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RESEARCH ARTICLE

Adaptive Finite Element method for a coefficient inverse problem for the Maxwell's system

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We consider a coefficient inverse problems for the Maxwell' system in 3-D. The coefficient of interest is the dielectric permittivity function. Only backscattering single measurement data are used. The problem is formulated as an optimization problem. The key idea is to use the adaptive finite element method for the solution. Both analytical and numerical results are presented. Similar ideas for inverse problems for the complete time dependent Maxwell's system were not considered in the past.

Keywords: Time-dependent inverse electromagnetic scattering, adaptive finite element methods, a posteriori error estimation.

AMS Subject Classification: 65M15, 65M32, 65M50, 65M60

1. Introduction

In this work we consider an adaptive hybrid finite element/difference method for an electromagnetic coefficient inverse problem (CIP) in the form of a parameter identification problem. Our goal is reconstruct dielectric permittivity ϵ of the media under condition that magnetic permeability $\mu = 1$. We consider the case of a single measurement and use the backscattering data only to reconstruct this coefficient ϵ. Potential applications of our algorithm are in airport security, imaging of land mines, imaging of defects in non-destructive testing, etc.. This is because the dielectric constants of explosives are much higher than ones of regular materials, see tables in http://www.clippercontrols.com/info/dielectric_constants.html.

To solve our inverse problem numerically, we seek to minimize the Tikhonov functional:

$$F(E, \epsilon) = \frac{1}{2}\|E - \tilde{E}\|^2 + \frac{1}{2}\gamma\|\epsilon - \epsilon_0\|^2. \tag{1.1}$$

Here E is the vector of the electric field satisfying Maxwell´s equations and $\tilde{E}$ is observed data at a finite set of observation points at the backscattering side of the boundary, ϵ_0 is the initial guess for ϵ, γ is regularization parameter (Tikhonov regularization), and $\|\cdot\|$ is the discrete L_2 norm. The data $\tilde{E}$ in our computations are generated in experiments, where short electromagnetic impulses are emitted

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on the part of the boundary of the surrounding media. The goal is to recover the unknown spatially distributed function ϵ from the recorded boundary data $\tilde{E}$.

The minimization problem is reformulated as the problem of finding a stationary point of a Lagrangian involving a forward equation (the state equation), a backward equation (the adjoint equation) and an equation expressing that the gradient with respect to the coefficient ϵ vanishes. To approximately obtain the value of ϵ, we arrange an iterative process via solving in each step the forward and backward equations and updating the coefficient ϵ. In our numerical example the regularization parameter γ [13, 32, 33] is chosen experimentally on the basis of the best performance. An analytical study of the question of the choice of the regularization parameter is outside of the scope of this publication. We refer to [17] for a detailed analysis of this interesting topic for the adaptivity technique.

The aim of this work is to derive *a posteriori* error estimate for our CIP and present a numerical example of an accurate reconstruction using adaptive error control. Following Johnson et al. [4, 5, 14, 16, 22], and related works, we shall derive *a posteriori* error estimate for the Lagrangian involving the residuals of the state equation, adjoint state equation and the gradient with respect to ϵ. In this work we use the called all-at-once approach to find Frechét derivative for the Tikhonov functional. Rigorous derivation of the Frechét derivatives for state and adjoint problems as well as of the Frechét derivative of the Tikhonov functional with respect to the coefficient can be performed similarly with [7, 8] and will be done in a forthcoming publication.

Given a finite element mesh, *a posteriori* error analysis shows subdomains where the biggest error of the computed solution is. Thus, one needs to refine mesh in those subdomains. It is important that a posteriori error analysis does not need *a priori* knowledge of the solution. Instead it uses only an upper bound of the solution. In the case of classic forward problems, upper bounds are obtained from *a priori* estimates of solutions [1]. In the case of CIPs, upper bounds are assumed to be known in advance, which goes along well with the Tikhonov concept for ill-posed problems [13, 33].

A posteriori error analysis addresses the main question of the adaptivity: *Where to refine the mesh?* In the case of classic forward problems this analysis provides upper estimates for differences between computed and exact solutions locally, in subdomains of the original domain, see, e.g. [1, 14–16, 31]. In the case of a forward problem, the main factor enabling to conduct *a posteriori* error analysis is the well-posedness of this problem. However, every CIP is non-linear and ill-posed. Because of that, an estimate of the difference between computed and exact coefficients is replaced by a posteriori estimate of the accuracy of either the Lagrangian [3, 6, 17] or of the Tikhonov functional [7]. Nevertheless, it was shown in the recent publications [4, 8] that an estimate of the accuracy of the reconstruction of the unknown coefficient is possible in CIPs (in particular, see subsection 2.3 and Theorems 7.3 and 7.4 of [8]).

An outline of the work is following: in Section 2.1 we recall Maxwell´s equations and in Section 2.2 we present the constrained formulation of Maxwell´s equations. In Section 3 we formulate our CIP and in Section 4 we introduce the finite element discretization. In Section 5 we present a fully discrete version used in the computations. Next, in Section 6 we establish *a posteriori* error estimate and formulate the adaptive algorithm. Finally, in Section 7 we present computational results demonstrating the effectiveness of the adaptive finite element/difference method on an inverse scattering problem in three dimensions.

2. Statements of forward and inverse problems

2.1. *Maxwell's equations*

The electromagnetic equations in an inhomogeneous isotropic case in the bounded domain $\Omega \subset \mathbb{R}^d,\ d = 2, 3$ with boundary $\partial\Omega$ are described by the first order system of partial differential equations

$$
\begin{aligned}
\frac{\partial D}{\partial t} - \nabla \times H &= -J, \qquad \text{in } \ \Omega \times (0, T),\\
\frac{\partial B}{\partial t} + \nabla \times E &= 0, \qquad \text{in } \ \Omega \times (0, T),\\
D &= \epsilon E,\\
B &= \mu H,\\
E(x, 0) &= E_0(x),\\
H(x, 0) &= H_0(x),
\end{aligned}
\tag{2.1}
$$

where $E(x,t), H(x,t), D(x,t), B(x,t)$ are the electric and magnetic fields and the electric and magnetic inductions, respectively, while $\epsilon(x) > 0$ and $\mu(x) > 0$ are the dielectric permittivity and magnetic permeability that depend on $x \in \Omega$, t is the time variable, T is some final time, and $J(x,t) \in \mathbb{R}^d$ is a (given) current density.

The electric and magnetic inductions satisfy the relations

$$
\nabla \cdot D = \rho, \ \nabla \cdot B = 0 \qquad \text{in } \ \Omega \times (0, T), \tag{2.2}
$$

where $\rho(x,t)$ is a given charge density.

Eliminating B and D from (2.1), we obtain two independent second order systems of partial differential equations

$$
\epsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\mu^{-1} \nabla \times E) = -j, \tag{2.3}
$$

$$
\mu \frac{\partial^2 H}{\partial t^2} + \nabla \times (\epsilon^{-1} \nabla \times H) = \nabla \times (\epsilon^{-1} J), \tag{2.4}
$$

where $j = \frac{\partial J}{\partial t}$. System (2.3)-(2.4) should be completed with appropriate initial and boundary conditions.

2.2. *Constrained formulation of Maxwell´s equations*

To discretize Maxwell's equations are available different formulation. Examples are the edge elements of Nédélec [27], the node-based first-order formulation of Lee and Madsen [24], the node-based curl-curl formulation with divergence condition of Paulsen and Lynch [29], the node-based interior-penalty discontinuous Galerkin FEM [18]. Edge elements are probably the most satisfactory from a theoretical point of view [25]; in particular, they correctly represent singular behavior at reentrant corners. However, they are less attractive for time dependent computations, because the solution of a linear system is required at every time iteration. Indeed, in the case of triangular or tetrahedral edge elements, the entries of the diagonal matrix resulting from mass-lumping are not necessarily strictly positive [12]; therefore, explicit time stepping cannot be used in general. In contrast, nodal elements naturally lead to a fully explicit scheme when mass-lumping is applied [12, 23].

In this work we consider Maxwell's equations in convex geometry without reentrant corners and with smooth coefficient ϵ where value of ϵ does not varies much. Since we consider applications of our method in airport security and imaging of land mines such assumptions are natural. Thus, we are able use the node-based curl-curl formulation with divergence condition of Paulsen and Lynch [29]. Direct application of standard piecewise continuous $[H^1(\Omega)]^3$- conforming FE for the numerical solution of Maxwell's equations can result in spurious solutions. Following [29] we supplement divergence equations for electric and magnetic fields to enforce the divergence condition and reformulate Maxwell equations as a constrained system:

$$\epsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\nabla \times E) - s\nabla(\nabla \cdot E) = -j, \tag{2.5}$$

and

$$\frac{\partial^2 H}{\partial t^2} + \nabla \times (\epsilon^{-1}\nabla \times H) - s\nabla(\epsilon^{-1}\nabla \cdot H) = \nabla \times (\epsilon^{-1}J), \tag{2.6}$$

respectively, where $s > 0$ denotes the penalty factor. Here and below we assume that electric permeability $\mu = 1$.

For simplicity, we consider the system (2.5) – (2.6) with homogeneous initial conditions

$$\frac{\partial E}{\partial t}(x,0) = E(x,0) = 0, \qquad \text{in } \ \Omega, \tag{2.7}$$

$$\frac{\partial H}{\partial t}(x,0) = H(x,0) = 0, \qquad \text{in } \ \Omega, \tag{2.8}$$

and perfectly conducting boundary conditions

$$E \times n = 0, \qquad \text{on } \ \partial\Omega \times (0,T), \tag{2.9}$$

$$H \cdot n = 0. \qquad \text{on } \ \partial\Omega \times (0,T), \tag{2.10}$$

where n is the outward normal vector on $\partial\Omega$. The choice of the parameter s depends on how much emphasis one places on the gauge condition; the optimal choice is $s = 1$ [21, 29].

2.3. *Statements of forward and inverse problems*

In this work as the forward problem we consider Maxwell equation for electric field with homogeneous initial conditions and perfectly conducting boundary conditions

$$\begin{aligned}
\epsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\nabla \times E) - s\nabla(\nabla \cdot E) = -j, \ x \in \Omega, \ 0 < t < T, \\
\nabla \cdot (\epsilon E) = 0, \ x \in \Omega, \ 0 < t < T, \\
\frac{\partial E}{\partial t}(x,0) = E(x,0) = 0, \qquad \text{in } \ \Omega, \\
E \times n = 0, \qquad \text{on } \ \partial\Omega \times (0,T).
\end{aligned} \tag{2.11}$$

The inverse problem for (2.6), (2.8), (2.10) can be formulated similarly and is not considered in this work. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with the boundary $\partial\Omega \in C^3$. We assume that the coefficient $\epsilon(x)$ of equation (2.11) is such that

$$\epsilon(x) \in [1, d], d = const. > 1, \epsilon(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega, \tag{2.12}$$

$$\epsilon(x) \in C^2(\mathbb{R}^3). \tag{2.13}$$

We consider the following

Inverse Problem. *Suppose that the coefficient $\epsilon(x)$ satisfies (2.12) and (2.13), where the number $d > 1$ is given. Assume that the function $\epsilon(x)$ is unknown in the domain Ω. Determine the function $\epsilon(x)$ for $x \in \Omega$, assuming that the following function $\tilde{E}(x,t)$ is known*

$$E(x,t) = \tilde{E}(x,t), \forall (x,t) \in \partial\Omega \times (0,\infty). \tag{2.14}$$

A priori knowledge of upper and lower bounds of the coefficient $\epsilon(x)$ corresponds well with the Tikhonov concept about the availability of a priori information for an ill-posed problem [13, 33]. In applications the assumption $\epsilon(x) = 1$ for $x \in \mathbb{R}^3 \setminus \Omega$ means that the target coefficient $\epsilon(x)$ has a known constant value outside of the medium of interest Ω. The function $\tilde{E}(x,t)$ models time dependent measurements of the electric wave field at the boundary of the domain of interest. In practice measurements are performed at a number of detectors. In this case the function $\tilde{E}(x,t)$ can be obtained via one of standard interpolation procedures, a discussion of which is outside of the scope of this publication.

3. Tikhonov functional and optimality conditions

We reformulate our inverse problem as an optimization problem, where one seek the permittivity $\epsilon(x)$, which result in a solution of equations (2.11) with best fit to time domain observations $\tilde{E}$, measured at a finite number of observation points. Denote $Q_T = \Omega \times (0,T), S_T = \partial\Omega \times (0,T)$. Our goal is minimize Tikhonov functional

$$F(E,\epsilon) = \frac{1}{2}\int_{S_T} (E|_{S_T} - \tilde{E})^2 z_\delta(t) dx dt + \frac{1}{2}\gamma \int_\Omega (\epsilon - \epsilon_0)^2 \ dx, \tag{3.1}$$

where $\tilde{E}$ is the observed electric field, E satisfies the equations (2.11) and thus depends on ϵ, and γ is regularization parameter. Here $z_\delta(t)$ is a cut-off function, which is introduced to ensure that compatibility conditions at $\overline{S}_T \cap \{t = T\}$ are satisfied, and $\delta > 0$ is a small number. So, we choose such a function z_δ that

$$z_\delta \in C^\infty[0,T], \ z_\delta(t) = \begin{cases} 1 & \text{for} t \in [0, T-\delta], \\ 0 & \text{for } t \in \left(T - \frac{\delta}{2}, T\right], \\ 0 < z_\delta < 1 & \text{for } t \in \left(T-\delta, T-\frac{\delta}{2}\right). \end{cases}$$

To solve this minimization problem we introduce the Lagrangian

$$
\begin{aligned}
L(u) = F(E,\epsilon) - \int_{\Omega_T} \epsilon \frac{\partial \lambda}{\partial t}\frac{\partial E}{\partial t}\ dxdt + \int_{\Omega_T} (\nabla \times E)(\nabla \times \lambda)\ dxdt \\
+ \int_{\Omega_T} \nabla \cdot (\epsilon E)\lambda\ dxdt + s\int_{\Omega_T} (\nabla \cdot E)(\nabla \cdot \lambda)\ dxdt + \int_{\Omega_T} j\lambda\ dxdt,
\end{aligned}
\tag{3.2}
$$

where $u = (E, \lambda, \epsilon)$, and search for a stationary point with respect to u satisfying $\forall \bar{u} = (\bar{E}, \bar{\lambda}, \bar{\epsilon})$

$$
L'(u; \bar{u}) = 0, \tag{3.3}
$$

where $L'(u; \cdot)$ is the Jacobian of L at u.

We assume that $\lambda(x,T) = \partial_t \lambda(x,T) = 0$ and seek to impose such conditions on the function λ that in (3.2) $L(E,\lambda,\epsilon) := L(u) = F(E,\epsilon)$. In other words, the sum of integral terms in (3.2) should be equal to zero. Then we will come up with the formulation of the so-called adjont problem for the function λ.

To proceed further we use the fact that $\lambda(x,T) = \frac{\partial \lambda}{\partial t}(x,T) = 0$ and $E(x,0) = \frac{\partial E}{\partial t}(x,0) = 0$, together with perfectly conducting boundary conditions $n \times E = n \times \lambda = 0$ and $n \cdot (\nabla \cdot E) = n \cdot E = n \cdot (\epsilon E) = 0$ and $n \cdot (\nabla \cdot \lambda) = n \cdot \lambda = 0$ on $\partial\Omega$. The equation (3.3) expresses that for all $\bar{u}$,

$$
\begin{aligned}
0 = \frac{\partial L}{\partial \lambda}(u)(\bar{\lambda}) = -\int_{\Omega_T} \epsilon \frac{\partial \bar{\lambda}}{\partial t}\frac{\partial E}{\partial t}\ dxdt + \int_{\Omega_T} (\nabla \times E)(\nabla \times \bar{\lambda})\ dxdt \\
+ s\int_{\Omega_T} (\nabla \cdot E)(\nabla \cdot \bar{\lambda})\ dxdt + \int_{\Omega_T} \nabla \cdot (\epsilon E)\bar{\lambda}\ dxdt + \int_{\Omega_T} j\bar{\lambda}\ dxdt,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
0 = \frac{\partial L}{\partial E}(u)(\bar{E}) &= \int_{\Omega_T} (E - \tilde{E})\ \bar{E}\ z_\delta\ dxdt \\
&- \int_{\Omega_T} \epsilon \frac{\partial \lambda}{\partial t}\frac{\partial \bar{E}}{\partial t}\ dxdt + \int_{\Omega_T} (\nabla \times \lambda)(\nabla \times \bar{E})\ dxdt \\
&+ s\int_{\Omega_T} (\nabla \cdot \lambda)(\nabla \cdot \bar{E})\ dxdt - \int_{\Omega_T} \epsilon \nabla \lambda \bar{E}\ dxdt,
\end{aligned}
\tag{3.5}
$$

$$
0 = \frac{\partial L}{\partial \epsilon}(u)(\bar{\epsilon}) = -\int_{\Omega_T} \frac{\partial \lambda}{\partial t}\frac{\partial E}{\partial t}\bar{\epsilon}\ dxdt - \int_{\Omega_T} E\nabla\lambda\bar{\epsilon}\ dxdt + \gamma\int_{\Omega} (\epsilon - \epsilon_0)\bar{\epsilon}\ dx,\ x \in \Omega,
\tag{3.6}
$$

The equation (3.4) is the weak formulation of the state equation (2.5) and the equation (3.5) is the weak formulation of the following adjoint problem

$$
\begin{aligned}
\epsilon\frac{\partial^2\lambda}{\partial t^2} + \nabla\times(\nabla\times\lambda) - s\nabla(\nabla\cdot\lambda) &= -(E-\tilde{E})z_\delta,\ x\in\Omega,\ 0<t<T,\\
\nabla\cdot(\epsilon\lambda) &= 0,\ x\in\Omega,\ 0<t<T,\\
\lambda(\cdot,T) &= \frac{\partial\lambda}{\partial t}(\cdot,T) = 0,\\
\lambda\times n &= 0 \text{ on } S_T.
\end{aligned}
\tag{3.7}
$$

Further, (3.6) expresses stationarity with respect to ϵ.

4. Finite element discretization

We discretize $\Omega\times(0,T)$ denoting by $K_h=\{K\}$ a partition of the domain Ω into tetrahedra K ($h=h(x)$ being a mesh function defined as $h|_K=h_K$ representing the local diameter of the elements), and we let J_k be a partition of the time interval $(0,T)$ into time intervals $J=(t_{k-1},t_k]$ of uniform length $\tau=t_k-t_{k-1}$. We assume also a minimal angle condition on the K_h [9].

To formulate the finite element method for (3.3) we introduce the finite element spaces V_h, W_h^E and W_h^λ defined by :

$$
\begin{aligned}
V_h &:= \{v\in L_2(\Omega): v\in P_0(K), \forall K\in K_h\},\\
W^E &:= \{w\in[H^1(\Omega\times I)]^3 : w(\cdot,0)=0, w\times n|_{\partial\Omega}=0\},\\
W_h^E &:= \{w\in W^E : w|_{K\times J}\in[P_1(K)\times P_1(J)]^3, \forall K\in K_h, \forall J\in J_k\},\\
W^\lambda &:= \{w\in[H^1(\Omega\times I)]^3 : w(\cdot,T)=0, w\times n|_{\partial\Omega}=0\},\\
W_h^\lambda &:= \{w\in W^\lambda : w|_{K\times J}\in[P_1(K)\times P_1(J)]^3, \forall K\in K_h, \forall J\in J_k\},
\end{aligned}
$$

where $P_1(K)$ and $P_1(J)$ are the set of continuous piecewise linear functions on K and J, respectively.

We define $U_h=W_h^E\times W_h^\lambda\times V_h$. The finite element method now reads: Find $u_h\in U_h$, such that

$$
L'(u_h)(\bar{u}) = 0\ \forall\bar{u}\in U_h. \tag{4.1}
$$

5. Fully discrete scheme

We expand E,λ in terms of the standard continuous piecewise linear functions $\varphi_i(x)$ in space and $\psi_i(t)$ in time and substitute this into (2.11) and (3.7) to obtain the following system of linear equations:

$$
\begin{aligned}
M(\mathbf{E}^{k+1}-2\mathbf{E}^k+\mathbf{E}^{k-1}) &= -\tau^2F^k - \tau^2K\mathbf{E}^k - s\tau^2C\mathbf{E}^k - \tau^2B\mathbf{E}^k,\\
M(\boldsymbol{\lambda}^{k+1}-2\boldsymbol{\lambda}^k+\boldsymbol{\lambda}^{k-1}) &= -\tau^2S^k - \tau^2K\boldsymbol{\lambda}^k - s\tau^2C\boldsymbol{\lambda}^k - \tau^2B\boldsymbol{\lambda}^k,
\end{aligned}
\tag{5.1}
$$

with initial conditions :

$$E(\cdot,0) = \frac{\partial E}{\partial t}(\cdot,0) = 0, \tag{5.2}$$
$$\lambda(\cdot,T) = \frac{\partial \lambda}{\partial t}(\cdot,T) = 0. \tag{5.3}$$

Here, M is the block mass matrix in space, K is the block stiffness matrix corresponding to the rotation term, C and B are the stiffness matrices corresponding to the divergence terms, F^k and S^k are the load vectors at time level t_k, $\mathbf{E}^k$ and $\boldsymbol{\lambda}^k$ denote the nodal values of $E(\cdot,t_k)$ and $\lambda(\cdot,t_k)$, respectively, τ is the time step.

The explicit formulas for the entries in system (5.1) at each element e can be given as:

$$\begin{aligned}
M^e_{i,j} &= (\epsilon\ \varphi_i, \varphi_j)_e,\\
K^e_{i,j} &= (\frac{1}{\mu}\ \nabla\times\varphi_i, \nabla\times\varphi_j)_e,\\
C^e_{i,j} &= (\frac{1}{\mu}\ \nabla\cdot\varphi_i, \nabla\cdot\varphi_j)_e,\\
B^e_{i,j} &= (\nabla\cdot(\epsilon\varphi_i), \varphi_j)_e,\\
F^e_{j,m} &= ((j, \varphi_j\psi_m))_{e\times J},\\
S^e_{j,m} &= ((E-\bar{E}, \varphi_j\psi_m))_{e\times J},
\end{aligned} \tag{5.4}$$

where $(\cdot,\cdot)_e$ denotes the $L_2(e)$ scalar product.

To obtain an explicit scheme we approximate M with the lumped mass matrix M^L – see [10, 20, 23]. Next, we multiply (5.1) with $(M^L)^{-1}$ and get the following explicit method:

$$\begin{aligned}
\mathbf{E}^{k+1} =& -\tau^2(M^L)^{-1}F^k + 2\mathbf{E}^k - \tau^2(M^L)^{-1}K\mathbf{E}^k\\
&- s\tau^2(M^L)^{-1}C\mathbf{E}^k - \tau^2(M^L)^{-1}B\mathbf{E}^k - \mathbf{E}^{k-1},\\
\boldsymbol{\lambda}^{k-1} =& -\tau^2(M^L)^{-1}S^k + 2\boldsymbol{\lambda}^k - \tau^2(M^L)^{-1}K\boldsymbol{\lambda}^k\\
&- s\tau^2(M^L)^{-1}C\boldsymbol{\lambda}^k - \tau^2(M^L)^{-1}B\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k+1}.
\end{aligned} \tag{5.5}$$

Finally, to approximate coefficient ϵ can be used one of the gradient-like methods with an appropriate initial guess value ϵ_0. The discrete version of gradient with respect to the coefficient (3.6) takes the form:

$$g_h = -\int_0^T \frac{\partial \lambda_h^k}{\partial t}\frac{\partial E_h^k}{\partial t}\ dxdt - \int_0^T E_h^k \nabla\lambda_h^k dt + \gamma(\epsilon_h^k - \epsilon_0). \tag{5.6}$$

Here, λ_h^k and E_h^K are computed values of the adjoint and forward problems at time moment k using explicit scheme (5.5), and ϵ_h^k is approximated value of the coefficient.

6. An a posteriori error estimate for the Lagrangian and an adaptive algorithm

6.1. *A posteriori error estimate*

Following [3] we now present the main framework in the proof of an a posteriori error estimate for the Lagrangian. Let C denote various constants of moderate size. We write an equation for the error e in the Lagrangian as

$$\begin{aligned} e &= L(u) - L(u_h) \\ &= \int_0^1 \frac{d}{d\epsilon} L(u\epsilon + (1-\epsilon)u_h) d\epsilon \\ &= \int_0^1 L'(u\epsilon + (1-\epsilon)u_h)(u - u_h) d\epsilon \\ &= L'(u_h)(u - u_h) + R, \end{aligned} \tag{6.1}$$

where R denotes (a small) second order term. For full details of the arguments we refer to [2] and [14].

Next, we use the splitting $u - u_h = (u - u_h^I) + (u_h^I - u_h)$ where u_h^I denotes an interpolant of u, the Galerkin orthogonality (4.1) and neglect the term R to get the following error representation:

$$e \approx L'(u_h)(u - u_h^I) = (I_1 + I_2 + I_3), \tag{6.2}$$

where

$$\begin{aligned} I_1 = &- \int_{\Omega_T} \left(\epsilon_h \frac{\partial (\lambda - \lambda_h^I)}{\partial t} \frac{\partial E_h}{\partial t} \, dxdt + \int_{\Omega_T} (\nabla \times (\lambda - \lambda_h^I))(\nabla \times E_h) \, dxdt \right. \\ &+ s \int_{\Omega_T} (\nabla \cdot E_h)(\nabla \cdot (\lambda - \lambda_h^I)) \, dxdt \\ &+ \int_{\Omega_T} \nabla \cdot (\epsilon_h E_h)(\lambda - \lambda_h^I) \, dxdt + \int_{\Omega_T} j(\lambda - \lambda_h^I) \, dxdt, \end{aligned} \tag{6.3}$$

$$\begin{aligned} I_2 = &\int_{S_T} (E_h - \tilde{E}) \, (E - E_h^I) \, z_\delta \, dxdt - \int_{\Omega_T} \epsilon_h \frac{\partial \lambda_h}{\partial t} \frac{\partial (E - E_h^I)}{\partial t} \, dxdt \\ &+ \int_{\Omega_T} (\nabla \times \lambda_h)(\nabla \times (E - E_h^I)) \, dxdt \\ &- \int_{\Omega_T} \epsilon_h \nabla \lambda_h (E - E_h^I) \, dxdt + s \int_{\Omega_T} (\nabla \cdot \lambda_h)(\nabla \cdot (E - E_h^I)) \, dxdt, \end{aligned} \tag{6.4}$$

$$I_3 = -\int_{\Omega_T} \frac{\partial \lambda_h}{\partial t} \frac{\partial E_h}{\partial t} (\epsilon - \epsilon_h^I) \, dxdt - \int_{\Omega_T} E_h \nabla \lambda_h (\epsilon - \epsilon_h^I) \, dxdt + \gamma \int_\Omega (\epsilon_h - \epsilon_0)(\epsilon - \epsilon_h^I) \, dx. \tag{6.5}$$

To estimate (6.3) we integrate by parts in the first, second and third terms to get:

$$I_1 = \int_{\Omega_T} \Big(\epsilon_h \frac{\partial^2 E_h}{\partial t^2} + \nabla \times (\nabla \times E_h) - s \nabla (\nabla \cdot E_h) + \nabla \cdot (\epsilon_h E_h) + j \Big) (\lambda - \lambda_h^I) \ \ dx\ dt$$
$$+ \sum_k \int_\Omega \epsilon_h \Big[\frac{\partial E_h}{\partial t}(t_k) \Big] \ (\lambda - \lambda_h^I)(t_k)\ dx - \sum_K \int_0^T \int_{\partial K} \Big(n_K \times (\nabla \times E_h) \Big) \ (\lambda - \lambda_h^I) \ \ dsdt$$
$$+ s \sum_K \int_0^T \int_{\partial K} (\nabla \cdot E_h)\ (n_K \cdot (\lambda - \lambda_h^I)) \ \ dS\ dt = J_1 + J_2 + J_3 + J_4, \tag{6.6}$$

where $J_i, i = 1, ..., 4$ denote integrals that appear on the right of (6.6). In particular, J_2, J_3 result from integration by parts in space, whereas $\left[\frac{\partial E_h}{\partial t}\right]$ appears during the integration by parts in time and denotes the jump of the derivative of E_h in time. Here n_K denotes the exterior unit normal to element K.

To estimate J_3 we sum over the element boundaries where each internal side $S \in S_h$ occurs twice. Let E_s denote the function E_h in one of the normal directions of each side S and n_s is outward normal vector on S. Then we can write

$$\sum_K \int_{\partial K} \Big(n_K \times (\nabla \times E_h)\Big) \ (\lambda - \lambda_h^I) \ \ dS = \sum_S \int_S \Big[n_S \times (\nabla \times E_S) \Big] \ (\lambda - \lambda_h^I) \ \ dS, \tag{6.7}$$

where $\Big[n_S \times (\nabla \times E_s)\Big]$ is the tangential jump of $\nabla \times E_h$ computed from the two elements sharing S. We distribute each jump equally to the two sharing triangles and return to a sum over all element edges ∂K as :

$$\sum_S \int_S \Big[n_S \times (\nabla \times E_s) \Big] \ (\lambda - \lambda_h^I) \ \ dS = \sum_K \frac{1}{2} h_K^{-1} \int_{\partial K} \Big[n_S \times (\nabla \times E_s) \Big] \ (\lambda - \lambda_h^I) h_K \ \ dS. \tag{6.8}$$

We formally set $dx = h_K dS$ and replace the integrals over the element boundaries ∂K by integrals over the elements K, to get:

$$\Big| \sum_K \frac{1}{2} h_K^{-1} \int_{\partial K} \Big[n_S \times (\nabla \times E_s) \Big] \ (\lambda - \lambda_h^I) h_K \ \ dS \Big| \le C \int_\Omega \max_{S \subset \partial K} h_K^{-1} \Big| \Big[n_K \times (\nabla \times E_h) \Big] \Big| \cdot \Big| \lambda - \lambda_h^I \Big| \ dx, \tag{6.9}$$

with $\Big[n_K \times (\nabla \times E_h) \Big] \Big|_K = \max_{S \subset \partial K} \Big[n_S \times (\nabla \times E_s) \Big] \Big|_S$. Here and later we denote by C different constants of moderate size.

In a similar way we can estimate J_4 in (6.6):

$$\begin{aligned} J_4 &= s \sum_K \int_{\partial K} (\nabla \cdot E_h)\ (n_K \cdot (\lambda - \lambda_h^I)) \ \ as \\ &= s \sum_S \int_S [\nabla \cdot E_s]\ [n_S \cdot (\lambda - \lambda_h^I)]\ dS \\ &= s\ \sum_K \frac{1}{2} h_K^{-1} \int_{\partial K} [\nabla \cdot E_s]\ [n_S \cdot (\lambda - \lambda_h^I)] h_K\ dS. \end{aligned} \tag{6.10}$$

Again, replacing the integrals over the boundaries by integrals over the elements we get the following estimate for J_4:

$$\Big|J_4\Big| \leq s\ C \int_\Omega \max_{S\subset\partial K} h_K^{-1}\Big|[\nabla\cdot E_h]\Big| \ \cdot \Big|[n_K\cdot(\lambda-\lambda_h^I)]\Big|\ dx, \tag{6.11}$$

with $[\nabla\cdot E_h]\Big|_K = \max_{S\subset\partial K}[\nabla\cdot E_s]\Big|_S$. J_2 is estimated similarly with J_3, J_4.

We substitute expressions for J_2, J_3 and J_4 in (6.6) to get:

$$\begin{aligned}
\big|I_1\big| \leq & \int_{\Omega_T} \Big|\Big(\epsilon_h\frac{\partial^2 E_h}{\partial t^2} + \nabla\times(\nabla\times E_h) - s\nabla(\nabla\cdot E_h) + \nabla\cdot(\epsilon_h E_h) + j\Big)\Big|\cdot\Big|\lambda-\lambda_h^I\Big|\ \ dx\ dt \\
&+ C\int_{\Omega_T} \epsilon_h\tau^{-1}\cdot\big|[\partial E_{ht}]\big|\cdot\big|\lambda-\lambda_h^I\big|\ dxdt \\
&+ C\int_{\Omega_T} \max_{S\subset\partial K} h_K^{-1}\cdot\Big|\Big[n_K\times(\nabla\times E_h)\Big]\Big|\cdot\Big|\lambda-\lambda_h^I\Big|\ dxdt \\
&+ s\ C\int_{\Omega_T} \max_{S\subset\partial K} h_K^{-1}\cdot\big|[\nabla\cdot E_h]\big|\cdot\Big|\Big[n_K\cdot(\lambda-\lambda_h^I)\Big]\Big|\ dxdt.
\end{aligned} \tag{6.12}$$

where

$$[\partial E_{ht}] = [\partial E_{ht_k}] \text{ on } J_k$$

and $[\partial E_{ht_k}]$ is defined as the maximum of the two jumps in time on each time interval J_k:

$$[\partial E_{ht_k}] = \max_{J_k}\left(\left[\frac{\partial E_h}{\partial t}(t_k)\right],\ \left[\frac{\partial E_h}{\partial t}(t_{k+1})\right]\right). \tag{6.13}$$

Next, we use a standard interpolation estimate [3] for $\lambda-\lambda_h^I$ to get

$$\begin{aligned}
\big|I_1\big| \leq & \int_{\Omega_T} \Big|\Big(\epsilon_h\frac{\partial^2 E_h}{\partial t^2} + \nabla\times(\nabla\times E_h) - s\nabla(\nabla\cdot E_h) + \nabla\cdot(\epsilon_h E_h) + j\Big)\Big|\cdot\Big(\tau^2\Big|\frac{\partial^2\lambda}{\partial t^2}\Big| + h^2\big|D_x^2\lambda\big|\Big)\ \ dx\ dt \\
&+ C\int_{\Omega_T} \epsilon_h\tau^{-1}\cdot\big|[\partial E_{ht}]\big|\cdot\Big(\tau^2\Big|\frac{\partial^2\lambda}{\partial t^2}\Big| + h^2\big|D_x^2\lambda\big|\Big)\ dxdt \\
&+ C\int_{\Omega_T} \max_{S\subset\partial K} h_K^{-1}\cdot\Big|\Big[n_K\times(\nabla\times E_h)\Big]\Big|\cdot\Big(\tau^2\Big|\frac{\partial^2\lambda}{\partial t^2}\Big| + h^2\big|D_x^2\lambda\big|\Big)\ dxdt \\
&+ s\ C\int_{\Omega_T} \max_{S\subset\partial K} h_K^{-1}\cdot\big|[\nabla\cdot E_h]\big|\cdot\Big[n_K\cdot\Big(\tau^2\Big|\frac{\partial^2\lambda}{\partial t^2}\Big| + h^2\big|D_x^2\lambda\big|\Big)\Big]\ dxdt.
\end{aligned} \tag{6.14}$$

Next, in (6.14) the terms $\frac{\partial^2 E_h}{\partial t^2}, \nabla\times(\nabla\times E_h), \nabla(\nabla\cdot E_h)$ vanish, since E_h is continuous piecewise linear function. We then estimate $\frac{\partial^2\lambda}{\partial t^2} \approx \frac{\left[\frac{\partial\lambda_h}{\partial t}\right]}{\tau}$ and $D_x^2\lambda \approx$

$\frac{\left[\frac{\partial \lambda_h}{\partial n}\right]}{h}$ to get:

$$
\begin{aligned}
\left|I_1\right| \leq & C \int_{\Omega_T} \left|j + \nabla \cdot (\epsilon_h E_h)\right| \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial n}\right]}{h}\right|\right) \, dxdt \\
& + C \int_{\Omega_T} \epsilon_h \tau^{-1} \left|\left[\partial E_{ht}\right]\right| \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial n}\right]}{h}\right|\right) \, dxdt \\
& + C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot \left|\left[n_K \times (\nabla \times E_h)\right]\right| \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial n}\right]}{h}\right|\right) \, dxdt \\
& + s\ C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot \left|\left[\nabla \cdot E_h\right]\right| \cdot \left[n_K \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial \lambda_h}{\partial n}\right]}{h}\right|\right)\right] \, dxdt.
\end{aligned}
\tag{6.15}
$$

We estimate I_2 similarly:

$$
\begin{aligned}
\left|I_2\right| \leq & \int_{\Omega_T} \left| \left(\epsilon_h \frac{\partial^2 \lambda_h}{\partial t^2} + \nabla \times (\nabla \times \lambda_h) - \epsilon_h \lambda_h - s \nabla (\nabla \cdot \lambda_h)\right) \right| \cdot \left|E - E_h^I\right| \, dxdt \\
& + \int_{S_T} \left|E_h - \tilde{E}\right| \cdot \left|E - E_h^I\right| \, z_\delta \, dxdt + \left| \sum_k \int_\Omega \epsilon_h \left[\frac{\partial \lambda_h}{\partial t}(t_k)\right] \, (E - E_h^I)(t_k) \, dx \right| \\
& + \left| \sum_K \int_0^T \int_{\partial K} \left(n_K \times (\nabla \times \lambda_h)\right) \, (E - E_h^I) \, \, dSdt \right| \\
& + s \sum_K \int_0^T \int_{\partial K} (\nabla \cdot \lambda_h) \, (n_K \cdot (E - E_h^I)) \, \, dS \, dt \Big|.
\end{aligned}
\tag{6.16}
$$

Next, we can estimate (6.16) similarly with (6.15) as

$$
\begin{aligned}
\left|I_2\right| \leq & \int_{\Omega_T} \left| \left(\epsilon_h \frac{\partial^2 \lambda_h}{\partial t^2} + \nabla \times (\nabla \times \lambda_h) - \epsilon_h \nabla \lambda_h - s \nabla (\nabla \cdot \lambda_h)\right) \right| \cdot \left|E - E_h^I\right| \, dxdt \\
& + \int_{S_T} \left|E_h - \tilde{E}\right| \cdot \left|E - E_h^I\right| \, z_\delta \, dxdt \\
& + C \int_{\Omega_T} \epsilon_h \tau^{-1} \cdot \left|\left[\partial \lambda_{ht}\right]\right| \cdot \left|E - E_h^I\right| \, dxdt \\
& + C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot \left|\left[n_K \times (\nabla \times \lambda_h)\right]\right| \cdot \left|E - E_h^I\right| \, dxdt \\
& + s\ C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot \left|\left[\nabla \cdot \lambda_h\right]\right| \cdot \left|\left[n_K \cdot (E - E_h^I)\right]\right| \, dxdt.
\end{aligned}
\tag{6.17}
$$

Again, the terms $\frac{\partial^2 \lambda_h}{\partial t^2}, \nabla \times (\nabla \times \lambda_h), \nabla(\nabla \cdot \lambda_h)$ vanish, since λ_h is also continuous piecewise linear function. Finally we get

$$
\begin{aligned}
\big|I_2\big| \leq & \int_{\Omega_T} \big|\epsilon_h \nabla \lambda_h\big| \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial E_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial E_h}{\partial n}\right]}{h}\right|\right) \, dxdt \\
& + \int_{S_T} \big|E_h - \tilde{E}\big| \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial E_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial E_h}{\partial n}\right]}{h}\right|\right) z_\delta \, dxdt \\
& + C \int_{\Omega_T} \epsilon_h \tau^{-1} \cdot \big|\left[\partial \lambda_{ht}\right]\big| \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial E_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial E_h}{\partial n}\right]}{h}\right|\right) \, dxdt \\
& + C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot \Big|\Big[n_K \times (\nabla \times \lambda_h)\Big]\Big| \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial E_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial E_h}{\partial n}\right]}{h}\right|\right) \, dxdt \\
& + s \, C \int_{\Omega_T} \max_{S \subset \partial K} h_K^{-1} \cdot \big|\left[\nabla \cdot \lambda_h\right]\big| \cdot \left[n_K \cdot \left(\tau^2 \left|\frac{\left[\frac{\partial E_h}{\partial t}\right]}{\tau}\right| + h^2 \left|\frac{\left[\frac{\partial E_h}{\partial n}\right]}{h}\right|\right)\right] \, dxdt.
\end{aligned} \tag{6.18}
$$

To estimate I_3 we use a standard approximation estimate in the form $\epsilon - \epsilon_h^I \approx hD_x\epsilon$ to get:

$$
\begin{aligned}
\big|I_3\big| \leq & \, C \int_{\Omega_T} \left|\frac{\partial \lambda_h}{\partial t}\right| \cdot \left|\frac{\partial E_h}{\partial t}\right| \cdot h\big|D_x\epsilon\big| \, dxdt + C \int_{\Omega_T} \big|E_h\big| \cdot \big|\nabla \lambda_h\big| \cdot h\big|D_x\epsilon\big| \, dxdt \\
& + \gamma \, C \int_{\Omega} |\epsilon_h - \epsilon_0| \cdot h\big|D_x\epsilon\big| \, dx \\
& \leq C \int_0^T \int_{\Omega} \left|\frac{\partial \lambda_h}{\partial t}\right| \cdot \left|\frac{\partial E_h}{\partial t}\right| \cdot h\left|\frac{[\epsilon_h]}{h}\right| \, dxdt + C \int_{\Omega_T} \big|E_h\big| \cdot \big|\nabla \lambda_h\big| \cdot h\left|\frac{[\epsilon_h]}{h}\right| \, dxdt \\
& + \gamma \, C \int_{\Omega} |\epsilon_h - \epsilon_0| \cdot h\left|\frac{[\epsilon_h]}{h}\right| \, dx \\
& \leq C \int_0^T \int_{\Omega} \left|\frac{\partial \lambda_h}{\partial t}\right| \cdot \left|\frac{\partial E_h}{\partial t}\right| \cdot \big|[\epsilon_h]\big| \, dxdt + C \int_{\Omega_T} \big|E_h\big| \cdot \big|\nabla \lambda_h\big| \cdot \big|[\epsilon_h]\big| \, dxdt \\
& + \gamma \, C \int_{\Omega} |\epsilon_h - \epsilon_0| \cdot \big|[\epsilon_h]\big| \, dx.
\end{aligned} \tag{6.19}
$$

We therefore obtain the following:

Theorem 5.1 Let $L(u) = L(E, \lambda, \epsilon)$ be the Lagrangian defined in (3.2), and let $L(u_h) = L(E_h, \lambda_h, \epsilon_h)$ be the approximation of $L(u)$. Then the following error representation formula for the error $e = L(u) - L(u_h)$ in the Lagrangian holds:

$$
\begin{aligned}
|e| \leq & \sum_{i=1}^{3} \int_{\Omega_T} R_{E_i} \sigma_{\lambda_1} \, dxdt + \int_{\Omega_T} R_{E_4} \sigma_{\lambda_2} \, dxdt + \int_{S_T} R_{\lambda_1} \sigma_{E_1} z_\delta \, dxdt \\
& + \sum_{i=2}^{4} \int_{\Omega_T} R_{\lambda_i} \sigma_{E_1} \, dxdt + \int_{\Omega_T} R_{\lambda_5} \sigma_{E_2} \, dxdt + \sum_{i=1}^{3} \int_{\Omega_T} R_{\epsilon_i} \sigma_{\epsilon} \, dxdt,
\end{aligned}
\tag{6.20}
$$

where residuals are defined by

$$
\begin{aligned}
& R_{E_1} = \left| j + \nabla \cdot (\epsilon_h E_h) \right|, \qquad R_{E_2} = \epsilon_h \tau^{-1} \left| [\partial E_{ht}] \right|, \qquad R_{E_3} = \max_{S \subset \partial K} \ h_K^{-1} \left| \left[n_K \times (\nabla \times E_h) \right] \right|, \\
& R_{E_4} = s \ \max_{S \subset \partial K} \ h_K^{-1} \left| [\nabla \cdot E_h] \right|, \\
& R_{\lambda_1} = \left| E_h - \tilde{E} \right|, \qquad R_{\lambda_2} = |\epsilon_h \nabla \lambda_h|, \qquad R_{\lambda_3} = \epsilon_h \tau^{-1} \left| [\partial \lambda_{ht}] \right|, \\
& R_{\lambda_4} = \max_{S \subset \partial K} \ h_K^{-1} \left| \left[n_K \times (\nabla \times \lambda_h) \right] \right|, R_{\lambda_5} = s \ \max_{S \subset \partial K} \ h_K^{-1} \left| [\nabla \cdot \lambda_h] \right|, \\
& R_{\epsilon_1} = \left| \frac{\partial \lambda_h}{\partial t} \right| \cdot \left| \frac{\partial E_h}{\partial t} \right|, \qquad R_{\epsilon_2} = |E_h| \cdot |\nabla \lambda_h|, R_{\epsilon_3} = \gamma \ |\epsilon_h - \epsilon_0|,
\end{aligned}
$$

and interpolation errors are

$$
\begin{aligned}
& \sigma_{\lambda_1} = C \left(\tau \left| \left[\frac{\partial \lambda_h}{\partial t} \right] \right| + h \left| \left[\frac{\partial \lambda_h}{\partial n} \right] \right| \right), \ \sigma_{\lambda_2} = C \ \left[n_K \ \cdot \left(\tau \left| \left[\frac{\partial \lambda_h}{\partial t} \right] \right| + h \left| \left[\frac{\partial \lambda_h}{\partial n} \right] \right| \right) \right], \\
& \sigma_{E_1} = C \left(\tau \left| \left[\frac{\partial E_h}{\partial t} \right] \right| + h \left| \left[\frac{\partial E_h}{\partial n} \right] \right| \right), \ \sigma_{E_2} = C \ \left[n_K \ \cdot \left(\tau \left| \left[\frac{\partial E_h}{\partial t} \right] \right| + h \left| \left[\frac{\partial E_h}{\partial n} \right] \right| \right) \right], \\
& \sigma_\epsilon = C \left| [\epsilon_h] \right|.
\end{aligned}
$$

Remark 5.1

If solutions λ_h and E_h to the adjoint and state equations are computed with good accuracy, then we can neglect terms $\sum_{i=1}^{4} \int_{\Omega_T} R_{E_i} \sigma_{\lambda_1} \ dxdt + \sum_{i=1}^{5} \int_{\Omega_T} R_{\lambda_i} \sigma_{E_1} \ dxdt + \int_{\Omega_T} R_{\epsilon_2} \sigma_\epsilon \ dxdt$ in a posteriori error estimation (6.20). Thus the term

$$
N(\epsilon_h) = \left| \int_0^T \frac{\partial \lambda_h}{\partial t} \frac{\partial E_h}{\partial t} dt + \gamma \ (\epsilon_h - \epsilon_0) \right| \tag{6.21}
$$

dominates. This fact is also observed numerically (see next section) and will be explained analytically in forthcoming publication.

Mesh refinement recommendation

From the Theorem 5.1 and Remark 5.1 follows that the mesh should be refined in such subdomain of the domain Ω where values of the function $N(\epsilon_h)$ are close to the number

$$
\max_{\Omega} |N(\epsilon_h)| = \max_{\Omega} \left| \int_0^T \frac{\partial \lambda_h}{\partial t} \frac{\partial E_h}{\partial t} dt + \gamma \ (\epsilon_h - \epsilon_0) \right|. \tag{6.22}
$$

6.2. *The Adaptive algorithm*

In this section we outline our adaptive algorithm using the mesh refinement recommendation of section 5. So, on each mesh we should find an approximate solution of the equation $N(\epsilon_h) = 0$. In other words, we should approximately solve the following equation with respect to the function $\epsilon_h(x)$,

$$\int_0^T \frac{\partial \lambda_h}{\partial t}\frac{\partial E_h}{\partial t} dt + \gamma\left(\epsilon_h - \epsilon_0\right) = 0. \tag{6.23}$$

For each new mesh we first linearly interpolate the function $\epsilon_0(x)$ on it. On every mesh we iteratively update approximations ϵ_h^m of the function ϵ_h, where m is the number of iteration in optimization procedure. To do so, we use the quasi-Newton method with the classic BFGS update formula with the limited storage [28]. Denote

$$g^m(x) = \gamma(\epsilon_h^m - \epsilon_0)(x) + \int_0^T (E_{ht}\lambda_{ht})(x,t,\epsilon_h^m)\, dt,$$

where functions $E_h(x,t,\epsilon_h^m), \lambda_h(x,t,\epsilon_h^m)$ are computed via solving state and adjoint problems with $\epsilon := \epsilon_h^m$.

Based on the mesh refinement recommendation of section 5, we use the following adaptivity algorithm in our computations:

Adaptive algorithm

Step 0. Choose an initial mesh K_h in Ω and an initial time partition J_0 of the time interval $(0,T)$. Start with the initial approximation $\epsilon_h^0 = \epsilon_0$ and compute the sequence of ϵ_h^m via the following steps:

Step 1. Compute solutions $E_h(x,t,\epsilon_h^m)$ and $\lambda_h(x,t,\epsilon_h^m)$ of state and adjoint problems of (2.11) and (3.7) on K_h and J_k.

Step 2. Update the coefficient $\epsilon_h := \epsilon_h^{m+1}$ on K_h and J_k using the quasi-Newton method, see details in [3, 28]

$$\epsilon_h^{m+1} = \epsilon_h^m + \alpha g^m(x),$$

where α is step-size in gradient update [30].

Step 3. Stop computing ϵ_h^m and obtain the function ϵ_h if either $||g^m||_{L_2(\Omega)} \leq \theta$ or norms $||g^m||_{L_2(\Omega)}$ are stabilized. Otherwise set $m := m+1$ and go to step 1. Here θ is the tolerance in quasi-Newton updates.

Step 4. Compute the function $B_h(x)$,

$$B_h(x) = \left| \int_0^T \frac{\partial \lambda_h}{\partial t}\frac{\partial E_h}{\partial t} dt + \gamma\left(\epsilon_h - \epsilon_0\right)\right|.$$

Next, refine the mesh at all points where

$$B_h(x) \geq \beta_1 \max_{\overline{\Omega}} B_h(x). \tag{6.24}$$

Here the tolerance number $\beta_1 \in (0,1)$ is chosen by the user.

Step 5. Construct a new mesh K_h in Ω and a new time partition J_k of the time interval $(0,T)$. On J_k the new time step τ should be chosen in such a way that the CFL condition is satisfied. Interpolate the initial approximation ϵ_0 from the previous mesh to the new mesh. Next, return to step 1 and perform all above steps on the new mesh.

Step 6. Stop mesh refinements if norms defined in step 3 either increase or stabilize, compared with the previous mesh.

7. Numerical example

We test the performance of the adaptive algorithm formulated above on the solution of an inverse electromagnetic scattering problem in three dimensions. In our computational example we consider the domain $\Omega = [-9.0, 9.0] \times [-10.0, -12.0] \times [-9.0, 9.0]$ with an unstructured mesh consisting of tetrahedra. The domain Ω is split into inner domain Ω_1 which contains scatterer, and surrounding outer domain Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. The spherical part of the boundary of the domain Ω_1 we denote as $\partial\Omega_1$ and the boundary of the domain Ω we denote as $\partial\Omega$. The domain Ω_1 is a cylinder covered by spherical surface from top, see Figure 1-a). We set $\epsilon(x) = 10$ inside of the inclusion depicted on Figure 1-b) and $\epsilon(x) = 1$ outside of it. Hence, the inclusion/background contrast in the dielectric permittivity coefficient is $10 : 1$. In our computational test we chose a time step τ according to the Courant-Friedrichs-Levy (CFL) stability condition

$$\tau \leq \frac{\sqrt{\epsilon_{max}} h}{\sqrt{3}}, \tag{7.1}$$

where h is the minimal local mesh size, ϵ_{max} is an upper bound for the coefficient ϵ.

The forward problem in our test is

$$\begin{aligned}
\epsilon \frac{\partial^2 E}{\partial t^2} + \nabla \times (\nabla \times E) - s\nabla(\nabla \cdot E) = 0,\ x \in \Omega,\ 0 < t < T, \\
\nabla \cdot (\epsilon E) = 0,\ x \in \Omega,\ 0 < t < T, \\
\frac{\partial E}{\partial t}(x,0) = E(x,0) = 0, \qquad \text{in } \Omega, \\
E \times n = f(t), \qquad \text{on } \partial\Omega_1 \times [0, t_1], \\
E \times n = 0, \qquad \text{on } \partial\Omega_1 \times (t_1, T], \\
E \times n = 0, \qquad \text{on } \partial\Omega \times [0, T].
\end{aligned} \tag{7.2}$$

Let $\Omega_1 \subset \mathbb{R}^3$ be a convex bounded domain which is split into upper Ω_{up} and lower Ω_{down} domains such that $\Omega_1 = \Omega_{up} \cup \Omega_{down}$. We assume that we need to reconstruct coefficient $\epsilon(x)$ only in Ω_{down} from back reflected data at $\partial\Omega_1$. In other words we assume that the coefficient $\epsilon(x)$ of equation (7.2) is such that

$$\epsilon(x) \in [1, d], d = const. > 1, \epsilon(x) = 1 \text{ for } x \in \mathbb{R}^3 \setminus \Omega_{down}, \tag{7.3}$$

$$\epsilon(x) \in C^2(\mathbb{R}^3). \tag{7.4}$$

In the following example we consider the electrical field which is given as

$$f(t) = -((\sin(100t - \pi/2) + 1)/10) \times n,\ 0 \leq t \leq \frac{2\pi}{100}. \tag{7.5}$$

We initialize (7.5) at the spherical boundary $\partial\Omega_1$ and propagate it into Ω. The observation points are placed on $\partial\Omega_1$. We note, that in actual computations applying adaptive algorithm the number of the observations points on $\partial\Omega_1$ increases from coarse to finer mesh.

As follows from Theorem 5.1, to estimate the error in the Lagrangian we need to compute approximated values of $(E_h, \lambda_h, \epsilon_h)$ together with residuals and interpolation errors. Since the residuals $R_{\epsilon_1}, R_{\epsilon_3}$ dominate we neglect computations of others terms in a posteriori error estimate appearing in (6.20), see also Remark 5.1. We seek the solution of the optimization problem in an iterative process, where we start with a coarse mesh shown in Fig. 1, refine this mesh as in step 6 of Algorithm in section 6, and construct a new mesh and a new time partition.

To generate the data at the observation points, we solve the forward problem (7.2), with function $f(t)$ given by (7.5) in the time interval $t = [0, 36.0]$ with the exact value of the parameters $\epsilon = 10.0, \mu = 1$ inside scatterer, and $\epsilon = \mu = 1.0$ everywhere else in Ω. We start the optimization algorithm with guess values of the parameter $\epsilon = 1.0$ at all points in Ω. The solution of the inverse problem needs to be regularized since different coefficients can correspond to similar wave reflection data on $\partial\Omega_1$. We regularize the solution of the inverse problem by introducing an regularization parameter γ (small).

The computations were performed on four adaptively refined meshes. In Fig. 2-b) we show a comparison of R_{ϵ_1} over the time interval $[25, 36]$ on different adaptively refined meshes. Here, the smallest values of the residual R_{ϵ_1} are shown on the corresponding meshes.

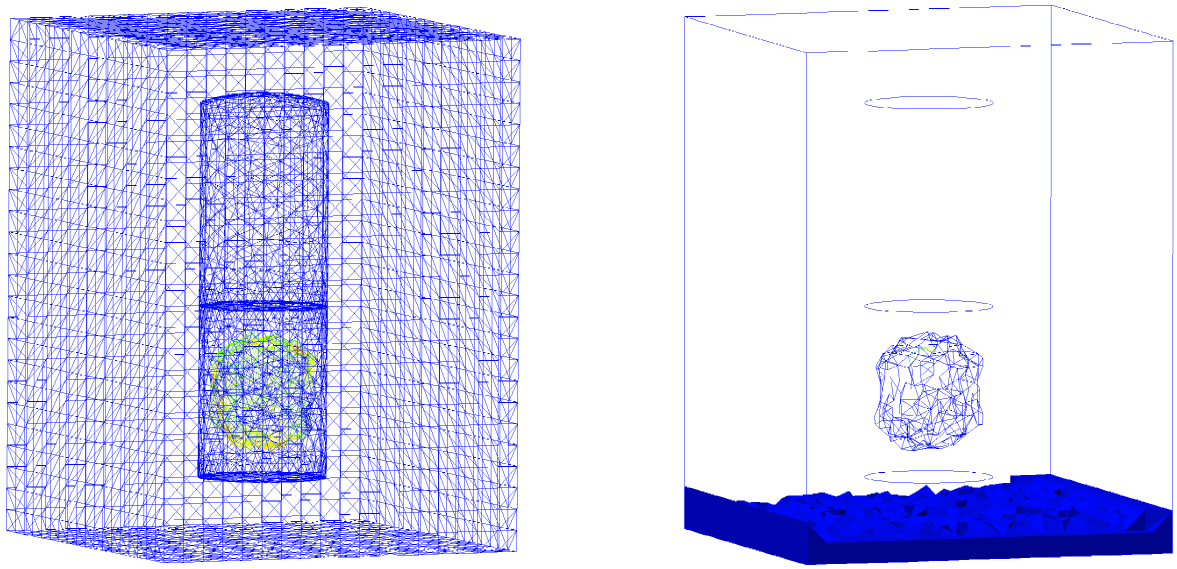
The L_2-norms in space of the adjoint solution λ_h on different optimization iterations on adaptively refined meshes are shown in Fig. 2-a). Here, we solved the adjoint problem backward in time from $t = 36.0$ down to $t = 0.0$. The L_2-norms are presented on the time interval $[25, 36]$ since the solution does not vary much on the time interval $[0, 25)$. We observe, that the norm of the adjoint solution decreases faster on finer meshes.

The reconstructed parameter ϵ on different adaptively refined meshes at the final optimization iteration is presented in Fig. 3. We show isosurfaces of the parameter field $\epsilon(x)$ with a given parameter value. We observe that the qualitative value of the reconstructed parameter ϵ is acceptable only using adaptive error control on finer meshes although the shape of the inclusion is reconstructed sufficiently good on the coarse mesh.

However, since the quasi-Newton method is only locally convergent, the values of the identified parameters are very sensitive to the guess values of the parameters in the optimization algorithm and also to the values of the regularization parameter γ. We use cut-off constrain on the computed parameter ϵ, as also a smoothness indicator to update new values of the parameter ϵ by local averaging over the neighbouring elements. Namely, minimal and maximal values of the coefficient ϵ in box constraints belongs to the following set of admissible parameters $\epsilon \in P = \{\epsilon \in C(\overline{\Omega}) | 1 \leq \epsilon(x) \leq 10\}$.

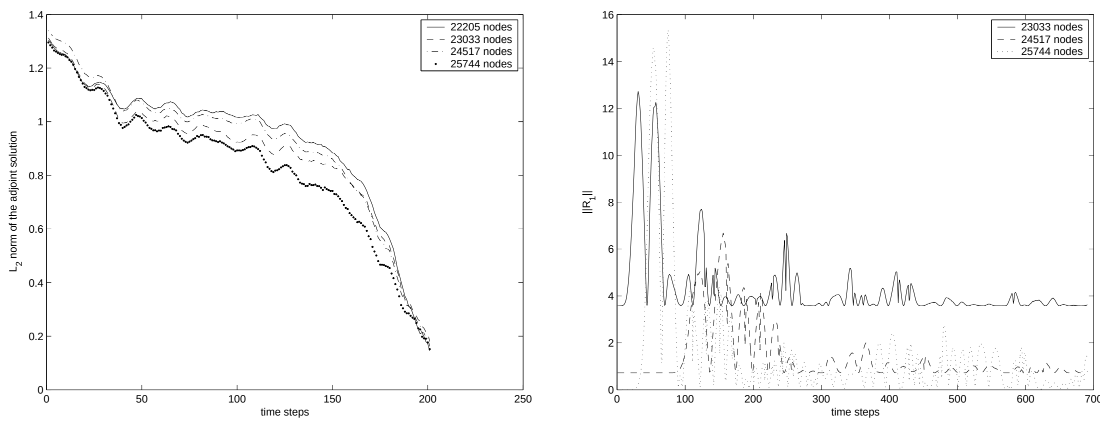
8. Conclusions

We present and adaptive finite element method for an inverse electromagnetic scattering problem. The adaptivity is based on a posteriori error estimate for the associated Lagrangian in the form of space-time integrals of residuals multiplied by weights. We illustrate usefulness of a posteriori error indicator on an inverse electromagnetic scattering problem in three dimensions.



a) Computational mesh   b) Scatterer to be reconstructed

Figure 1. Computational domain Ω



a)   b)

Figure 2. Comparison of a) $||\lambda_h||$ and b) R_{ϵ_1} on different adaptively refined meshes. Here the horizontal x-axis denotes time steps.

9. Acknowledgements

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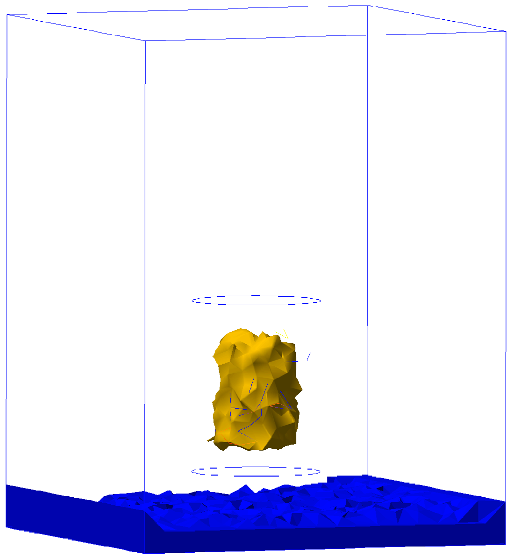
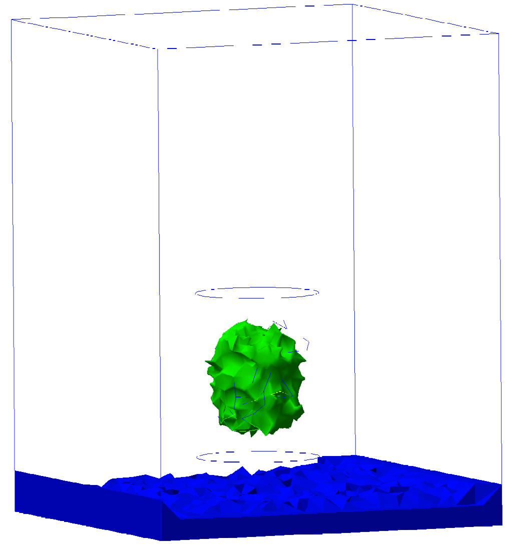
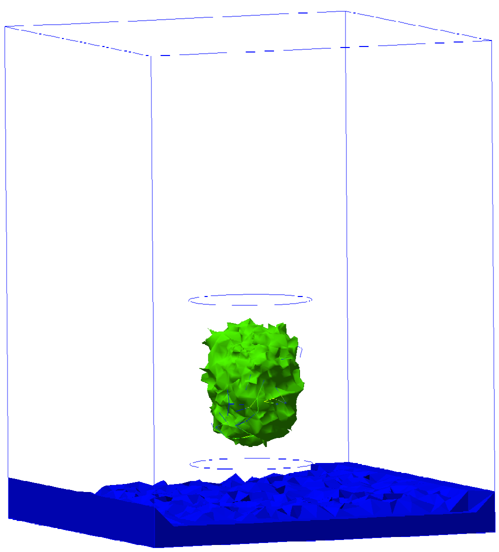


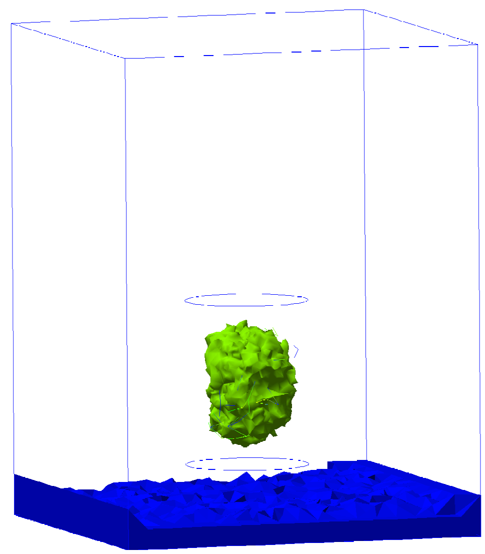
a) 22205 nodes, $\epsilon \approx 3.19$



b) 23033 nodes, $\epsilon \approx 4.84$



c) 24517 nodes, $\epsilon \approx 6.09$



d) 25744 nodes, $\epsilon \approx 7$

Figure 3. Isosurfaces of the parameter ϵ on different adaptively refined meshes.

References

[1] M. Ainsworth and J. Oden, *A Posteriori Error Estimation in Finite Element Analysis*, New York: Wiley, 2000.

[2] R.Becker. Adaptive finite elements for optimal control problems, *Habilitation thesis*, 2001.

[3] L. Beilina and C. Johnson. A hybrid FEM/FDM method for an inverse scattering problem. In *Numerical Mathematics and Advanced Applications - ENUMATH 2001*. Springer-Verlag, 2001.

[4] L.Beilina, C.Johnson. A posteriori error estimation in computational inverse scattering, *Math.Models and Methods in Applied Sciences*, 15(1), 23-35, 2005.

[5] L. Beilina, Adaptive finite element/difference method for inverse elastic scattering waves, *Appl. Comput. Math.*, **2**, 119-134, 2003.

[6] L. Beilina and C. Clason, An adaptive hybrid FEM/FDM method for an inverse scattering problem in scanning acoustic microscopy. *SIAM J. Sci. Comp.*, **28**, 382-402, 2006.

[7] L. Beilina and M. V. Klibanov. A posteriori error estimates for the adaptivity technique for the Tikhonov functional and global convergence for a coefficient inverse problem, *Inverse Problems*, 26, 045012, 2010.

[8] L. Beilina, M. V. Klibanov and M. Yu. Kokurin, Adaptivity with relaxation for ill-posed problems and global convergence for a coefficient inverse problem,*Journal of Mathematical Sciences*, 167 (3) 279-325, 2010.

[9] S. C. Brenner and L. R. Scott, *The Mathematical theory of finite element methods*, Springer-Verlag, Berlin, 1994.

[10] G. C. Cohen, *Highre order numerical methods for transient wave equations*, Springer-Verlag, 2002.

[11] J. E. Dennis and J. J. More, Quasi-Newton methods, motivation and theory, *SIAM Rev.*, 19, 46-89, 1977.

[12] A. Elmkies and P. Joly, Finite elements and mass lumping for Maxwell's equations: the 2D case, *Numerical Analysis, C. R. Acad. Sci. Paris*, 324, 1287-1293, 1997.

[13] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems* (Boston: Kluwer Academic Publishers), 2000.

[14] K. Eriksson and D. Estep and C. Johnson, *Computational Differential Equations*, Studentlitteratur, Lund, 1996.

[15] K. Eriksson and D. Estep and C. Johnson, *Applied Mathematics: Body and Soul. Calculus in Several Dimensions*, Berlin, Springer-Verlag, 2004.

[16] K. Eriksson and D. Estep and C. Johnson, Introduction to adaptive methods for differential equations, *Acta Numeric.*, 105-158, 1995.

[17] A. Griesbaum, B. Kaltenbacher and B. Vexler, Efficient computation of the Tikhonov regularization parameter by goal-oriented adaptive discretization, *Inverse Problems*, 24(025025), 2008.

[18] M. J. Grote, A. Schneebeli, D. Schötzau, Interior penalty discontinuous Galerkin method for Maxwell's equations: Energy norm error estimates, *Journal of Computational and Applied Mathematics*, Elsevier Science Publishers, 204(2) 375-386, 2007.

[19] S. He and S. Kabanikhin, An optimization approach to a three-dimensional acoustic inverse problem in the time domain, *J. Math. Phys.*, 36(8), 4028-4043, 1995.

[20] T.J.R.Hughes, *The finite element method*, Prentice Hall, 1987.

[21] J.Jin, *The finite element methods in electromagnetics*, Wiley, 1993.

[22] C. Johnson, Adaptive computational methods for differential equations, *ICIAM99*, Oxford University Press, 96-104, 2000.

[23] P. Joly, *Variational methods for time-dependent wave propagation problems*, Lecture Notes in Computational Science and Engineering, Springer-Verlag, 2003.

[24] R. L. Lee and N. K. Madsen, A mixed finite element formulation for Maxwell's equations in the time domain, *Journal of Computational Physics*, 88, 284-304, 1990.

[25] P. B. Monk, *Finite Element methods for Maxwell's equations*, Oxford University Press, 2003.

[26] P. B. Monk, A comparison of three mixed methods, *J.Sci.Statist.Comput.*, 13, 1992.

[27] J.C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$, *NUMMA*, 50, 57-81, 1986.

[28] J. Nocedal, Updating quasi-Newton matrices with limited storage, *Mathematics of Comp.*, V.35, N.151, 773–782, 1991.

[29] K. D. Paulsen and D. R. Lynch, Elimination of vector parasities in Finite Element Maxwell solutions, *IEEE Trans.Microwave Theory Tech.*, 39, 395-404, 1991.

[30] O.Pironneau, *Optimal shape design for elliptic systems*, Springer-Verlag, Berlin, 1984.

[31] S. I. Repin, *A Posteriori Estimates for Partial Differential Equations*, Berlin: de Gruiter, 2008.

[32] U. Tautenhahn, Tikhonov regularization for identification problems in differential equations, *Int.Conf. Parameter Identification and Inverse problems in Hydrology, Geology and Ecology*, Dodrecht, Kluwer, 261-270, 1996.

[33] A. N. Tikhonov, A. V. Goncharsky, V. V. Stepanov and A. G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems* (London: Kluwer), 1995.