Bounds on the Discrete Spectrum of Lattice Schrödinger Operators

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Abstract

We discuss the the validity of the Weyl asymptotics – in the sense of two-sided bounds – for the size of the discrete spectrum of (discrete) Schrödiger operators on the d–dimensional, $d \geq 1$, cubic lattice $\mathbb{Z}^d$ at large couplings. We show that the Weyl asymptotics can be violated in any spatial dimension $d \geq 1$ – even if the semi-classical number of bound states is finite. Furthermore, we prove for all dimensions $d \geq 1$ that, for potentials well-behaved at infinity and fulfilling suitable decay conditions, the Weyl asymptotics always hold. These decay conditions are mild in the case $d \geq 3$, while stronger for $d = 1, 2$. It is well-known that the semi-classical number of bound states is – up to a constant – always an upper bound on the size of the discrete spectrum of Schrödinger operators if $d \geq 3$. We show here how to construct general upper bounds on the exact number of bound states of Schrödinger operators on $\mathbb{Z}^d$ from semi-classical quantities in all space dimensions $d \geq 1$ and independently of the positivity-improving property of the free Hamiltonian.

1 Introduction

Let $V \in L^{d/2}(\mathbb{R}^d, \mathbb{R}_0^+)$ be a non-negative potential in the d–dimensional space with $d \geq 3$. From standard results of spectral theory [1] it follows that the negative spectrum $\sigma[-\Delta - \lambda V(x)] \cap \mathbb{R}^-$ of the corresponding self-adjoint Schrödinger operator

$$-\Delta_{\mathbb{R}^d} - \lambda V(x) \tag{1}$$

on $L^2(\mathbb{R}^d)$ is purely discrete, i.e., consists only of isolated eigenvalues of finite multiplicity. Here, $\Delta_{\mathbb{R}^d} = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplacian on $\mathbb{R}^d$ and V acts as a multiplication operator, $[V\varphi](x) := V(x)\varphi(x)$. By a well-known theorem – first established by Weyl [2, 3] for the case of Dirichlet Laplacian in a bounded domain – the number $N^{cont}[\lambda V]$ of negative eigenvalues of $-\Delta_{\mathbb{R}^d} - \lambda V$ (counting multiplicities) is asymptotically

$$N^{cont}[\lambda V] \; := \; \mathrm{Tr}\big\{\mathbf{1}[-\Delta_{\mathbb{R}^d} - \lambda V < 0]\big\} \; \sim \; N^{cont}_{sc}[\lambda V] \tag{2}$$

as $\lambda \to \infty$. The right-hand side of (2) is the volume

$$N^{cont}_{sc}[V] \; := \; \int \mathbf{1}[p^2 - V(x) < 0] \, \frac{d^d x \, d^d p}{(2\pi)^d} \tag{3}$$

these bound states occupy in phase space $\mathbb{R}^d \times (\mathbb{R}^*)^d = \mathbb{R}^{2d}$ according to semiclassical analysis. This so-called Weyl asymptotics (2) is complemented by the celebrated *non-asymptotic* bound of Rozenblum [4], Lieb [5], and Cwikel [6] on the number $N^{cont}[V]$ of bound states of $-\Delta_{\mathbb{R}^d} - V$ of the form

$$N^{cont}[V] \; \leq \; C_{RLC}(d) \, N^{cont}_{sc}[V] \tag{4}$$

for some $C_{RLC}(d) \geq 1$. Lieb [7, Eq. (4.5)] has shown that the optimal choice for $C_{RLC}(3)$ is smaller than $6,9$. Note that

$$N^{cont}_{sc}[V] \; = \; \frac{|S^{d-1}|}{d\,(2\pi)^d} \int V^{d/2}(x) \, d^d x, \tag{5}$$

where $|S^{d-1}|$ is the volume of the $(d-1)$–sphere.

In the present paper, we replace the Euclidean d–dimensional space $\mathbb{R}^d$ by the d–dimensional hypercubic lattice $\Gamma = \mathbb{Z}^d$ and study the discrete analogues of the Weyl asymptotics (2) and the Rozenblum-Lieb-Cwikel bound (4). For a given *potential* $V \in \ell^\infty(\Gamma, \mathbb{R}_0^+)$, the discrete Schrödinger operator corresponding to (1) is

$$-\Delta_\Gamma - \lambda V(x), \tag{6}$$

where V acts again as a multiplication operator and Δ_Γ is the discrete Laplacian defined by

$$[\Delta_\Gamma \varphi](x) \; = \; \sum_{|v|=1} \big\{\varphi(x+v) - \varphi(x)\big\}. \tag{7}$$

More generally, we assume to be given a Morse function $\mathfrak{e} \in C^2(\Gamma^*, \mathbb{R})$ on the d–dimensional torus (Brillouin zone) $\Gamma^* = \big(\mathbb{R}/2\pi\mathbb{Z}\big)^d = [-\pi, \pi)^d$, the dual group of Γ. Given such a function $\mathfrak{e}$, we consider the self-adjoint operator

$$H(\mathfrak{e}, V) \; := \; h(\mathfrak{e}) - V(x), \tag{8}$$

on $\ell^2(\Gamma)$, where $h(\mathfrak{e}) \in \mathcal{B}[\ell^2(\Gamma)]$ is the hopping matrix (convolution operator) corresponding to the dispersion relation $\mathfrak{e}$, i.e.,

$$\big[\mathcal{F}^*\big(h(\mathfrak{e})\varphi\big)\big](p) \;=\; \mathfrak{e}(p)\,[\mathcal{F}^*(\varphi)](p) \tag{9}$$

for all $\varphi \in L^2(\Gamma^*)$ and all $p \in \Gamma^*$. Here,

$$\mathcal{F}^* :\; \ell^2(\Gamma) \to L^2(\Gamma^*), \quad [\mathcal{F}^*(\varphi)](p) \;:=\; \sum_{x\in\Gamma} e^{-i\langle p,x\rangle}\varphi(x) \tag{10}$$

is the usual discrete Fourier transformation with inverse

$$\mathcal{F} :\; L^2(\Gamma^*) \to \ell^2(\Gamma), \quad [\mathcal{F}(\psi)](x) \;:=\; \int_{\Gamma^*} e^{i\langle p,x\rangle}\psi(p)\,d\mu^*(p). \tag{11}$$

Here, μ^* is the (normalized) Haar measure on the torus, $d\mu^*(p) = \frac{d^dp}{(2\pi)^d}$. Put differently, $h(\mathfrak{e}) = \mathcal{F}\mathfrak{e}\mathcal{F}^*$ is the Fourier multiplier corresponding to $\mathfrak{e}$. We assume w.l.o.g. that the minimum of $\mathfrak{e}$ is 0, so

$$\mathfrak{e}(\Gamma^*) \;=\; [0, \mathfrak{e}_{\max}(\mathfrak{e})], \tag{12}$$

and we refer to a Morse function $\mathfrak{e} \in C^2(\Gamma^*, \mathbb{R})$ obeying (12) as an *admissible dispersion relation*. Note that $-\Delta_\Gamma = h(\mathfrak{e}_{\mathrm{Lapl}})$, with

$$\mathfrak{e}_{\mathrm{Lapl}}(p) \;:=\; \sum_{i=1}^{d}\big(1-\cos(p_i)\big) \tag{13}$$

being admissible. We require that V decays at infinity,

$$V \;\in\; \ell_0^\infty(\Gamma, \mathbb{R}_0^+) \;:=\; \Big\{V:\Gamma\to\mathbb{R}_0^+ \;|\; \lim_{|x|\to\infty} V(x) = 0\Big\}, \tag{14}$$

or sometimes even that V has bounded support. Note that $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ is compact as a multiplication operator on $\ell^2(\Gamma)$ and by a(nother) theorem of Weyl,

$$\sigma_{\mathrm{ess}}[H(\mathfrak{e}, V)] \;=\; \sigma_{\mathrm{ess}}[H(\mathfrak{e}, 0)] \;=\; [0, \mathfrak{e}_{\max}], \tag{15}$$

where $\mathfrak{e}_{\max} \equiv \mathfrak{e}_{\max}(\mathfrak{e})$. From the positivity of V and the min-max principle we further obtain that all isolated eigenvalues of finite multiplicity lie below 0,

$$\sigma_{\mathrm{disc}}[H(\mathfrak{e}, V)] \;\subseteq\; \mathbb{R}^- \;:=\; (-\infty, 0). \tag{16}$$

We note in passing that – different to Schrödinger operators on $\mathbb{R}^d$ – discrete Schrödinger operators possibly have positive eigenvalues when changing the sign of the potential. Counting the number of positive eigenvalues, however, can be traced back to the one treated here by replacing $\mathfrak{e}(p)$ by $\mathfrak{e}_{\max} - \mathfrak{e}(p)$.

Our goal in this paper is to give – in all dimensions – both asymptotic and non-asymptotic bounds on the number

$$N[\mathfrak{e}, V] \;:=\; \mathrm{Tr}\left\{\mathbf{1}[H(\mathfrak{e}, V) < 0]\right\} \tag{17}$$

of negative eigenvalues of $H(\mathfrak{e}, V)$ in terms of the corresponding semi-classical quantity

$$N_{sc}[\mathfrak{e}, V] \;\;:=\;\; \sum_{x\in\Gamma} \int_{\Gamma^*} \mathbf{1}[\mathfrak{e}(p) - V(x) < 0] d\mu^*(p) \tag{18}$$

$$= \;\; \int_{\Gamma^*} \mathcal{L}_V[\mathfrak{e}(p)] d\mu^*(p), \tag{19}$$

where the sizes $\mathcal{L}_V[\alpha]$ of the level sets of V are defined by

$$\mathcal{L}_V[\alpha] \;:=\; \#\big\{x \in \Gamma \;\big|\; V(x) \geq \alpha\big\} \tag{20}$$

for $\alpha > 0$. Note that $\mathcal{L}_V[\alpha]$ is independent of the localization properties of V. This lets us introduce the notion of rearrangements of V. Given $V, \widetilde{V} \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$, we say that

$$\widetilde{V} \text{ is a } \textit{rearrangement of } V \quad :\Longleftrightarrow \quad \forall_{\alpha>0} :\; \mathcal{L}_{\widetilde{V}}[\alpha] = \mathcal{L}_V[\alpha]. \tag{21}$$

In other words, $\widetilde{V}|_{\mathrm{supp}\, \widetilde{V}} = V \circ J$ for some bijection $J : \mathrm{supp}\, \widetilde{V} \to \mathrm{supp}\, V$. A key condition for many of our results is the following:

Hypothesis (H-1). *The admissible dispersion $\mathfrak{e}$ is dominated by a positivity preserving admissible dispersion $\tilde{\mathfrak{e}}$, i.e.,*

$$\forall_{\, t\geq 0} : \quad \big|\big\langle \delta_x \big| \exp(-t\, h(\mathfrak{e}))\, \delta_y \big\rangle\big| \;\leq\; \big\langle \delta_x \big| \exp(-t\, h(\tilde{\mathfrak{e}}))\, \delta_y \big\rangle, \tag{22}$$

where $\delta_x(y) := \delta_{x,y}$ is an element of the canonical ONB $\{\delta_x | x \in \Gamma\} \subseteq \ell^2(\Gamma)$.

1.1 Non-Asymptotic Semi-classical Bounds

We first formulate our non-asymptotic results which correspond to the Rozenblum-Lieb-Cwikel bound (4) in the continuum case.

Theorem 1.1 (Non-asymptotic upper bound with (H-1)). *Let $d \geq 3$ and $\mathfrak{e}$ an admissible dispersion fulfilling (H-1). Then there exists a constant $C_{1.1}(d, \mathfrak{e}) \in [1, \infty)$ such that*

$$N[\mathfrak{e}, V] \;\leq\; C_{1.1}(d, \mathfrak{e})\, N_{sc}[\mathfrak{e}, V] \;<\; \infty \tag{23}$$

for all $V \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$.

If $\mathfrak{e}$ does not satisfy (H-1), the following weighted version of the non-asymptotic bound on $N[\mathfrak{e}, V]$ still holds:

Theorem 1.2 (Non-asymptotic upper bound without (H-1))**.** *Let $d \geq 1$ and $\mathfrak{e}$ be any admissible dispersion. If $d = 1, 2$ assume, moreover, that $\mathfrak{e} \in C^3(\Gamma^*, \mathbb{R}_0^+)$. Then there is a constant $C_{1.2}(d, \mathfrak{e}) < \infty$ such that, for any potential $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$,*

$$N[\mathfrak{e}, V] \; \leq \; C_{1.2}(d, \mathfrak{e})\Big(1 + N_{sc}\big[\mathfrak{e}, \, |x|^{\alpha_d} V\big]\Big). \tag{24}$$

Here, $\alpha_{d=1,2} := d + 5$, $\alpha_{d \geq 3} := d - 1 - (2/d)$.

Our results show that the right quantity to compare the number of eigenvalues to is the phase space volume $N_{sc}[\mathfrak{e}, V]$ of the set $\{(p, x) \mid \mathfrak{e}(p) - V(x) < 0\}$ and not (the $\frac{d}{2}^{th}$ power of) the $\ell^{d/2}$–norm of V. In the case of Schrödinger operators on $\mathbb{R}^d$, these quantities agree up to a multiplicative constant, see (5). While it is possible to bound $N_{sc}[\mathfrak{e}, V]$ and hence also $N[\mathfrak{e}, V]$ by a multiple of $|V|_{d/2}^{d/2} = \sum_x V^{d/2}(x)$, this bound grossly overestimates the number of eigenvalues in the limit of large couplings. For example, if $\Lambda \subset \Gamma$ is a finite subset then

$$N_{sc}[\mathfrak{e}, \lambda \, \mathbf{1}_\Lambda] \; = \; |\Lambda| \; \ll \; \lambda^{d/2} \, |\Lambda| \; = \; |\lambda \, \mathbf{1}_\Lambda|_{d/2}^{d/2} \tag{25}$$

for sufficiently large $\lambda > 0$.

In Sect. 3.2 we prove the optimality of Theorem 1.1 with respect to the class $\ell^{d/2}(\Gamma, \mathbb{R}_0^+) \ni V$.

Theorem 1.3 (Optimality of $\ell^{d/2}(\Gamma, \mathbb{R}_0^+) \ni V$ in Thm. 1.1)**.** *Let $d \geq 1$ and $\mathfrak{e}$ be an admissible dispersion for which $|h(\mathfrak{e})_{0,x}| \leq \mathrm{const} \, \langle x \rangle^{-2(d+1)}$ for some $\mathrm{const} < \infty$. Then, for any $\varepsilon > 0$, there exists a potential $V_\varepsilon \in \ell^{(d/2)+\varepsilon}(\Gamma, \mathbb{R}_0^+) \setminus \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$ such that $N[\mathfrak{e}, V_\varepsilon] = N_{sc}[\mathfrak{e}, V_\varepsilon] = \infty$.*

Here, $h(\mathfrak{e})_{x,y} := \big|\big\langle \delta_x \big| \, h(\mathfrak{e}) \delta_y \big\rangle\big|$ are the matrix elements w.t.r. to the canonical basis of $\ell^2(\Gamma)$ of the hopping matrix $h(\mathfrak{e})$ of the dispersion relation $\mathfrak{e}$, and $\langle x \rangle := 1 + |x|$.

This does not, however, imply that $N[\mathfrak{e}, V] = \infty$ whenever $N_{sc}[\mathfrak{e}, V] = \infty$. For instance, if $V(x) = \langle x \rangle^{-2} (\log \langle x \rangle)^{-\eta}$ for some $\eta \in (0, 2/d)$ then $N[\mathfrak{e}_{\mathrm{Lapl}}, V] < \infty$ but $N_{sc}[\mathfrak{e}_{\mathrm{Lapl}}, V] = \infty$. See the example in [8, Section 5.2].

We complement the non-asymptotic upper bounds by corresponding lower bounds:

Theorem 1.4 (Non-Asymptotic Lower Bound without (H-1))**.** *Let $d \geq 1$ and $\mathfrak{e}$ be an admissible dispersion. Then, for any potential $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ and all $c > \mathfrak{e}_{\max}$,*

$$N[\mathfrak{e}, V] \; \geq \; \mathcal{L}_V[c] \; = \; \#\big\{x \in \Gamma \mid V(x) \geq c\big\}. \tag{26}$$

From Theorems 1.1 and 1.4 emerges the interesting question, whether $N_{sc}[\mathfrak{e}, V]$ or $\mathcal{L}_V[\mathfrak{e}_{\max}]$ (or both) are saturated in certain limits. It turns out that $\mathcal{L}_V[\eta(\mathfrak{e})]$ correctlly describes $N[\mathfrak{e}, V]$ for sparse potentials V, see, for instance, Lemma 3.8 and proof of Corollary 4.6. Here, $0 \leq \eta(\mathfrak{e}) < \mathfrak{e}_{\max}$ is defined by

$$\frac{1}{\eta(\mathfrak{e})} \;:=\; \int_{\Gamma^*} \frac{d\mu^*(p)}{\mathfrak{e}(p)} \tag{27}$$

for $d \geq 3$, and $\eta(\mathfrak{e}) := 0$ for $d = 1, 2$. Observe that, as $\eta(\mathfrak{e}) \leq \mathfrak{e}_{\max}$, $\mathcal{L}_V[\mathfrak{e}_{\max}] \leq \mathcal{L}_V[\eta(\mathfrak{e})]$. Since for any $\delta > 0$ there is a potential $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ for which $\mathcal{L}_V[\eta(\mathfrak{e})]/\mathcal{L}_V[\mathfrak{e}_{\max}] < 1 + \delta$, the following theorem implies the optimality – w.r.t. rearrangements – of the lower bound in Theorem 1.4.

Theorem 1.5 (Optimality of Thm. 1.4 under rearrangements)**.** *Let $d \geq 3$, $\mathfrak{e}$ be an admissible dispersion. Given $\varepsilon \in (0, 1)$ and a potential $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$, there exists a rearrangement $\widetilde{V} \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ of V such that*

$$N[\mathfrak{e}, \widetilde{V}] \;\leq\; \mathcal{L}_{\widetilde{V}}[(1-\varepsilon)\eta(\mathfrak{e})] \;=\; \#\big\{x \in \Gamma \,\big|\, V(x) \geq (1-\varepsilon)\eta(\mathfrak{e})\big\}. \tag{28}$$

The semi-classical number of bound states $N_{sc}[\mathfrak{e}, \lambda V]$ is not in general – even up to prefactors – a lower bound on $N[\mathfrak{e}, \lambda V]$. This is illustrated by the following theorem.

Theorem 1.6. *Let $d \geq 3$ and $\mathfrak{e}$ be an admissible dispersion. Then there exists a potential $V \notin \bigcup_{p \geq 1} \ell^p(\Gamma)$ with $N[\mathfrak{e}, V] = 0$.*

1.2 (Weyl-)Asymptotic Semi-classical Bounds

The Weyl asymptotics (2) states that, for all fixed potentials $V \in L^{d/2}(\mathbb{R}^d)$,

$$\lim_{\lambda \to \infty} \frac{N^{cont}[\lambda V]}{N_{sc}^{cont}[\lambda V]} \;=\; 1, \tag{29}$$

and that $N_{sc}^{cont}[\lambda V] = \lambda^{d/2} N_{sc}^{cont}[V]$. For discrete Schrödinger operators, only weaker statements hold true, as is illustrated by the following lemma.

Lemma 1.7. *Assume $d \geq 3$, (H-1) and $V \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$. Then*

$$\lim_{\lambda \to \infty} \big\{\lambda^{-d/2} N[\mathfrak{e}, \lambda V]\big\} \;=\; \lim_{\lambda \to \infty} \big\{\lambda^{-d/2} N_{sc}[\mathfrak{e}, \lambda V]\big\} \;=\; 0. \tag{30}$$

For a precise formulation of our asymptotic bounds, we introduce the numbers

$$g_+(V) \;:=\; \sup_{r>0} \limsup_{\ell \to \infty} \frac{2}{d\,r}\Big(\ln \mathcal{L}_V\big[e^{-\ell-r}\big] - \ln \mathcal{L}_V\big[e^{-\ell}\big]\Big), \tag{31}$$

$$g_-(V) \;:=\; \inf_{r>0} \liminf_{\ell \to \infty} \frac{2}{d\,r}\Big(\ln \mathcal{L}_V\big[e^{-\ell-r}\big] - \ln \mathcal{L}_V\big[e^{-\ell}\big]\Big), \tag{32}$$

built from the level sets of V. While the significance of $g_-(V)$ is made clear in Section 4.1, $g_+(V)$ directly enters the following theorem.

Theorem 1.8 (Asymptotic bounds with (H-1))**.** *Assume* $d \geq 3$*, (H-1) and* $V \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$*. Then there are constants* $0 < C_{1.8s}(d, \mathfrak{e}) \leq C_{1.8g}(d, \mathfrak{e}) < \infty$ *such that*

$$\begin{aligned} \big(1 - g_+(V)\big)\, C_{1.8s}(d, \mathfrak{e}) \;&\leq\; \liminf_{\lambda \to \infty} \left\{ \frac{N[\mathfrak{e}, \lambda V]}{N_{sc}[\mathfrak{e}, \lambda V]} \right\} \\ &\leq\; \limsup_{\lambda \to \infty} \left\{ \frac{N[\mathfrak{e}, \lambda V]}{N_{sc}[\mathfrak{e}, \lambda V]} \right\} \;\leq\; C_{1.8g}(d, \mathfrak{e}). \end{aligned} \tag{33}$$

A somewhat weaker form of Theorem 1.8 still holds in case that $\mathfrak{e}$ does not fulfill (H-1).

Theorem 1.9 (Asymptotic Bounds without (H-1))**.** *Assume that* $d \geq 1$ *and* $V \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$*. Then there are constants* $0 < C_{1.9s}(d, \mathfrak{e}) \leq C_{1.9g}(d, \mathfrak{e}) < \infty$ *such that*

$$\big(1 - g_+(V)\big)\, C_{1.9s}(d, \mathfrak{e}) \;\leq\; \liminf_{\lambda \to \infty} \left\{ \frac{N[\mathfrak{e}, \lambda V]}{N_{sc}[\mathfrak{e}, \lambda V]} \right\}, \tag{34}$$

$$\limsup_{\lambda \to \infty} \left\{ \frac{N[\mathfrak{e}, \lambda V]}{1 + N_{sc}[\mathfrak{e}, \lambda |x|^{\alpha_d} V]} \right\} \;\leq\; C_{1.9g}(d, \mathfrak{e}).$$

Here, $\alpha_{d=1,2} = d + 5$ *and* $\alpha_{d \geq 3} := d - 1 - (2/d)$.

We remark that if V is rapidly decaying then typically $g_+(V) = 0$. For instance, if

$$c_1\, e^{-\alpha_1 |x|} \;\leq\; V(x) \;\leq\; c_2\, e^{-\alpha_2 |x|} \tag{35}$$

for some constants $c_1, \alpha_1, \alpha_2 > 0$, $c_2 < \infty$, and all $x \in \Gamma$, then $g_+(V) = 0$. Moreover, by the bounds proven here, in this case the usual Weyl semi-classical asymptotics hold true in all dimensions $d \geq 1$ and for all admissible dispersion relations (not necessarily satisfying (H-1)), in the sense that

$$\lim_{\lambda \to \infty} \left\{ \frac{N[\mathfrak{e}, \lambda V]}{N_{sc}[\mathfrak{e}, \lambda V]} \right\} \;=\; \lim_{\lambda \to \infty} \left\{ \frac{N[\mathfrak{e}, \lambda V]}{1 + N_{sc}[\mathfrak{e}, \lambda |x|^{\alpha_d} V]} \right\} \;=\; 1. \tag{36}$$

We further remark that if V behaves at infinity like an inverse power of $|x|$, i.e., if the limit

$$\lim_{|x| \to \infty} \left\{ \frac{-\log[V(x)]}{\log |x|} \right\} \;=\; \beta \tag{37}$$

exists, then $g_+(V) = g_-(V) = 2\beta/d$. In particular, in this case $g_+(V) < 1$ implies $V \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$, and $g_-(V) > 1$ implies $V \notin \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$.

In contrast to the continuum case, the boundedness of V in $\ell^{d/2}$ alone does not suffice to ensure the semi-classical asymptotic behavior of $N[\mathfrak{e}, \lambda V]$, but details of the behavior of V at infinity enter, too, as is illustrated by the following theorem.

Theorem 1.10. *Let $d \geq 3$ and $\mathfrak{e}$ be an admissible dispersion satisfying (H-1). Then there exists a potential V with $N_{sc}[\mathfrak{e}, \lambda V] < \infty$ for all $\lambda > 0$ for which*

$$\liminf_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda V]}{N[\mathfrak{e}, \lambda V]} < \infty \text{ and } \limsup_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda V]}{N[\mathfrak{e}, \lambda V]} = \infty.$$

In fact, potentials on the lattice can be so peculiar that their eigenvalue asymptotics assumes any prescribed behavior in the sense of the following theorem.

Theorem 1.11. *Let $d \geq 3$ and $\mathfrak{e}$ be any admissible dispersion. Let further $F : [1, \infty) \to \mathbb{N}$ be an arbitrary monotonically increasing, positively integer-valued, right-continuous function. Then, for any $\varepsilon \in (0, 1/2)$, there exists a potential $V_{F,\varepsilon} \in \ell_0^\infty(\Gamma)$ such that*

$$\forall_{\lambda\geq 2} : \qquad F\big((1-\varepsilon)\lambda\big) \;\leq\; N\big[\mathfrak{e}, \lambda V_{F,\varepsilon}\big] \;\leq\; F\big((1+\varepsilon)\lambda\big). \tag{38}$$

We give an overview on where to find the proofs of the theorems above:
• Theorem 1.1 is proved as Theorem 3.4 in Section 3.1.
• Theorem 1.2 is proved as Theorem 3.5 in Section 3.1 in the case $d \geq 3$, and as Corollary 5.5 in the case $d = 1, 2$.
• Theorem 1.3 is proved as Theorem 3.6 in Section 3.2 .
• Theorem 1.4 is repeated as Lemma 3.7 in Section 3.2 and proven there.
• Theorem 1.5 is proved as Corollary 4.6 in Section 4.2.
• Lemma 1.7 is proven as Lemma 4.2 in Section 4.2.
• Theorems 1.8 and 1.9 are proven at the end Section 4.1.
• Theorems 1.6, 1.10 and 1.11 are repeated as Corollary 4.5, Theorem 4.8 and Theorem 4.7, respectively, in Section 4.2 and proven there.

1.3 Validity of Hypothesis (H-1)

The main example of a positivity preserving dispersion $\mathfrak{e}$ is given in the following lemma:

Lemma 1.12 (Markovian hoppings satisfy (H-1))**.** *Let $\mathfrak{e}$ be any admissible dispersion. Assume that $h(\mathfrak{e})_{x,y} = h(\mathfrak{e})_{0,x-y} \leq 0$, for all $x, y \in \Gamma$, $x \neq y$. Then $\sum_{x\in\Gamma} |h(\mathfrak{e})_{0,x}| < \infty$, and, for all $t \geq 0$, $e^{-t\,h(\mathfrak{e})}$ is positivity preserving. Moreover, $\mathfrak{e}$ satisfies (H-1).*

This result is standard. Its proof is given in Appendix A.1 for completeness. An admissible dispersion $\mathfrak{e}$ is called *Markovian* if it fulfills the assumptions of Lemma 1.12. In particular, $\mathfrak{e}_{\text{Lapl}}$ is Markovian and satisfies (H-1). There are other physically relevant dispersions fulfilling Hypothesis (H-1) which are, however, not of the type described by Lemma 1.12:

Lemma 1.13 (Non-Markovian dispersions satisfying (H-1))**.** *Let $\mathfrak{e}$ be any admissible dispersion. For each $K \in \Gamma^*$ define the non-negative function $\mathfrak{e}^{(K)} : \Gamma \to \mathbb{R}$ by*

$$\mathfrak{e}^{(K)}(p) \;=\; \mathfrak{e}\,(p) + \mathfrak{e}\,(K-p) - \min_{p' \in \Gamma^*} \left\{ \mathfrak{e}\,(p') + \mathfrak{e}\,(K-p') \right\}. \tag{39}$$

Then, for $\mathfrak{e} \equiv \mathfrak{e}_{\mathrm{Lapl}}$ and all $K \in (-\pi,\pi)^d \subset \Gamma^*$, $\mathfrak{e}^{(K)} \equiv \mathfrak{e}^{(K)}_{\mathrm{Lapl}}$ is an admissible dispersion satisfying (H-1).

Proof: Let $K \in (-\pi,\pi)^d$ and consider the admissible dispersion

$$\tilde{\mathfrak{e}}^{(K)}_{\mathrm{Lapl}}(p) \;=\; \sum_{j=1}^{d} 2\cos(K_j/2)\big(1-\cos(p_j)\big). \tag{40}$$

It follows from straightforward computations using the Trotter-formula that the positivity preserving admissible dispersion $\tilde{\mathfrak{e}}^{(K)}_{\mathrm{Lapl}} \in C^\infty(\Gamma^*, \mathbb{R})$ dominates $\mathfrak{e}^{(K)}_{\mathrm{Lapl}}$. $\square$

Dispersions of the form (39) come about in the analysis of systems of two particles on the lattice Γ both having the same dispersion $\mathfrak{e}$ and interacting by a (translation invariant) potential $V(x_1 - x_2)$. Indeed, the two-particle Hamiltonian is unitarily equivalent to the direct integral

$$\int_{\Gamma^*}^{\oplus} H(\mathfrak{e}^{(K)}, V)\, d\mu^*(K). \tag{41}$$

The function $\mathfrak{e}^{(K)}$ is viewed as the (effective) dispersion of a pair of particles travelling with total quasi-momentum $K \in \Gamma^*$.

Observe that in Lemma 1.13 above the fact that $\tilde{\mathfrak{e}}^{(K)}_{\mathrm{Lapl}}$ dominates $\mathfrak{e}^{(K)}_{\mathrm{Lapl}}$ holds true due to special properties of trigonometric functions, which are not fulfilled by arbitrary dispersions $\mathfrak{e}$ – even if $h(\mathfrak{e})$ is Markovian. Thus, it is not obvious that if a (one-particle) dispersion $\mathfrak{e}$ satisfies (H-1) then so do the (effective two-particle) dispersions $\mathfrak{e}^{(K)}$.

2 Birman-Schwinger Principle and the Rozenblum-Lieb-Cwikel Bound

Operators of the form (8) are known to have eigenvalues below their essential spectra (i.e. below 0). This fact follows from the following lemma, for whose formulation we recall that $\eta(\mathfrak{e}) := 0$, for $d = 1, 2$, and $0 < \eta(\mathfrak{e}) < \mathfrak{e}_{\max}$ for $d \geq 3$, where

$$\frac{1}{\eta(\mathfrak{e})} \;:=\; \int_{\Gamma^*} \frac{d\mu^*(p)}{\mathfrak{e}(p)}. \tag{42}$$

The integral above is finite because all critical points of $\mathfrak{e}$ are non-degenerate.

2.1 The Birman-Schwinger Principle

Lemma 2.1. *Let $d \geq 1$ and $\mathfrak{e}$ be an admissible dispersion relation. Then $N[\mathfrak{e}, V] \geq 1$ if and only if*

$$\sup_{\rho > 0} \sup_{\varphi \in \ell^2(\Gamma),\, |\varphi|_2 = 1} \left\langle \varphi \,\middle|\, V^{1/2} \left[\rho + h(\mathfrak{e})\right]^{-1} V^{1/2} \varphi \right\rangle \; > \; 1. \tag{43}$$

In particular, if $d \geq 3$ and $\max_{x \in \Gamma} V(x) > \eta(\mathfrak{e})$ or if $d = 1, 2$ and $V \neq 0$ then $N[\mathfrak{e}, V] \geq 1$.

This result follows immediately from the min-max principle for compact operators and the Birman-Schwinger principle given below:

Lemma 2.2 (Birman-Schwinger principle). *Let $d \geq 1$, $\mathfrak{e}$ be an admissible dispersion relation and $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$. For any $\rho > 0$, define the compact, self-adjoint, non-negative* Birman-Schwinger operator *by*

$$B(\rho) \; = \; B(\rho, \mathfrak{e}, V) \; := \; V^{1/2} \left[\rho + h(\mathfrak{e})\right]^{-1} V^{1/2}. \tag{44}$$

Then the following assertions (i)–(iv) hold true.

- *(i) If $\varphi \in \ell^2(\Gamma)$ solves $H(\mathfrak{e}, V)\varphi = -\rho\varphi$ then $\psi := V^{1/2}\varphi \in \ell^2(\Gamma)$ solves $\psi = B(\rho)\psi$.*
- *(ii) If $\psi \in \ell^2(\Gamma)$ solves $\psi = B(\rho)\psi$ then $\varphi = [\rho + h(\mathfrak{e})]^{-1} V^{1/2}\psi \in \ell^2(\Gamma)$ solves $H(\mathfrak{e}, V)\varphi = -\rho\varphi$.*
- *(iii) $-\rho$ is an eigenvalue of $H(\mathfrak{e}, V)$ of multiplicity M if and only if 1 is an eigenvalue of $B(\rho)$ of multiplicity M.*
- *(iv) Counting multiplicities, the number of eigenvalues of $H(\mathfrak{e}, V)$ less or equal than $-\rho$ equals the number of eigenvalues of $B(\rho)$ greater or equal than 1.*

Proof: We recall that, due to the compactness of V, the Birman-Schwinger operator $B(\rho)$ is compact and has only discrete spectrum above 0. Similarly, the spectrum of $H(\mathfrak{e}, V)$ below 0 is discrete because $-V = H(\mathfrak{e}, V) - H(\mathfrak{e}, 0)$ is compact.

Suppose that $-\rho < 0$ is an eigenvalue of $H(\mathfrak{e}, V)$ of multiplicity $M \in \mathbb{N}$ and let $\{\varphi_1, \ldots, \varphi_M\} \subseteq \ell^2(\Gamma)$ be an ONB of the corresponding eigenspace. Set

$$\psi_1 := V^{1/2}\varphi_1, \ldots, \psi_M := V^{1/2}\varphi_M. \tag{45}$$

Then $\psi_m \in \ell^2(\Gamma)$ since $V \in \ell^\infty(\Gamma)$. Moreover,

$$\varphi_m \; = \; [\rho + h(\mathfrak{e})]^{-1} V \varphi_m \; = \; [\rho + h(\mathfrak{e})]^{-1} V^{1/2} \psi_m, \tag{46}$$

and the boundedness of $[\rho + h(\mathfrak{e})]^{-1} V^{1/2}$ implies that $\{\psi_1, \ldots, \psi_M\} \subseteq \ell^2(\Gamma)$ is linearly independent. Clearly, (45) and (46) also yield

$$B(\rho)\psi_m \;=\; V^{1/2}[\rho + h(\mathfrak{e})]^{-1} V^{1/2}\psi_m \;=\; \psi_m, \tag{47}$$

and hence the eigenspace of $B(\rho)$ corresponding to the eigenvalue 1 has at least dimension M.

Conversely, if $\{\psi_1, \ldots, \psi_L\} \subseteq \ell^2(\Gamma)$ is an ONB of the eigenspace of $B(\rho)$ corresponding to the eigenvalue 1 then we set

$$\varphi_1 := [\rho + h(\mathfrak{e})]^{-1} V^{1/2}\psi_1, \ldots, \varphi_L := [\rho + h(\mathfrak{e})]^{-1} V^{1/2}\psi_L. \tag{48}$$

Since $[\rho + h(\mathfrak{e})]^{-1} V^{1/2}$ is bounded, $\varphi_\ell \in \ell^2(\Gamma)$. Moreover,

$$\psi_\ell \;=\; B(\rho)\psi_\ell \;=\; V^{1/2}\varphi_\ell, \tag{49}$$

and the boundedness of $V^{1/2}$ implies that $\{\varphi_1, \ldots, \varphi_L\} \subseteq \ell^2(\Gamma)$ is linearly independent. Clearly, (48) and (49) also yield

$$H(\mathfrak{e}, V)\varphi_\ell \;=\; -\rho\varphi_\ell, \tag{50}$$

and hence the eigenspace of $H(\mathfrak{e}, V)$ corresponding to the eigenvalue $-\rho$ has at least dimension L.

These arguments prove (i) and (ii) and, furthermore, $M = L$ and thus (iii), i.e.,

$$\forall_{\rho>0} : \quad \dim\ker\big[H(\mathfrak{e}, V) + \rho\big] \;=\; \dim\ker\big[B(\rho) - 1\big]. \tag{51}$$

Observe that for all ρ',ρ with $\rho' \geq \rho > 0$: $B(\rho') \leq B(\rho)$. As the map $\rho \mapsto B(\rho)$ is norm continuous on $\mathbb{R}^+$ and $\lim_{\rho\to\infty} B(\rho) = 0$, by the min-max principle, if $z_k > 1$ is the k–th eigenvalue of $B(\rho)$ counting from above with multiplicities, then there is a $\rho_k > \rho$ such that 1 is the k–th eigenvalue of $B(\rho_k)$ (counting from above with multiplicities). Clearly, $\rho_{k'} \leq \rho_k$, whenever $k' \geq k$. By (iii), this implies that $H(\mathfrak{e}, V)$ has at least as many eigenvalues less or equal $-\rho$ as $B(\rho)$ has eigenvalues greater or equal 1. By similar arguments, $B(\rho)$ has at least as many eigenvalues greater or equal 1 as $H(\mathfrak{e}, V)$ has eigenvalues less or equal $-\rho$. □

Corollary 2.3. *Let $d \geq 1$ and $\mathfrak{e}$ be an admissible dispersion relation, $V \geq 0$ a potential decaying at infinity, and $\rho > 0$. Then, for all $m \in \mathbb{N}$, the compact operator*

$$K_{m,\rho} \;:=\; \sum_{k=0}^{m} (-1)^k \binom{m}{k} V^{1/2}\,[kV + \rho + h(\mathfrak{e})]^{-1}\,V^{1/2}. \tag{52}$$

is nonnegative, and the number $N_\rho[\mathfrak{e}, V]$ *of eigenvalues of* $H(\mathfrak{e}, V)$ *below* $-\rho$ *(counting multiplicities) is bounded by*

$$N_\rho[\mathfrak{e}, V] \;\le\; \dim \mathrm{Ran}\left\{\mathbf{1}\left[K_{m,\rho} > (m+1)^{-1}\right]\right\} \;\le\; (m+1)\ \mathrm{Tr}\left\{K_{m,\rho}\right\}. \tag{53}$$

Proof: Our proof follows Lieb's argument, patterned after the proof of Theorem XIII.12 in [9]. We first introduce

$$F_m[u] \;:=\; \sum_{k=0}^{m} (-1)^k \binom{m}{k} (1+ku)^{-1} \tag{54}$$

for all $u \ge 0$ and observe that

$$K_{m,\rho} \;=\; F_m[B(\rho)]\, B(\rho). \tag{55}$$

To check Identity (55), we first replace V by $V_\delta := V \cdot \mathbf{1}[V \ge \delta]$ and observe that $\delta \le V_\delta \le |V|_\infty$ and $\delta(\rho + \mathfrak{e}_{\max})^{-1} \le B_\delta(\rho) \le |V|_\infty \rho^{-1}$, where V_δ and $B_\delta(\rho) := V_\delta^{1/2}[\rho + h(\mathfrak{e})]^{-1} V_\delta^{1/2}$ act as a bounded operators with bounded inverses on $\ell^2(\mathrm{supp}\, V_\delta)$. Thus we have

$$\begin{aligned} K_{m,\rho,\delta} \;&=\; \sum_{k=0}^{m} (-1)^k \binom{m}{k} V_\delta^{1/2} \left[kV_\delta + \rho + h(\mathfrak{e})\right]^{-1} V_\delta^{1/2} \\ &=\; \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left[k + V_\delta^{-1/2}\big(\rho + h(\mathfrak{e})\big) V_\delta^{-1/2}\right]^{-1} \\ &=\; \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left[1 + kB_\delta(\rho)\right]^{-1} B_\delta(\rho) \;=\; F_m[B_\delta(\rho)]\, B_\delta(\rho). \end{aligned} \tag{56}$$

A limiting argument establishes (55), as $\delta \to 0$. Note that

$$F_m[u] \;=\; \int_0^\infty e^{-t}\big(1 - e^{-u\,t}\big)^m\, dt, \tag{57}$$

as is easily checked by using

$$(1+ku)^{-1} \;=\; \int_0^\infty e^{-t}\, e^{-k\,u\,t}\, dt. \tag{58}$$

Due to (57), the function F_m is obviously nonnegative. Since $B(\rho) \ge 0$,

$$K_{m,\rho} \;=\; F_m[B(\rho)]\, B(\rho) \;\ge\; 0. \tag{59}$$

Moreover, F_m is strictly monotonically increasing and so is $u \mapsto uF_m[u]$. Therefore the number $N_\rho[\mathfrak{e}, V]$ of eigenvalues of $H(\mathfrak{e}, V)$ below $-\rho$ equals the number of eigenvalues of $B(\rho)$ above 1 which, in turn, equals the number of eigenvalues of $K_{m,\rho}$ above $1 \cdot F_m[1] = F_m[1] = (m+1)^{-1}$. This establishes $N_\rho[\mathfrak{e}, V] \le \dim \mathrm{Ran}\{\mathbf{1}[K_{m,\rho} > (m+1)^{-1}]\}$. The second inequality in (53) follows from the positivity of $K_{m,\rho}$ and hence $\mathbf{1}[K_{m,\rho} > (m+1)^{-1}] \le (m+1)K_{m,\rho}$, in the sense of quadratic forms. □

2.2 The Rozenblum-Lieb-Cwikel Bound

We derive two bounds on the number of negative eigenvalues from the Birman-Schwinger principle. The first is a simple upper bound stated in Lemma 2.4 below, the second is the Rozenblum-Lieb-Cwikel bound given in Theorem 2.6 which is preceded by a preparatory lemma entering its proof.

Lemma 2.4 (a priori upper bound on $N[e, V]$)**.** *Let $d \geq 3$, $\mathfrak{e}$ be an admissible dispersion relation and $V \in \ell^1(\Gamma, \mathbb{R}_0^+)$ a summable potential. Then*

$$N[\mathfrak{e}, V] \;\leq\; \frac{|V|_1}{\eta(\mathfrak{e})}, \tag{60}$$

where $|V|_1 := \sum_{x\in\Gamma} V(x)$ denotes the ℓ^1-norm of V.

Proof: Let $\rho > 0$. By Lemma 2.2, the number $N_\rho[\mathfrak{e}, V]$ of eigenvalues of $H(\mathfrak{e}, V)$ (counting multiplicities) lying below $-\rho$ is $N_\rho = \mathrm{Tr}\left\{\mathbf{1}[B(\rho) > 1]\right\}$. Observe that, for all $\rho > 0$,

$$\mathbf{1}[B(\rho) > 1] \;\leq\; B(\rho), \tag{61}$$

as a quadratic form. Thus, for all $\rho > 0$,

$$N_\rho \;\leq\; \sum_{x\in\Gamma} \langle \delta_x |\, B(\rho) \delta_x \rangle \;=\; \Big(\sum_{x\in\Gamma} V(x) \Big) \Big(\int_{\Gamma^*} \frac{d\mu^*(p)}{\rho + \mathfrak{e}(p)} \Big) \;=\; \frac{|V|_1}{\eta(\mathfrak{e})}, \tag{62}$$

where we recall that $\delta_x(y) := \delta_{x,y}$. □

The following bound on the time-decay of the semi-groups $e^{-t\,h(\mathfrak{e})}$ is an important ingredient of the proof of Theorem 2.6.

Lemma 2.5. *Let $d \geq 1$ and $\mathfrak{e}$ be an admissible dispersion. Then there is a constant $C_{2.5}(d, \mathfrak{e}) < \infty$ such that, for all $x \in \Gamma$ and all $t \geq 0$:*

$$\big\langle \delta_x \big| e^{-t\,h(\mathfrak{e})} \delta_x \big\rangle \;\leq\; C_{2.5}(d, \mathfrak{e}) \; \langle t \rangle^{-d/2} \,. \tag{63}$$

Proof: Let $\mathrm{Min}(\mathfrak{e}) = \{p^{(1)}, p^{(2)}, \ldots, p^{(N)}\} \subset \Gamma^*$ be the set of points at which the minimum of $\mathfrak{e}$ is attained, i.e. $\mathrm{Min}(\mathfrak{e}) = \mathfrak{e}^{-1}(0)$. Observe that, as $\mathfrak{e}$ is a Morse function and Γ^* is compact, this set is finite. Clearly,

$$\big\langle \delta_x \big| e^{-t\,h(\mathfrak{e})} \delta_x \big\rangle \;=\; \int_{\Gamma^*} e^{-t\,\mathfrak{e}(p)} dp. \tag{64}$$

Again by the fact that $\mathfrak{e}$ is a Morse function and Γ^* is compact, there is a constant $\alpha > 0$ such that

$$\forall_{p\in\Gamma^*} : \quad \mathfrak{e}(p) \;\geq\; \alpha \; \mathrm{dist}(M, p)^2. \tag{65}$$

Thus

$$\langle \delta_x | e^{-t\,h(\mathfrak{e})} \delta_x \rangle \;\le\; \int_{\Gamma^*} \exp(-t\,\alpha\,\mathrm{dist}(M,p)^2)\, d\mu^*(p) \tag{66}$$

$$\le\; C_{2.5}(d,\mathfrak{e})\,\langle t\rangle^{-d/2} \tag{67}$$

for some $C_{2.5}(d,\mathfrak{e}) < \infty$. □

The upper bound on $N[\mathfrak{e}, V]$ in Lemma 2.4 has the advantage of not imposing any condition on the dispersion $\mathfrak{e}$. If the dispersion $\mathfrak{e}$ satisfies (H-1), however, it follows from Theorem 2.6 below that the a priori upper bound (60) overestimates $N[\mathfrak{e}, V]$ in case of slowly decaying potentials and $d \ge 3$.

Theorem 2.6 (Rozenblum-Lieb-Cwikel bound)**.** *Let $d \ge 3$ and $\mathfrak{e}$ be any admissible dispersion satisfying (H-1). Then, for some constant $C_{2.6}(d,\mathfrak{e}) < \infty$,*

$$N[\mathfrak{e}, V] \;\le\; C_{2.6}(d,\mathfrak{e})\,|V|_{d/2}^{d/2}. \tag{68}$$

This kind of upper bound is known to be true in the continuous case. See, for instance, [9, Theorem XIII.12] or [10, Theorem 9.3]. It was proven by Rozenblum [4], Lieb [5], and Cwikel [6] by three different methods in the continuous case. More recently, it was shown by Rozenblum and Solomyak [11, 12] that the Rozenblum-Lieb-Cwikel bound is not only true for Schrödinger Operators of the form (1), but also for a very large class of operators including, in particular, lattice Schrödinger operators.

Proof: Our proof of Theorem 2.6 is an adaption of Lieb's original method [7] based on path integrals for lattice Hamiltonians $H(\mathfrak{e}, V)$ with $\mathfrak{e}$ satisfying (H-1). We first observe that, due to the monotonicity of $V \mapsto N_\rho[\mathfrak{e}, V]$ and of $\rho \mapsto N_\rho[\mathfrak{e}, V]$, we have that

$$N_{2\rho}[\mathfrak{e}, V] \;=\; N_\rho[\mathfrak{e}, V-\rho] \;\le\; N_\rho\big[\mathfrak{e}, (V-\rho)_+\big] \;\le\; N\big[\mathfrak{e}, V\,\mathbf{1}[V \ge \rho]\big], \tag{69}$$

where $N_\rho[\mathfrak{e}, V]$ denotes the number of eigenvalues of $H(\mathfrak{e}, V)$ below $-\rho$ (counting multiplicities) and $(f)_+ := \max\{f, 0\}$ denotes the positive part. Thus

$$N[\mathfrak{e}, V] \;=\; \lim_{\rho \searrow 0} N_\rho\big[\mathfrak{e}, V\,\mathbf{1}[V \ge \rho]\big] \tag{70}$$

and it suffices to prove

$$N_\rho[\mathfrak{e}, V] \;\le\; C_{2.6}(d,\mathfrak{e})\,|V|_{d/2}^{d/2}. \tag{71}$$

uniformly in $\rho > 0$ and for all $V : \Gamma \to \mathbb{R}_0^+$ with finite support, $\#\,\mathrm{supp}(V) < \infty$. To this end we observe that, thanks to Corollary 2.3, we have

$$N_\rho[\mathfrak{e}, V] \;\le\; (m+1)\;\mathrm{Tr}\left\{ \int_0^\infty L_{m,\rho}^{(t)}\,dt \right\}, \tag{72}$$

where

$$L_{m,\rho}^{(t)} \;:=\; e^{-\rho\, t} \sum_{k=0}^{m} (-1)^k \binom{m}{k} V^{1/2} \, \exp\big\{ -t[kV + h(\mathfrak{e})]\big\}\, V^{1/2}. \tag{73}$$

Note that the trace above is a finite sum, thus

$$N_\rho[\mathfrak{e}, V] \;\leq\; (m+1) \int_0^\infty \operatorname{Tr}\big\{ L_{m,\rho}^{(t)} \big\}\, dt. \tag{74}$$

By the Trotter product formula we have, for all $t \geq 0$,

$$\big\langle \delta_x \;\big|\; V\, e^{-t(kV + h(\mathfrak{e}))}\, \delta_x \big\rangle \;=\; \lim_{n\to\infty} \big\langle \delta_x \;\big|\; V \left[e^{-t\,k\,V/n}\, e^{-t\,h(\mathfrak{e})/n} \right]^n \delta_x \big\rangle. \tag{75}$$

Hence, the cyclicity of the trace and the finiteness of the support of V imply that,

$$\begin{aligned} &\operatorname{Tr}\big\{ V^{1/2}\, e^{-t[kV + h(\mathfrak{e})]}\, V^{1/2} \big\} \\ &= \sum_{x\in\Gamma} \lim_{n\to\infty} \big\langle \delta_x \;\big|\; V \left[e^{-t\,kV/n}\, e^{-t\,h(\mathfrak{e})/n} \right]^n \delta_x \big\rangle \\ &= \lim_{n\to\infty} \sum_{x\in\Gamma} \big\langle \delta_x \;\big|\; V \left[e^{-t\,k\,V/n}\, e^{-t\,h(\mathfrak{e})/n} \right]^n \delta_x \big\rangle \\ &= \lim_{n\to\infty} \operatorname{Tr}\big\{ V \left[e^{-t\,k\,V/n}\, e^{-t\,h(\mathfrak{e})/n} \right]^n \big\} \\ &= \lim_{n\to\infty} \operatorname{Tr}\big\{ \left[e^{-t\,k\,V/n}\, e^{-t\,h(\mathfrak{e})/n} \right]^{j_n} V \left[e^{-t\,k\,V/n}\, e^{-t\,h(\mathfrak{e})/n} \right]^{(n-j_n)} \big\}, \end{aligned} \tag{76}$$

for all $j_n \in \{0, 1, \ldots, n\}$.

Next, we fix $x \in \Gamma$ and $t > 0$ and define a measure $\mu_{\mathfrak{e},x,t}^{(n)}$ on $\Gamma^{n-1} \ni \underline{\omega} := (\omega_1, \ldots, \omega_{n-1})$, for each $n \in \mathbb{N}$, by

$$\mu_{\mathfrak{e},x,t}^{(n)}(\underline{\omega}) \;:=\; \big\langle \delta_x \big| e^{-t\,h(\mathfrak{e})} \delta_{\omega_1} \big\rangle \big\langle \delta_{\omega_1} \big| e^{-t\,h(\mathfrak{e})} \delta_{\omega_2} \big\rangle \cdots \big\langle \delta_{\omega_{n-1}} \big| e^{-t\,h(\mathfrak{e})} \delta_x \big\rangle. \tag{77}$$

Note that $\mu_{\mathfrak{e},x,t}^{(n)}$ is complex, in general, but dominated by the positive measure $\mu_{\tilde{\mathfrak{e}},x,t}^{(n)}$, due to (H-1), i.e.

$$|\mu_{\mathfrak{e},x,t}^{(n)}(\underline{\omega})| \;\leq\; \mu_{\tilde{\mathfrak{e}},x,t}^{(n)}(\underline{\omega}). \tag{78}$$

Moreover, using $h(\tilde{\mathfrak{e}}) \geq 0$ and $V \geq 0$, we have that

$$\begin{aligned} &\sum_{x\in\Gamma} \sum_{\underline{\omega}\in\Gamma^{n-1}} V(\omega_j)\, \exp\left(-\frac{t}{n} V(x) - \frac{kt}{n} \sum_{\ell=1}^{n-1} V(\omega_\ell) \right) \mu_{\tilde{\mathfrak{e}},x,t}^{(n)}(\underline{\omega}) \\ &\leq \; \sum_{x\in\Gamma} V(x) \;<\; \infty, \end{aligned} \tag{79}$$

and we obtain from (76) that

$$\begin{aligned} &\mathrm{Tr}\left\{V^{1/2}\, e^{-t[kV+h(\mathfrak{e})]}\, V^{1/2}\right\} \qquad (80)\\ &= \lim_{n\to\infty} \sum_{x\in\Gamma} \sum_{\underline{\omega}\in\Gamma^{n-1}} \\ &\quad \left(\frac{1}{n}V(x) + \frac{1}{n}\sum_{\ell=1}^{n-1} V(\omega_\ell)\right) \exp\left(- \frac{k\,t}{n}V(x) - \frac{k\,t}{n}\sum_{\ell=1}^{n-1} V(\omega_\ell)\right) \mu^{(n)}_{\mathfrak{e},x,t}(\underline{\omega}) \\ &= \lim_{n\to\infty} \sum_{x\in\Gamma} \frac{1}{t} \int \left(\int_0^t V(\omega(\tau))\, d\tau\right) \exp\left(- k \int_0^t V(\omega(\tau))\, d\tau\right) d\mu^{(n)}_{\mathfrak{e},x,t}(\omega), \end{aligned}$$

where, for given $t > 0$, $x \in \Gamma$, and $n \in \mathbb{N}$, we identify any element $\underline{\omega} \in \Gamma^{n-1}$ with the piecewise constant function $\omega : [0,t) \to \Gamma$ defined by

$$\omega(s) \;=\; x \cdot \mathbf{1}[s \in I_0 \cup I_n] \;+\; \sum_{\ell=1}^{n-1} \omega_\ell \cdot \mathbf{1}[s \in I_\ell], \qquad (81)$$

with $I_\ell := \big[(\ell-1/2)t/n\;,\;(\ell+1/2)t/n\big)$, $\ell = 1,2,\ldots n-1$, $I_0 := \big[0, t/2n\big)$, $I_n := \big[t - 1/2n, t\big)$, and we write

$$\sum_{\underline{\omega}\in\Gamma^{n-1}} f(\underline{\omega})\, \mu^{(n)}_{\mathfrak{e},x,t}(\underline{\omega}) \;=:\; \int f(\omega)\, d\mu^{(n)}_{\mathfrak{e},x,t}(\omega). \qquad (82)$$

Eqs. (80) and (73) yield

$$\mathrm{Tr}\left\{L^{(t)}_{m,\rho}\right\} \;=\; \lim_{n\to\infty} \sum_{x\in\Gamma} \int \frac{e^{-\rho t}}{t}\, G_m\left(\int_0^t t\, V(\omega(\tau))\, \frac{d\tau}{t}\right) d\mu^{(n)}_{\mathfrak{e},x,t}(\omega), \qquad (83)$$

where

$$G_m(u) \;:=\; \sum_{k=0}^{m} (-1)^k \binom{m}{k}\, u\, e^{-k\,u} \;=\; u\left(1 - e^{-u}\right)^m \;\geq\; 0 \qquad (84)$$

for all $u \geq 0$. Using this positivity and (78), we obtain an upper bound on $\mathrm{Tr}\{L^{(t)}_{m,\rho}\}$ by replacing $\mathfrak{e}$ by $\tilde{\mathfrak{e}}$ on the right-hand side of (83),

$$\mathrm{Tr}\left\{L^{(t)}_{m,\rho}\right\} \;\leq\; \lim_{n\to\infty} \sum_{x\in\Gamma} \int \frac{e^{-\rho t}}{t}\, G_m\left(\int_0^t t\, V(\omega(\tau))\, \frac{d\tau}{t}\right) d\mu^{(n)}_{\tilde{\mathfrak{e}},x,t}(\omega). \qquad (85)$$

Next we observe that $1 - e^{-u} \leq \min\{1, u\}$ which implies that

$$G_m(u) \;\leq\; \min\left\{u^{m+1}\,,\, u\right\} \;\leq\; \widetilde{G}_m(u) \;:=\; \min\left\{u^{m+1}\,,\, (m+1)u\right\} \qquad (86)$$

for all $u \geq 0$. As $\widetilde{G}_m : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is convex, we obtain from Jensen's inequality that

$$\mathrm{Tr}\left\{L_{m,\rho}^{(t)}\right\} \;\leq\; \lim_{n\to\infty} \sum_{x\in\Gamma} \int \frac{e^{-\rho t}}{t^2} \int_0^t \widetilde{G}_m\big(t\,V(\omega(\tau)]\big)\,d\tau\,d\mu_{\tilde{\mathfrak{e}},x,t}^{(n)}(\omega). \tag{87}$$

We observe that

$$\begin{aligned}
&\sum_{x\in\Gamma} \int \int_0^t \widetilde{G}_m\big(t\;V(\omega(\tau))\big)\,d\tau\,d\mu_{\tilde{e},x,t}^{(n)}(\omega) \\
&\quad= \int \int_0^t \sum_{x\in\Gamma} \widetilde{G}_m\big(t\,V(x+\omega(\tau))\big)\,d\tau\,d\mu_{\tilde{\mathfrak{e}},0,t}^{(n)}(\omega) \\
&\quad\leq C_{2.5}(d,\tilde{\mathfrak{e}})t^{1-(d/2)}\left(\sum_{x\in\Gamma} \widetilde{G}_m\big(t\,V(x)\big)\right),
\end{aligned} \tag{88}$$

using

$$\int d\mu_{\tilde{\mathfrak{e}},0,t}^{(n)}(\omega) \;=\; \big\langle \delta_0 \big|\, e^{-t\,h(\tilde{\mathfrak{e}})}\,\delta_0 \big\rangle \tag{89}$$

and Lemma 2.5 in the last inequality. Inserting this estimate into (87) and then the resulting inequality into (72), we arrive at

$$\begin{aligned}
N_\rho[\mathfrak{e},V] \;&\leq\; (m+1)\,C_{2.5}(d,\tilde{\mathfrak{e}}) \int_0^\infty \left(\sum_{x\in\Gamma} \widetilde{G}_m\big(t\,V(x)\big)\right) \frac{dt}{t^{1+(d/2)}} \\
&=\; (m+1)\,C_{2.5}(d,\tilde{\mathfrak{e}})\left(\int_0^\infty \frac{\widetilde{G}_m(t)\,dt}{t^{1+(d/2)}}\right) \sum_{x\in\Gamma} V(x)^{d/2} \\
&=\; (m+1)\,C_{2.5}(d,\tilde{\mathfrak{e}})\left(\frac{2}{2m+1-d} + \frac{2m+2}{d-2}\right) \sum_{x\in\Gamma} V(x)^{d/2},
\end{aligned} \tag{90}$$

using that $d \geq 3$ and assuming that we choose $2m > d-1$ which is guaranteed by $m := \frac{d+1}{2}$, for $d = 3, 5, 7, \ldots$, and $m := \frac{d}{2} + 1$, for $d = 4, 6, 8, \ldots$ □

3 Non-Asymptotic Semi-classical Bounds

3.1 Derivation of Non-Asymptotic Bounds

Now we are in position to use Theorem 2.6 to yield a semi-classical bound, i.e., a bound on $N[\mathfrak{e}, V]$ by multiples of $N_{sc}[\mathfrak{e}, V]$. The following lemma is a standard estimate on the size of the discrete spectrum of a sum of self-adjoint operators. Its proof is given for completeness.

Lemma 3.1. *Let $A = A^*, B = B^* \in \mathcal{B}[\mathcal{H}]$ be two bounded self-adjoint operators on a separable Hilbert space $\mathcal{H}$. Then*

$$N[A+B] \ \le \ N[A] + N[B]. \tag{91}$$

$N[Q] := \mathrm{Tr}\,\big\{\mathbf{1}[Q<0]\big\}$ denotes the number of negative eigenvalues of a bounded self-adjoint operator $Q \in \mathcal{B}[\mathcal{H}]$. $N[Q] := \infty$ if $\sigma_{\mathrm{ess}}(Q) \cap \mathbb{R}^- \neq \varnothing$.

Proof: We assume that $N[B], N[A] < \infty$, otherwise there is nothing to prove. As $N[A+B] \le N[A-B_-]$ and $N[B] = N[-B_-]$, it suffices to show that

$$N[A-B_-] \ \le \ N[A] + N[-B_-].$$

Here, $B_- := |B| 1[B<0]$. Let $M := N[B] = \dim\,\mathrm{Ran}(B_-)$ and assume that $A - B_-$ has at least $N[A] + M + 1$ eigenvalues (counting multiplicities) below 0. Then, by the min-max principle, there is a subspace $X \subset \mathcal{H}$, $\dim\, X = N[A] + M + 1$, for which

$$\sup_{\psi \in X,\, |\psi|_2 = 1} \langle \psi \,|\, (A-B_-)(\psi) \rangle < 0.$$

Hence

$$\sup_{\psi \in X \cap \ker(B_-),\, |\psi|_2 = 1} \langle \psi \,|\, (A-B_-)(\psi) \rangle = \sup_{\psi \in X \cap \ker(B_-),\, |\psi|_2 = 1} \langle \psi \,|\, A(\psi) \rangle < 0.$$

$\dim\, X \cap \ker(B_-) \ge \dim\, X - M = N[A] + 1$. Again by the min-max principle, this would then imply that $N[A] \ge N[A] + 1$. $\square$

A simple application of Lemma 3.1, with $A := H(\mathfrak{e}, V_1)$, $B = V_2$, and $A + B = H(\mathfrak{e}, V_1 + V_2)$, is the following corollary.

Corollary 3.2. *Let $d \ge 1$, $\mathfrak{e}$ be an admissible dispersion relation, and $V_1, V_2 \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ be two potentials. Then*

$$N[\mathfrak{e}, V_1 + V_2] \ \le \ N[\mathfrak{e}, V_1] + \#\,\mathrm{supp}\{V_2\}. \tag{92}$$

In order to compare the contributions $N[\mathfrak{e}, V_1]$ and $\#\,\mathrm{supp}\{V_2\}$ on the right-hand side of (92) to $N_{sc}[\mathfrak{e}, V]$, we use the following definition.

Definition 3.3. *Let $d \ge 1$. Given a dispersion relation $\mathfrak{e}$ and a potential $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$, we define:*

$$N_{sc}^{>}[\mathfrak{e}, V] \ := \ \#\big\{x \in \Gamma \,\big|\, V(x) \ge \mathfrak{e}_{\max}\big\}, \tag{93}$$

$$N_{sc}^{<}[\mathfrak{e}, V] \ := \ \sum_{x \in \Gamma} \mathbf{1}[V(x) < \mathfrak{e}_{\max}]\, V^{d/2}(x). \tag{94}$$

Observe that, because dispersion relations are Morse functions, there are constants $0 < c_1(\mathfrak{e}) \leq c_2(\mathfrak{e}) < \infty$ such that for any potential $V \geq 0$,

$$c_1(\mathfrak{e})\Big(N_{sc}^{>}[\mathfrak{e},V]+N_{sc}^{<}[\mathfrak{e},V]\Big) \;\leq\; N_{sc}[\mathfrak{e},V] \;\leq\; c_2(\mathfrak{e})\Big(N_{sc}^{>}[\mathfrak{e},V]+N_{sc}^{<}[\mathfrak{e},V]\Big). \quad (95)$$

Corollary 3.2, (95), and the Rozenblum-Lieb-Cwikel bound immediately imply Theorem 1.1:

Theorem 3.4 (Thm. 1.1)**.** *Let $d \geq 3$ and $\mathfrak{e}$ an admissible dispersion fulfilling (H-1). Then there exists a constant $C_{3.4}(d,\mathfrak{e}) \in [1,\infty)$ such that*

$$N[\mathfrak{e},V] \;\leq\; C_{3.4}(d,\mathfrak{e})\, N_{sc}[\mathfrak{e},V] \;<\; \infty \quad (96)$$

for all $V \in \ell^{d/2}(\Gamma,\mathbb{R}_0^+)$.

Proof: We apply Corollary 3.2 to $V = V_1+V_2$, with $V_1(x) := V(x)\mathbf{1}[V(x) <\mathfrak{e}_{\max}]$ and $V_2(x) := V(x)\mathbf{1}[V(x) \geq\mathfrak{e}_{\max}]$, and then Theorem 2.6 to $N[\mathfrak{e},V_1]$. This gives

$$N[\mathfrak{e},V] \;\leq\; N[\mathfrak{e},V_1] \;+\; \#\,\mathrm{supp}\,V_2 \quad (97)$$

$$\leq\; C_{2.6}\, N_{sc}^{<}[\mathfrak{e},V] \;+\; N_{sc}^{>}[\mathfrak{e},V] \;\leq\; \frac{C_{2.6}+1}{c_1(\mathfrak{e})}\, N_{sc}[\mathfrak{e},V]. \quad (98)$$

□

Similarly, Corollary 3.2, (95), and Lemma 2.4 imply Theorem 1.2 in case that $d \geq 3$:

Theorem 3.5 (Thm. 1.2 for $d \geq 3$)**.** *Let $d \geq 3$ and $\mathfrak{e}$ be any admissible dispersion. Then*

$$N[\mathfrak{e},V] \;\leq\; \#\big\{x\in\Gamma \,\big|\, V(x) \geq \mathfrak{e}_{\max}\big\} \;+\; \sum_{x\in\Gamma} \mathbf{1}[V(x) < \mathfrak{e}_{\max}]\; V(x). \quad (99)$$

Moreover, there is a constant $C_{3.5}(d,\mathfrak{e}) < \infty$ such that

$$N[\mathfrak{e},V] \;\leq\; C_{3.5}(d,\mathfrak{e})\,\Big(1 + N_{sc}\big[\mathfrak{e},\; |x-x_0|^{d-1-(2/d)}\, V\big]\Big) \quad (100)$$

for any potential $V \in \ell_0^\infty(\Gamma,\mathbb{R}_0^+)$ and any point $x_0 \in \Gamma$.

Proof: By shifting the origin, we may clearly assume that $x_0 = 0$. Just as in the proof of Theorem 3.4, we obtain (99) from Corollary 3.2 – this time, however, in connection with Lemma 2.4. We then use Hölder's Inequality to obtain

$$|V_1|_1 \;\leq\; \mathfrak{e}_{\max} \;+\; \Big(\sum_{x\in\Gamma\setminus\{0\}} |x|^{-\kappa q}\Big)^{1/q} \Big(\sum_{x\in\Gamma} |x|^{\kappa p} V^p(x)\Big)^{1/p}, \quad (101)$$

where $\frac{1}{q} + \frac{1}{p} = 1$ and $\kappa > 0$ is such that $\kappa q > d$. We choose $p := \frac{d}{2}$, $q := \frac{d}{d-2}$, and $\kappa := \frac{d+1}{q} = \frac{d^2-d-2}{d}$. Then $|x|^{-\kappa q} = |x|^{-d-1}$ is summable and hence

$$\begin{aligned} |V_1|_1 \;&\leq\; C\Big[1 \,+\, \Big(\sum_{x\in\Gamma} |x|^{\frac{1}{2}(d^2-d-2)}\, V^{d/2}(x)\Big)^{2/d}\Big] \\ &\leq\; C'\Big[1 \,+\, \sum_{x\in\Gamma} \Big(|x|^{\frac{1}{d}(d^2-d-2)}\, V(x)\Big)^{d/2}\Big] \end{aligned} \tag{102}$$

for suitable constants $0 < C \leq C' < \infty$. □

3.2 Saturation of the Non-Asymptotic Semi-classical Bounds

We discuss in the following the optimality of the bound in Theorem 1.1 in three different situations: For slowly decaying potentials, for strong and finitely supported potentials, and for weak potentials which are slowly varying in space.

We first show that if V decays slower than $|x|^{-2}$ then 0 is an accumulation point of the discrete spectrum of $H(\mathfrak{e}, V)$ and, in particular, $H(\mathfrak{e}, V)$ has infinitely many negative eigenvalues, i.e. $N[\mathfrak{e}, V] = N_{sc}[\mathfrak{e}, V] = \infty$. To formulate the statement, we recall that $h_{x,y} = h(\mathfrak{e})_{x,y} := \langle \delta_x \mid h(\mathfrak{e})\, \delta_y \rangle$ denotes matrix elements of $h(\mathfrak{e})$.

Theorem 3.6 ($N[e, V] = \infty$ for slowly decaying potentials)**.** *Let $\mathfrak{e}$ be an admissible dispersion relation with hopping matrix $h(\mathfrak{e})$ and $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$. Assume that there are constants* $\mathrm{const} < \infty$ *and* $\mathrm{const}', \alpha, \alpha' > 0$ *with* $\alpha < \min\{\alpha', 2\}$ *such that, for all* $x \in \Gamma\backslash\{0\}$*,*

$$V(x) \geq \mathrm{const}'\, |x|^{-\alpha}, \quad |h_{0,x}| \leq \mathrm{const}\, |x|^{-(2d+\alpha')}. \tag{103}$$

Then $H(\mathfrak{e}, V)$ has infinitely many eigenvalues below 0*.*

The proof of this theorem is a bit lengthy and is given in Appendix A.2. For the case $\mathfrak{e} = \mathfrak{e}_{\mathrm{Lapl}}$ and $d = 1$, see also [13].

Note that – assuming $\alpha' \geq 2$ – Theorem 3.6 together with the bound (68) implies that the case $V(x) \sim |x|^{-2}$ is critical in dimension $d \geq 3$ in the sense that

$$\exists_{\varepsilon>0}\ \sup_{x\in\Gamma} \left\{ \frac{V(x)}{|x|^{2+\epsilon}} \right\} < \infty \implies N[\mathfrak{e}, V],\ N_{sc}[\mathfrak{e}, V] < \infty, \tag{104}$$

$$\exists_{\varepsilon>0}\ \inf_{x\in\Gamma} \left\{ \frac{V(x)}{|x|^{2-\epsilon}} \right\} > 0 \implies N[\mathfrak{e}, V] = N_{sc}[\mathfrak{e}, V] = \infty. \tag{105}$$

Observe also that Theorem 1.3 follows from Theorem 3.6.

Lemma 3.7 (Lower bound on $N[e,V]$ without (H-1) and for $d \geq 1$)**.** *Let $d \geq 1$ and $\mathfrak{e}$ be an admissible dispersion relation. Furthermore let $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ be a potential decaying at ∞. Then, for all $c > \mathfrak{e}_{\max}$,*

$$N[\mathfrak{e},V] \;\geq\; \mathcal{L}_V[c] \;=\; \big\{x \in \Gamma \,\big|\, V(x) \geq c\big\}. \tag{106}$$

Proof: For all $\rho > 0$,

$$B(\rho) \;:=\; V^{1/2}\frac{1}{\rho + h(\mathfrak{e})}V^{1/2} \geq \frac{1}{\mathfrak{e}_{\max}}V. \tag{107}$$

By the min-max principle and Lemma 2.2 (Birman-Schwinger principle), we hence obtain that

$$N[\mathfrak{e},V] \geq \; \mathcal{L}_V[c], \; c > \mathfrak{e}_{\max}. \tag{108}$$

□

The following (stronger) result holds for sparse potentials:

Lemma 3.8 (Lower bound on $N[e,V]$ for sparse potentials)**.** *Let $d \geq 3$ and $\mathfrak{e}$ be an admissible dispersion relation. Let $0 < \eta(\mathfrak{e}) < \mathfrak{e}_{\max}$ be defined by*

$$\frac{1}{\eta(\mathfrak{e})} = \int [\mathfrak{e}(p)]^{-1}\, d\mu^*(p).$$

Furthermore, let $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ be a potential which is sparse in the sense that

$$\eta(\mathfrak{e}) \sup_{\rho>0} \sup_{x \in \mathrm{supp}\, V} \sum_{y \in \mathrm{supp}\, V \setminus \{x\}} |\langle \delta_x|\, (\rho + h(\mathfrak{e}))^{-1}\delta_y\rangle| < \frac{\varepsilon}{1+\varepsilon} < 1$$

for some $0 < \varepsilon < \infty$. Then

$$N[\mathfrak{e},V] \;\geq\; \mathcal{L}_V[(1+\varepsilon)\eta(\mathfrak{e})] \;=\; \big\{x \in \Gamma \,\big|\, V(x) \geq (1+\varepsilon)\eta(\mathfrak{e})\big\}. \tag{109}$$

Proof: Observe that $N[\mathfrak{e},V] \geq N[\mathfrak{e},V']$ with $V'(x) := \max\{V(x), (1+\varepsilon)\eta(\mathfrak{e})\}$. Let $\rho > 0$ and $x \in \Gamma$. Similarly to (62), we have

$$\langle \delta_x|\, B(\rho,\mathfrak{e},V')\delta_x\rangle \;=\; V'(x)\bigg(\int_{\Gamma^*} \frac{d\mu^*(p)}{\rho + \mathfrak{e}(p)}\bigg). \tag{110}$$

Observe that, by the assumption on V and the Schur bound, for all $\psi \in \ell^2(\Gamma)$,

$$\sup_{\rho>0} \langle \psi|\, B(\rho,\mathfrak{e},V')\psi\rangle > \sum_{x\in\Gamma,\, V(x)\geq(1+\varepsilon)\eta(\mathfrak{e})} |\psi_x|^2(1+\varepsilon) - \varepsilon.$$

By Lemma 2.2 (Birman-Schwinger principle) and the min-max principle, we hence obtain that

$$N[\mathfrak{e},V'] \;\geq\; \mathcal{L}_V[(1+\varepsilon)\eta(\mathfrak{e})]. \tag{111}$$

□

Note that Lemma 3.7 together with Corollary 3.2 and $N[\mathfrak{e},0]=0$ implies that, for finitely supported potentials V, we have

$$\lim_{\lambda\to\infty} N[\mathfrak{e},\lambda V] = \lim_{\lambda\to\infty} N_{sc}[\mathfrak{e},\lambda V] = \operatorname{supp} V, \tag{112}$$

and thus the semi-classical upper bound on $N[\mathfrak{e},\lambda V]$ saturates when $\lambda\to\infty$.

Observe further that, on one hand, theorem 3.10 below implies that the lower bound on $N[\mathfrak{e},V]$ given in Lemma 3.7 strongly underestimates the size of the discrete spectrum of $H(\mathfrak{e},V)$ in the case where V is slowly varying in space. $N_{sc}[\mathfrak{e},V]$ describes – in this precise case – the behavior of $N[\mathfrak{e},V]$ more correctly. On the other hand, it seems that there is no other simple candidate for a lower bound on $N[\mathfrak{e},V]$ holding in general and based on quantities like $N_{sc}[\mathfrak{e},V]$ or $|V|_p^p$. See Corollary 4.5 and remark thereafter.

For any continuous function $f:\mathbb{R}^d\to\mathbb{R}_0^+$ define for all $M\in\mathbb{N}_0$ the step functions $f_-^{(M)}:\mathbb{R}^d\to\mathbb{R}_0^+$ by:

$$f_-^{(M)}(x) := \sum_{X\in\mathbb{Z}^d} \mathbf{1}[x\in 2^{-M}X+[0,2^{-M})^d]\min\{\, f(x') \,|\, x'\in 2^{-M}X+[0,2^{-M})^d\}. \tag{113}$$

Lemma 3.9. *Let $v\in C_0(\mathbb{R}^d,\mathbb{R}_0^d)$ be compactly supported. For all $L>0$ define the potential $V_L:\Gamma\to\mathbb{R}_0^+$ by:*

$$V_L(x) := L^{-2}v(L^{-1}x). \tag{114}$$

Let $\mathfrak{e}$ be any admissible dispersion relation from $C^3(\Gamma^*,\mathbb{R})$. Assume, moreover, that for some $D<\infty$ and some $\alpha>2$, for all $x\in\Gamma$,

$$|h(\mathfrak{e})_{0,x}| \le D\,\langle x\rangle^{2d+\alpha}. \tag{115}$$

Then there are constants $\mathrm{const}'>0$, $\mathrm{const}<\infty$, depending only $\mathfrak{e}$ such that for all $M\in\mathbb{N}_0$,

$$\liminf_{L\to\infty} N[\mathfrak{e},V_L] \ge \mathrm{const}' \int_{\mathbb{R}^d} v_-^{(M)}(x)^{d/2}\,\mathbf{1}[v_-^{(M)}(x)>\mathrm{const}^{2M}]d^dx. \tag{116}$$

We prove this by standard arguments using coherent states, see Appendix A.2. The following result is an immediate consequence of the lemma above.

Theorem 3.10. *Let $\mathfrak{e}$ be any admissible dispersion relation from $C^3(\Gamma^*,\mathbb{R})$ and $v\in C_0(\mathbb{R}^d,\mathbb{R}_0^d)$ be compactly supported. Let the potentials $V_L=V_L(v)$ be defined as above. Then, for some constant $\mathrm{const}>0$ depending only on $\mathfrak{e}$,*

$$\liminf_{\lambda\to\infty}\liminf_{L\to\infty} N[\mathfrak{e},\lambda V_L] \ge \mathrm{const}\,\lambda^{d/2}\int_{\mathbb{R}^d} v(x)^{d/2}d^dx. \tag{117}$$

Observe, moreover, that from Theorem 3.10: $N[\mathfrak{e},\lambda V_L] \geq \mathrm{const}\; N_{sc}[\mathfrak{e},\lambda V_L]$ for some $\mathrm{const} > 0$ and sufficiently large $\lambda > 0$ and $L > 0$. Thus, as expected, like in the continuous case: $N[\mathfrak{e},\lambda V_L] \sim N_{sc}[\mathfrak{e},\lambda V_L]$ at large $\lambda > 0$ and $L > 0$.

4 Asymptotics of $N[\mathfrak{e},\lambda V]$ for large λ

In this section we investigate the question whether the semi-classical number of bound states $N_{sc}[\mathfrak{e},\lambda V]$ describes $N[\mathfrak{e},\lambda V]$ correctly in the limit $\lambda \to \infty$ or not. This leads us to the proof of Theorems 1.8 and 1.9.

Equally interesting, however, is the observation made in this section that an asymptotic comparison of $N[\mathfrak{e},\lambda V]$ to $N_{sc}[\mathfrak{e},\lambda V]$ does not always make much sense. Namely, in Theorem 4.7 below, we prove that $\lambda \mapsto N[\mathfrak{e},\lambda V]$ may approximate any given continuous and monotonically increasing function $F(\lambda)$ of λ. More precisely, given F, we can always find a potential V_F such that $N[\mathfrak{e},\lambda V_F] = F(\lambda)$ up to a small error.

4.1 Potentials with Semi-classical Asymptotic Behavior of $N[\mathfrak{e},\lambda V]$ at large λ

This subsection is devoted to the proof of Theorems 1.8 and 1.9. To this end, we recall that

$$g_+(V) := \quad \sup_{r>0} \limsup_{\ell\to\infty} \frac{2}{d\,r}\Big(\ln \mathcal{L}_V\big[e^{-\ell-r}\big] - \ln \mathcal{L}_V\big[e^{-\ell}\big]\Big), \tag{118}$$

$$g_-(V) := \quad \inf_{r>0} \liminf_{\ell\to\infty} \frac{2}{d\,r}\Big(\ln \mathcal{L}_V\big[e^{-\ell-r}\big] - \ln \mathcal{L}_V\big[e^{-\ell}\big]\Big). \tag{119}$$

The following lemma illustrates that, for potentials with $g_+(V) < 1$, the main contribution to $N_{sc}[\mathfrak{e},\lambda V]$ is given by $\#\{\lambda V \geq \mathfrak{e}_{\max}\}$, and that this actually defines a borderline in the sense that if $g_-(V) \geq 1$ then this assertion is reversed.

Lemma 4.1. *Assume $d \geq 1$ and $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$.*

(i) Then there is a constant $C_{4.1}(d,\mathfrak{e}) > 0$ such that

$$\liminf_{\lambda\to\infty} \left\{ \frac{N_{sc}^{>}[\mathfrak{e},\lambda V]}{N_{sc}[\mathfrak{e},\lambda V]} \right\} \;\geq\; \big(1 - g_+(V)\big)\, C_{4.1}(d,\mathfrak{e}). \tag{120}$$

(ii) Conversely, if $g_-(V) \geq 1$ then

$$\lim_{\lambda\to\infty} \left\{ \frac{N_{sc}^{>}[\mathfrak{e},\lambda V]}{N_{sc}[\mathfrak{e},\lambda V]} \right\} \;=\; 0, \tag{121}$$

where $N_{sc}^{>}[\mathfrak{e},V] = \mathcal{L}_V[\mathfrak{e}_{\max}] = \#\{V \geq \mathfrak{e}_{\max}\}$ is defined in Definition 3.3.

Proof: We first fix $x \in \Gamma$, set $\rho_x := \min\big\{1,\ \lambda V(x)/\mathfrak{e}_{\max}\big\}$, and observe that

$$c_1\,\rho_x^{d/2} \;\le\; \int_{\Gamma^*} \mathbf{1}[\mathfrak{e}(p) < \lambda V(x)] d\mu^*(p) \;\le\; C_1\,\rho_x^{d/2}, \tag{122}$$

for some $0 < c_1 \equiv c_1(d, \mathfrak{e}) < C_1 \equiv C_1(d, \mathfrak{e}) < \infty$, since $\mathfrak{e}(p)$ is a Morse function. Furthermore, we have that

$$\begin{aligned} N_{sc}[\mathfrak{e}, \lambda V] \;&=\; \sum_{x\in\Gamma} \int_{\Gamma^*} \mathbf{1}[\mathfrak{e}(p) < \lambda V(x)]\, d\mu^*(p) \qquad (123)\\ &=\; \sum_{x\in\Gamma} \int_{\Gamma^*} \mathbf{1}[\mathfrak{e}(p) < \lambda V(x) \le \mathfrak{e}_{\max}]\, d\mu^*(p) + \mathcal{L}_V[\lambda^{-1}\mathfrak{e}_{\max}]. \end{aligned}$$

Using that

$$\rho_x^{d/2} \;=\; \frac{d}{2} \int_0^\infty \mathbf{1}\big[e^{-r} < \rho_x\big]\, e^{-dr/2}\, dr \tag{124}$$

and $\ell_\lambda := \log(\lambda) - \log(\mathfrak{e}_{\max})$, we hence obtain

$$\begin{aligned} &N_{sc}[\mathfrak{e}, \lambda V] - \mathcal{L}_V\big[e^{-\ell_\lambda}\big] \\ =\;& \sum_{x\in\Gamma} \int_{\Gamma^*} \mathbf{1}[\mathfrak{e}(p) < \lambda V(x) < e_{\max}] d\mu^*(p) \qquad (125)\\ \le\;& \frac{dC_1}{2} \sum_{x\in\Gamma} \int_0^\infty \Big\{\mathbf{1}\big[e^{-r} \le \lambda \mathfrak{e}_{\max}^{-1} V(x)\big] - \mathbf{1}\big[1 \le \lambda \mathfrak{e}_{\max}^{-1} V(x)\big]\Big\}\, e^{-dr/2}\, dr, \\ =\;& \frac{dC_1}{2} \int_0^\infty \Big\{\mathcal{L}_V\big[e^{-\ell_\lambda - r}\big] - \mathcal{L}_V\big[e^{-\ell_\lambda}\big]\Big\}\, e^{-dr/2}\, dr, \\ =\;& \frac{d\,C_1\,\mathcal{L}_V[e^{-\ell_\lambda}]}{2} \int_0^\infty \left\{\frac{\mathcal{L}_V[e^{-\ell_\lambda - r}]}{\mathcal{L}_V[e^{-\ell_\lambda}]}\right\} e^{-dr/2}\, dr - C_1\,\mathcal{L}_V\big[e^{-\ell_\lambda}\big]. \qquad (126) \end{aligned}$$

and similarly

$$\begin{aligned} & N_{sc}[\mathfrak{e}, \lambda V] - \mathcal{L}_V\big[e^{-\ell_\lambda}\big] \qquad (127)\\ \ge\;& \frac{d\,c_1\,\mathcal{L}_V[e^{-\ell_\lambda}]}{2} \int_0^\infty \left\{\frac{\mathcal{L}_V[e^{-\ell_\lambda - r}]}{\mathcal{L}_V[e^{-\ell_\lambda}]}\right\} e^{-dr/2}\, dr - c_1\,\mathcal{L}_V\big[e^{-\ell_\lambda}\big]. \end{aligned}$$

Defining

$$g_\ell(r) \;:=\; \frac{2}{d\,r}\Big(\ln \mathcal{L}_V\big[e^{-\ell - r}\big] - \ln \mathcal{L}_V\big[e^{-\ell}\big]\Big), \tag{128}$$

we hence have

$$\frac{d\,C_1}{2}\int_0^\infty \exp\left(-[1-g_{\ell_\lambda}(r)]\,\frac{d}{2}r\right)\,dr \;\;\geq\;\; \frac{N_{sc}[\mathfrak{e},\lambda V]}{\mathcal{L}_V[e^{-\ell_\lambda}]}-1+C_1. \quad (129)$$

$$\frac{d\,c_1}{2}\int_0^\infty \exp\left(-[1-g_{\ell_\lambda}(r)]\,\frac{d}{2}r\right)\,dr \;\;\leq\;\; \frac{N_{sc}[\mathfrak{e},\lambda V]}{\mathcal{L}_V[e^{-\ell_\lambda}]}-1+c_1, \quad (130)$$

Now, an application of Fatou's Lemma yields

$$\begin{aligned}\limsup_{\lambda\to\infty}\frac{N_{sc}[\mathfrak{e},\lambda V]}{\mathcal{L}_V[e^{-\ell_\lambda}]} \;&\leq\; 1-C_1+\frac{d\,C_1}{2}\int_0^\infty \exp\left(-[1-g_+(V)]\right)\,r dr\\ &=\; 1-C_1+\frac{dC_1}{[1-g_+(V)]}, \qquad (131)\end{aligned}$$

which implies (i). Assertion (ii) is similar, for if $g_-(V)\geq 1$ then another application of Fatou's Lemma gives

$$\liminf_{\lambda\to\infty}\frac{N_{sc}[\mathfrak{e},\lambda V]}{\mathcal{L}_V[e^{-\ell_\lambda}]} \;\geq\; 1-c_1+\frac{d\,c_1}{2}\int_0^\infty \exp\left([g_-(V)-1]\,\frac{d}{2}r\right)\,dr \;=\; \infty. \quad (132)$$

□

Proof of Theorems 1.8 and 1.9: By Theorem 1.4 and Definition 3.3, we have

$$\frac{N[\mathfrak{e},\lambda V]}{N_{sc}[\mathfrak{e},\lambda V]} \;\geq\; \frac{\mathcal{L}_V[\lambda^{-1}\mathfrak{e}_{\max}]}{N_{sc}[\mathfrak{e},\lambda V]} \;=\; \frac{N_{sc}^{>}[\mathfrak{e},\lambda V]}{N_{sc}[\mathfrak{e},\lambda V]}. \quad (133)$$

Now, the left-hand inequality in (33) and the first inequality in (34) follow directly from Lemma 4.1 (i). The right-hand inequality in Eqs. (33) follows from Theorem 1.1, while the second inequality in (34) is a consequence of Theorem 1.2.

□

4.2 Failure of Semi-classical Asymptotic Behavior of $N[\mathfrak{e},\lambda V]$ at large λ

For the continuum Schrödinger operator $-\Delta-\lambda V(x)$ on $\mathbb{R}^d$, the number of negative eigenvalues is asymptotically homogeneous of degree $d/2$ in λ, i.e., $N_{sc}^{cont}[\lambda V] \;=\; \lambda^{d/2}\,N_{sc}^{cont}[V]$. For discrete Schrödinger operators, only weaker statements hold true, as is illustrated by the following lemma. See also [8, Section 5.2].

Lemma 4.2 (Lemma 1.7)**.** *Assume $d\geq 3$, (H-1) and $V\in\ell^{d/2}(\Gamma,\mathbb{R}_0^+)$. Then*

$$\lim_{\lambda\to\infty}\left\{\lambda^{-d/2}\,N[\mathfrak{e},\lambda V]\right\} \;=\; \lim_{\lambda\to\infty}\left\{\lambda^{-d/2}\,N_{sc}[\mathfrak{e},\lambda V]\right\} \;=\; 0. \quad (134)$$

Proof: It suffices to prove the second equality, since $N[\mathfrak{e}, \lambda V] \leq C_{2.6}(d, \mathfrak{e}) N_{sc}[\mathfrak{e}, \lambda V]$, by Theorem 1.1. By (95), we have that

$$\lambda^{-d/2} \, N[\mathfrak{e}, \lambda V] \; \leq \; c_2(\mathfrak{e}) \, \lambda^{-d/2} \Big(N_{sc}^{>}[\mathfrak{e}, \lambda V] + N_{sc}^{<}[\mathfrak{e}, \lambda V] \Big), \tag{135}$$

and

$$\begin{aligned} &\lambda^{-d/2} \, (N_{sc}^{>}[\mathfrak{e}, \lambda V] + N_{sc}^{<}[\mathfrak{e}, \lambda V]) \qquad (136) \\ &= \; \lambda^{-d/2} \sum_{x \in \Gamma} \min \big\{ \mathfrak{e}_{\max}, \lambda^{d/2} \, V^{d/2}(x) \big\} \; = \; \sum_{x \in \Gamma} \min \big\{ \lambda^{-d/2} \, \mathfrak{e}_{\max}, V^{d/2}(x) \big\}. \end{aligned}$$

Since $\lim_{\lambda \to \infty} \min\{\lambda^{-d/2} \, \mathfrak{e}_{\max}, V^{d/2}(x)\} = 0$, for every $x \in \Gamma$ and $\min\{\lambda^{-d/2} \, \mathfrak{e}_{\max}, V^{d/2}\}$ is dominated by $V^{d/2} \in \ell^1(\Gamma)$, the assertion follows from the dominated convergence theorem. □

Lemma 4.3. *Let $d \geq 3$ and $\mathfrak{e}$ be an admissible dispersion relation. Then there is a constant $C_{4.3}(d, \mathfrak{e}) < \infty$ such that, for all $\rho \in (0, 1]$ and all $x, y \in \Gamma$, $x \neq y$,*

$$\big| \big\langle \delta_x \, \big| \, \big(\rho + h(\mathfrak{e}) \big)^{-1} \delta_y \big\rangle \big| \; \leq \; \frac{C_{4.3}(d, \mathfrak{e})}{|x - y|^{1/2}}. \tag{137}$$

Proof: Let $\mathrm{Min}(\mathfrak{e}) := \{\xi \in \Gamma^* \mid \mathfrak{e}(\xi) = 0\}$ be the set of points in Γ^* for which $\mathfrak{e}$ is minimal. We construct a partition of unity localizing on the Voronoi cells

$$\mathcal{V}(\xi) \; := \; \big\{ p \in \Gamma^* \, \big| \, \gamma(p, \xi) = \min_{\tilde{\xi} \in \mathrm{Min}(\mathfrak{e})} \gamma(p, \tilde{\xi}) \big\}, \tag{138}$$

where $\xi \in \mathrm{Min}(\mathfrak{e})$ and $\gamma : \Gamma^* \times \Gamma^* \to \mathbb{R}_0^+$ is the natural metric on $\Gamma^* = (\mathbb{R}/2\pi\mathbb{Z})^d$. Denote by $r > 0$ the largest radius, such that $B_\gamma(\xi, 2r) \subseteq \mathcal{V}(\xi)$, for all $\xi \in \mathrm{Min}(\mathfrak{e})$, and choose $j \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_0^+)$ such that $\mathrm{supp}\, j \subseteq B(0, 1)$ and $\int_{\mathbb{R}^d} j(p) \, d^d p = 1$. We then set $j_r(p) := r^{-d} j(p/r)$ for $p \in \Gamma^*$ (which makes sense because $r > 0$ is sufficiently small), and

$$\chi_\xi \; := \; j_r * \mathbf{1}_{\mathcal{V}(\xi)}. \tag{139}$$

We list a few properties of this partition in combination with the dispersion $\mathfrak{e}$ deriving from the fact that $\mathfrak{e}$ is a Morse function.

$$\begin{aligned} \forall_{p \in \Gamma^*} : \sum_{\xi \in \mathrm{Min}(\mathfrak{e})} \chi_\xi(p) \quad &= \quad 1, \qquad (140) \\ \forall_{p \in \Gamma^*} \forall_{\xi, \tilde{\xi} \in \mathrm{Min}(\mathfrak{e}), \, \xi \neq \tilde{\xi}} : \chi_\xi(p) > 0 \quad &\Longrightarrow \quad \gamma(p, \tilde{\xi}) > r, \\ \exists_{c_1 > 0} \, \forall_{p \in \Gamma^*} \forall_{\xi \in \mathrm{Min}(\mathfrak{e})} : \nabla_p \chi_\xi(p) > 0 \quad &\Longrightarrow \quad \mathfrak{e}(p) \geq c_1, \\ \exists_{c_2 > 0} \, \forall_{p \in \Gamma^*} \forall_{\xi \in \mathrm{Min}(\mathfrak{e})} : \chi_\xi(p) > 0 \quad &\Longrightarrow \quad \mathfrak{e}(p) \geq c_2 (p - \xi)^2, \\ \exists_{c_3 < \infty} \, \forall_{p \in \Gamma^*} \forall_{\xi \in \mathrm{Min}(\mathfrak{e})} : \chi_\xi(p) > 0 \quad &\Longrightarrow \quad |\nabla \mathfrak{e}(p)| \leq c_3 |p - \xi|. \end{aligned}$$

By translation invariance, it suffices to prove (137) for $y = 0$ and $x \neq 0$. We observe that

$$|x|^2 \left|\left\langle \delta_x \,\middle|\, \big(\rho + h(\mathfrak{e})\big)^{-1} \delta_0 \right\rangle\right| \;=\; \left| \int_{\Gamma^*} \frac{x \cdot \nabla_p\big(e^{ip\cdot x}\big)\, d\mu^*(p)}{\rho + \mathfrak{e}(p)} \right| \tag{141}$$

$$= \left| \sum_{\xi \in \mathrm{Min}(\mathfrak{e})} \int_{\Gamma^*} x \cdot \nabla_p\big(e^{i(p-\xi)\cdot x} - 1\big)\, \frac{\chi_\xi(p)\, d\mu^*(p)}{\rho + \mathfrak{e}(p)} \right|$$

$$= \left| \sum_{\xi \in \mathrm{Min}(\mathfrak{e})} \int_{\Gamma^*} \big(e^{i(p-\xi)\cdot x} - 1\big) \left\{ \frac{x \cdot \nabla_p \chi_\xi(p)}{\rho + \mathfrak{e}(p)} - \frac{\chi_\xi(p)\, x \cdot \nabla_p \mathfrak{e}(p)}{[\rho + \mathfrak{e}(p)]^2} \right\} d\mu^*(p) \right|.$$

Now we use (140), $|e^{i(p-\xi)\cdot x} - 1| \leq 2$, and $|e^{i(p-\xi)\cdot x} - 1| \leq 2\,|x|^{1/2}\,|p-\xi|^{1/2}$ to obtain

$$|x|^{1/2} \left|\left\langle \delta_x \,\middle|\, \big(\rho + h(\mathfrak{e})\big)^{-1} \delta_0 \right\rangle\right| \tag{142}$$

$$\leq \sum_{\xi \in \mathrm{Min}(\mathfrak{e})} \int_{\Gamma^*} \left\{ \frac{2\,|\nabla_p \chi_\xi(p)|}{c_1} + \frac{\chi_\xi(p)\, c_3}{c_2^2 (p-\xi)^{5/2}} \right\} d\mu^*(p) \;\leq\; C_4,$$

for some constant $C_4 < \infty$, since $|p-\xi|^{-5/2}$ is locally integrable for $d \geq 3$. We remark that we may have improved this estimate to $\mathcal{O}(|x|^{\beta-1})$, for any $\beta > 0$, by using $|e^{i(p-\xi)\cdot x} - 1| \leq 2\,|x|^\beta\,|p-\xi|^\beta$. ☐

Lemma 4.4. *Let $d \geq 3$ and $\mathfrak{e}$ be an admissible dispersion. Let $\underline{r} := (r_k)_{k=0}^\infty$ be an increasing sequence of positive integers with $9r_k \leq r_{k+1}$ for all $k \geq 0$, and define $\omega(\underline{r}) := \{x_0, x_1, x_2, \ldots\} \subseteq \Gamma$ by*

$$x_k \;:=\; (r_k, 0, \ldots, 0). \tag{143}$$

If $V \in \ell^\infty(\Gamma)$ with $\operatorname{supp} V \subseteq \omega(\underline{r})$ and

$$|V|_\infty \;<\; \eta(\mathfrak{e}) - \frac{1}{4}\, C_{4.3}(d, \mathfrak{e})\, \eta(\mathfrak{e})^2\, r_0^{-1/2}, \tag{144}$$

then $N[\mathfrak{e}, V] = 0$.

Proof: For any normalized $\psi = (\psi_x)_{x\in\Gamma} \in \ell^2(\Gamma)$ and all $\rho > 0$, we have that

$$\left\langle \psi \,\middle|\, V^{1/2}\, (\rho + h(\mathfrak{e}))^{-1}\, V^{1/2} \psi \right\rangle \tag{145}$$

$$\leq \frac{1}{\eta(\mathfrak{e})} \sum_{x \in \omega(\underline{r})} V(x)\, |\psi_x|^2 \tag{146}$$

$$+ \sum_{x,y \in \omega(\underline{r}),\; x \neq y} \overline{\psi_x} \psi_y\, [V(x)V(y)]^{1/2} \left\langle \delta_x \,\middle|\, \big(\rho + h(\mathfrak{e})\big)^{-1} \delta_y \right\rangle$$

$$\leq |V|_\infty \left(\frac{1}{\eta(\mathfrak{e})} + \sup_{x \in \omega(\underline{r})} \left\{ \sum_{y \in \omega(\underline{r}) \setminus \{x\}} \left|\left\langle \delta_x \,\middle|\, \big(\rho + h(\mathfrak{e})\big)^{-1} \delta_y \right\rangle\right| \right\} \right),$$

by the Schur bound. From Lemma 4.3 it follows that

$$\sup_{x\in\omega(\underline{r})}\Big\{\sum_{y\in\omega(\underline{r})\setminus\{x\}}\big|\big\langle\delta_x\,\big|\,\big(\rho+h(\mathfrak{e})\big)^{-1}\delta_y\big\rangle\big|\Big\}\Big)\;\le\;C_{4.3}(d,\mathfrak{e})\,\sup_{k\ge 0}\big\{X_k+Y_k\big\},\tag{147}$$

where

$$X_k\;:=\;\sum_{\ell=0}^{k-1}|r_k-r_\ell|^{-1/2}\quad\text{and}\quad Y_k\;:=\;\sum_{\ell=k+1}^{\infty}|r_k-r_\ell|^{-1/2}.\tag{148}$$

For $\ell<k$, we have that $|r_k-r_\ell|\ge 8r_k\ge 8\cdot 9^k\,r_0$ and hence

$$X_k\;\le\;\frac{k\,3^{-k}}{\sqrt{8\,r_0}}.\tag{149}$$

Similarly, we have that $|r_k-r_\ell|\ge 8r_\ell\ge 8\cdot 9^\ell\,r_0$ for $\ell>k$, and thus

$$Y_k\;\le\;\frac{3^{-k}}{3\,(1-\frac{1}{3})\,\sqrt{8\,r_0}}\;=\;\frac{3^{-k}}{2\,\sqrt{8\,r_0}}.\tag{150}$$

We hence conclude that

$$\sup_{x\in\omega(\underline{r})}\Big\{\sum_{y\in\omega(\underline{r})\setminus\{x\}}\big|\big\langle\delta_x\,\big|\,\big(\rho+h(\mathfrak{e})\big)^{-1}\delta_y\big\rangle\big|\Big\}\Big)\;\le\;\frac{C_{4.3}(d,\mathfrak{e})}{2\,\sqrt{8\,r_0}}.\tag{151}$$

Thus, the operator norm of the Birman-Schwinger operator is strictly smaller than one,

$$\big\|V^{1/2}\,(\rho+h(\mathfrak{e}))^{-1}\,V^{1/2}\big\|\;\le\;|V|_\infty\left(\frac{1}{\eta(\mathfrak{e})}\;+\;\frac{C_{4.3}(d,\mathfrak{e})}{2\,\sqrt{8\,r_0}}\right)\;<\;1,\tag{152}$$

for all $\rho>0$, which implies that $N[\mathfrak{e},V]=0$. □

The last lemma has the following immediate consequences.

Corollary 4.5 (Thm. 1.6)**.** *Let $d\ge 3$ and $\mathfrak{e}$ be an admissible dispersion. Then there exists a potential $V\notin\bigcup_{p\ge 1}\ell^p(\Gamma)$ with $N[\mathfrak{e},V]=0$.*

Proof: Fix $r_0\in\mathbb{N}$, choose $r_k:=9^k\,r_0$, $x_k:=(r_k,0,\ldots,0)$, and set

$$V(x)\;:=\;\sum_{j=0}^{\infty}\mathbf{1}_{\{x_j\}}(x)\,\frac{\eta(\mathfrak{e})}{\ln(4+j)}.\tag{153}$$

Note that $V \in \ell_0^\infty(\Gamma)$ but that, for all $p \geq 1$, the p-norm of V diverges, $|V|_p = \eta(\mathfrak{e})^p \sum_{j=0}^\infty \big[\ln(4+j)\big]^{-p} = \infty$. Moreover, $|V|_\infty = \frac{1}{\ln(4)}\eta(\mathfrak{e}) < \eta(\mathfrak{e})$, and Lemma 4.4 implies that $N[\mathfrak{e}, V] = 0$ provided $r_0 \in \mathbb{N}$ is chosen sufficiently large such that $C_{4.3}(d, \mathfrak{e})\, \eta(\mathfrak{e}) r_0^{-1/2} < 4\left(1 - \frac{1}{\ln(4)}\right)$. ☐

We remark that $N_{sc}[\mathfrak{e}, V] = \infty$ in Corollary 4.5, since $V \notin \bigcup_{p\geq 1} \ell^p(\Gamma)$. Thus, a *lower* bound on $N[\mathfrak{e}, V]$ in terms of ℓ^p-norms or in multiples of $N_{sc}[\mathfrak{e}, V]$ cannot possibly hold true. See also [8].

Corollary 4.6 (Thm. 1.5)**.** *Let $d \geq 3$, $\mathfrak{e}$ be an admissible dispersion. Given $\varepsilon \in (0,1)$ and a potential $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$, there exists a rearrangement $\widetilde{V} \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ of V such that*

$$N[\mathfrak{e}, \widetilde{V}] \;\leq\; \mathcal{L}_{\widetilde{V}}[(1-\varepsilon)\eta(\mathfrak{e})] \;=\; \#\big\{x \in \Gamma \,\big|\, V(x) \geq (1-\varepsilon)\eta(\mathfrak{e})\big\}. \tag{154}$$

Proof: We write $V = V^{(>)} + V^{(<)}$ with

$$V^{(>)} \;=\; V \cdot \mathbf{1}\big[V \geq (1-\varepsilon)\eta(\mathfrak{e})\big] \quad \text{and} \quad V^{(<)} \;=\; V \cdot \mathbf{1}\big[V < (1-\varepsilon)\eta(\mathfrak{e})\big]. \tag{155}$$

Note that $V^{(>)}$ has bounded support. Thus, choosing $\widetilde{V}^{(<)}$ to be a rearrangement of $V^{(<)}$ with

$$\operatorname{supp} \widetilde{V}^{(<)} \;\subset\; \big\{(r_k, 0, \ldots, 0) \,\big|\, r_k := 9^k\, r_0\,,\ k \in \mathbb{N}_0\big\} \tag{156}$$

and $r_0 \in \mathbb{N}$ chosen sufficiently large, we find that

$$\big\|\widetilde{V}^{(<)}\big\|_\infty \;=\; (1-\varepsilon)\eta(\mathfrak{e}) \;<\; \eta(\mathfrak{e}) - \frac{1}{4}\, C_{4.3}(d, \mathfrak{e})\, \eta(\mathfrak{e})^2 r_0^{-1/2}, \tag{157}$$

and Lemma 4.4 implies that $N[\mathfrak{e}, \widetilde{V}^{(<)}] = 0$. Hence, defining $\widetilde{V} := V^{(>)} + \widetilde{V}^{(<)}$, we have for sufficiently large $r_0 \in \mathbb{N}$ that $\operatorname{supp} V^{(>)} \cap \operatorname{supp} \widetilde{V}^{(<)} = \varnothing$, $\widetilde{V}$ is a rearrangement of V, and

$$\begin{aligned} N[\mathfrak{e}, \widetilde{V}] \;&\leq\; \#\operatorname{supp} V^{(>)} \;+\; N[\mathfrak{e}, \widetilde{V}^{(<)}] && (158)\\ &=\; \#\operatorname{supp} V^{(>)} \;=\; \mathcal{L}_V[(1-\varepsilon)\eta(\mathfrak{e})], && (159) \end{aligned}$$

by Corollary 3.2. ☐

The next theorem illustrates for $d \geq 3$ that – opposed to the continuum case – the asymptotics of $N[\mathfrak{e}, \lambda V]$ as $\lambda \to \infty$ can be prescribed arbitrarily.

Theorem 4.7 (Thm. 1.11)**.** *Let $d \geq 3$ and $\mathfrak{e}$ be any admissible dispersion. Let further $F : [1, \infty) \to \mathbb{N}$ be an arbitrary monotonically increasing, positively integer-valued, right-continuous function. Then, for any $\varepsilon \in (0, 1/2)$, there exists a potential $V_{F,\varepsilon} \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ such that*

$$\forall_{\lambda \geq 2} : \qquad F\big((1-\varepsilon)\lambda\big) \;\leq\; N\big[\mathfrak{e}, \lambda V_F\big] \;\leq\; F\big((1+\varepsilon)\lambda\big). \tag{160}$$

Proof: For the proof, we abbreviate $\eta := \eta(\mathfrak{e})$. Since $F : [1, \infty) \to \mathbb{N}$ is monotonically increasing and right-continuous, there is a monotonically increasing sequence $1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ such that

$$F(\lambda) \;=\; \sum_{j=1}^{\infty} \mathbf{1}[\lambda_j \leq \lambda]. \tag{161}$$

Note that the monotonicity of F is not necessarily strict, and possibly $\lambda_j = \lambda_{j+1}$. For a sequence $\underline{r} = (r_k)_{k=0}^{\infty}$ of positive integers, with $9r_k \leq r_{k+1}$, to be further specified later, and $x_k = (r_k, 0, \ldots, 0) \in \Gamma$, we set

$$V_{F,\varepsilon}(x) \;:=\; \sum_{j=1}^{\infty} \frac{\eta}{\lambda_j} \mathbf{1}_{\{x_j\}}(x). \tag{162}$$

Let $\varepsilon' > 0$ be such that $(1+\varepsilon')^{-1} > 1 - \varepsilon$. Choosing $r_0 > 0$ large enough such that

$$\eta \sup_{\rho>0} \sup_{x \in \text{supp } V_{F,\varepsilon}} \sum_{y \in \text{supp } V_{F,\varepsilon} \setminus \{x\}} |\langle \delta_x | \, (\rho + h(\mathfrak{e}))^{-1} \delta_y \rangle| < \frac{\varepsilon'}{1+\varepsilon'}$$

we observe that

$$\begin{aligned} \mathcal{L}_{\lambda V_{F,\varepsilon}}((1+\varepsilon')\eta) \;&=\; \mathcal{L}_{V_{F,\varepsilon}}((1+\varepsilon')\eta/\lambda) & (163)\\ &=\; \#\Big\{ x \in \Gamma \,\Big|\, V_{F,\varepsilon}(x) \geq (1+\varepsilon')\frac{\eta}{\lambda} \Big\} & (164)\\ &=\; \sum_{j=1}^{\infty} \mathbf{1}\Big[\frac{\eta}{\lambda_j} \geq (1+\varepsilon')\frac{\eta}{\lambda} \Big] \;=\; F((1+\varepsilon')^{-1}\lambda). \end{aligned}$$

Thanks to Lemma 3.8, we have thus established the lower bound on $N[\mathfrak{e}, \lambda V_{F,\varepsilon}]$ in (160),

$$F((1-\varepsilon)\lambda) \;\leq F((1+\varepsilon')^{-1}\lambda) \;\leq\; N\big[\mathfrak{e}, \lambda V_{F,\varepsilon}\big] \tag{165}$$

for all $\lambda \geq 2$. Choose now $\varepsilon' > 0$ such that $(1-\varepsilon')^{-1} < 1 + \varepsilon$. For the proof of the upper bound in (160) we write $\lambda V_{F,\varepsilon} = V_{F,\lambda}^{(>)} + V_{F,\lambda}^{(<)}$, where

$$\begin{aligned} V_{F,\lambda}^{(>)}(x) \;&:=\; \lambda\, V_{F,\varepsilon}\, \mathbf{1}\Big[V_{F,\varepsilon}(x) \geq (1-\varepsilon')\frac{\eta}{\lambda} \Big] & (166)\\ &=\; \sum_{j=1}^{\infty} \mathbf{1}_{\{x_j\}}(x)\, \mathbf{1}\big[\lambda_j \leq (1-\varepsilon')^{-1}\lambda\big] \frac{\eta\,\lambda}{\lambda_j}, \\ V_{F,\lambda}^{(<)}(x) \;&:=\; \lambda\, V_{F,\varepsilon}\, \mathbf{1}\Big[V_{F,\varepsilon} < (1-\varepsilon')\frac{\eta}{\lambda} \Big] \\ &=\; \sum_{j=1}^{\infty} \mathbf{1}_{\{x_j\}}(x)\, \mathbf{1}\big[\lambda_j > (1-\varepsilon')^{-1}\lambda\big] \frac{\eta\,\lambda}{\lambda_j}, \end{aligned}$$

Observe that, due to (166)

$$\#\,\mathrm{supp}\, V_{F,\lambda}^{(>)} \;=\; \#\Big\{ x \in \omega(\underline{r}) \,\Big|\, V_{F,\varepsilon}(x) \geq (1-\varepsilon')\frac{\eta}{\lambda} \Big\} \;=\; F\big((1-\varepsilon')^{-1}\lambda\big). \quad (167)$$

Hence, Corollary 3.2 yields

$$N\big[\mathfrak{e}, \lambda V_{F,\varepsilon}\big] \;\leq\; F\big((1+\varepsilon)\lambda\big) + N\big[\mathfrak{e}, V_{F,\lambda}^{(<)}\big], \quad (168)$$

and it remains to fix the sequence $\underline{r}$ so that

$$N\big[\mathfrak{e}, V_{F,\lambda}^{(<)}\big] \;=\; 0, \quad (169)$$

for all $\lambda \geq 1$. To this end, we first note that

$$\big\| V_{F,\lambda}^{(<)} \big\|_\infty \;\leq\; \eta\,(1-\varepsilon'). \quad (170)$$

From Lemma 4.4, (169) holds by choosing $r_0 > 0$ large enough and the right-hand inequality in (160) follows. □

A similar result in proven in [8, Section 6]. Observe, however, that we do not assume here that $\lambda_j/\lambda_{j+1} \to 1$ as $j \to \infty$ for the asymptotics of eigenvalues.

Assume that for a given potential $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$, $N[\mathfrak{e}, \lambda V] \sim N_{sc}[\mathfrak{e}, \lambda V] < \infty$ at large $\lambda > 0$, i.e., that $N[\mathfrak{e}, \lambda V]$ is finite and obeys the Weyl asymptotics at large λ. Then it would follow that $N[\mathfrak{e}, \lambda V] = O(\lambda^{d/2})$. By the last theorem, for any $\alpha > 0$, there are potentials $V_\alpha \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ such that $N[\mathfrak{e}, \lambda V]$ behaves like λ^α as $\lambda \to \infty$. In particular, the semi-classical asymptotics cannot hold for V_α with $\alpha > d/2$. See also [8]. Observe, however, that in such a case, by the semi-classical upper bound on $N[\mathfrak{e}, \lambda V_\alpha]$ (Theorem 1.1), $N_{sc}[\mathfrak{e}, \lambda V_\alpha] = \infty$ (whereas $N[\mathfrak{e}, \lambda V_\alpha] < \infty$) for all $\lambda > 0$ and speaking about semi-classical behavior does not really make sense. We discuss below another kind of example for which the semi-classical asymptotics – in the sense of two–side bounds – is violated, even if $N_{sc}[\mathfrak{e}, \lambda V] < \infty$ for all $\lambda > 0$.

Theorem 4.8 (Thm. 1.10)**.** *Let $d \geq 3$ and $\mathfrak{e}$ be any admissible dispersion relation fulfilling (H-1). There is a potential $V \geq 0$, $V \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$, such that*

$$\liminf_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda V]}{N[\mathfrak{e}, \lambda V]} < \infty, \quad \limsup_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda V]}{N[\mathfrak{e}, \lambda V]} = \infty. \quad (171)$$

Proof: Define the potentials $V_1, V_2 \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+)$ by

$$V_1(x) := \frac{1}{\langle x\rangle^2 \ln\langle x\rangle}, \;\; V_2(x) := e^{-|x|}.$$

Clearly, $g_-(V_1) = 1$ and $g_+(V_2) = 0$. By Lemma 4.1,

$$\lim_{\lambda\to\infty} \frac{N_{sc}^>[\mathfrak{e}, \lambda V_1]}{N_{sc}[\mathfrak{e}, \lambda V_1]} = 0, \ \lim_{\lambda\to\infty} \frac{N_{sc}^>[\mathfrak{e}, \lambda V_2]}{N_{sc}[\mathfrak{e}, \lambda V_2]} > 0. \tag{172}$$

For any monotonically increasing sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ of positive real numbers define $\beta_\alpha : \Gamma \to \{0, 1\}$ by $\beta_\alpha(x) := 1$ if $\alpha_{1+2n} \leq |x| \leq \alpha_{2+2n}$ for some $n \in \mathbb{N}_0$, and $\beta_\alpha(x) := 0$ else. Consider potentials of the form $\tilde{V} = \tilde{V}_\alpha := \beta_\alpha(V_1 - V_2) + V_2 \geq 0$. By (172), there exists a sequence α such that:

$$\liminf_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda\tilde{V}]}{N_{sc}^>[\mathfrak{e}, \lambda\tilde{V}]} < \infty, \ \limsup_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda\tilde{V}]}{N_{sc}^>[\mathfrak{e}, \lambda\tilde{V}]} = \infty. \tag{173}$$

By (173) and Lemma 3.7, for any rearrangement V of $\tilde{V}$,

$$\liminf_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda V]}{N[\mathfrak{e}, \lambda V]} < \infty.$$

Observe that, by Corollary 3.2 and Lemma 4.4, there is a rearrangement V of $\tilde{V}$ such that

$$\limsup_{\lambda\to\infty} \frac{N_{sc}^>[\mathfrak{e}, \lambda(2\mathfrak{e}_{\max}/\eta(\mathfrak{e}))V]}{N[\mathfrak{e}, \lambda V]} \geq 1. \tag{174}$$

To conclude the proof use that for some $1 < C < \infty$,

$$C^{-1} N_{sc}[\mathfrak{e}, \lambda V] \leq N_{sc}[\mathfrak{e}, \lambda(2\mathfrak{e}_{\max}/\eta(\mathfrak{e}))V] \leq C\, N_{sc}[\mathfrak{e}, \lambda V] \tag{175}$$

for all $\lambda > 0$. This together with (173) and (174) imply

$$\limsup_{\lambda\to\infty} \frac{N_{sc}[\mathfrak{e}, \lambda V]}{N[\mathfrak{e}, \lambda V]} = \infty.$$

Note that we have used above the invariance of the semi-classical quantities $N_{sc}^>[\mathfrak{e}, \tilde{V}]$ and $N_{sc}[\mathfrak{e}, \tilde{V}]$ w.r.t. rearrangements of $\tilde{V}$. □

5 One and two dimensions

We start this section by showing (Corollary 5.3) that the semi-classical upper bound, as stated in Theorem 1.1 for instance, cannot be valid in less than three dimensions.

Lemma 5.1. *Let $d = 1, 2$, $\mathfrak{e}$ be an admissible dispersion relation, and $V \geq 0$ be a potential with finite support. For all $\rho > 0$ and all rearrangements $\widetilde{V}$ of V define the compact self-adjoint operator*

$$K(\rho, \widetilde{V}) \ = \ P_{\operatorname{Ran}\widetilde{V}}\, \widetilde{V}^{1/2}\, (\rho + h(\mathfrak{e}))^{-1}\, \widetilde{V}^{1/2}\, P_{\operatorname{Ran}\widetilde{V}} \ - \ P_{\operatorname{Ran}\widetilde{V}}. \tag{176}$$

Then there exist $\rho > 0$ and a rearrangement $\widetilde{V}$ of V such that $K(\rho, \widetilde{V}) > 0$.

Proof: If $\operatorname{supp} V = \varnothing$ there is nothing to prove, so we assume that $V \neq 0$. Let $\widetilde{V} \geq 0$ be a rearrangement of V. Then for all $\rho > 0$ and all $\psi = (\psi_x)_{x\in\Gamma} \in \operatorname{Ran} \widetilde{V}$,

$$\begin{aligned} \langle \psi \,|\, K(\rho, \widetilde{V})\, \psi \rangle \;&=\; -|\psi|_2^2 + \sum_{x \in \operatorname{supp} \widetilde{V}} \widetilde{V}(x) |\psi_x|^2 \int_{\Gamma^*} \frac{d\mu^*(p)}{\rho + \mathfrak{e}(p)} \\ &\quad + \sum_{x,y \in \operatorname{supp} \widetilde{V},\; x\neq y} [\widetilde{V}(x)\widetilde{V}(y)]^{1/2} \big\langle \delta_x \big| (\rho + h(\mathfrak{e}))^{-1} \delta_y \big\rangle \overline{\psi_x} \psi_y, \end{aligned} \tag{177}$$

and thus

$$\begin{aligned} K(\rho, \widetilde{V}) \;&\geq\; -1 + \min_{x\in\operatorname{supp} V} V(x) \int_{\Gamma^*} \frac{d\mu^*(p)}{\rho + \mathfrak{e}(p)} \\ &\quad - |V|_\infty \sup_{\psi \in \operatorname{Ran}\widetilde{V},\, |\psi|_2 = 1} \sum_{x,y\in\operatorname{supp}\widetilde{V},\; x\neq y} \big| \big\langle \delta_x \big| (\rho + h(\mathfrak{e}))^{-1} \delta_y \big\rangle \overline{\psi_x}\psi_y \big|. \end{aligned} \tag{178}$$

Choose $\rho > 0$ such that

$$\min_{x\in\operatorname{supp} V} V(x) \int_{\Gamma^*} \frac{d\mu^*(p)}{\rho + \mathfrak{e}(p)} \;>\; 2. \tag{179}$$

This is always possible since $d \leq 2$. For any fixed $\rho > 0$, we have that

$$\langle \delta_x | (\rho + h(\mathfrak{e}))^{-1} \delta_y \rangle \to 0$$

as $|x - y| \to \infty$. This follows from the Riemann-Lebesgue Lemma since $\langle \delta_x | (\rho + h(\mathfrak{e}))^{-1} \delta_y \rangle$ is the Fourier transform of the integrable function $(\rho + \mathfrak{e})^{-1} \in L^1(\Gamma^*)$. In particular, there is a rearrangement $\widetilde{V}$ of V such that

$$|V|_\infty \sup_{\psi \in \operatorname{Ran}\widetilde{V},\, |\psi|_2 = 1} \sum_{x,y\in\operatorname{supp}\widetilde{V},\; x\neq y} \big| \big\langle \delta_x \big| (\rho + h(\mathfrak{e}))^{-1} \delta_y \big\rangle \overline{\psi_x}\psi_y \big| \;\leq\; 1. \tag{180}$$

For such $\rho > 0$ and $\widetilde{V}$ we hence have that $K(\rho, \widetilde{V}) > 0$. □

Theorem 5.2. *Let $d = 1, 2$ and $\mathfrak{e}$ be an admissible dispersion relation. Then, for any finitely supported potential V, there is a rearrangement $\widetilde{V}$ of V such that*

$$N[\mathfrak{e}, \widetilde{V}] \;=\; \#\operatorname{supp} \widetilde{V} \;=\; \#\operatorname{supp} V. \tag{181}$$

Proof: Clearly, for any rearrangement $\widetilde{V}$ of V, we have $N[\mathfrak{e}, \widetilde{V}] \leq \#\operatorname{supp} V$, as follows, e.g., from Corollary 3.2 and the fact that $N[\mathfrak{e}, 0] = 0$. Let $\rho > 0$ and the rearrangement $\widetilde{V}$ of V be as in the lemma above. Then, by the min-max principle and the bound $K(\rho, \widetilde{V}) > 0$, the compact operator $(\widetilde{V})^{1/2}(\rho + h(\mathfrak{e}))^{-1}(\widetilde{V})^{1/2}$ has at least $\dim\ \mathrm{Ran}\,\widetilde{V} = |\operatorname{supp}\,\widetilde{V}|$ discrete eigenvalues above 1. By Lemma 2.2, it follows from this that $N[\mathfrak{e}, \widetilde{V}] \geq \#\operatorname{supp}\widetilde{V}$. □

Observing that the semi-classical number of bound states $N_{sc}[\mathfrak{e}, V]$ is invariant w.r.t. rearrangements of the potential V, the following corollary follows immediately:

Corollary 5.3 (Breakdown of the semi-classical upper bound in $d = 1, 2$)**.** *Let $d = 1, 2$ and $\mathfrak{e}$ be any admissible dispersion. Then, for all $\epsilon > 0$,*

$$\sup\left\{ \frac{N[\mathfrak{e}, V]}{N_{sc}[\mathfrak{e}, V]} \;\middle|\; V,\ N_{sc}[\mathfrak{e}, V] < \epsilon \right\} \;=\; \infty. \tag{182}$$

The last corollary implies in one or two dimensions that multiples of $N_{sc}[\mathfrak{e}, V]$ cannot be, in general, an upper bound on $N[\mathfrak{e}, V]$. The discussion above shows, more precisely, that $\mathrm{const}\, N_{sc}[\mathfrak{e}, V]$ fails to be such an upper bound in the case of sparse potentials, i.e. in the situation where the distance between points in the support of the potential V is large. Hence, any quantity $Q(V)$ which is supposed to be an upper bound on $N[\mathfrak{e}, V]$ should keep track of the behavior of V in space. This motivates the use of the weighted semi-classical quantities $N_{sc}\big[\mathfrak{e}, \tilde{V}(V)\big]$ – as stated in Theorem 1.2 – as upper bounds on $N[\mathfrak{e}, V]$ in one and two dimensions.

The a priori upper bound on $N[\mathfrak{e}, V]$ given in Lemma 2.4 – used to derive (weighted) semi-classical bounds on the number of bound states $N[\mathfrak{e}, V]$ in three or more dimensions – is useless for $d = 1, 2$ since in this case $\eta(\mathfrak{e}) = 0$. Indeed, observe that, by Theorem 3.6, for $d = 1$, the inequality $N[\mathfrak{e}, V] \leq \mathrm{const}\,|V|_1$ can be violated for any $\mathrm{const} < \infty$. It is also expected to be the case for $d = 2$. However, Lemma 5.4 – a similar result to Lemma 2.4 – holds for $d = 1, 2$ by replacing the ℓ_1 norm with a stronger homogeneous (of degree one) functional.

For any $p > 0$, $m \geq 0$, and any function $V : \Gamma \to \mathbb{R}_0^+$ define

$$|V|_{p,m} \;:=\; \left(\sum_{x \in \Gamma} V^p(x)\, \langle x \rangle^m \right)^{1/p}. \tag{183}$$

Observe that $|\cdot|_{p,m}$ is not a norm, for $p \in (0, 1)$, but only a homogeneous functional of degree one. For any function $\mathfrak{e} \in C^m(\Gamma^*, \mathbb{C})$ and $m \in \mathbb{N}_0$, define the C^m-(semi)norms by

$$\|\mathfrak{e}\|_{C^m} \;:=\; \max_{\underline{n} \in \mathbb{N}_0^d,\ |\underline{n}| = m}\ \max_{p \in \Gamma^*} \big|\partial_p^{\underline{n}} \mathfrak{e}(p)\big|. \tag{184}$$

Let $\mathfrak{e}$ be an admissible dispersion relation. We denote the set of all critical points of $\mathfrak{e}$ by

$$\mathrm{Crit}(\mathfrak{e}) \;:=\; \big\{ p \in \Gamma^* \;\big|\; \nabla \mathfrak{e}(p) = 0 \big\}. \tag{185}$$

Recall that, as Γ^* is compact, dispersion relations have at most finitely many critical points. $\mathrm{Min}(\mathfrak{e}) \subset \mathrm{Crit}(\mathfrak{e})$ denotes the set of points on which the minimum of $\mathfrak{e}$ is taken.

Let $\mathfrak{e}''(p)$ be the Hessian matrix of $\mathfrak{e}$ at $p \in \mathrm{Crit}(\mathfrak{e})$. Define the *minimal curvature of* $\mathfrak{e}$ *at* $p \in \mathrm{Crit}(\mathfrak{e})$ by

$$K(\mathfrak{e}, p) \ := \ \min \left\{ |\lambda|^{1/2} \ \middle| \ \lambda \in \sigma[\mathfrak{e}''(p)] \right\} > 0. \tag{186}$$

Define also the *minimal (critical) curvature of* $\mathfrak{e}$ by

$$K(\mathfrak{e}) \ := \ \min \left\{ K(\mathfrak{e}, p) \ \middle| \ p \in \mathrm{Crit}(\mathfrak{e}) \right\} > 0. \tag{187}$$

Lemma 5.4 (A priori upper bound on $N(e, V)$, $d = 1, 2$)**.** *Let* $\mathfrak{e}$ *be any dispersion relation from* $C^3(\Gamma^*, \mathbb{R})$*. Let* $C < \infty$ *and* $K > 0$ *be such that* $\|\mathfrak{e}\|_{C^3} < C$ *and* $K(\mathfrak{e}) > K$*. Define* $\delta := \min\{\mathfrak{e}(p) \mid p \in \mathrm{Crit}(\mathfrak{e}) \backslash \mathrm{Min}(\mathfrak{e})\} > 0$*.*

(i) There is a constant $C_{5.4(i)} < \infty$ *depending only on* $\mathfrak{e}, C, K, \#\mathrm{Min}(\mathfrak{e})$*, and* δ *such that* $N[\mathfrak{e}, V] \leq \#\mathrm{Min}(\mathfrak{e})$ *whenever* $|V|_{1/2,1} < C_{5.4a}$*.*

(ii) There is a constant $C_{5.4(ii)} < \infty$ *depending only on* $\mathfrak{e}, C, K, \#\mathrm{Min}(\mathfrak{e})$*, and* δ *such that*

$$N[\mathfrak{e}, V] \ \leq \ C_{5.4(ii)} \, |V|_{1/2,2} \ + \ \#\mathrm{Min}(\mathfrak{e}). \tag{188}$$

Proof:
Let $C^1(\Gamma^*)$ be the Banach space of all continuously differentiable functions $\Gamma^* \to \mathbb{C}$ with norm $\| \cdot \|_{C_1}$. Observe that if $|V|_{1/2,1}$ is finite $\mathcal{F}^* \circ V^{1/2}$ defines a continuous linear map $\ell^2(\Gamma) \to C_1(\Gamma^*)$ with

$$\| \mathcal{F}^* \circ V^{1/2} \|_{\mathcal{B}[\ell^2(\Gamma), C_1(\Gamma^*)]} \leq |V|_{1/2,1}^{1/2}. \tag{189}$$

Let $\mathrm{Min}(\mathfrak{e}) = \{p^{(1)}, \ldots, p^{(m)}\}$, $m = \#\mathrm{Min}(\mathfrak{e})$, and define the linear functionals ζ_i, $i = 1, 2, \ldots, m$, on $\ell^2(\Gamma)$ by $\zeta_i(\varphi) := \mathcal{F}^* \circ V^{1/2}(\varphi)(p^{(i)})$. By (189), the functionals ζ_i are continuous. Let $X = \bigcap_{i=1}^m \ker \zeta_i$. Assume that $H(\mathfrak{e}, V)$ has more than m eigenvalues (counting multiplicities) below 0. Then, by Lemma 2.2 and the min-max principle, there is some $\rho > 0$ and some $(m+1)$–dimensional subspace $S \subset \ell^2(\Gamma)$ with

$$\min_{\varphi \in S, \ |\varphi|_2 = 1} \langle \varphi \, | \, V^{1/2} (\rho + h(\mathfrak{e}))^{-1} V^{1/2} \varphi \rangle > 1. \tag{190}$$

Observe that for all $\varphi \in \ell^2(\Gamma)$,

$$\langle \varphi \, | \, V^{1/2} (\rho + h(\mathfrak{e}))^{-1} V^{1/2} \varphi \rangle = \int_{\Gamma^*} \frac{|\mathcal{F}^* \circ V^{1/2}(\varphi)(p)|^2}{\rho + \mathfrak{e}(p)} d\mu^*(p). \tag{191}$$

As the dimension of S is larger than m, there is a vector $\tilde{\varphi} \in S \cap X$, $|\tilde{\varphi}|_2 = 1$. Notice that in this case there is a constant $\mathrm{const} < \infty$ depending only on C and m such that for all $p \in \Gamma^*$,

$$|\mathcal{F}^* \circ V^{1/2}(\tilde{\varphi})(p)|^2 \leq \mathrm{const}\, |V|_{1/2,1} \prod_{i=1}^{m} (1 - \cos(p - p^{(i)})), \qquad (192)$$

where for each $q = (q_1, \ldots, q_d) \in \Gamma_d^*$,

$$\cos(q) := d^{-1}(\cos(q_1) + \ldots + \cos(q_d)).$$

It means that

$$1 < \mathrm{const}\, |V|_{1/2,1} \int_{\Gamma^*} \frac{\prod_{i=1}^{m}(1 - \cos(p - p^{(i)}))}{\rho + \mathfrak{e}(p)} d\mu^*(p). \qquad (193)$$

Observing that the integral on the right-hand side of (193) is bounded by a constant depending only on C, K and m this concludes the proof of (i).

Now we prove (ii). For any $q \in \Gamma^*$ define the linear maps $\zeta'_q : \ell^2(\Gamma) \to \mathbb{C} \times \mathbb{C}^d$ by

$$\zeta'_q(\varphi) = \Big((\mathcal{F}^* \circ V^{1/2})(\varphi)(q)\, ,\, (\nabla \mathcal{F}^* \circ V^{1/2})(\varphi)(q) \Big). \qquad (194)$$

By $|V|_{1/2,1} \leq |V|_{1/2,2} < \infty$ it follows that ζ'_q is continuous.

There is a constant $\mathrm{const} < \infty$ such that, for any fixed $\mu > 0$ small enough, there is a set of points $\{q_1, \ldots, q_{n(\mu)}\}$ from Γ^* containing $\mathrm{Min}(\mathfrak{e})$ with the property that $n(\mu) \leq \mu^{-1}$ and, for all $q \in \Gamma^*$, $\min_{i=1,2,\ldots,n(\mu)} |q - q_i| \leq \mathrm{const}\, \mu^{1/d}$. If the subspace $S \subset \ell^2(\Gamma)$ has dimension larger than $(d+1)\mu^{-1}$ then there is a vector $\tilde{\varphi} \in S$ with $|\tilde{\varphi}|_2 = 1$ and

$$\tilde{\varphi} \in \bigcap_{j=1}^{n(\mu)} \mathrm{Ker}\, \zeta'_{q_j}. \qquad (195)$$

By Taylor expansions, for such a vector $\tilde{\varphi}$ we have, similarly as in the proof of (i), that for some constant $\mathrm{const} < \infty$ and all $p \in \Gamma^*$:

$$|\mathcal{F}^* \circ V^{1/2}\tilde{\varphi}(p)| \leq \mathrm{const}\, |V|_{1/2,2}^{1/2} \prod_{i=1}^{m} (1 - \cos(p - p_i)), \qquad (196)$$

$$|\mathcal{F}^* \circ V^{1/2}\tilde{\varphi}(p)| \leq \mathrm{const}\, \mu\, |V|_{1/2,2}^{1/2}. \qquad (197)$$

Using the last two inequalities we get

$$\begin{aligned} & |\langle \tilde{\varphi} \,|\, V^{1/2} h(\mathfrak{e})^{-1} V^{1/2} \tilde{\varphi} \rangle| \\ \leq\ & |\mathcal{F}^* \circ V^{1/2}\tilde{\varphi}|_\infty \int_{\Gamma^*} \frac{|\mathcal{F}^* \circ V^{1/2}\tilde{\varphi}(p)|}{\mathfrak{e}(p)} d\mu^*(p) \\ \leq\ & \mathrm{const}\ \mu |V|_{1/2,2}. \end{aligned} \qquad (198)$$

Thus, by (i), Lemma 2.2 and the min-max principle, for some $\mathrm{const} < \infty$, $H(\mathfrak{e}, V)$ has at most $(\mathrm{const}\,|V|_{1/2,2} + m)$ eigenvalues below 0. □

Corollary 5.5 (Semi-classical upper bound on $N[e, V]$, Thm. 1.2 for $d = 1, 2$)**.** *Let $d = 1, 2$ and $\mathfrak{e}$ be any admissible dispersion relation from $C^3(\Gamma^*)$. Then there is a constant $c(\mathfrak{e}) < \infty$ such that for all potentials $V \geq 0$,*

$$N[\mathfrak{e}, V] \leq c(\mathfrak{e})(1 + N_{sc}[\mathfrak{e}, \tilde{V}]), \tag{199}$$

where the effective potential $\tilde{V}$ is given by $\tilde{V}(x) := V(x)|x|^{d+5}$.

Proof: From Lemma 5.4 and Corollary 3.2:

$$\begin{aligned} N[\mathfrak{e}, V] \quad \leq \quad & |\{x \in \Gamma \,|\, \langle x \rangle^{d+5} V(x) \geq \mathfrak{e}_{\max}\}| + \#\mathrm{Min}(\mathfrak{e}) \\ & + C_{5.4(ii)} \left(\sum_{x \in \Gamma,\, \langle x \rangle^{d+5} V(x) < \mathfrak{e}_{\max}} \langle x \rangle^{-\frac{d+1}{2}} [\langle x \rangle^{d+1} \langle x \rangle^4 V(x)]^{1/2} \right)^2 . \end{aligned}$$

Thus, by the Cauchy-Schwarz inequality:

$$\begin{aligned} N[\mathfrak{e}, V] \quad \leq \quad & |\{x \in \Gamma \,|\, \langle x \rangle^{d+5} V(x) \geq \mathfrak{e}_{\max}\}| + \#\mathrm{Min}(\mathfrak{e}) \\ & + C_{5.4(ii)} \left(\sum_{x \in \Gamma} \langle x \rangle^{-(d+1)} \right) \left(\sum_{x \in \Gamma,\, \langle x \rangle^{d+5} V(x) < \mathfrak{e}_{\max}} \langle x \rangle^{d+5} V(x) \right) . \end{aligned}$$

As $\mathfrak{e}$ is a Morse function this implies (199) in the case $d = 2$. Observing that $\langle x \rangle^{d+5} V(x) \leq [\mathfrak{e}_{\max} \langle x \rangle^{d+5} V(x)]^{1/2}$, whenever $\langle x \rangle^{d+5} V(x) \leq \mathfrak{e}_{\max}$, the case $d = 1$ follows from the last inequality as well. □

A Appendix

A.1 Proof of Lemma 1.12

Lemma A.1. *Let $\mathfrak{e}$ be any admissible dispersion relation with $h(\mathfrak{e})_{x,y} \leq 0$ for all $x \neq y$. Then, for all $x \in \Gamma$,*

$$\sum_{y \in \Gamma \setminus \{x\}} h(\mathfrak{e})_{x,y} = -h(\mathfrak{e})_{x,x} = -h(\mathfrak{e})_{0,0}.$$

In particular,

$$\max_{x \in \Gamma} \sum_{y \in \Gamma \setminus \{x\}} |h(\mathfrak{e})_{x,y}| = h(\mathfrak{e})_{0,0}.$$

Proof: By translation invariance of $h(\mathfrak{e})$ it suffices to show $\sum_{y\in\Gamma\setminus\{0\}} h(\mathfrak{e})_{0,y} = -h(\mathfrak{e})_{0,0}$. Since $h(\mathfrak{e})_{0,y} \le 0$ for all $y \ne 0$,

$$\begin{aligned}\sum_{y\in\Gamma\setminus\{0\}} h(\mathfrak{e})_{0,y} &= -h(\mathfrak{e})_{0,0} + \lim_{L\to\infty} \sum_{y\in\Gamma,\,|y|\le L} h(\mathfrak{e})_{0,y} \\ &= -h(\mathfrak{e})_{0,0} + \lim_{L\to\infty} \int_{\Gamma^*} \mathfrak{e}(-p) \sum_{y\in\Gamma,\,|y|\le L} e^{ip\cdot y}\, d\mu^*(p).\end{aligned}$$

As $\mathfrak{e}$ is twice continuously differentiable:

$$\lim_{L\to\infty} \int_{\Gamma^*} \mathfrak{e}(-p) \sum_{y\in\Gamma,\,|y|\le L} e^{ip\cdot y}\, d\mu^*(p) = \mathfrak{e}(0).$$

Finally, by the assumption $\mathfrak{e}$ is admissible, $\mathfrak{e}(0) = 0$. □

Proof of Lemma 1.12: The first part of Lemma 1.12, i.e.,

$$\sum_{y\in\Gamma} |h(\mathfrak{e})_{x,y}| < \infty,$$

follows immediately from the lemma above.

By the Yosida approximation for semi-groups, for all $u \in \ell^2(\Gamma)$ and all $t \ge 0$,

$$e^{-t\,h(\mathfrak{e})}u = \lim_{s\to\infty} e^{-s\,t} \exp\left(s^2 t (s + h(\mathfrak{e}))^{-1}\right) u.$$

Thus it suffices to prove that, for all $s > 0$ and all $u \in \ell^2(\Gamma)$, $u \ge 0$ implies $(s + h(\mathfrak{e}))^{-1} u \ge 0$.

Consider any positive real number $s > 0$ and any vector $u \in \ell^2(\Gamma)$, $u \ge 0$. Let $w := (s + h(\mathfrak{e}))^{-1}u$. Then, for any $v \in \ell^2(\Gamma)$, $v \ge 0$,

$$\begin{aligned}&\langle w + v \mid (s + h(\mathfrak{e}))(w + v)\rangle \\ =\ &\langle w \mid (s + h(\mathfrak{e}))w\rangle + \langle v \mid (s + h(\mathfrak{e}))v\rangle + 2\,\langle u \mid v\rangle \\ \ge\ &\langle w \mid (s + h(\mathfrak{e}))w\rangle + \langle v \mid (s + h(\mathfrak{e}))v\rangle\end{aligned} \tag{200}$$

Observing that the function w is real valued (since $h(\mathfrak{e})_{x,y}$ as only real entries and u is real valued) and choosing $v = |w| - w$, it follows from the inequality above that

$$\langle |w| \mid (s + h(\mathfrak{e}))|w|\rangle - \langle w \mid (s + h(\mathfrak{e}))w\rangle \ge \langle v \mid (s + h(\mathfrak{e}))v\rangle\,.$$

Notice that the assumption $h(\mathfrak{e})_{x,y} \le 0$ for all $x \ne y$ and the fact that $\sum_{y\in\Gamma} |h(\mathfrak{e})_{x,y}| < \infty$ imply

$$\langle |w| \mid (s + h(\mathfrak{e}))|w|\rangle \le \langle w \mid (s + h(\mathfrak{e}))w\rangle\,.$$

These two last inequalities together imply that

$$\langle v \mid (s + h(\mathfrak{e}))v\rangle \leq 0.$$

As the operator $(s + h(\mathfrak{e}))$ is (per assumption) strictly positive, it follows that $v = |w| - w = 0$, i.e., $(s + h(\mathfrak{e}))^{-1}u \geq 0$.

By the Trotter formula, for all $u \in \ell^2(\Gamma)$,

$$e^{-t\,h(\mathfrak{e})}u = e^{-t\,h(\mathfrak{e})_{0,0}} \lim_{n\to\infty} (1 - (h(\mathfrak{e}) - h(\mathfrak{e})_{0,0})/n)^n u. \tag{201}$$

Again by $h(\mathfrak{e})_{x,y} \leq 0$ for all $x \neq y$ and $\sum_{y\in\Gamma} |h(\mathfrak{e})_{x,y}| < \infty$:

$$|e^{-t\,h(\mathfrak{e})}u| \leq e^{-t\,h(\mathfrak{e})_{0,0}} \lim_{n\to\infty} (1 - (h(\mathfrak{e}) - h(\mathfrak{e})_{0,0})/n)^n |u| = e^{-t\,h(\mathfrak{e})}|u|, \tag{202}$$

i.e., $\mathfrak{e}$ dominates itself and is positivity preserving. □

A.2 Proof of Theorem 3.6 and Lemma 3.9

For any $\chi \in C^\infty(\mathbb{R}^d, \mathbb{R})$, define its Gevrey norms by:

$$\|\chi\|_{s,R} := \sum_{\underline{n}\in\mathbb{N}_0^d} \frac{R^{|\underline{n}|}}{(\underline{n}!)^s} \sup_{p\in\mathbb{R}^d} |\partial_p^{\underline{n}}\chi(p)|, \quad s \geq 1,\ R > 0. \tag{203}$$

The function χ is called s-*Gevrey* is for some $R > 0$, $\|\chi\|_{s,R} < \infty$.

Lemma A.2. *Let* $\chi \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$. *Then, for all* $p \in \mathbb{R}^d$,

$$|\hat{\chi}(p)| \leq \|\chi\|_{R,s}\, |\mathrm{supp}\,\chi|\ \exp\left(1 - (e^{-1}R|p|)^{\frac{1}{s}}\right).$$

Here, $|p| := \max\{|p_1|, |p_2|, \ldots, |p_d|\}$, $\hat{\chi}$ *is the Fourier transform of* χ, *and* $|\mathrm{supp}\,\chi|$ *is the volume of the support of the function* χ.

Proof: The bound above holds clearly, if $e^{-1}R|p| \leq 1$. We consider thus only the case $e^{-1}R|p| > 1$. By assumption, for all $n \in \mathbb{N}$:

$$\begin{aligned} |\hat{\chi}(p)| &\leq \frac{(n!)^s}{(R\max\{|p_1|, |p_2|, \ldots, |p_d|\})^n} \|\chi\|_{R,s}\, |\mathrm{supp}\,\chi| \\ &\leq \frac{n^{sn}}{R^n|p|^n} \|\chi\|_{R,s}\, |\mathrm{supp}\,\chi|. \end{aligned} \tag{204}$$

Now use that for all r with $e^{-1}r > 1$

$$\min_{n\in\mathbb{N}} \frac{n^{sn}}{r^n} \leq \max_{\xi\in[-1,0]+(e^{-1}r)^{\frac{1}{s}}} e^{\xi(s\log(\xi)-\log(r))}.$$

□

Lemma A.3 (Poisson summation formula)**.** *Let* $\chi : \mathbb{R}^d \to \mathbb{R}$ *be smooth and assume that* $\operatorname{supp}\chi$ *is compact. Define* $\tilde{\chi} : \Gamma_d^* \to \mathbb{C}$ *by*

$$\tilde{\chi}([p]) := \sum_{x \in \mathbb{Z}^d} \chi(x) e^{ip\cdot x}.$$

For all $p \in [-\pi, \pi)^d$*,*

$$\tilde{\chi}([p]) = (2\pi)^{d/2} \sum_{q \in (2\pi\mathbb{Z})^d} \hat{\chi}(p+q).$$

Corollary A.4. *For all* $p \in [-\pi, \pi)^d$*, all* $R > 1$*, and all* s*,* $1 \le s < \infty$*,*

$$\begin{aligned} |\tilde{\chi}([p]) - (2\pi)^{\frac{d}{2}}\hat{\chi}(p)| &\le \|\chi\|_{R,s} \, |\operatorname{supp}\chi| \, e^{1-R^{\frac{1}{s}}} \\ &\quad (2\pi)^{\frac{d}{2}} \left(\sum_{p' \in \mathbb{Z}^d} e^{-|p'|^{\frac{1}{s}}} \right) \\ &\le \text{const} \, \|\chi\|_{R,s} \, |\operatorname{supp}\chi| \; e^{-R^{\frac{1}{s}}}, \end{aligned}$$

where $\text{const} < \infty$ *is a constant depending only on* s *and* d*.*

Proof of Theorem 3.6: For simplicity, we temporarily assume that the hopping matrix $h(\mathfrak{e})$ has finite range. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be any Gevrey function with: $0 \le \chi(x) \le 1$ for all $x \in \mathbb{R}$; $\chi(x) = 1$ for all x, $|x| \le 1$; and $\chi(x) = 0$ for all x, $|x| \ge 2$. Such a s-Gevrey function exists for any $s > 1$. For each $L, \Delta L > 0$ define the Gevrey function $\tilde{\Phi}_{L,\Delta L} : \mathbb{R}^d \to \mathbb{R}$,

$$\tilde{\Phi}_{L,\Delta L}(x) \; := \; \chi\big((x_1 + L)/\Delta L\big)\chi(x_2/\Delta L)\cdots\chi(x_d/\Delta L). \tag{205}$$

If χ is a s-Gevrey function, by definition of the Gevrey norms, for some $\text{const} < \infty$, some $\Delta L_0 > 0$, and all $L, \Delta L > 0$:

$$\|\tilde{\Phi}_{L,\Delta L}\|_{s,\Delta L/\Delta L_0} \le \text{const}. \tag{206}$$

Let $p^{(0)} \in \operatorname{Min}(\mathfrak{e})$, i.e. $\mathfrak{e}(p^{(0)}) = 0$. Define for each $L, \Delta L > 0$, the vector $\Phi_{L,\Delta L} \in \ell^2(\Gamma)$,

$$\Phi_{L,\Delta L}(x) = e^{ip_0\cdot x}\tilde{\Phi}_{L,\Delta L}(x), \quad x \in \Gamma. \tag{207}$$

By (206), Lemma A.2 and Corollary A.4, for some constant $\text{const} < \infty$ depending only on $\mathfrak{e}$ and all $L, \Delta L \ge 1$:

$$|\langle \Phi_{L,\Delta L} \,|\, h(\mathfrak{e})\Phi_{L,\Delta L}\rangle| \le \text{const} \; (\Delta L)^{-2} \, |\Phi_{L,\Delta L}|_2^2. \tag{208}$$

Observe that, by the assumption (103), for some constant $\mathrm{const} > 0$ and all $L, \Delta L \geq 1$:

$$\langle \Phi_{L,\Delta L} \,|\, V \Phi_{L,\Delta L} \rangle \geq \mathrm{const}\, (L + \Delta L)^{-\alpha} |\Phi_{L,\Delta L}|_2^2. \tag{209}$$

Let $R < \infty$ be the range of the hopping matrix $h(\mathfrak{e})$. Notice that, for all $L, \Delta L > 0$ and all $L', \Delta L' > 0$ with $L + 2\Delta L + R < L' - 2\Delta L' - R$,

$$\langle \Phi_{L,\Delta L} \,|\, H(\mathfrak{e}, V) \Phi_{L',\Delta L'} \rangle = 0. \tag{210}$$

For any fixed $N \in \mathbb{N}$ and $L > 0$, define $L_k, \Delta L_k$, $k = 1, 2, \ldots, N$, by:

$$L_k = k\, L, \quad \Delta L_k = L/8. \tag{211}$$

Then, for L sufficiently large, (210) is satisfied for all $(L, \Delta L) = (L_k, \Delta_k)$, $(L', \Delta L') = (L_l, \Delta_l)$, $k \neq l$. Furthermore, by (208) and (209), as $\alpha < 2$, for L large enough:

$$\langle \Phi_{L_k,\Delta L_k} \,|\, H(\mathfrak{e}, V) \Phi_{L_k,\Delta L_k} \rangle < 0, \quad k = 1, 2, \ldots, N. \tag{212}$$

It follows by the min-max principle that for all $N \in \mathbb{N}$, $N[\mathfrak{e}, V] \geq N$.

Now assume that $h(\mathfrak{e})$ is not necessarily finite range, but still satisfies the bound in (103). Then, for some $\mathrm{const} < \infty$ not depending on L and all $k, l = 1, 2 \ldots, N$, $k \neq l$,

$$\begin{aligned} |\langle \Phi_{L_k,\Delta L_k} \,|\, H(\mathfrak{e}, V) \Phi_{L_l,\Delta L_l} \rangle| &< \mathrm{const}\, L^{-\alpha'} |\Phi_{L_k,\Delta L_k}|_2 |\Phi_{L_l,\Delta L_l}|_2 \\ &= \mathrm{const}\, L^{-\alpha'} |\Phi_{L_1,\Delta L_1}|_2^2. \end{aligned} \tag{213}$$

It follows from this bound, (208), and (209) that

$$\max_{\varphi \in \mathrm{span}\{\Phi_{L_1,\Delta L_1}, \ldots, \Phi_{L_N,\Delta L_N}\},\, |\varphi|_2 = 1} \langle \varphi \,|\, H(\mathfrak{e}, V) \varphi \rangle \leq \mathrm{const}' L^{-\alpha'} - \mathrm{const}\, L^{-\alpha}$$

for some $\mathrm{const} > 0$, $\mathrm{const}' < \infty$ depending on N but not on L. As, by assumption, $\alpha < \alpha'$, the right-hand side of the equation above is strictly negative for L sufficiently large. Thus, by the min-max principle, for all $N \in \mathbb{N}$, $N[\mathfrak{e}, V] \geq N$. □

Proof of Lemma 3.9: Let $\chi : \mathbb{R} \to \mathbb{R}_0^+$ be a smooth function with $\chi(x) = 1$ if $|x - 1/2| \leq 1/2$, and $\chi(x) = 0$ if $|x - 1/2| \geq 3/4$. We will assume that χ is a s–Gevrey function for some $s > 1$. For all $M, m \in \mathbb{N}_0$, all $X \in \mathbb{Z}^d$, and all $\underline{k} \in \{0, 1, \ldots, 2^m - 1\}^d$ define the function $\Phi(M, m \mid X, \underline{k}) : \mathbb{R}^d \to \mathbb{R}_0^+$ by

$$\Phi(M, m \,|\, X, \underline{k})(y) := \prod_{i=1}^{d} \chi \left(2^{M+m} (y_i - 2^{-M} X_i - 2^{-M-m} k_i) \right). \tag{214}$$

Clearly, if $(X,\underline{k}) \neq (X',\underline{k}')$,

$$\mathrm{dist}\,(\mathrm{supp}\,\Phi(M,m\,|\,X,\underline{k})\,,\mathrm{supp}\,\Phi(M,m\,|\,X',\underline{k}')) \geq 2^{-(M+m+2)}. \qquad (215)$$

Let $p^{(0)} \in \mathrm{Min}(\mathfrak{e})$ and let $c_0 < \infty$ be some constant such that for some $\epsilon > 0$ and all $p \in B(p_0,\epsilon)$, $\mathfrak{e}(p) \leq c_0|p-p^{(0)}|^2$. Let further c_1 be a constant with

$$\int_{\mathbb{R}^d} |p|^2\,|\hat{\Phi}(p)|^2\,d^dp \leq c_1 \int_{\mathbb{R}^d} |\hat{\Phi}(p)|^2\,d^dp, \qquad (216)$$

where $\hat{\Phi}$ is the Fourier transform of $\Phi(0,0\,|\,0,0)$.

Let $\mathbf{X} := \{X_1,\ldots,X_N\}$ be the set of points from $\mathbb{Z}^d$ on which

$$2c_0c_1[2^{M+m_n}]^2 < v_-^{(M)}(2^{-M}X_n)\ \mathrm{for\,some}\ m_n \geq 0. \qquad (217)$$

For all $n \in \{1,\ldots,N\}$ let $m_n \in \mathbb{N}_0$ be the largest integer satisfying (217).

For all $L > 0$ define the functions $\Phi^{(L)}_{n,\underline{k}} \in \ell^2(\Gamma)$, $n = \{1,2,\ldots,N\}$, $\underline{k} \in \{0,1,\ldots,2^{m_n}-1\}^d$ by

$$\Phi^{(L)}_{n,\underline{k}}(x) := e^{ip_0\cdot x}\Phi(M,m_n\,|\,X_n,\underline{k})(L^{-1}x). \qquad (218)$$

Using Lemma A.3 we see that, by construction, for all $n = \{1,2,\ldots,N\}$ and all $\underline{k} \in \{0,1,\ldots,2^{m_n}-1\}^d$,

$$\begin{aligned}&\langle\Phi^{(L)}_{n,\underline{k}}\,|\,H(\mathfrak{e},V_L)\Phi^{(L)}_{n,\underline{k}}\rangle\\ &\leq \left[-\frac{1}{2}L^{-2}v_-^{(M)}(2^{-M}X_n)+O(L^{-3})\right]|\Phi^{(L)}_{n,\underline{k}}|_2^2.\end{aligned} \qquad (219)$$

Furthermore, for all $(n,\underline{k})$, $(n',\underline{k}')$, $n,n' \in \{1,2,\ldots,N\}$, $\underline{k} \in \{0,1,\ldots,2^{m_n}-1\}^d$, $\underline{k}' \in \{0,1,\ldots,2^{m_n'}-1\}^d$ with $(n,\underline{k}) \neq (n',\underline{k}')$, we have, for some $\mathrm{const} < \infty$ not depending on L, the following estimate:

$$|\langle\Phi^{(L)}_{n,\underline{k}}\,|\,H(\mathfrak{e},V_L)\Phi^{(L)}_{n',\underline{k}'}\rangle| \leq \mathrm{const}\,L^{-\alpha}|\Phi^{(L)}_{n,\underline{k}}|_2|\Phi^{(L)}_{n',\underline{k}'}|_2. \qquad (220)$$

Finally, (116) follows by using the min-max principle and observing that, by the choice of the numbers m_n, for some $\mathrm{const}' > 0$,

$$2^{dM}2^{dm_n} \geq \mathrm{const}'[v_-^{(M)}(2^{-M}X_n)]^{d/2}.$$

□

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