ESTIMATES FOR EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A COMPLEX POTENTIAL

OLEG SAFRONOV

ABSTRACT. We study the distribution of eigenvalues of the Schrödinger operator with a complex valued potential V. We prove that if $|V|$ decays faster than the Coulomb potential, then all eigenvalues are in a disc of a finite radius.

1. INTRODUCTION

We consider the Schrödinger operator $H = -\Delta + V$ with a complex potential V and then we study the distribution of eigenvalues of H in the complex plane.

Our work in this direction was motivated by the question of E.B. Davies about an integral estimate for eigenvalues of H (see [1] and [2]). If $d = 1$ then all eigenvalues of H which do not belong to $\mathbb{R}_+ = [0, \infty)$ satisfy

$$|\lambda| \leq \frac{1}{4}\Big(\int_{\mathbb{R}} |V(x)|dx\Big)^2.$$

The question is whether something similar holds in dimension $d \geq 2$. We prove the following result related directly to this matter.

Theorem 1.1. *Let $V : \mathbb{R}^d \mapsto \mathbb{C}$ satisfy the condition*

$$|V(x)| \leq \frac{L}{(1+|x|^2)^{p/2}}, \qquad 1 < p < 3,$$

with a constant $L > 0$. Let $\varkappa = (p-1)/2$ and let $\epsilon > 0$ be an arbitrarily small number that belongs to the intersection of the intervals $(0, (1-\varkappa)/2) \cap (0, 1/2)$. Then any eigenvalue $\lambda \notin \mathbb{R}_+$ of H with $\Re\lambda > 0$ satisfies one of the conditions:

$$1) \text{ either } \quad |\lambda| \leq 1$$

or

$$2) \quad 1 \leq CL\Big(|\Re\lambda|^{(\varkappa+2\varepsilon-1)/2} + |\lambda|^{\varepsilon-1/2} + \frac{1+|\lambda|^{\varepsilon}}{(|\lambda|-1)}\Big)$$

where the constant C depends on the dimension d and on the parameters p and ε. In particular, it means that all non-real eigenvalues are in a disc of a finite radius.

2000 *Mathematics Subject Classification.* 47F05.

Key words and phrases. Eigenvalue estimates, complex potentials, Schrödinger operators.

The study of eigenvalue estimates for operators with a complex potential already has a bibliography. Besides [1] and [2], we would like to mention the papers [3] and [4]. The main result of [3] tells us, that for any $t > 0$, the eigenvalues z_j of H lying outside the sector $\{z : \ |\Im z| < t\, \Re z\}$ satisfy the estimate

$$\sum |z_j|^\gamma \le C \int |V(x)|^{\gamma+d/2} dx, \qquad \gamma \ge 1,$$

where the constant C depends on t, γ and d (see also [5] for the case when V is real).

The paper [4] deals with natural question that appears in relation to the main result of [3]: what estimates are valid for the eigenvalues situated inside the conical sector $\{z : \ |\Im z| < t\Re z\}$, where the eigenvalues might be close to the positive half-line? Theorems of the article [4] provide some information about the rate of accumulation of eigenvalues to the set $\mathbb{R}_+ = [0, \infty)$. Namely, [4] gives sufficient conditions on V that guarantee convergence of the sum

$$\sum_{a<\Re z_j<b} |\Im z_j|^\gamma < \infty$$

for $0 \le a < b < \infty$. Moreover, the following result is also proven in [4]:

Theorem 1.2. *Let V be a function from $L^p(\mathbb{R}^d)$, where $p \ge d/2$, if $d \ge 3$¿; $p > 1$, if $d = 2$, and $p \ge 1$, if $d = 1$. Then every eigenvalue λ of the operator $H = -\Delta + V$ with the property $\Re\lambda > 0$ satisfies the estimate*

$$|\Im\lambda|^{p-1} \le |\lambda|^{d/2-1} C \int_{\mathbb{R}^d} |V|^p dx. \tag{1.1}$$

The constant C in this inequality depends only on d and p. Moreover, $C = 1/2$ for $p = d = 1$.

2. Proof of Theorem 1.1

Consider first the case $L = 1$. For the sake of convenience we introduce the notations $W = |V|^{1/2}$ and $l = p/2$. According to the Birman-Schwinger principle, a number $\lambda \notin \mathbb{R}_+$ is an eigenvalue of the operator $H = -\Delta + V(x)$ if and only if the number -1 is an eigenvalue of the operator

$$X_0 = W(-\Delta - \lambda)^{-1} W \frac{V}{|V|}.$$

Therefore if λ is a point of the spectrum of the operator H, then $||X_0|| \ge 1$. On the other side, since multiplication by the function $\frac{V}{|V|}$ represents a unitary operator, the condition $||X_0|| \ge 1$ implies that the norm of the operator

$$X = W(-\Delta - \lambda)^{-1} W$$

is also not less than 1.

In order to estimate the norm of the operator X from above, we consider its kernel

$$(2\pi)^{-d}W(x)\int \frac{e^{i\xi(x-y)}}{\xi^2-\lambda}d\xi\, W(y)$$

It follows from this formula that X can be represented in the form

$$X=\int_0^\infty \frac{\Gamma_\rho^*\Gamma_\rho}{\rho^2-\lambda}d\rho,$$

where Γ_ρ is the operator mapping $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{S}_\rho)$, and $\mathbb{S}_\rho$ is the sphere of radius ρ with the center at the point 0:

$$\Gamma_\rho u(\theta)=(2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-i\rho(\theta x)}W(x)u(x)dx$$

The main properties of this operator follow from Sobolev's embedding theorems. Suppose that $W(x)\leq(1+|x|^2)^{-l/2}$ and $u\in L^2(\mathbb{R}^d)$. Then the Fourier transformation of the function $W(x)u(x)$ belongs to the class $H^l(\mathbb{R}^d)$, moreover the norm $||\hat{Wu}||_{H^l}$ is estimated by the norm $||u||_{L^2}$. According to Sobolev's theorems, the embedding of the class $H^l(\mathbb{R}^d)$ into the class $L^2(\mathbb{S}_\rho)$ is continuous under the condition $l>1/2$. Moreover, the norm of the embedding operator depends in a weak manner on the parameter $\rho\geq 1$. Indeed, suppose that the inequality

$$\int_{\mathbb{S}_1}|\phi(\theta)|^2d\theta\leq C\int_{\mathbb{R}^d}\Big(|\nabla^l\phi|^2+|\phi|^2\Big)dx$$

holds for any function $\phi\in H^l(\mathbb{R}^d)$. Then setting $\phi(x)=u(\rho x)$ we obtain that

$$\int_{\mathbb{S}_1}|u(\rho\theta)|^2d\theta\leq C\int_{\mathbb{R}^d}\Big(\rho^{2l}|\nabla^l u(\rho x)|^2+|u(\rho x)|^2\Big)dx.$$

Multiplying both sides of this inequality by ρ^{d-1}, we obtain that

$$\int_{\mathbb{S}_\rho}|u(x)|^2dS\leq C\int_{\mathbb{R}^d}\Big(\rho^{2l-1}|\nabla^l u(x)|^2+\rho^{-1}|u(x)|^2\Big)dx.$$

If l is close to $1/2$ then ρ^{2l-1} practically behaves as a constant. Anyway, without loss of generality we can assume that for $\rho>1$

$$\int_{\mathbb{S}_\rho}|u(x)|^2dS\leq C_\varepsilon\rho^{2\varepsilon}||u||^2_{H^l}$$

where $\varepsilon>0$ is an arbitrary small number. It implies that

$$||\Gamma_\rho||\leq C_\varepsilon\rho^\varepsilon,\qquad \rho\geq 1. \tag{2.1}$$

Moreover, Γ_ρ depends continuously on the parameter ρ in the following sense. Let us introduce the operator U_ρ that transforms functions on the sphere $\mathbb{S}_\rho$ into functions on the sphere $\mathbb{S}=\mathbb{S}_1$. according to the rule

$$U_\rho u(\theta)=u(\rho\theta)\rho^{(d-1)/2}.$$

This operator is unitary and therefore its norm equals 1. Define now the operator $Y_\rho = U_\rho \Gamma_\rho$. Our statement is that

$$||Y_{\rho'} - Y_\rho|| \leq C|\rho' - \rho|^\alpha \rho^\delta(\rho^\varepsilon + (\rho')^\varepsilon)$$

where $\alpha < l - 1/2,\ \delta = l - \alpha - 1/2$ and $\rho' > \rho \geq 1$. Our arguments are similar to those we used in the proof of the inequality (2.1). If we assume that the inequality

$$\int_{\mathbb{S}_1} |\phi((1+h)\theta) - \phi(\theta)|^2 d\theta \leq Ch^{2\alpha} \int_{\mathbb{R}^d} \Big(|\nabla^l \phi|^2 + |\phi|^2\Big) dx$$

holds for any function $\phi \in H^l(\mathbb{R}^d)$. Then the substitution $\phi(x) = u(\rho x)$ will lead to the inequality

$$\int_{\mathbb{S}_1} |u((1+h)\rho\theta) - u(\rho\theta)|^2 d\theta \leq Ch^{2\alpha} \int_{\mathbb{R}^d} \Big(\rho^{2l}|\nabla^l u(\rho x)|^2 + |u(\rho x)|^2\Big) dx.$$

Multiplying both sides of this inequality by ρ^{d-1} and denoting $\rho' = (1+h)\rho$, we obtain that

$$\int_{\mathbb{S}_\rho} |u(\rho^{-1}\rho' x) - u(x)|^2 dS \leq C|\rho' - \rho|^{2\alpha} \int_{\mathbb{R}^d} \Big(\rho^{2\delta}|\nabla^l u(x)|^2 + \rho^{-2l}|u(x)|^2\Big) dx.$$

provided that $\rho' > \rho \geq 1$. This leads to

$$||\Big(\frac{\rho}{\rho'}\Big)^{(d-1)/2} Y_{\rho'} - Y_\rho|| \leq C|\rho' - \rho|^\alpha \rho^\delta.$$

We apply now the triangle inequality to estimate the norm of the difference $Y_{\rho'} - Y_\rho$ for $\rho' > \rho \geq 1$

$$||Y_{\rho'} - Y_\rho|| \leq \Big|\Big(\frac{\rho}{\rho'}\Big)^{(d-1)/2} - 1\Big|\, ||Y_{\rho'}|| + C|\rho' - \rho|^\alpha \rho^\delta \leq C_0(\rho^\varepsilon + (\rho')^\varepsilon)|\rho' - \rho|^\alpha \rho^\delta.$$

To be more convincing, we mention that

$$\Big|\Big(\frac{\rho}{\rho'}\Big)^{(d-1)/2} - 1\Big| \leq \min\{2^{-1}(d-1)|\rho' - \rho|,\, 2\}.$$

Introduce now the notation $G_\rho = \Gamma_\rho^* \Gamma_\rho$. Obviously, G_ρ aslo has representation $G_\rho = Y_\rho^* Y_\rho$. Consequently,

$$||G_{\rho'} - G_\rho|| \leq ||Y_{\rho'}^* - Y_\rho^*|| \cdot ||Y_{\rho'}|| + ||Y_\rho^*|| \cdot ||Y_{\rho'} - Y_\rho|| \leq C(\rho^\varepsilon + (\rho')^\varepsilon)^2 |\rho' - \rho|^\alpha \rho^\delta.$$

Let us summarize the results. The operator X can be written in the form

$$X = \int_0^\infty \frac{G_\rho d\rho}{\rho^2 - \lambda},$$

where

$$||G_\rho|| \leq C\rho^{2\varepsilon}, \qquad \rho \geq 1,$$

and

$$||G_{\rho'} - G_\rho|| \leq C(\rho^\varepsilon + (\rho')^\varepsilon)^2 |\rho' - \rho|^\alpha \rho^\delta, \qquad \rho' > \rho \geq 1.$$

Now, since the integral representation for the operator X can be also rewritten in the form

$$X = \int_1^\infty \frac{(G_\rho - G_\tau)d\rho}{\rho^2 - \lambda} + \int_1^\infty \frac{G_\tau d\rho}{\rho^2 - \lambda} + W(-\Delta - \lambda)^{-1}E[0,1]W$$

where $\tau = |\Re\lambda|^{1/2}$ and $E[0,1]$ is the spectral projection of the operator $-\Delta$ corresponding to the interval $[0,1]$, we obtain that

$$||X|| \leq \int_1^\infty \frac{||G_\rho - G_\tau||d\rho}{|\rho^2 - \lambda|} + \frac{\pi||G_\tau||}{2|\lambda|^{1/2}} + \frac{||V||_{L^\infty} + ||G_\tau||}{(|\lambda| - 1)},$$

for $|\lambda| > 1$. Consequently,

$$||X|| \leq C\int_0^\infty \frac{|\rho - \tau|^\alpha(\rho^\delta + |\Re\lambda|^{\delta/2})(\rho^\varepsilon + |\Re\lambda|^{\varepsilon/2})^2}{|\rho^2 - \Re\lambda|}d\rho + C\frac{\tau^{2\varepsilon}}{|\lambda|^{1/2}} + \frac{||V||_{L^\infty} + C\tau^{2\varepsilon}}{(|\lambda| - 1)},$$

which leads to

$$1 \leq ||X|| \leq C\Big(|\Re\lambda|^{(\alpha+\delta+2\varepsilon-1)/2} + |\lambda|^{\varepsilon-1/2} + \frac{1 + |\lambda|^\varepsilon}{(|\lambda| - 1)}\Big).$$

We proved the statement of the theorem for the case $L = 1$. If $L \neq 1$ then this inequality takes the form

$$1 \leq CL\Big(|\Re\lambda|^{(\alpha+\delta+2\varepsilon-1)/2} + |\lambda|^{\varepsilon-1/2} + \frac{1 + |\lambda|^\varepsilon}{(|\lambda| - 1)}\Big).$$

The proof is completed. □

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E-mail address: osafrono@uncc.edu