ON A SUM RULE FOR SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

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ABSTRACT. We study the distribution of eigenvalues of the one-dimensional Schrödinger operator with a complex valued potential V. We prove that if $|V|$ decays faster than the Coulomb potential, then the series of imaginary parts of square roots of eigenvalues is convergent.

1. INTRODUCTION

Let $V : [0,\infty) \mapsto \mathbb{C}$ be a complex valued potential. The object of our investigation is the one-dimensional Schrödinger operator

$$H = -\frac{d^2}{dx^2} + V(x)$$

on the half-line with the Dirichlet boundary condition at zero. Denote by λ_j the eigenvalues of the operator H lying outside of the interval $\mathbb{R}_+ = [0,\infty)$.

We shall consider only potentials from the space $L^1(\mathbb{R}_+)$. It is interesting, that in this case, all non-real eigenvalues λ of H satisfy the estimate

$$|\lambda| \le \Big(\int_0^\infty |V| dx\Big)^2.$$

The proof of this result can be found in [1] (see also [2]). Recently, this result was (partially) generalized to the multi-dimensional case. It was proven in [7], that the condition $|V| \le C(1+|x|)^{-q}$ with $q > 1$implies that all non-real eigenvalues of $-\Delta + V$ are situated in a disk of a finite radius. However, the estimate

$$|\lambda| \le C\Big(\int_{\mathbb{R}^d} (1+|x|)^{1-d} |V| dx\Big)^2$$

has not been proven.

The paper [3] treats the multi-dimensional case. (Everywere below, $\Re z$ and $\Im z$ denote the real and the imaginary parts of z.) The one-dimensional version of the main result of [3] tells us, that for any $t > 0$, the eigenvalues λ_j of H lying outside the sector $\{\lambda : \ |\Im\lambda| < t\,\Re\lambda\}$ satisfy the estimate

$$\sum |\lambda_j|^\gamma \le C \int |V(x)|^{\gamma+1/2} dx, \qquad \gamma \ge 1, \tag{1.1}$$

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where the constant C depends on t and γ (see also [6] for the case when V is real).

Finally, we would like to mention the paper [5]. It deals with the natural question that appears in relation to the main result of [3]: what estimates are valid for the eigenvalues situated inside the conical sector $\{\lambda : \ |\Im\lambda| < t\Re\lambda\}$, where the eigenvalues might be close to the positive half-line? Theorems of the article [5] provide some information about the rate of accumulation of eigenvalues to the set $\mathbb{R}_+ = [0,\infty)$. Namely, [5] gives sufficient conditions on V that guarantee convergence of the sum

$$\sum_{a<\Re\lambda_j<b} |\Im\lambda_j|^\gamma < \infty \tag{1.2}$$

for $0 \le a < b < \infty$.

Both exponents γ in (1.1) and in (1.2) are not less than 1. We suggest a method that allows one to study the case $\gamma = 1/2$.

Theorem 1.1. *Let $V : \mathbb{R}_+ \mapsto \mathbb{C}$ satisfy the condition*

$$\int_0^\infty (1+|x|^p)|V(x)|dx < \infty,$$

for some $p \in (0,1)$. Then

$$\sum_j |\Im\sqrt{\lambda_j}| \le C\Big(\int_0^\infty |x|^p|V(x)|\,dx + \log_+(2||V||_{L^1})\int_0^\infty |V(x)|\,dx\Big),$$

where the positive constant C depends on p, but is independent of V.

2. Proof of Theorem 1.1

1. Before proving the theorem we will acquaint the reader with our notations. As it was already mentioned $\Re z$ and $\Im z$ denote the real and the imaginary parts of z. The class of compact operators T having the property

$$||T||^q_{\mathfrak{S}_q} := \text{tr } (T^*T)^{q/2} < \infty, \qquad q \ge 1,$$

is called the Neumann-Schatten class $\mathfrak{S}_q$. The functional $||T||_{\mathfrak{S}_q}$ is a norm on $\mathfrak{S}_q$. For $T \in \mathfrak{S}_1$ one can introduce $\det(I+T)$ as the product of eigenvalues of $I+T$. Note that

$$|\det(I+T)| \le \exp(||T||_{\mathfrak{S}_1}.$$

Besides $\det(I+T)$, one can introduce the second determinant by setting

$$\det{}_2(I+T) = \det(I+T)e^{-\text{tr } T}.$$

The advantage of this definition is illustrated by the estimate

$$|\det{}_2(I+T)| \le \exp(C||T||_{\mathfrak{S}_2}).$$

2. The basic tool of the proof is the trace formula involving the eigenvalues λ_j and the perturbation determinant $\det(I+VR(z))$ where $R(z) = (-d^2/dx^2 - z)^{-1}$. It is known that the eigenvalues of the operator H are zeros of the function $d(z) = \det(I + VR(z))$. Traditionally, one writes z in the form $z = k^2$ and one considers the function $a(k) = d(k^2)$ with $k \in \mathbb{C}_+$ instead of $d(z)$.

Denote by k_j the zeros of the function $a(k)$ lying in the upper half-plane $\mathbb{C}_+$. We construct the Blaschke product $B(k)$ having the same zeros as $a(k)$

$$B(k) = \prod_j \frac{k - k_j}{k - \overline{k_j}} \frac{k_j}{|k_j|}.$$

It is pretty obvious that the ratio $a(k)/B(k)$ does not have zeros and therefore the function $\log(a(k)/B(k))$ is well defined in the upper half-plane. Moreover, the ratio $a(k)/B(k)$ has the nice property that

$$\Big|\frac{a(k)}{B(k)}\Big| = |a(k)| \qquad \text{if } k \in \mathbb{R}.$$

The trace formula is a relation that involves an integral of the function $\log|a(k)|$ and the zeros k_j. The Blaschke product allows one to separate the contribution of zeros into the trace formula from other contributions. Indeed, since

$$\log B(k) = \log(\prod_j \frac{k_j}{|k_j|}) - 2i\sum_j \frac{\Im k_j}{k} - i\sum_j \frac{\Im k_j^2}{k^2} - 2i\sum_j \frac{\Im k_j^3}{3k^3} + O(k^{-4})$$

as $k \to \infty$, we obtain that the real part of the integral

$$\int_{C_R} \log(B(k))\rho(k)dk, \qquad \rho(k) = (R^2 - k^2),$$

over the contour, consisting of the interval $[-R, R]$ and the half-circle of radius R, equals

$$2\pi R^2 \sum_j \Im k_j - \frac{2\pi}{3} \sum_j \Im k_j^3.$$

for a sufficiently large $R > 0$. It is also clear that

$$\int_{C_R} \log\Big(\frac{a(k)}{B(k)}\Big)\rho(k)dk = 0,$$

since the function $\log\Big(\frac{a(k)}{B(k)}\Big)$ is analytic in the upper half-plane. Thus, we obtain that

$$\int_{C_R} \log(B(k))\rho(k)dk = \int_{C_R} \log(a(k))\rho(k)dk,$$

which implies the equality

$$2\pi R^2 \sum_j \Im k_j - \frac{2\pi}{3} \sum_j \Im k_j^3 = \Re \int_{C_R} \log(a(k))\rho(k)dk.$$

Choose now $R = 2\int |V|dx$. We will shortly see how convenient this choice is, and now we will obtain an estimate of the quantity $\log(a(k))$.

We have to estimate this quantity twice: first time, we have to estimate the absolute value $|\log(a(k))|$ under the condition that $|k| = R$; second time, we will establish an upper estimate of $\log |a(k)|$ on the interval $[-R, R]$.

Let us carry out the computations for $|k| = R$. The arguments are borrowed from [4]. Let us estimate the derivative of the function $\psi(z) = a(k)$, $z = k^2$. We have

$$\psi'(z) = \text{tr } (H - z)V(-d^2/dx^2 - z)^{-1} =$$

$$\sum_{j=0}^{\infty}(-1)^j \text{tr } \Big[(-d^2/dx^2-z)^{-1}WU(W(-d^2/dx^2-z)^{-1}WU)^jW(-d^2/dx^2-z)^{-1}\Big]$$

where $U = V/|V|$ and $W = \sqrt{|V|}$. Since, for $|k| = R$,

$$||W(-d^2/dx^2 - z)^{-1}W|| \leq \frac{\int |V|dx}{|k|} \leq \frac{1}{2},$$

we obtain that

$$\Big|\psi'(z)\Big| \leq C \int |V|dx \int_{-\infty}^{\infty} \frac{d\xi}{|\xi^2 - z|^2} \leq \frac{C_1 \int |V|dx}{|\Im z|^{3/2}}$$

Integrating along the vertical line we will obtain that

$$|\psi(z)| \leq \frac{C_0 \int |V|dx}{|\Im z|^{1/2}}$$

Consequently, for $\phi = Arg(z)$,

$$|\psi(z)||\rho(k)| \leq \frac{C_0 \int |V|dx}{|R\sin(\phi)|^{1/2}}|R^2(1 - e^{i2\phi})| \leq CR \int |V|dx$$

on the circle $\{k : \ |k| = R, \Im k > 0\}$. It implies the following estimate for the integral

$$\Big|\int_{|k|=R, \Im k>0} \log(a(k))\rho(k)dk \ \Big| \leq C\pi R^2 \int |V|dx.$$

Assume now that $k = \bar{k}$. Let us estimate the quantity $\log |a(k)| = \log |\det(I + VR(z))|$ from above. We already know that

$$(2.1) \qquad ||WR(z)W||_{\mathfrak{S}_2} \leq \frac{\int |V|dx}{|k|} \implies \log |a(k)| \leq C\frac{\int |V|dx}{|k|}$$

however this estimate is not suitable for $k \to 0$. Therefore we have to conduct our reasoning in a more delicate way. Consider the integral kernel of the operator $X = WR(z)W$. It is a function of the form

$$cW(x) \int_{-\infty}^{\infty} \frac{\sin(\xi x)\sin(\xi y)}{\xi^2 - z} W(y)d\xi.$$

It follows clearly from this formula that X is representable as the integral

$$X = c \int_{-\infty}^{\infty} \frac{l_\xi^* l_\xi}{\xi^2 - z} d\xi,$$

where the linear functional l_ξ is defined by the relation

$$l_\xi(u) = \int_0^\infty \sin(\xi y) W(y) u(y) dy$$

and acts from $L^2(\mathbb{R}_+)$ to $\mathbb{C}$.

It is obvious that

$$||l_\xi||^2 \le |\xi|^p \int |x|^p |V|\, dx, \quad 0 < p < 1.$$

Moreover $||l_\xi - l_\eta||$ can be estimated in the following way. Since

$$|\sin(\xi y) - \sin(\eta y)| \le 2 |\sin\Big((\frac{\xi - \eta}{2}) y \Big)| \le C |\xi - \eta|^{p/2} |y|^{p/2},$$

we obtain that

$$||l_\xi - l_\eta|| \le C|\xi - \eta|^{p/2} \Big(\int |x|^p |V(x)|\, dx \Big)^{1/2}.$$

Consider now the operator $G_\xi = l_\xi^* l_\xi$. It is clear that

$$||G_\xi||_{\mathfrak{S}_1} \le |\xi|^p \int |x|^p |V|\, dx, \quad 0 < p < 1.$$

Moreover,

$$||G_\xi - G_\eta||_{\mathfrak{S}_1} \le ||l_\xi - l_\eta|| (||l_\xi|| + ||l_\eta||) \le$$
$$\le C|\xi - \eta|^{p/2} (|\xi|^{p/2} + |\eta|^{p/2}) \Big(\int |x|^p |V(x)|\, dx \Big)$$

Therefore the following representation of the operator X

$$X = c\Big(\int_{-\infty}^\infty \frac{G_\xi - G_\eta}{\xi^2 - z} d\xi + \frac{\pi i G_\eta}{k} \Big), \qquad \eta = |\Re z|^{1/2}$$

implies that

$$||X||_{\mathfrak{S}_1} \le C \Big(\int_{-\infty}^\infty \frac{|\xi - \eta|^{p/2} (|\xi|^{p/2} + |\eta|^{p/2})}{|\xi^2 - \eta^2|} d\xi + \frac{\eta^p}{|k|} \Big) \int_0^\infty |x|^p |V(x)|\, dx.$$

If $k \in \mathbb{R}$ is real , then we obtain that

$$||X||_{\mathfrak{S}_1} \le \frac{C}{|k|^{1-p}} \int_0^\infty |x|^p |V(x)|\, dx. \tag{2.2}$$

Combining (2.1) with (2.2) we derive the following estimates

$$\log |a(k)| \le \begin{cases} \frac{C}{|k|^{1-p}} \int_0^\infty |x|^p |V(x)| dx \ \text{ for } \ |k| \le 1, \\ \\ \frac{C}{|k|} \int_0^\infty |V(x)|\, dx \ \text{ for } \ |k| > 1. \end{cases}$$

Therefore,

$$\int_{-R}^R \log |a(k)| \rho(k) dk \le R^2 \int_{-R}^R \log |a(k)| dk \le$$

$$R^2C\Big(\int_0^\infty |x|^p|V(x)|\,dx+\log_+(2||V||_{L^1})\int_0^\infty |V(x)|\,dx\Big).$$

Let us summarize the results: we proved that

$$\sum_j \Im k_j - \frac{1}{3R^2}\sum_j \Im k_j^3 \le C\Big(\int_0^\infty |x|^p|V(x)|\,dx+\log_+(2||V||_{L^1})\int_0^\infty |V(x)|\,dx\Big)$$

It remains to notice that $|k_j|\le\int|V|dx=R/2$, which implies that

$$\frac{1}{3R^2}\Im k_j^3\le\frac14\Im k_j.$$

The proof is completed.

REFERENCES

[1] Abramov, A.A., Aslanyan, A., Davies, E.B.: *Bounds on complex eigenvalues and resonances*. J. Phys. A **34**, 57–72 (2001)

[2] Davies, E.B., Nath, J.: *Schrödinger operators with slowly decaying potentials*. J. Comput. Appl. Math. **148** (1), 1–28 (2002)

[3] Frank, Rupert L.; Laptev, Ari; Lieb, Elliott H.; Seiringer, Robert *Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials*. Lett. Math. Phys. **77** (2006), no. 3, 309–316.

[4] Killip, R.: *Perturbations of one-dimensional Schrödinger operators preserving the absolutely continuous spectrum*, Int. Math. Res. Not. 2002, **no. 38**, 2029–2061.

[5] Laptev, A. and Safronov, O: *Eigenvalue estimates for Schrödinger operators with complex potentials*, submitted.

[6] Lieb, E. H. and Thirring, W.:*Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, in Studies in Mathematical Physics (Essays in Honor of Valentine Bargmann), 269–303. Princeton Univ. Press, Princeton, NJ, 1976.

[7] Safronov, O: *Estimates for eigenvalues of the Schrödinger operator with a complex potential*, submitted.

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