frequency vector of the unperturbed motion, while $0 < \varepsilon \ll 1$ is the perturbation parameter. It is supposed that I ranges in some finite domain G of the n-dimensional Euclidean space $\mathbb{R}^n$. In the unperturbed system, evolution of the action variables I is absent: $\dot{I} = 0$.

Theorem. Let the unperturbed Hamilton function H_0 satisfy in G a certain nondegeneracy condition called steepness. Then there exist positive constants a, b, R_*, K_*, and ε_* such that for any solution $\varphi(t), I(t)$ of system (1), the inequality

$$\left(\sum_{j=1}^{n} \left[I_j(t) - I_j(0) \right]^2 \right)^{1/2} \leq R_* \varepsilon^b \quad \text{for} \quad |t| \leq \exp(K_* \varepsilon^{-a}) \tag{2}$$

holds whenever $0 < \varepsilon \leq \varepsilon_*$.

This theorem was announced by N.N.Nekhoroshev in 1971 [N1], a detailed proof (rather complicated) was published in 1977–79 [N3, N4]. The constants a and b are called the *stability exponents*; of them, exponent a is the most important one. These exponents depend on the number n of degrees of freedom and the geometric properties of the function H_0 and tend to zero as $n \to \infty$. One usually calls the quantity $T(\varepsilon) = \exp(K_* \varepsilon^{-a})$ the *stability time*, the distance $R(\varepsilon) = R_* \varepsilon^b$ the *radius of confinement*, and the constant ε_* the *threshold of validity*. Having estimate (2) in view, one sometimes speaks of *effective stability of the action variables* in system (1). The larger the exponents a and b, the "more stable" are the action variables. Analyticity of the Hamilton function H is essential for the existence of exponential estimate (2), although there are classes of non-analytic Hamilton functions for which this estimate holds as well [MS].

The steepness condition is very weak: the nonsteep Hamilton functions H_0 for which evolution of the action variables in the perturbed system (1) can proceed fast (e.g., with a rate of the order of ε) constitute an extremely rare exception among all the analytic functions defined in domain G [N2, N3]. The linear functions are typical examples of such nonsteep functions. On the other hand, among all the *steep Hamilton functions* H_0, the "steepest" ones for which the values of the stability exponents a and b are maximal (for a fixed number n of degrees of freedom) are convex and quasi-convex functions.

Definition. An unperturbed Hamilton function $H_0\colon G \to \mathbb{R}$, for which the frequency vector $\omega(I)$ nowhere vanishes, is said to be *convex* if there exists a constant $c > 0$ such that

$$\left|\left\langle \eta,\ \frac{\partial \omega(I)}{\partial I}\eta \right\rangle\right| \geq c \sum_{j=1}^{n} \eta_j^2 \tag{3}$$

for all $I \in G$ and for all vectors $\eta \in \mathbb{R}^n$ (the angular brackets here denote the scalar product). A Hamilton function H_0 is said to be *quasi-convex* if there exists a constant $c > 0$ such that inequality (3) holds for all $I \in G$ and $\eta \in \mathbb{R}^n$ such that $\langle \omega(I), \eta \rangle = 0$.

For instance, the function $H_0(I) = \frac{1}{2}\sum_{j=1}^{n} I_j^2$ is convex whereas the function $H_0(I) = I_1 + \frac{1}{2}\sum_{j=2}^{n} I_j^2$ is quasi-convex but not convex. For quasi-convex unperturbed Hamilton functions H_0, estimate (2) holds for $a = b = 1/(2n)$. Moreover, for any $0 < \mu \leq 1$ one can take [L2, P]

$$a = \frac{\mu}{2n}, \qquad b = \frac{1-\mu}{2} + \frac{\mu}{2n}.$$

These values of the stability exponents seem to be optimal, they are known to be at least very close to being so [LM].

All the unperturbed systems with convex Hamilton functions H_0 are **Kolmogorov nondegenerate** (the converse is not true). All the unperturbed systems with quasi-convex (and a fortiori convex) Hamilton functions H_0 are **isoenergetically nondegenerate** (the converse for $n \geq 3$ is not true).

Combining the Nekhoroshev theorem with the results of **KAM theory**, we arrive at the following picture of the evolution of the action variables in Hamiltonian systems close to integrable ones. Let, for instance, the unperturbed Hamilton function H_0 be quasi-convex. For sufficiently small perturbation parameter ε, system (1) possesses **Kolmogorov invariant n-tori** close to the tori $I = \text{const}$. The **motions** on these invariant tori are **quasi-periodic**, and the Lebesgue measure of the complement to the union of the tori is $O\big(\sqrt{\varepsilon}\,\big)$. For a trajectory lying on one of these tori, the values of the action variables $I_j(t)$ will remain forever close to their initial values $I_j(0)$, $1 \leq j \leq n$. In the case $n = 2$, the same holds for trajectories in the **resonant zones** between the tori as well. In the case $n \geq 3$, on the other hand, the action variables $I_j(t)$ along trajectories in the resonant zones remain close to their initial values $I_j(0)$ during a time period exponentially large with respect to the inverse perturbation parameter.

In 1992, P.Lochak published an essentially new proof of the Nekhoroshev theorem for the case of quasi-convex Hamilton functions H_0 [L1]. There are generalizations of estimate (2) on the evolution rate of the action variables I to time-periodic perturbations, as well as to systems with Hamilton functions of the form $H_0(I)+\varepsilon H_1(I,\varphi,\varepsilon t,\varepsilon)$ and $H_0(I)+\varepsilon H_1(I,\varphi,p,q,\varepsilon)$ [N3]. Here $p \in \mathbb{R}^m$, $q \in \mathbb{T}^m$ ($\mathbb{T}^m$ denoting the m-torus), while the symplectic 2-form in the case where the variables p and q are present is

$$\sum_{j=1}^{n} dI_j \wedge d\varphi_j + \sum_{s=1}^{m} dp_s \wedge dq_s.$$

There exist also analogues of the Nekhoroshev theorem for infinite dimensional Hamiltonian systems. The Nekhoroshev theorem and its generalizations have numerous applications in the problems of Celestial Mechanics.

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[Sau] D.Sauzin, *Nekhoroshev estimates and instability for Gevrey class Hamiltonians*, in "Dynamical Systems. Part I. Hamiltonian Systems and Celestial Mechanics", Pubblicazioni del Centro di Ricerca Matematica Ennio de Giorgi, Scuola Normale Superiore, Pisa, 2003, 199–217.

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Reversible system is a dynamical system that becomes the system with the reverse time direction under the action of a certain phase space involution (i.e., a mapping whose square is the identity transformation). Thus, a differential equation $\dot{x} = V(x,t)$ is reversible with respect to a smooth involution G, if this equation is invariant under the change $t \mapsto -t$, $x \mapsto G(x)$, i.e., if $G\bigl(x(-t)\bigr)$ is a solution of this equation whenever $x(t)$ is. A one-to-one mapping A of a space onto itself is reversible with respect to an involution G, if $GA = A^{-1}G$.

For instance, any system of Newtonian equations $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$ is reversible with respect to the involution $G\colon (\mathbf{r}, \dot{\mathbf{r}}) \mapsto (\mathbf{r}, -\dot{\mathbf{r}})$ of the phase space. It is entirely inessential here whether the "forces" $\mathbf{F}$ are conservative. A more general system of equations of motion $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ is still reversible with respect to the same involution G provided that the "forces" $\mathbf{F}$ are even in the velocities: $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) \equiv \mathbf{F}(\mathbf{r}, -\dot{\mathbf{r}})$. A still more general example: a system $\dot{p} = P(p,q)$, $\dot{q} = Q(p,q)$ with $p = (p_1, \ldots, p_{n-k})$, $q = (q_1, \ldots, q_k)$ is reversible with respect to the involution $G\colon (p,q) \mapsto (-p,q)$ if and only if $P(p,q) \equiv P(-p,q)$, $Q(p,q) \equiv -Q(-p,q)$. The manifold of fixed points of this involution G is the k-dimensional surface $p = 0$.

As a rule, reversible systems of differential equations are not Hamiltonian. Moreover, it is typical for reversible systems that in their phase space, there coexist domains where the general picture of the motion is the same as that in Hamiltonian systems, and domains with dynamics characteristic for dissipative systems. There are also known examples of Hamiltonian systems that are not reversible. On the other hand, the similarity between reversible systems (in some domains of the phase space) and Hamiltonian systems is really striking and concerns, in particular:

1) the spectra, normal forms, and versal unfoldings of linear systems, as well as parametric resonance in linear systems,

2) the properties and bifurcations of periodic solutions (closed trajectories) of nonlinear systems,

3) Birkhoff-type normal forms of nonlinear systems in neighborhoods of equilibria and closed trajectories,

4) the properties of homoclinic and heteroclinic trajectories,

5) **invariant tori** carrying **quasi-periodic motions** (such tori are studied in **KAM theory**).

For instance, closed trajectories of an autonomous Hamiltonian system are generically not isolated in the phase space; instead, they are organized into smooth one-parameter families (the parameter being the value of the Hamilton function). Now consider an autonomous reversible system in an n-dimensional space, and let the manifold Σ of fixed points of the reversing involution G be of dimension $k < n$. It turns out that if $2k \geq n$ then G-invariant closed trajectories of this system (each of such trajectories intersects Σ at exactly two points) are generically not isolated in the phase space either: they are organized into smooth $(2k-n+1)$-parameter families (V.I.Arnold [A]). If λ is a multiplier of a closed trajectory, so is λ^{-1} (in both the Hamiltonian and reversible cases).

The relation between reversible and symplectic (or canonical, in more traditional terminology) mappings is similar to that between reversible and Hamiltonian differential equations.

As a generalization of reversible systems, one may consider *weakly reversible systems* that become the systems with the reverse time direction under the action of a certain—not necessarily involutive—transformation of the phase space. The concept of a weakly reversible system was introduced by V.I.Arnold in 1984 [A].

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Invariant torus is an invariant manifold of a dynamical system (i.e., of a mapping or a system of ordinary differential equations) that is diffeomorphic to the n-torus $\mathbb{T}^n$. Invariant tori are a mathematical description of multi-frequency oscillatory processes in dynamical systems.

An invariant zero-dimensional torus (0-torus) is an equilibrium of a system of differential equations or a fixed point of a mapping. An invariant one-dimensional torus (1-torus) is a closed trajectory of a system of differential equations or an invariant closed curve of a mapping. In the general theory of invariant tori, one usually considers invariant n-tori of nonlinear systems of differential equations for $n \geq 2$ and of nonlinear mappings for $n \geq 1$, because for smaller values of n as well as in the linear case, most of the aspects of this theory become trivial. The theory of invariant tori studies such questions as the conditions for the existence of an invariant torus and its stability under small perturbations of the system, the behavior of trajectories on an invariant torus and in its neighborhood, bifurcations of invariant tori and their families.

The properties of invariant tori in dynamical systems with various symmetries or conservation laws (for instance, in Hamiltonian or **reversible systems**) and in general dynamical systems differ drastically. For Hamiltonian or reversible systems, the situation is typical where invariant tori of a given dimension occur in the phase space in large quantities and constitute everywhere discontinuous families with a complicated structure. The **motions** on these tori are **conditionally periodic** with **strongly incommensurable frequencies** (see **KAM theory**). The smoothness of an invariant torus is only slightly less than the smoothness of the system itself, and if the latter is infinitely differentiable or analytic, so are the invariant tori. On the contrary, invariant tori of general systems are, as a rule, isolated in the phase space. The motions on the tori are usually not conditionally periodic, and these tori are often finitely smooth only, even if the system itself is infinitely differentiable or analytic.

One of the widespread bifurcations of invariant tori is the birth of $(n+1)$-tori from n-tori, e.g., the birth of an invariant 2-torus from a limit cycle. Another typical bifurcation consists in that an invariant torus loses its smoothness gradually, turns to a so-called strange attractor and, finally, breaks-up with the onset of stochastic oscillations.

A special branch of the theory of invariant tori explores invariant tori in infinite dimensional systems.

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See also the bibliography to the article “**KAM theory**”.

Ergodic hypothesis in Hamiltonian Mechanics consists in that a generic Hamiltonian system with $n \geq 2$ degrees of freedom is ergodic on typical connected components of the energy levels (a measure-preserving dynamical system with a compact phase space is said to be ergodic if each invariant set of this system is either of measure zero or of full measure). This conjecture was regarded as quite plausible for a long time, up to the middle of the 20-th century. Nevertheless, it fails for any n, which is one of the most important corollaries of **KAM theory**. Indeed, consider an **isoenergetically nondegenerate completely integrable Hamiltonian system** with n degrees of freedom whose energy level hypersurfaces are compact and connected, for instance, the system with Hamilton function $H_0(I) = \frac{1}{2}(I_1^2 + I_2^2 + \ldots + I_n^2)$ in the domain $0 < H_0(I) < C$, where I denotes the action variables and C is a positive constant. Consider also a sufficiently small (of the order of $\varepsilon \ll 1$) generic Hamiltonian perturbation of such a system. On each energy level hypersurface of the perturbed system, there are **Kolmogorov invariant n-tori**. The measure of their union is positive, the measure of the complement to the union is also positive (and does not exceed a quantity of the order of $\sqrt{\varepsilon}$). Thus, the perturbed system is not ergodic on any energy level.

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Quasi-ergodic hypothesis in Hamiltonian Mechanics consists in that a generic Hamiltonian system with $n \geq 2$ degrees of freedom is quasi-ergodic on typical connected components of the energy levels (a measure-preserving dynamical system with a compact phase space is said to be quasi-ergodic if it admits an everywhere dense trajectory). **KAM theory** implies that this conjecture fails for any n. Indeed, consider an **isoenergetically nondegenerate completely integrable Hamiltonian system** with two degrees of freedom whose energy level hypersurfaces are compact and connected, for instance, the system with Hamilton function $H_0(I) = \frac{1}{2}(I_1^2 + I_2^2)$ in the domain $0 < H_0(I) < C$, where I denotes the action variables and C is a positive constant. Consider also a sufficiently small generic Hamiltonian perturbation of such a system. On each three-dimensional energy level hypersurface of the perturbed system, there are **Kolmogorov invariant 2-tori**. They divide this hypersurface into **resonant zones** which do not overlap each other, and any trajectory lies either on one of the Kolmogorov tori or inside one of the resonant zones. Thus, the perturbed system is not quasi-ergodic on any energy level. For $n \geq 3$, one should consider, instead of a completely integrable Hamiltonian system, a Hamiltonian system whose phase space is foliated into invariant tori of dimension $2n-2$. Then one may apply the results by I.O.Parasyuk and M.R.Herman on small perturbations of such systems. The reason for the absence of quasi-ergodicity in the case $n \geq 3$ is the same as in the case $n = 2$: the invariant $(2n-2)$-tori of the perturbed system divide a $(2n-1)$-dimensional energy level hypersurface into resonant zones which do not overlap each other.

On the other hand, the symplectic 2-form in the Parasyuk–Herman theorems is not exact. The question whether the quasi-ergodic hypothesis is valid for Hamiltonian systems with $n \geq 3$ degrees of freedom under the condition that the symplectic 2-form is exact remains open. The studies of **Arnold diffusion** suggest that the answer to this question is most likely affirmative.

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