Computation of derivatives of the rotation number for parametric families of circle diffeomorphisms

Alejandro Luque Jordi Villanueva

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Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya,
Diagonal 647, 08028 Barcelona (Spain)
`alejandro.luque@upc.edu`, `jordi.villanueva@upc.es`

Abstract

In this paper we present a numerical method to compute derivatives of the rotation number for parametric families of circle diffeomorphisms with high accuracy. Our methodology is an extension of a recently developed approach to compute rotation numbers based on suitable averages of the iterates of the map and Richardson extrapolation. We focus on analytic circle diffeomorphisms, but the method also works if the maps are differentiable enough. In order to justify the method, we also require the family of maps to be differentiable with respect to the parameters and the rotation number to be Diophantine. In particular, the method turns out to be very efficient for computing Taylor expansions of Arnold Tongues of families of circle maps. Finally, we adapt these ideas to study invariant curves for parametric families of planar twist maps.

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Contents

1 Introduction

The rotation number, introduced by Poincaré, is an important topological invariant in the study of the dynamics of circle maps and, by extension, invariant curves for maps or two dimensional invariant tori for vector fields. For this reason, several numerical methods for approximating rotation numbers have been developed during the last years. We refer to the works [3, 4, 8, 13, 14, 21, 24, 31] as examples of methods of different nature and complexity. This last ranges from pure definition of the rotation number to sophisticated and involved methods like frequency analysis. The efficiency of these methods varies depending if the approximated rotation number is rational or irrational. Moreover, even though some of them can be very accurate in many cases, they are not adequate for every kind of application, for example due to violation of their assumptions or due to practical reasons, like the required amount of memory.

Recently, a new method for computing Diophantine rotation numbers of circle diffeomorphisms with high precision at low computational cost has been introduced in [26]. This method is built assuming that the circle map is conjugate to a rigid rotation in a sufficiently smooth way and, basically, it consists in averaging the iterates of the map together with Richardson extrapolation. This construction takes advantage of the geometry and the dynamics of the problem, so it is very efficient in multiple applications. The method is specially suited if we are able to compute the iterates of the map with high precision, for example if we can work with computer arithmetic having a large number of decimal digits.

The goal of this paper is to extend the method of [26] in order to compute derivatives of the rotation number with respect to parameters in families of circle diffeomorphisms. We follow the same averaging-extrapolation process applied to the derivatives of the iterates of the map. To this end, we require the family to be differentiable with respect to parameters. Hence, we are able to obtain accurate variational information at the same time that we approximate the rotation number. Consequently, the method allows us to study parametric families of circle maps from a point of view that is not given by any of the previously mentioned methods.

From a practical point of view, circle diffeomorphisms appear in the study of quasi-periodic invariant curves for maps. In particular, for planar twist maps, any such a curve induces a circle diffeomorphism in a direct way just by projecting the iterates on the angular variable. Then, using the approximated derivatives of the rotation number, we can continue numerically these invariant curves with respect to parameters by means of the Newton method. The methodology presented is an alternative to more common approaches based on solving numerically the invariance equation, interpolation of the map or approximation by periodic orbits (see for example [5, 7, 12, 28]). Furthermore, using the variational information obtained, we are able to compute the asymptotic expansion relating parameters and initial conditions that correspond to curves of fixed rotation number.

Finally, we point out that the method can be formally extended to deal with maps of the torus with Diophantine rotation vector. However, in order to apply the method to the study of quasi-periodic tori for symplectic maps in higher dimension, there is not an analogue of the twist condition to guarantee a well defined projection of the iterates on the standard torus. Then, the immediate interest is focused in the generalization of the method to the case of non-twist maps and deal with folded invariant curves (for example, the so-called meanderings [29]). These and other extensions will be object of future research [22].

The contents of the paper are organized as follows. In section 2 we recall some fundamental facts about circle maps and we briefly review the method of [26]. In section 3 we describe the method for the computation of derivatives of the rotation number. The rest of the paper is devoted to illustrate the method through several applications. Concretely, in section 4 we study the Arnold family of circle maps. Finally, in section 5 we focus on the computation and continuation of invariant curves for planar twist maps and, in particular, we present some computations for the conservative Hénon map.

2 Notation and previous results

All the results presented in this section can be found in the bibliography, but we include them for self-consistence of the text. Concretely, in subsection 2.1 we recall the basic definitions, notations and properties of circle maps that we need in the paper (we refer to [9, 18] for more details and proofs). On the other hand, in subsection 2.2 we review briefly the method of [26] for computing rotation numbers of circle diffeomorphisms.

2.1 Circle diffeomorphisms

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the real circle which inherits both a group structure and a topology by means of the natural projection $\pi : \mathbb{R} \to \mathbb{T}$ (also called the universal cover of $\mathbb{T}$). We denote by $\mathrm{Diff}_+^r(\mathbb{T})$, $r \in [0, +\infty) \cup \{\infty, \omega\}$, the group of orientation-preserving homeomorphisms of $\mathbb{T}$ of class $\mathcal{C}^r$. Concretely, if $r = 0$ it is the group of homeomorphisms of $\mathbb{T}$; if $r \geq 1$, $r \in (0, \infty)\backslash\mathbb{N}$, it is the group of $\mathcal{C}^{\lfloor r \rfloor}$-diffeomorphisms whose $\lfloor r \rfloor$-th derivative verifies a Hölder condition with exponent $r - \lfloor r \rfloor$; if $r = \omega$ it is the group of real analytic diffeomorphisms.

Given $f \in \mathrm{Diff}_+^r(\mathbb{T})$, we can lift f to $\mathbb{R}$ by π obtaining a $\mathcal{C}^r$ map $\tilde{f}$ that makes the following diagram commute

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{T} & \xrightarrow{f} & \mathbb{T} \end{array} \qquad \pi \circ \tilde{f} = f \circ \pi.$$

Moreover, we have $\tilde{f}(x + 1) - \tilde{f}(x) = 1$ (since f is orientation-preserving) and the lift is unique if we ask for $\tilde{f}(0) \in [0, 1)$. Accordingly, from now on we choose the lift with this normalization so we can omit the tilde without any ambiguity.

Definition 2.1. *Let f be the lift of an orientation-preserving homeomorphism of the circle such that $f(0) \in [0, 1)$. Then the* rotation number of f *is defined as the limit*

$$\rho(f) := \lim_{|n| \to \infty} \frac{f^n(x_0) - x_0}{n},$$

that exists for all $x_0 \in \mathbb{R}$, is independent of x_0 and satisfies $\rho(f) \in [0, 1)$.

Let us remark that the rotation number is invariant under orientation-preserving conjugation, i.e., for every $f, h \in \mathrm{Diff}_+^0(\mathbb{T})$ we have that $\rho(h^{-1} \circ f \circ h) = \rho(f)$. Furthermore, given $f \in \mathrm{Diff}_+^2(\mathbb{T})$ with $\rho(f) \in \mathbb{R}\backslash\mathbb{Q}$, Denjoy's theorem ensures that f is topologically conjugate to the rigid rotation $R_{\rho(f)}$, where $R_\theta(x) = x + \theta$. That is, there exists $\eta \in \mathrm{Diff}_+^0(\mathbb{T})$ making the following diagram commute

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{f} & \mathbb{T} \\ {\scriptstyle\eta}\uparrow & & \uparrow{\scriptstyle\eta} \\ \mathbb{T} & \xrightarrow{R_{\rho(f)}} & \mathbb{T} \end{array} \qquad f \circ \eta = \eta \circ R_{\rho(f)}. \tag{1}$$

In addition, if we require $\eta(0) = x_0$, for fixed x_0, then the conjugacy η is unique.

All the ideas and algorithms described in this paper make use of the existence of such conjugation and its regularity. Let us remark that, although smooth or even finite differentiability is enough, in this paper we are concerned with the analytic case. Moreover, it is well known that the regularity of the conjugation depends also on the rational approximation properties of $\rho(f)$, so we will focus on Diophantine numbers.

Definition 2.2. *Given $\theta \in \mathbb{R}$, we say that θ is a* Diophantine number *of (C, τ) type if there exist constants $C > 0$ and $\tau \geq 1$ such that*

$$\left|1 - \mathrm{e}^{2\pi \mathrm{i} k\theta}\right|^{-1} \leq C|k|^{\tau}, \qquad \forall k \in \mathbb{Z}_*.$$

We will denote by $\mathcal{D}(C, \tau)$ the set of such numbers and by $\mathcal{D}$ the set of Diophantine numbers of any type.

Although Diophantine sets are Cantorian (i.e., compact, perfect and nowhere dense) a remarkable property is that $\mathbb{R} \backslash \mathcal{D}$ has zero Lebesgue measure. For this reason, this condition fits very well in practical issues and we do not resort to other weak conditions on small divisors such as the Brjuno condition (see [33]).

The first result on the regularity of the conjugacy (1) is due to Arnold [2] but we also refer to [16, 19, 30, 33] for later contributions. In particular, the theoretical support of the methodology is provided by the following result:

Theorem 2.3 (Katznelson and Ornstein [19])**.** *If $f \in \mathrm{Diff}_+^r(\mathbb{T})$ has Diophantine rotation number $\rho(f) \in \mathcal{D}(C, \tau)$ for $\tau + 1 < r$, then f is conjugated to $R_{\rho(f)}$ by means of a conjugacy $\eta \in \mathrm{Diff}_+^{r-\tau-\varepsilon}(\mathbb{T})$, for any $\varepsilon > 0$. Note that $\mathrm{Diff}_+^{\omega}(\mathbb{T}) = \mathrm{Diff}_+^{\omega-\tau-\varepsilon}(\mathbb{T})$ while the domain of analyticity is reduced.*

2.2 Computing rotation numbers by averaging and extrapolation

We review here the method developed in [26] for computing Diophantine rotation numbers of analytic circle diffeomorphisms (the $\mathcal{C}^r$ case is similar). This method is highly accurate with low computational cost and it turns out to be very efficient when combined with multiple precision arithmetic routines. The reader is referred there for a detailed discussion and several applications.

Let us consider $f \in \mathrm{Diff}_+^\omega(\mathbb{T})$ with rotation number $\theta = \rho(f) \in \mathcal{D}$. Notice that we can write the conjugacy of theorem 2.3 as $\eta(x) = x + \xi(x)$, ξ being a 1-periodic function normalized in such a way that $\xi(0) = x_0$, for a fixed $x_0 \in [0, 1)$. Now, by using the fact that η conjugates f to a rigid rotation, we can write the following expression for the iterates under the lift:

$$f^n(x_0) = f^n(\eta(0)) = \eta(n\theta) = n\theta + \sum_{k\in\mathbb{Z}} \hat{\xi}_k \mathrm{e}^{2\pi \mathrm{i} k n\theta}, \qquad \forall n \in \mathbb{Z}, \tag{2}$$

where the sequence $\{\hat{\xi}_k\}_{k\in\mathbb{Z}}$ denotes the Fourier coefficients of ξ. Then, the above expression gives us the following formula

$$\frac{f^n(x_0) - x_0}{n} = \theta + \frac{1}{n} \sum_{k\in\mathbb{Z}_*} \hat{\xi}_k (\mathrm{e}^{2\pi \mathrm{i} k n\theta} - 1),$$

to compute θ modulo terms of order $\mathcal{O}(1/n)$. Unfortunately, this order of convergence is very slow for practical purposes, since it requires a huge number of iterates if we want to compute θ with high precision. Nevertheless, by averaging the iterates $f^n(x_0)$ in a suitable way, we can manage to decrease the order of this quasi-periodic term.

As a motivation, let us start by considering the sum of the first N iterates under f, that has the following expression (we use (2) to write the iterates)

$$S_N^1(f) := \sum_{n=1}^{N} (f^n(x_0) - x_0) = \frac{N(N+1)}{2}\theta - N \sum_{k\in\mathbb{Z}_*} \hat{\xi}_k + \sum_{k\in\mathbb{Z}_*} \hat{\xi}_k \frac{\mathrm{e}^{2\pi \mathrm{i} k\theta}(1 - \mathrm{e}^{2\pi \mathrm{i} k N\theta})}{1 - \mathrm{e}^{2\pi \mathrm{i} k\theta}},$$

and we observe that the new factor multiplying θ grows quadratically with the number of iterates, while it appears a linear term in N with constant $A_1 = -\sum_{k\in\mathbb{Z}_*} \hat{\xi}_k$. Moreover, the quasi-periodic sum remains uniformly bounded since θ is Diophantine and η is analytic (use lemma 2.4 with $p = 1$). Thus, we obtain

$$\frac{2}{N(N+1)} S_N^1(f) = \theta + \frac{2}{N+1} A_1 + \mathcal{O}(1/N^2), \tag{3}$$

that allows us to extrapolate the value of θ with an error $\mathcal{O}(1/N^2)$ if, for example, we compute $S_N(f)$ and $S_{2N}(f)$.

In general, we introduce the following *recursive sums* for $p \in \mathbb{N}$

$$S_N^0(f) := f^N(x_0) - x_0, \qquad S_N^p(f) := \sum_{j=1}^{N} S_j^{p-1}(f). \tag{4}$$

Then, the result presented in [26] says that under the above hypotheses, the following *averaged sums of order* p

$$\tilde{S}_N^p(f) := \binom{N+p}{p+1}^{-1} S_N^p(f) \tag{5}$$

satisfy the expression

$$\tilde{S}_N^p(f) = \theta + \sum_{l=1}^{p} \frac{A_l^p}{(N+p-l+1)\cdots(N+p)} + E^p(N), \tag{6}$$

where the coefficients A_l^p depend on f and p but are independent of N. Furthermore, we have the following expressions for them

$$A_l^p = (-1)^l (p-l+2)\cdots(p+1) \sum_{k\in\mathbb{Z}_*} \hat{\xi}_k \frac{\mathrm{e}^{2\pi \mathrm{i} k(l-1)\theta}}{(1-\mathrm{e}^{2\pi\mathrm{i}k\theta})^{l-1}},$$

$$E^p(N) = (-1)^{p+1} \frac{(p+1)!}{N\cdots(N+p)} \sum_{k\in\mathbb{Z}_*} \hat{\xi}_k \frac{\mathrm{e}^{2\pi\mathrm{i}kp\theta}(1-\mathrm{e}^{2\pi\mathrm{i}kN\theta})}{(1-\mathrm{e}^{2\pi\mathrm{i}k\theta})^p}.$$

Finally, the remainder $E^p(N)$ is uniformly bounded by an expression of order $\mathcal{O}(1/N^{p+1})$. This follows immediately from the next standard lemma on small divisors.

Lemma 2.4. *Let $\xi \in \mathrm{Diff}_+^\omega(\mathbb{T})$ be a circle map that can be extended analytically to a complex strip $B_\Delta = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \Delta\}$, with $|\xi(z)| \leq M$ up to the boundary of the strip. If we denote $\{\hat{\xi}_k\}_{k\in\mathbb{Z}}$ the Fourier coefficients of ξ and consider $\theta \in \mathcal{D}(C,\tau)$, then for any fixed $p \in \mathbb{N}$ we have*

$$\left| \sum_{k\in\mathbb{Z}_*} \hat{\xi}_k \frac{\mathrm{e}^{2\pi\mathrm{i}kp\theta}(1-\mathrm{e}^{2\pi\mathrm{i}kN\theta})}{(1-\mathrm{e}^{2\pi\mathrm{i}k\theta})^p} \right| \leq \frac{\mathrm{e}^{-\pi\Delta}}{1-\mathrm{e}^{-\pi\Delta}} 4MC^p \left(\frac{\tau p}{\pi\Delta\mathrm{e}}\right)^{\tau p}.$$

To conclude this survey, we describe the implementation of the method and discuss the expected behavior of the extrapolation error. In order to make Richardson extrapolation we assume, for simplicity, that the total number of iterates is a power of two. Concretely, we select an averaging order $p \in \mathbb{N}$, a maximum number of iterates $N = 2^q$, for some $q \geq p$, and compute the averaged sums $\{\tilde{S}_{N_j}^p(f)\}_{j=0,\ldots,p}$ with $N_j = 2^{q-p+j}$. Then, we can use formula (6) to obtain θ by neglecting the remainders $E^p(N_j)$ and solving the resulting linear set of equations for the unknowns $\theta, A_1^p, \ldots, A_p^p$.

However, let us point out that, due to the denominators $(N_j + p - l + 1)\cdots(N_j + p)$, the matrix of this linear system depends on q, and this is inconvenient if we want to repeat the computations using different number of iterates. Nevertheless, we note that expression (6) can be written alternatively as

$$\tilde{S}_N^p(f) = \theta + \sum_{l=1}^{p} \frac{\hat{A}_l^p}{N^l} + \hat{E}^p(N), \tag{7}$$

for certain $\{\hat{A}_l^p\}_{l=1,\ldots,p}$, also independent of N, and with a new remainder $\hat{E}^p(N)$ that differs from $E^p(N)$ only by terms of order $\mathcal{O}(1/N^{p+1})$. Then, by neglecting the remainder $\hat{E}^p(N)$ in (7), we can obtain θ by solving a new $(p+1)$-dimensional system of equations, independent of q, for the unknowns $\theta, \hat{A}_1^p/2^{1(q-p)}, \ldots, \hat{A}_p^p/2^{p(q-p)}$. Therefore, the rotation number can be computed as follows

$$\theta = \Theta_{q,p}(f) + \mathcal{O}(2^{-(p+1)q}), \tag{8}$$

where $\Theta_{q,p}$ is an *extrapolation operator*, that is given by

$$\Theta_{q,p}(f) := \sum_{j=0}^{p} c_j^p \tilde{S}^p_{2^{q-p+j}}(f), \tag{9}$$

and the coefficients $\{c_j^p\}_{j=0,\ldots,p}$ are

$$c_l^p = (-1)^{p-l} \frac{2^{l(l+1)/2}}{\delta(l)\delta(p-l)}, \tag{10}$$

where we define $\delta(n) := (2^n - 1)(2^{n-1} - 1) \cdots (2^1 - 1)$ for $n \geq 1$ and $\delta(0) := 1$.

Remark 2.5. *Note that the dimension of this linear system and the asymptotic behaviour of the error only depend on the averaging order p. For this reason, in [26] p is called the extrapolation order. However, this is not always the case when computing derivatives of the rotation number. As we discuss in section 3, the extrapolation order is in general less than the averaging order.*

As far as the behavior of the error is concerned, using (8) we have that

$$|\theta - \Theta_{q,p}(f)| \leq c/2^{q(p+1)},$$

for certain constant c, independent of q, that we estimate heuristically as follows. Let us compute $\Theta_{q-1,p}(f)$ and $\Theta_{q,p}(f)$. Since $\Theta_{q,p}(f)$ is a better approximation of θ, it turns out that

$$c \sim 2^{(q-1)(p+1)} |\Theta_{q,p}(f) - \Theta_{q-1,p}(f)|.$$

Then, we obtain the expression

$$|\theta - \Theta_{q,p}(f)| \leq \frac{\nu}{2^{p+1}} |\Theta_{q,p}(f) - \Theta_{q-1,p}(f)|, \tag{11}$$

where ν is a "safety parameter" whose role is to prevent the oscillations in the error as a function of q due to the quasi-periodic part. In every numerical computation we take $\nu = 10$. For more details on the behavior of the error we refer to [26].

Now, we comment two sources of error to take into account in the implementation of the method:

- The sums $S^p_{N_j}(f)$ are evaluated using the lift rather than the map itself. Of course, this makes the sums $S^p_{N_j}(f)$ to increase (actually they are of order $\mathcal{O}(N^{p+1})$) and is recommended to store separately their integer and decimal parts in order to keep the desired precision.

- If the required number of iterates increases, we have to be aware of round-off errors in the evaluation of the iterates. For this reason, when implementing the above scheme in a computer, we use multiple-precision arithmetics. The computations presented in this paper have been performed using a C++ compiler and the multiple arithmetic has been provided by the routines *double-double and quad-double package* of [17], which include a *double-double* data type of approximately 32 decimal digits and a *quadruple-double* data type of approximately 64 digits.

Along this section we have required the rotation number to be Diophantine. Of course, if $\theta \in \mathbb{Q}$ equation (6) is not valid since, in general, the dynamics of f is not conjugate to a rigid rotation. Anyway, we can compute the sums $S_N^p(f)$ and it turns out that the method works as well as for Diophantine numbers. We can justify this behavior from the known fact that, for any circle homeomorphism of rational rotation number, every orbit is either periodic or its iterates converge to a periodic orbit (see [9, 18]). Then, the iterates of the map tend toward periodic points, and for such points, one can see that the averaged sums $\tilde{S}_N^p(f)$ also satisfy an expression like (7) with an error of the same order, and this is all we need to perform the extrapolation. In fact, the worst situation appears when computing irrational rotation numbers that are "close" to rational ones (see also the discussion in subsection 4.1).

3 Derivatives of the rotation number with respect to parameters

Now we adapt the method already described in section 2 in order to compute derivatives of the rotation number with respect to parameters (assuming that they exist). For the sake of simplicity, we introduce the method for one-parameter families of circle diffeomorphisms, albeit the construction can be adapted to deal with multiple parameters (we discuss this situation in remark 3.3). Thus, consider $\mu \in I \subset \mathbb{R} \mapsto f_\mu \in \mathrm{Diff}_+^\omega(\mathbb{T})$ depending on μ in a regular way. The rotation numbers of the family $\{f_\mu\}_{\mu \in I}$ induce a function $\theta : I \to [0, 1)$ given by $\theta(\mu) = \rho(f_\mu)$. Then, our goal is to approximate numerically the derivatives of θ at a given point μ_0.

Let us remark that the function θ is only continuous in the $\mathcal{C}^0$-topology and, actually, the rotation number depends on μ in a very non-smooth way: generically, there exist a family of disjoint open intervals of I, with dense union, such that θ takes constant rational values on these intervals (a so-called Devil's Staircase). However, $\theta'(\mu)$ is defined for any μ such that $\theta(\mu) \notin \mathbb{Q}$ (see [15]).

For what refers to higher order derivatives, they are defined in "many" points in the sense of Whitney. Concretely, let $J \subset I$ be the subset of parameters such that $\theta(\mu) \in \mathcal{D}$ (typically a Cantor set). Then, from theorem 2.3, there exists a family of conjugacies $\mu \in J \mapsto \eta_\mu \in \mathrm{Diff}_+^\omega(\mathbb{T})$, satisfying $f_\mu \circ \eta_\mu = \eta_\mu \circ R_{\theta(\mu)}$, that is unique if we fix $\eta_\mu(0) = x_0$. Then, if f_μ is $\mathcal{C}^d$ with respect to μ, the Whitney derivatives $D_\mu^j \eta_\mu$ and $D_\mu^j \theta$, for $j = 1, \ldots, d$, can be computed by taking formal derivatives with respect to μ on the conjugacy equation and solving the small divisors equations thus obtained. Actually, we know that, if we define $J(C,\tau)$ as the subset of J such that $\theta(\mu) \in \mathcal{D}(C,\tau)$, for certain $C > 0$ and $\tau \geq 1$, then the maps $\mu \in J(C,\tau) \mapsto \eta_\mu$ and $\mu \in J(C,\tau) \mapsto \theta$ can be extended to $\mathcal{C}^s$ functions on I, where s depends on d and τ, provided that d is big enough (see [32]).

As it is shown in subsection 3.1, when we extend the method for computing the d-th derivative of θ, in general, we are forced to select an averaging order $p > d$ and the remainder turns out to be of order $\mathcal{O}(1/N^{p-d+1})$. Nevertheless, if the rotation number is known to be constant as a function of the parameters, we can avoid the previous limitations. Concretely, in this case we can select any averaging order p, independent of d, since the remainder is now of order $\mathcal{O}(1/N^{p+1})$. Of course, if the rotation number is constant, then the derivatives of θ are all zero and the fact that we can obtain them with better precision seems to be irrelevant. However, from the computation of these vanishing derivatives, we can derive information about other involved objects. This is the case of many applications in which this methodology turns out to be very useful (two examples are worked out in subsections 4.3 and 5.3).

3.1 Computation of the first derivative

We start by explaining how to compute the first derivative of θ. For notational convenience, from now on we fix μ_0 such that $\theta(\mu_0) \in \mathcal{D}$ and we omit the dependence on μ as a subscript in families of circle maps. In addition, let us recall that we can write any conjugation as $\eta(x) = x + \xi(x)$ and denote by $\{\hat{\xi}_k\}_{k\in\mathbb{Z}}$ the Fourier coefficients of ξ. Finally, we denote the first derivatives as $\theta' = D_\mu\theta$ and $\hat{\xi}'_k = D_\mu\hat{\xi}_k$.

As we did in subsection 2.2, we begin by computing the first averages (of the derivatives of the iterates) in order to illustrate the idea of the method. Thus, we proceed by formally taking derivatives with respect to μ at both sides of equation (2)

$$D_\mu f^n(x_0) = n\theta' + \sum_{k\in\mathbb{Z}} \hat{\xi}'_k \mathrm{e}^{2\pi \mathrm{i} kn\theta} + 2\pi \mathrm{i} n\theta' \sum_{k\in\mathbb{Z}} k\hat{\xi}_k \mathrm{e}^{2\pi \mathrm{i} kn\theta}, \qquad \forall n \in \mathbb{Z}.$$

Then, notice that a factor n appears multiplying the second quasi-periodic sum. However, if we perform the recursive sums, we can still manage to control the growth of this term due to the quasi-periodic part. Let us compute the sum

$$\begin{aligned} D_\mu S^1_N(f) &:= \sum_{n=1}^{N} D_\mu(f^n(x_0) - x_0) \\ &= \frac{N(N+1)}{2}\theta' - N\sum_{k\in\mathbb{Z}_*}\hat{\xi}'_k + \sum_{k\in\mathbb{Z}_*}\hat{\xi}'_k \frac{\mathrm{e}^{2\pi \mathrm{i} k\theta}(1-\mathrm{e}^{2\pi \mathrm{i} kN\theta})}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}} \\ &\quad +2\pi \mathrm{i}\theta' \sum_{k\in\mathbb{Z}_*} k\hat{\xi}_k \frac{N\mathrm{e}^{2\pi \mathrm{i} k(N+2)\theta} - (N+1)\mathrm{e}^{2\pi \mathrm{i} k(N+1)\theta} + \mathrm{e}^{2\pi \mathrm{i} k\theta}}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^2}. \end{aligned}$$

Hence, we observe that the method is still valid, even though for $\theta' \neq 0$ the quasi-periodic sum is bigger than expected a priori. Indeed, we obtain the following formula

$$\frac{2}{N(N+1)} D_\mu S_N^1(f) = \theta' + \mathcal{O}(1/N), \tag{12}$$

that is similar to equation (3), but notice that the term $2A_1/(N+1)$ has been included in the remainder since there are oscillatory terms of the same order. Proceeding like in subsection 2.2, we introduce *recursive sums* for the derivatives of the iterates

$$D_\mu S_N^p(f) := D_\mu(f^N(x_0) - x_0), \qquad D_\mu S_N^p(f) := \sum_{j=1}^{N} D_\mu S_j^{p-1}(f),$$

and their corresponding *averaged sums of order* p

$$D_\mu \tilde{S}_N^p(f) := \binom{N+p}{p+1}^{-1} D_\mu S_N^p(f).$$

Finding an expression like (12) for $p > 1$ is quite cumbersome to do directly, since the computations are very involved. However, the computation is straightforward if we take formal derivatives at both sides of equation (6). The resulting expression reads as

$$D_\mu \tilde{S}_N^p(f) = \theta' + \sum_{l=1}^{p} \frac{D_\mu A_l^p}{(N+p-l+1)\cdots(N+p)} + D_\mu E^p(N),$$

where the new coefficients are $D_\mu A_l^p = (-1)^l (p-l+2)\cdots(p+1) D_\mu A_l$ with

$$D_\mu A_l = \sum_{k \in \mathbb{Z}_*} \frac{\mathrm{e}^{2\pi \mathrm{i} k(l-1)\theta}}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^{l-1}} \left(\hat{\xi}_k' + \frac{2\pi \mathrm{i} k(l-1)\hat{\xi}_k \theta'}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}} \right),$$

and the new remainder is

$$\begin{aligned} D_\mu E^p(N) \quad &= \quad (-1)^{p+1} \frac{(p+1)!}{N\cdots(N+p)} \sum_{k \in \mathbb{Z}_*} \frac{\mathrm{e}^{2\pi \mathrm{i} kp\theta}}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^p} \Big\{ \hat{\xi}_k' (1-\mathrm{e}^{2\pi \mathrm{i} kN\theta}) \\ &\quad + 2\pi \mathrm{i} k \hat{\xi}_k \theta' \left(p \frac{1-\mathrm{e}^{2\pi \mathrm{i} kN\theta}}{1-\mathrm{e}^{2\pi \mathrm{i} k\theta}} - N\mathrm{e}^{2\pi \mathrm{i} kpN\theta} \right) \Big\}. \end{aligned}$$

Assuming that $\theta(\mu_0) \in \mathcal{D}$ and $\theta'(\mu_0) \neq 0$, we can obtain analogous bounds as those of lemma 2.4 and conclude that the remainder satisfies $D_\mu E^p(N) = \mathcal{O}(1/N^p)$. Moreover, we observe that the coefficient $D_\mu A_p^p$ corresponds to a term of the same order, so we have to redefine the remainder in order to include this term. Hence, as we did in equation (7), we can arrange the unknown terms and obtain

$$D_\mu \tilde{S}_N^p(f) = \theta' + \sum_{l=1}^{p-1} \frac{D_\mu \hat{A}_l^p}{N^l} + \mathcal{O}(1/N^p),$$

where the coefficients $\{D_\mu \hat{A}_l^p\}_{l=1,\ldots,p-1}$ are the derivatives of $\{\hat{A}_l^p\}_{l=1,\ldots,p-1}$ that appear in equation (7).

Finally, we can extrapolate an approximation to θ' using Richardson's method of order $p-1$ as in subsection 2.2. Concretely, if we compute $N = 2^q$ iterates, we can approximate the derivative of the rotation number by means of the following formula

$$\theta' = \sum_{j=0}^{p-1} c_j^{p-1} D_\mu \tilde{S}^p_{2^{q-p+1+j}}(f) + \mathcal{O}(2^{-pq}), \tag{13}$$

where the coefficients $\{c_j^{p-1}\}_{j=0,\ldots,p-1}$ are given by (10).

3.2 Computation of higher order derivatives

The goal of this section is to generalize formula (13) to approximate $D_\mu^d \theta$ for any d, when they exists. Then, we assume that the family $\mu \mapsto f \in \mathrm{Diff}_+^\omega(\mathbb{T})$ depends $\mathcal{C}^d$-smoothly with respect to the parameter. As usual, we define the recursive sums for the d-derivative and their averages of order p as

$$D_\mu^d S_N^0(f) := D_\mu^d(f^n(x_0) - x_0), \quad D_\mu^d S_N^p(f) := \sum_{j=0}^{N} D_\mu^d S_j^{p-1}(f),$$

and

$$D_\mu^d \tilde{S}_N^p(f) := \binom{N+p}{p+1}^{-1} D_\mu^d S_N^p(f),$$

respectively. As before, we relate these sums to $D_\mu^d \theta$ by taking formal derivatives in equation (6), thus obtaining

$$D_\mu^d \tilde{S}_N^p(f) = D_\mu^d \theta + \sum_{l=1}^{p} \frac{D_\mu^d A_l^p}{(N+p-l+1)\cdots(N+p)} + D_\mu^d E^p(N). \tag{14}$$

Is immediate to check that, if $\theta(\mu_0) \in \mathcal{D}$ and $D_\mu^d \theta(\mu_0) \neq 0$, the remainder $D_\mu^d E^p(N)$ is of order $\mathcal{O}(1/N^{p-d+1})$, so this expression makes sense if the averaging order satisfies $p > d$.

Remark 3.1. *Notice that, in order to work with reasonable computational time and round-off errors, p cannot be taken arbitrarily big. Consequently, there is a (practical) limitation in the computation of high order derivatives.*

In addition, as it was done for the first derivative, the remainder $D_\mu^d E^p(N)$ must be redefined in order to include the terms corresponding to $l \geq p-d+1$ in equation (14). Then we can extrapolate $D_\mu^d\theta$ by computing $N = 2^q$ iterates and solving the linear $(p-d+1)$-dimensional system associated to the following rearranged equation

$$D_\mu^d \tilde{S}_N^p(f) = D_\mu^d\theta + \sum_{l=1}^{p-d} \frac{D_\mu^d \hat{A}_l^p}{N^l} + \mathcal{O}(1/N^{p-d+1}). \quad (15)$$

Since the averaging order p and the extrapolation order $p-d$ do not coincide, let us define the *extrapolation operator of order m for the d-derivative* as

$$\Theta_{q,p,m}^d(f) := \sum_{j=0}^{m} c_j^m D_\mu^d \tilde{S}_{2^{q-m+j}}^p(f), \quad (16)$$

where the coefficients $\{c_j^m\}_{j=0,\ldots,m}$ are given by (10). Therefore, according to formula (15), we can approximate the d-th derivative of the rotation number as

$$D_\mu^d\theta = \Theta_{q,p,p-d}^d(f) + \mathcal{O}(2^{-(p-d+1)q}).$$

Furthermore, as explained in subsection 2.2, by comparing the approximations that correspond to 2^{q-1} and 2^q iterates, we obtain the following heuristic formula for the extrapolation error:

$$|D_\mu^d\theta - \Theta_{q,p,p-d}^d(f)| \leq \frac{\nu}{2^{p-d+1}} |\Theta_{q,p,p-d}^d(f) - \Theta_{q-1,p,p-d}^d(f)|, \quad (17)$$

where, once again, ν is a "safety parameter" that we take as $\nu = 10$.

Remark 3.2. *Up to this point we have assumed that $D_\mu^d\theta \neq 0$ at the computed point. However, if we know a priori that $D_\mu^r\theta = 0$ for $r = 1,\ldots,d$, then equation (14) holds with the following expression for the remainder:*

$$D_\mu^d E^p(N) = (-1)^{p+1} \frac{(p+1)!}{N\cdots(N+p)} \sum_{k\in\mathbb{Z}_*} D_\mu^d \hat{\xi}_k \frac{\mathrm{e}^{2\pi \mathrm{i} kp\theta}(1-\mathrm{e}^{2\pi \mathrm{i} kN\theta})}{(1-\mathrm{e}^{2\pi \mathrm{i} k\theta})^p},$$

which now is of order $\mathcal{O}(1/N^{p+1})$. As in section 2, this allows us to approximate $D_\mu^d\theta$ with the same extrapolation order as the averaging order p. Indeed, we obtain

$$0 = D_\mu^d\theta = \Theta_{q,p,p}^d(f) + \mathcal{O}(2^{-(p+1)q}),$$

and we observe that the order d is not limited by p anymore.

The case remarked above is very interesting since we know that many applications can be modeled as a family of circle diffeomorphisms of fixed rotation number. The possibilities of this approach are illustrated by computing the Taylor expansion of Arnold Tongues (subsection 4.3) and the continuation of invariant curves for the Hénon map (subsection 5.3).

3.3 Scheme for evaluating the derivatives of the averaged sums

Let us introduce a recursive way for computing the sums $D_\mu^d \tilde{S}_N^p(f)$ required to evaluate the extrapolation operator (16). First of all, notice that by linearity it suffices to compute $D_\mu^d(f^n(x_0))$ for any $n \in \mathbb{N}$.

To compute the derivatives of $f^n = f\circ \overset{(n)}{\cdots} \circ f$, we proceed inductively with respect to n and d. Thus, let us assume that the derivatives $D_\mu^r(f^{n-1}(x_0))$ are known for a given $n \geq 1$ and for any $r \leq d$. Then, if we denote $z := f^{n-1}(x_0)$, our goal is to compute $D_\mu^r(f(z))$ for $r \leq d$ by using the known derivatives of z.

For $d = 1$, a recursive formula appears directly by applying the chain rule

$$D_\mu(f(z)) = \partial_\mu f(z) + \partial_x f(z) D_\mu(z). \tag{18}$$

This formula can be implemented provided the partial derivatives $\partial_\mu f$ and $\partial_x f$ can be numerically evaluated at the point z.

In general, we can perform higher order derivatives and obtain the following expression

$$\begin{aligned} D_\mu^d(f(z)) &= D_\mu^{d-1}\Big(\partial_\mu f(z) + \partial_x f(z) D_\mu(z)\Big) \\ &= D_\mu^{d-1}(\partial_\mu f(z)) + \sum_{r=0}^{d-1} \binom{d-1}{r} D_\mu^r(\partial_x f(z)) D_\mu^{d-r}(z). \end{aligned}$$

This motivates the extension of recurrence (18), since for evaluating the previous formula we require to know the derivatives $D_\mu^r(\partial_x f(z))$ for $r < d$ and $D_\mu^{d-1}(\partial_\mu f(z))$. We note that these derivatives can also be computed recursively using similar expressions for the maps $\partial_x f$ and $\partial_\mu f$, respectively. Concretely, assuming that we can evaluate $\partial_{\mu,x}^{i,j} f(z)$ for any $(i,j) \in \mathbb{Z}_+^2$ such that $i + j \leq d$, we can use the following recurrences

$$D_\mu^r(\partial_{\mu,x}^{i,j} f(z)) = D_\mu^{r-1}(\partial_{\mu,x}^{i+1,j} f(z)) + \sum_{s=0}^{r-1} \binom{r-1}{s} D_\mu^s(\partial_{\mu,x}^{i,j+1} f(z)) D_\mu^{r-s}(z),$$

to compute in a tree-like order the corresponding derivatives. To prevent redundant computations in the implementation of the method, we store (in memory) the value of the "intermediate" derivatives $D_\mu^r(\partial_{\mu,x}^{i,j} f(z))$ so they only have to be computed one time. For this reason, this scheme turns out to be more efficient than evaluating explicit expressions such as Faà di Bruno formulas (see for example [20]). Figure 1 summarizes the recursive computations required and the convenience of storing these intermediate computations.

Remark 3.3. *The above scheme can be generalized immediately to the case of several parameters. For example, consider a two-parameter family* $(\mu_1, \mu_2) \mapsto f_{\mu_1,\mu_2} \in \mathrm{Diff}_+^\omega(\mathbb{T})$ *whose rotation number induces a map* $(\mu_1, \mu_2) \mapsto \theta(\mu_1, \mu_2)$*. Then, if* $\theta(\mu_1^0, \mu_2^0) \in \mathcal{D}$*, we can obtain a similar scheme to approximate* $D_{\mu_1,\mu_2}^{d_1,d_2} \theta(\mu_1^0, \mu_2^0)$*. In this context, note that the operator*

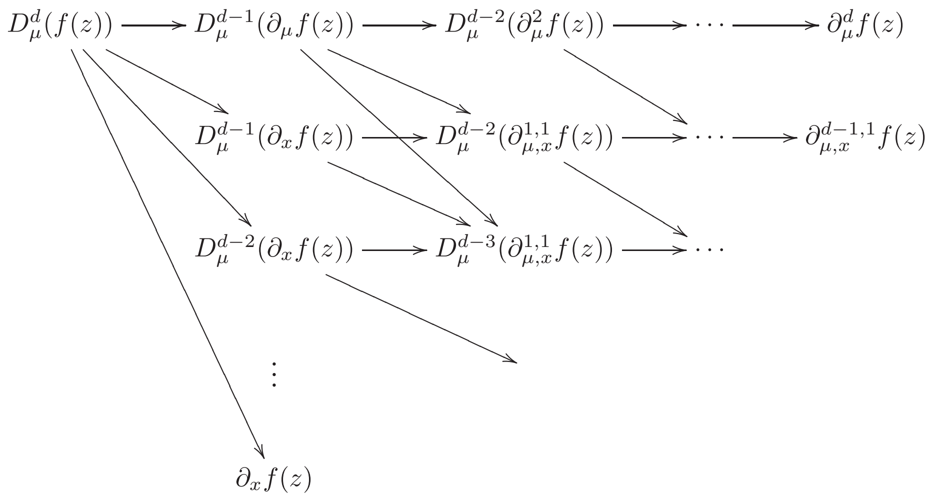

Figure 1: Schematic representation of the recurrent computations performed to evaluate $D^d_\mu(f(z))$.

$\Theta^{d_1,d_2}_{q,p,p-d_1-d_2}$ can be defined as (16), but averaging the derivatives $D^{d_1,d_2}_{\mu_1,\mu_2}(f^n(x_0))$. Finally, if we write $z := f^{n-1}(x_0)$, we can compute inductively the derivatives $D^{m,l}_{\mu_1,\mu_2}(f(z))$, for $m \leq d_1$ and $l \leq d_2$, using the following recurrences

$$\begin{aligned} D^{m,l}_{\mu_1,\mu_2}(\partial^{i,j,k}_{\mu_1,\mu_2,x} f(z)) &= D^{m-1,l}_{\mu_1,\mu_2}(\partial^{i+1,j,k}_{\mu_1,\mu_2,x} f(z)) \\ &+ \sum_{s=0}^{m-1} \sum_{r=0}^{l} \binom{m-1}{s} \binom{l}{r} D^{s,r}_{\mu_1,\mu_2}(\partial^{i,j,k+1}_{\mu_1,\mu_2,x} f(z)) D^{m-s,l-r}_{\mu_1,\mu_2}(z), \end{aligned}$$

if $m \neq 0$ and

$$D^{0,l}_{\mu_1,\mu_2}(\partial^{i,j,k}_{\mu_1,\mu_2,x} f(z)) = D^{0,l-1}_{\mu_1,\mu_2}(\partial^{i,j+1,k}_{\mu_1,\mu_2,x} f(z)) + \sum_{r=0}^{l-1} \binom{l-1}{r} D^{0,r}_{\mu_1,\mu_2}(\partial^{i,j,k+1}_{\mu_1,\mu_2,x} f(z)) D^{0,l-r}_{\mu_1,\mu_2}(z),$$

if $l \neq 0$. Of course, $D^{0,0}_{\mu_1,\mu_2}(\partial^{i,j,k}_{\mu_1,\mu_2,x} f(z)) = \partial^{i,j,k}_{\mu_1,\mu_2,x} f(z)$ corresponds to the evaluation of the partial derivative of the map.

4 Application to the Arnold family

As a first example, let us consider the Arnold family of circle maps, given by

$$\begin{aligned} f_{\alpha,\varepsilon} : \mathbb{S} &\longrightarrow \mathbb{S} \\ x &\longmapsto x + 2\pi\alpha + \varepsilon \sin(x), \end{aligned} \tag{19}$$

where $(\alpha, \varepsilon) \in [0,1) \times [0,1)$ are parameters and $\mathbb{S} = \mathbb{R}/(2\pi\mathbb{Z})$. Notice that this family satisfies $f_{\alpha,\varepsilon} \in \mathrm{Diff}^\omega_+(\mathbb{S})$ for any value of the parameters. Let us remark that (19) allows us to illustrate the method in a direct way, since there are explicit formulas for the partial derivatives $\partial^{i,j,k}_{\alpha,\varepsilon,x} f(x)$ of the map, for any $(i,j,k) \in \mathbb{Z}^3_+$. In section 5 we will consider another interesting application in which the studied family is not given explicitly.

For this family of maps, it is convenient to take the angles modulo 2π just for avoiding the loss of significant digits due to the factors $(2\pi)^{d-1}$ that would appear in the d-derivative of the map.

The contents of this section are organized as follows. First, in subsection 4.1 we compute the derivative of a Devil's Staircase, that corresponds to the variation of the rotation number of (19) with respect to α for a fixed ε. In subsection 4.2 we use the computation of derivatives of the rotation number to approximate the Arnold Tongues of the family (19) by means of the Newton method. Furthermore, we compute the asymptotic expansion of these tongues and obtain pseudo-analytical expressions for the first coefficients, as a function of the rotation number.

4.1 Stepping up to a Devil's Staircase

Let us fix the value of $\varepsilon \in [0,1)$ and consider the one-parameter family $\{f_\alpha\}_{\alpha\in[0,1)}$ given by equation (19), i.e. $f_\alpha := f_{\alpha,\varepsilon}$. Let us recall that we can establish an ordering in this family since the normalized lifts satisfy $f_{\alpha_1}(x) < f_{\alpha_2}(x)$ for all $x \in \mathbb{R}$ if and only if $\alpha_1 < \alpha_2$. Then, we conclude that the function $\alpha \mapsto \rho(f_\alpha)$ is monotone increasing. In particular, for $\alpha_1 < \alpha_2$ such that $\rho(f_{\alpha_1}) \in \mathbb{R}\backslash\mathbb{Q}$ we have $\rho(f_{\alpha_1}) < \rho(f_{\alpha_2})$. On the other hand, if $\rho(f_{\alpha_1}) \in \mathbb{Q}$, there is an interval containing α_1 giving the same rotation number. As the values of α for which f_α has rational rotation number are dense in $[0,1)$ (the complement is a Cantor set), there are infinitely many intervals where $\rho(f_\alpha)$ is locally constant. Therefore, the map $\alpha \mapsto \rho(f_\alpha)$ gives rise to a "staircase" with a dense number of stairs, that is usually called a Devil's Staircase (we refer to [9, 18] for more details).

To illustrate the behavior of the method we have computed the above staircase for $\varepsilon = 0.75$. The computations have been performed by taking 10^4 points of $\alpha \in [0,1)$, using 32-digit arithmetics (*double-double* data type from [17]), and a fixed averaging order $p = 8$. In addition, we estimate the error in the approximation of $\rho(f_\alpha)$ and $D_\alpha\rho(f_\alpha)$ using formulas (11) and (17), respectively. Then, we stop the computations for a tolerance of 10^{-26} and 10^{-24}, respectively, using at most $2^{22} = 4194304$ iterates.

Let us discuss the obtained results. First, we point out that only 11.4 % of the selected points have not reached the previous tolerances for 2^{22} iterates. Moreover, we observe that the rotation number for 98.8 % of the points has been obtained with an error less that 10^{-20}, while the estimated error in the derivatives is less than 10^{-18} for 97.7 % of the points. Let us focus in $\alpha = 0.3377$, that is one of the "bad" points. The estimated errors for the rotation number and the derivative at this point are of order 10^{-18} and 10^{-9}, respectively. We observe that, even though this rotation number is irrational (the derivative does not vanish), it is very close to the rational $105/317$, since $|317 \cdot \Theta_{22,9}(f_{0.3377}) - 105| \simeq 4.2 \cdot 10^{-6}$.

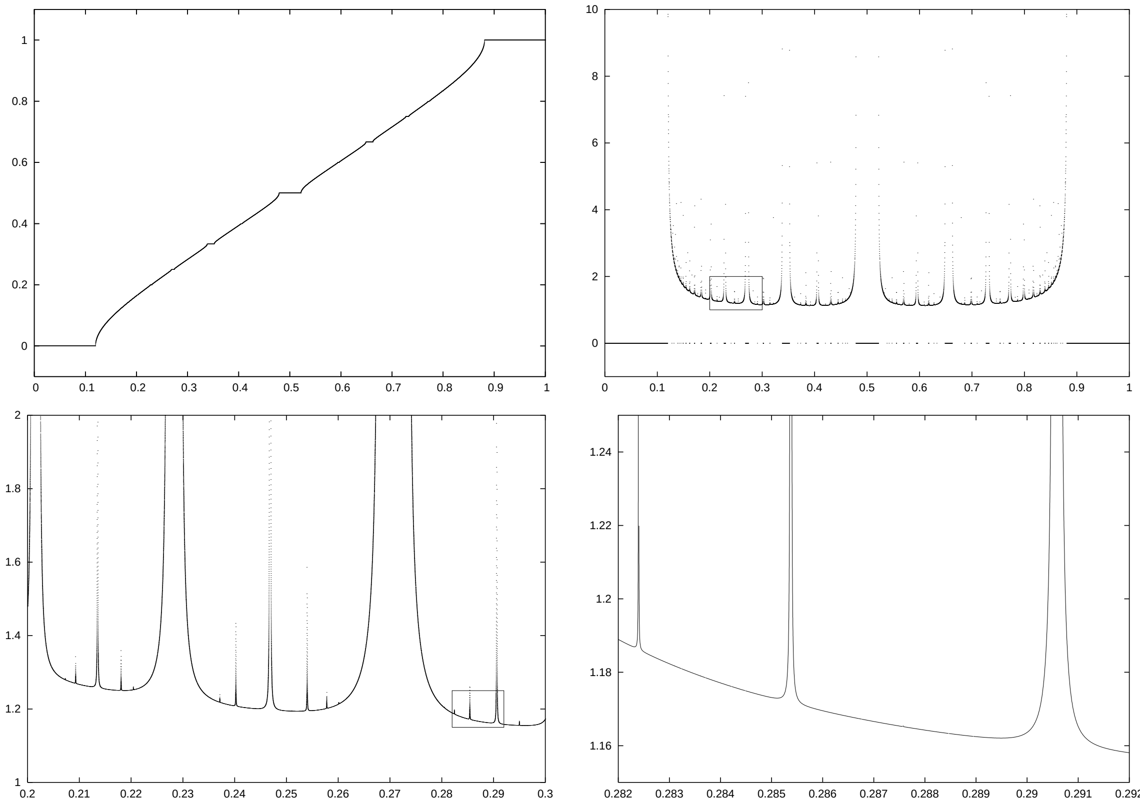

Figure 2: Devil's Staircase $\alpha \mapsto \rho(f_\alpha)$ (top-left) and its derivative (top-right) for the Arnold family with $\varepsilon = 0.75$. The plots in the bottom correspond to some magnifications of the top-right one.

In figure 2 we show $\alpha \mapsto \rho(f_\alpha)$ and its derivative $\alpha \mapsto D_\alpha\rho(f_\alpha)$ for those points that satisfy that the estimated error is less than 10^{-18} and 10^{-16}, respectively. We recall that the rational values of the rotation number correspond to the constant intervals in the top-left plot, and note that by looking at the derivative (top-right plot) we can visualize the density of the stairs better than looking at the staircase itself. We remark that both these rational rotation numbers and their vanishing derivatives have been computed as well as in the Diophantine case.

Moreover, at the bottom of the same figure, we plot some magnifications of the derivative to illustrate the non-smoothness of a Devil's Staircase. Concretely, the plot in the bottom-left corresponds to 10^5 values of $\alpha \in [0.2, 0.3]$ using the same implementation parameters as before. Once again, if the estimated error is bigger than 10^{-16} the point is not plotted. Finally, on the right plot we give another magnification for 10^6 values of $\alpha \in [0.282, 0.292]$ that are computed with $p = 7$, and allowing at most $2^{21} = 2097152$ iterates. In this case, the points that correspond to the branch in the left (i.e. close to $\alpha = 0.2825$), are typically computed with an error 10^{-10}.

4.2 Newton method for computing the Arnold Tongues

Since $f_{\alpha,\varepsilon} \in \mathrm{Diff}_+^\omega(\mathbb{S})$, we obtain a function $(\alpha, \varepsilon) \mapsto \rho(\alpha, \varepsilon) := \rho(f_{\alpha,\varepsilon})$ given by the rotation number. Then, the Arnold Tongues of (19) are defined as the sets $T_\theta = \{(\alpha, \varepsilon) : \rho(\alpha, \varepsilon) = \theta\}$, for any $\theta \in [0, 1)$. It is well known that if $\theta \in \mathbb{Q}$, then T_θ is a set with interior; otherwise, T_θ is a continuous curve which is the graph of a function $\varepsilon \mapsto \alpha(\varepsilon)$, with $\alpha(0) = \theta$. In addition, if $\theta \in \mathcal{D}$, the corresponding tongue is given by an analytic curve (see [25]).

Using the method described in subsection 2.2, some Arnold Tongues T_θ of Diophantine rotation number, were approximated in [26] by means of the secant method. Now, since we can compute derivatives of the rotation number, we are able to repeat the computations using a Newton method. To do that, we fix $\theta \in \mathcal{D}$ and solve the equation $\rho(\alpha, \varepsilon) - \theta = 0$ by continuing the known solution $(\theta, 0)$ with respect to ε. Indeed, we fix a partition $\{\varepsilon_j\}_{j=0,\ldots,K}$ of $[0, 1)$, and compute a numerical approximation α_j^* for every $\alpha(\varepsilon_j)$.

To this end, assume that we have a good approximation α_{j-1}^* to $\alpha(\varepsilon_{j-1})$ and let us first compute an initial approximation for $\alpha(\varepsilon_j)$. Taking derivative in the equation $\rho(\alpha(\varepsilon), \varepsilon) - \theta = 0$ we obtain

$$D_\alpha \rho(\alpha(\varepsilon), \varepsilon)\alpha'(\varepsilon) + D_\varepsilon \rho(\alpha(\varepsilon), \varepsilon) = 0. \tag{20}$$

Thus, we can approximate $\alpha'(\varepsilon_{j-1})$ by computing numerically the derivatives $D_\alpha\rho$ and $D_\varepsilon\rho$ at $(\alpha_{j-1}^*, \varepsilon_{j-1})$. Hence, we obtain an approximated value $\alpha_j^{(0)} = \alpha_{j-1}^* + \alpha'(\varepsilon_{j-1})(\varepsilon_j - \varepsilon_{j-1})$ for $\alpha(\varepsilon_j)$. Next, we apply the Newton method

$$\alpha_j^{(n+1)} = \alpha_j^{(n)} - \frac{\rho(\alpha_j^{(n)}, \varepsilon_j) - \theta}{D_\alpha \rho(\alpha_j^{(n)}, \varepsilon_j)},$$

and stop when we converge to a value α_j^* that approximates $\alpha(\varepsilon_j)$.

The computations are performed using 64 digits (*quadruple-double* data type from [17]) and, in order to compare with the results obtained in [26], we select the same parameters in the implementation. In particular, we take a partition $\varepsilon_j = j/K$ with $K = 100$ of the interval $[0, 1)$, we select an averaging order $p = 9$ and allow at most $2^{23} = 8388608$ iterates of the map. The required tolerances are taken as 10^{-32} for the computation of the rotation number (we use (11) to estimate the error) and 10^{-30} for the convergence of the Newton method. Let us remark that the computations are done without any prescribed tolerance for the computation of the derivatives $D_\alpha\rho$ and $D_\varepsilon\rho$, even though we check, using (17), that the extrapolation is done correctly.

Let us discuss the results obtained for $\theta = (\sqrt{5} - 1)/2$. As expected, the number of iterates of the Newton method is less than the ones required by the secant method. Concretely, we perform from 2 to 3 corrections as we approach the critical value $\varepsilon = 1$, while using the secant method we need at least 4 steps to converge. However, we observe that the computation of the derivatives $D_\alpha\rho$ and $D_\varepsilon\rho$ fails if we take $\varepsilon = 1$, even though the secant method converges after 18 iterations. This is totally consistent since we know that $f_{\alpha,1} \in \mathrm{Diff}_+^0(\mathbb{T})$ but is still an analytic map, and that the conjugation to a rigid rotation is only Hölder continuous (see [8, 34]).

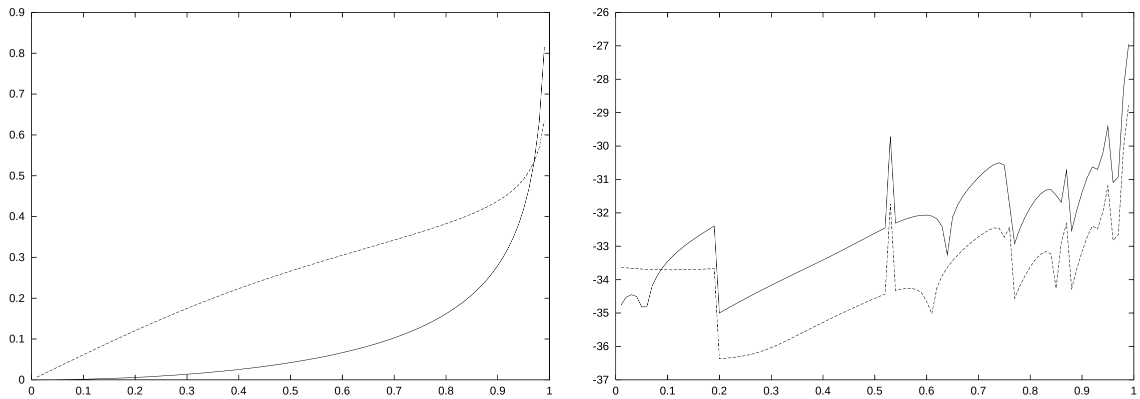

Figure 3: Left: Graph of the derivatives $\varepsilon \mapsto D_\alpha \rho(\alpha(\varepsilon), \varepsilon)$ and $\varepsilon \mapsto D_\varepsilon \rho(\alpha(\varepsilon), \varepsilon)$ along T_θ, for $\theta = (\sqrt{5}-1)/2$. The solid curve corresponds to $(D_\alpha \rho - 1)$ and the dashed one to $(20 \cdot D_\varepsilon \rho)$. Right: error (estimated using (11)) in $\log_{10}$ scale in the computation of these derivatives.

In figure 3 (left) we plot the derivatives $\varepsilon \mapsto D_\alpha \rho(\alpha(\varepsilon), \varepsilon)$ and $\varepsilon \mapsto D_\varepsilon \rho(\alpha(\varepsilon), \varepsilon)$ evaluated on the previous tongue. We observe that the derivatives have been normalized in order to fit together in the same plot. On the other hand, in the right plot we show the estimated error in the computation of these derivatives (obtained from equation (17)). In the worst case, $\varepsilon = 0.99$, we obtain errors of order 10^{-27} and 10^{-29} for $D_\alpha \rho$ and $D_\varepsilon \rho$, respectively.

4.3 Computation of the Taylor expansion of the Arnold Tongues

As we have mentioned in subsection 4.2, if $\theta \in \mathcal{D}$ then the Arnold Tongue T_θ of (19) is given by the graph of an analytic function $\alpha(\varepsilon)$, for $\varepsilon \in [0, 1)$. Then, we can expand α at the origin as

$$\alpha(\varepsilon) = \theta + \frac{\alpha'(0)}{1!}\varepsilon + \frac{\alpha''(0)}{2!}\varepsilon^2 + \cdots + \frac{\alpha^{(d)}(0)}{d!}\varepsilon^d + \mathcal{O}(\varepsilon^{d+1}), \tag{21}$$

and the goal now is to approximate numerically the terms in this expansion. We know that every odd derivative in this expansion vanishes, so the Taylor expansion can be written in terms of powers in ε^2 (see [27] for details). However, we do not use this symmetry, but instead we verify the accuracy of the computations according to this information (see the results presented in table 1).

First of all, we want to emphasize that the direct extension of the computations performed in the previous subsection is hopeless. Concretely, as we did for approximating $\alpha'(\varepsilon)$, we could take higher order derivatives with respect to ε at equation (20) and, after evaluating at the point $(\alpha, \varepsilon) = (\theta, 0)$, isolate the derivatives $\alpha^{(r)}(0)$, $1 \leq r \leq d$. For example, once we know $\alpha'(0)$, the computation of $\alpha''(0)$ would follow from the expression

$$D_\alpha \rho(\alpha(\varepsilon), \varepsilon)\alpha''(\varepsilon) + \left(D^2_\alpha \rho(\alpha(\varepsilon), \varepsilon)\alpha'(\varepsilon) + 2D_{\alpha,\varepsilon}\rho(\alpha(\varepsilon), \varepsilon) \right)\alpha'(\varepsilon) + D^2_\varepsilon \rho(\alpha(\varepsilon), \varepsilon) = 0, \tag{22}$$

that requires to compute the second order partial derivatives of the rotation number (see remark 3.3). Then, by induction, we would obtain recurrent formulas to compute the expansion (21) up to order d. However, this approach is highly inefficient due to the following reasons:

- As discussed in subsection 3.2, using this approach we are limited to compute $\alpha^{(r)}(0)$ up to order $p-1$, where p is the selected averaging order. Of course, the precision for $\alpha^{(r)}(0)$ decreases dramatically when r increases to p.

- Note that, for the Arnold family, we can write explicitly $D_\alpha(f^n_{\alpha,\varepsilon}(x_0))|_{(\theta,0)} = n$. Then, if we look at the formulas in remark 3.3, we expect the terms $D^{m,l}_{\alpha,\varepsilon}(f^n_{\alpha,\varepsilon}(x_0))|_{(\theta,0)}$ to grow very fast, since they contain factors of the previous expression. Actually, we find that these quantities depend polynomially on n, with a power that increases with the order of the derivative. On the other hand, we expect the sums $D^{m,l}_{\alpha,\varepsilon}\tilde{S}^p_N(f_{\alpha,\varepsilon})$ to converge, and therefore many cancelations are taking place in the computations. Consequently, when implementing this approach we unnecessarily lose a high amount of significant digits.

- Even if we could compute $D^{m,l}_{\alpha,\varepsilon}\rho(\theta,0)$ up to any order, it turns out that the generalization of equation (22) for computing $\alpha^{(r)}(0)$ is badly conditioned. Concretely, the derivatives of the rotation number increase with the order, giving rise to a big propagation of errors. Actually, the round-off errors increase so fast that, in practice, we cannot go beyond order 5 in the computation of (21) with the above methodology.

Therefore, we have to approach the problem in a different way. Concretely, our idea is to use the fact that the rotation number is constant on the tongue combined with remark 3.2. To this end, we consider the one-parameter family $\{f_{\alpha(\varepsilon),\varepsilon}\}_{\varepsilon\in[0,1)}$ of circle diffeomorphisms, where the graph of α parametrizes the tongue T_θ. For this family, we have $\rho(f_{\alpha(\varepsilon),\varepsilon}) = \theta$ for any $\varepsilon \in [0,1)$, and hence, from remark 3.2 we read the expression

$$0 = \Theta^d_{q,p,p}(f_{\alpha(\varepsilon),\varepsilon}) + \mathcal{O}(2^{-(p+1)q}), \tag{23}$$

where p is the averaging order, we use 2^q iterates and $\Theta^d_{q,p,p}$ is the extrapolation operator (16) that, in this case, depends on the derivatives of $\alpha(\varepsilon)$ up to order d. With this idea in mind, the aim of the next paragraphs is to show how we can isolate inductively these derivatives at $\varepsilon = 0$ from the previous equation.

Let us start by describing how to approximate the first derivative $\alpha'(0)$. As mentioned above, we have to write $\Theta^1_{q,p,p}(f_{\alpha(\varepsilon),\varepsilon})|_{\varepsilon=0}$ in terms of $\alpha'(0)$ and we note that, by linearity, it suffices to work with the expression $D_\varepsilon(f^n_{\alpha(\varepsilon),\varepsilon}(x_0))|_{\varepsilon=0}$. To do that, we write

$$f(x) = 2\pi\alpha(\varepsilon) + g(x), \qquad g(x) = x + \varepsilon\sin(x),$$

in order to uncouple the dependence on α in the circle map. Observe that, as usual, we omit the dependence on the parameter in the maps. Using this notation, we have:

$$\begin{aligned} D_\varepsilon(f(x_0)) &= 2\pi\alpha'(\varepsilon) + \partial_\varepsilon g(x_0), \\ D_\varepsilon(f^2(x_0)) &= 2\pi\alpha'(\varepsilon) + \partial_\varepsilon g(f(x_0)) + \partial_x g(f(x_0)) D_\varepsilon(f(x_0)) \\ &= 2\pi\alpha'(\varepsilon)\Big\{1 + \partial_x g(f(x_0))\Big\} + \partial_\varepsilon g(f(x_0)) + \partial_x g(f(x_0))\partial_\varepsilon g(x_0). \end{aligned}$$

Similarly, we can proceed inductively and split the derivative of the n-th iterate, $D_\varepsilon(f^n(x_0))$, in two parts, one of them having a factor $2\pi\alpha'(\varepsilon)$. Moreover, if we set $\varepsilon = 0$ in $D_\varepsilon(f^n(x_0))$, then it is clear that, with the exception of the previous factor, the resulting expression does not depend on $\alpha'(0)$ but only on $\alpha(0) = \theta$.

Now, we generalize the above argument to higher order derivatives. Let us assume that the values $\alpha'(0), \ldots, \alpha^{(d-1)}(0)$ are known, and isolate the derivative $\alpha^{(d)}(0)$ from $D^d_\varepsilon(f^n(x_0))|_{\varepsilon=0}$. We claim that the following formula holds

$$D^d_\varepsilon(f^n(x_0))|_{\varepsilon=0} = 2\pi n\alpha^{(d)}(0) + g^d_n, \tag{24}$$

where the factor $2\pi n$ comes from the fact that $\partial_x g|_{\varepsilon=0} = 1$, and $g^d := \{g^d_n\}_{n\in\mathbb{N}}$ is a sequence that only requires the known derivatives $\alpha^{(r)}(0)$, for $r < d$. Concretely, let us obtain the term g^d_n of the sequence by induction with respect to n. Once again, it is straightforward to write

$$\begin{aligned} D^d_\varepsilon(f^n(x_0)) &= D^{d-1}_\varepsilon\Big(2\pi\alpha'(\varepsilon) + \partial_\varepsilon g(f^{n-1}(x_0)) + \partial_x g(f^{n-1}(x_0)) D_\varepsilon(f^{n-1}(x_0))\Big) \\ &= 2\pi\alpha^{(d)}(\varepsilon) + D^{d-1}_\varepsilon(\partial_\varepsilon g(f^{n-1}(x_0)) \\ &\quad + \sum_{r=0}^{d-1} \binom{d-1}{r} D^r_\varepsilon(\partial_x g(f^{n-1}(x_0))) D^{d-r}_\varepsilon(f^{n-1}(x_0)). \end{aligned}$$

We note that the term $r = 0$ in this expression contains $D^d_\varepsilon(f^{n-1}(x_0))$. Then, if we set $\varepsilon = 0$ and replace inductively the previous term by equation (24), we find that

$$\begin{aligned} g^d_n &= D^{d-1}_\varepsilon(\partial_\varepsilon g(f^{n-1}(x_0))|_{\varepsilon=0} \\ &\quad + \sum_{r=1}^{d-1} \binom{d-1}{r} D^r_\varepsilon(\partial_x g(f^{n-1}(x_0))) D^{d-r}_\varepsilon(f^{n-1}(x_0))|_{\varepsilon=0} + g^d_{n-1} \end{aligned}$$

and let us remark that, as mentioned, this expression is independent of $\alpha^{(d)}(0)$.

We conclude the explanation of the method by describing the extrapolation process that allows us to approximate these derivatives. To this end, we introduce an extrapolation operator as (9) for the sequence g^d. Indeed, we extend the recursive sums (4) and the averaged sums (5)

d	$2\pi\alpha^{(d)}(0)$	e_1	e_2
0	3.883222077450933154693731259925391915269339787692096599014776434	-	-
1	5.289596087298835974306750728481413682115174017433159533705768026 $\cdot 10^{-54}$	$2{\cdot}10^{-50}$	$5{\cdot}10^{-54}$
2	-1.944003667801032197325141712953470682792841985057545477738933600 $\cdot 10^{-1}$	$7{\cdot}10^{-50}$	$2{\cdot}10^{-53}$
3	6.353866339253870417285870622952031667026712174414003758743809499 $\cdot 10^{-52}$	$3{\cdot}10^{-48}$	$6{\cdot}10^{-52}$
4	9.865443989835495993231949890783720243438883460505483297079900562 $\cdot 10^{-1}$	$2{\cdot}10^{-47}$	$5{\cdot}10^{-51}$
5	4.733853534850495777271526084574485398105534790325269345544052633 $\cdot 10^{-49}$	$2{\cdot}10^{-45}$	$5{\cdot}10^{-49}$
6	-1.451874181864020963416053802229271731186248529989217665545212404 $\cdot 10^{1}$	$6{\cdot}10^{-45}$	$1{\cdot}10^{-48}$
7	-1.986768674642925514096249083525472601734104441662711304098209993 $\cdot 10^{-47}$	$7{\cdot}10^{-44}$	$2{\cdot}10^{-47}$
8	1.673363822376717001078781931538386967523434046199355922539083323 $\cdot 10^{1}$	$8{\cdot}10^{-42}$	$2{\cdot}10^{-45}$
9	-5.559060362825539878039137008326038842079877436013501651866007318 $\cdot 10^{-44}$	$2{\cdot}10^{-40}$	$6{\cdot}10^{-44}$
10	1.974679484744669888248485084754876332689468886829840384314732615 $\cdot 10^{4}$	$2{\cdot}10^{-39}$	$4{\cdot}10^{-43}$
11	4.019718902900154426125206309959051888079502318143227318836414835 $\cdot 10^{-42}$	$1{\cdot}10^{-38}$	$4{\cdot}10^{-42}$
12	3.594891944526889578314748272295019294147597687816868847742850594 $\cdot 10^{5}$	$6{\cdot}10^{-37}$	-
13	-4.123166034989923032518732576715313341946051550138603536248010821 $\cdot 10^{-39}$	$2{\cdot}10^{-35}$	$4{\cdot}10^{-39}$
14	2.198602821435568153883567054383394767567371744732559263055644337 $\cdot 10^{6}$	$3{\cdot}10^{-33}$	-
15	1.307318024754974551233761145122558811543944190022138837513637182 $\cdot 10^{-35}$	$6{\cdot}10^{-32}$	$1{\cdot}10^{-35}$
16	-4.009257214040427899940043656551946700300230713255210114705187412 $\cdot 10^{10}$	$4{\cdot}10^{-31}$	-
17	-6.641638995605492204184114438636683272452899190211080822408603857 $\cdot 10^{-33}$	$4{\cdot}10^{-29}$	$7{\cdot}10^{-33}$
18	-2.582559893723659427522610275977697024396910000154382754643273110 $\cdot 10^{12}$	$1{\cdot}10^{-27}$	-
19	-4.366235264281358239242428788236090577328510850575386329987344515 $\cdot 10^{-30}$	$2{\cdot}10^{-26}$	$4{\cdot}10^{-30}$

Table 1: Derivatives of $2\pi\alpha(\varepsilon)$ at the origin for $\theta = (\sqrt{5}-1)/2$. The column e_1 corresponds to the estimated error using (11). The column e_2 is the real error, that for even derivatives is computed comparing with the analytic expressions (25) and (26) using the coefficients from table 2.

for this sequence, thus obtaining

$$\Theta_{q,p}(g^d) := \sum_{j=0}^{p} c_j^p \tilde{S}^p_{2^{q-p+j}}(g^d).$$

Recalling that $D^d_\varepsilon \theta$ vanishes, we obtain from equation (23) that

$$\Theta^d_{q,p,p}(f)|_{\varepsilon=0} = 2\pi\alpha^{(d)}(0) + \Theta_{q,p}(g^d) = \mathcal{O}(2^{-(p+1)q}).$$

Therefore, the Taylor expansion (21) follows from the sequential computation of $\alpha^{(d)}(0)$ by means of the expression

$$\alpha^{(d)}(0) = -\frac{1}{2\pi}\Theta_{q,p}(g^d) + \mathcal{O}(2^{-(p+1)q}).$$

Let us discuss some obtained results. The following computations are performed using 64 digits (*quadruple-double* data type from [17]). The implementation parameters are selected as $p = 11$, $q = 23$ and any tolerance is required in the extrapolation error (which is estimated by means of (11)).

In table 1 we show the computations of $2\pi\alpha^{(d)}(0)$, for $0 \leq d \leq 19$, that correspond to the Arnold Tongue associated to $\theta = (\sqrt{5}-1)/2$.

In addition, we use the above computations to obtain formulas, depending on θ, for the first coefficients of (21). To make this dependence explicit, we introduce the notation $\alpha_r(\theta) :=$

$\alpha^{(2r)}(0)$, where $(\varepsilon, \alpha(\varepsilon))$ parametrizes the Arnold Tongue T_θ. Analytic expressions for these coefficients can be found, for example, by solving the conjugation equation of diagram (1) using Lindstedt series. However, the complexity of the symbolic manipulations required for carrying the above computations is very big. In particular, the first two coefficients, whose computation is detailed in [27], are

$$\alpha_1(\theta) = \frac{\cos(\pi\theta)}{2^2\pi\sin(\pi\theta)}, \qquad \alpha_2(\theta) = -\frac{3\cos(4\pi\theta)+9}{2^5\pi\big(\sin(\pi\theta)\big)^2\sin(2\pi\theta)}. \tag{25}$$

From these formulas and a heuristic analysis of the small divisors equations to be solved for computing the remaining coefficients, we make the following guess for $\alpha_r(\theta)$:

$$\alpha_r(\theta) = \frac{P_r(\theta)}{2^{c(r)}\pi\big(\sin(\pi\theta)\big)^{2^{r-1}}\big(\sin(2\pi\theta)\big)^{2^{r-2}}\cdots\big(\sin((r-1)\pi\theta)\big)^2\sin(r\pi\theta)}, \tag{26}$$

where $c(r)$ is a natural number and P_r is a trigonometric polynomial of the form

$$P_r(\theta) = \sum_{j=1}^{d_r} a_j\cos(j\pi\theta),$$

with integer coefficients and degree $d_r = 2^{r+1} - r - 2$ that coincides with the degree of the denominator. In addition, the coefficients a_j vanish except for indexes j such that $j \equiv d_r (\mathrm{mod}\, 2)$.

In order to obtain the coefficients of P_r, we have computed the Taylor expansions of the Arnold Tongues for 120 different rotation numbers. Concretely, we have selected the quadratic irrationals $\theta_{a,b} = (\sqrt{b^2+4b/a} - b)/2$, for $1 \leq a \leq b \leq 5$, that have periodic continued fraction given by $\theta_{a,b} = [0; \widehat{a,b}]$. Then, we fix the value of $c(r)$ and perform minimum square fit for the coefficients a_j. We validate the computations if the solution corresponds to integer numbers, or we increase $c(r)$ otherwise. In order to detect if $a_j \in \mathbb{Z}$, we require an arithmetic precision higher than 64 digits. Then, these computations have been implemented in PARI-GP (available at [1]) using 100-digit arithmetics.

Following the above idea, we have obtained expressions for the next three coefficients. Concretely, we find the values $c(3) = 10$, $c(4) = 19$, and $c(5) = 38$. On the other hand, the corresponding polynomials P_r are given in table 2. The comparison between these pseudo-analytical coefficients and the values computed numerically for $\theta = (\sqrt{5}-1)/2$ is shown in column e_2 of table 1, obtaining a very good agreement. Let us observe that the coefficients of P_r grow very fast with respect to r, and the same occurs to $c(r)$. Indeed, the values that correspond to $r = 6$ are too big to be computed with the selected precision, due to the loss of significant digits.

Finally, we also compare the truncated Taylor expansions with the numerical approximation of the Arnold Tongue for $\theta = (\sqrt{5}-1)/2$, computed using Newton method. To this end, we perform the computation of subsection 4.2 for $\varepsilon \in [0, 0.1]$, using *quadruple-double* precision, an averaging order $p = 9$ and requiring tolerances of 10^{-42} for the computation of the rotation

P_3		P_4			
j	a_j	j	a_j	j	a_j
1	-105	0	-360150	14	-177625
3	825	2	40950	16	-14770
5	-465	4	469630	18	34755
7	-315	6	91140	20	49735
9	120	8	-378700	22	-53235
11	-60	10	67165	24	18900
		12	215355	26	-3150

P_5					
j	a_j	j	a_j	j	a_j
1	33992959770	21	46136915685	41	6059661930
3	-96457394880	23	-28888862310	43	-4422651975
5	107920471050	25	23182141500	45	1217211030
7	-47792873520	27	-24695086815	47	651686490
9	-1102024980	29	7313756940	49	-826836885
11	3276815850	31	14354738685	51	404729640
13	-38366469540	33	-20342636055	53	-112651560
15	97991931555	35	13721635620	55	17781120
17	-74144022120	37	-4249642635	57	-1270080
19	-11687638410	39	-3152375100		

Table 2: Coefficients for the trigonometric polynomials P_3, P_4 and P_5.

number, and 10^{-40} for the convergence of the Newton method. In all the computations, we allow at most 2^{23} iterates of the map. Then, in figure 4 we compare the approximated tongue with the Taylor expansions truncated at orders $2, 4, 6, 8$ and 10.

5 Study of invariant curves for planar twist maps

In this last section we deal with a classical problem in dynamical systems that arise in many applications: the study of quasi-periodic invariant curves for planar maps. Concretely, we focus on the context of so-called twist maps, because in this case we can easily make a link with circle diffeomorphisms. First of all, in subsection 5.1 we formalize the problem and fix some notation. Then, in subsection 5.2 we adapt our methodology to compute invariant curves and their evolution with respect to parameters by means of the Newton method. Finally, in subsection 5.3 we follow the ideas of subsection 4.3 and compute asymptotic expansions relating initial conditions and parameters that correspond to invariant curves of fixed rotation number. As an example, we study the neighborhood of the elliptic fixed point for the Hénon map, which appears generically in the study of area-preserving maps.

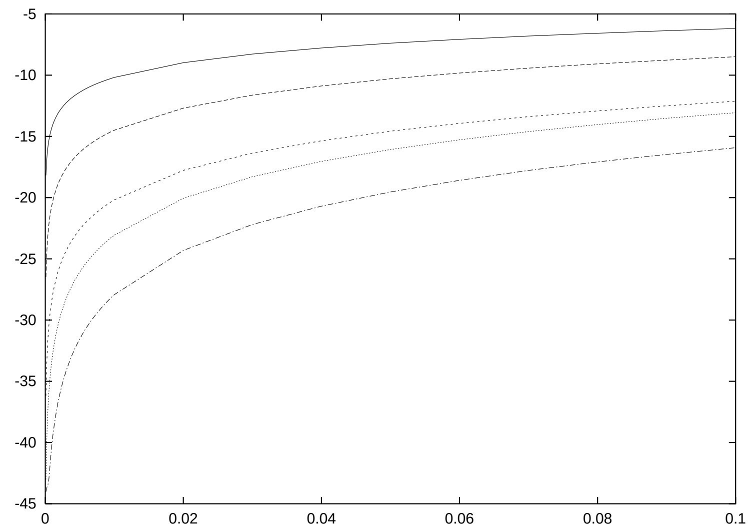

Figure 4: Comparison between the numerical expressions of $\alpha(\varepsilon)$ for the Arnold Tongue T_θ, with $\theta = (\sqrt{5}-1)/2$, obtained using the Newton method and the truncated Taylor expansion (21) up to order d. Concretely we plot, as a function of ε, the difference in $\log_{10}$ scale between these quantities. The curves from top to bottom correspond, respectively, to $d = 2, 4, 6, 8$ and 10.

5.1 Description of the problem

Let $\mathcal{A} = \mathbb{T} \times I$ be the real annulus, where I is any real interval, that can be lifted to the strip $A = \mathbb{R} \times I$ using the universal cover $\pi : A \to \mathcal{A}$. Let also $X : A \to \mathbb{R}$ and $Y : A \to I$ denote the canonical projections $X(x, y) = x$ and $Y(x, y) = y$.

In this section, we consider diffeomorphisms $F : \mathcal{A} \to \mathcal{A}$ and their lifts $\tilde{F} : A \to A$ given by $F \circ \pi = \pi \circ \tilde{F}$. Note that the lift is unique if we require $X(\tilde{F}(0, y_0)) \in [0, 1)$ for certain $y_0 \in I$, so we omit the tilde in the lift. In addition, we restrict to maps satisfying that $\partial(X \circ F)/\partial y$ does not vanish, a condition that is called *twist*.

Assume that $F : A \to A$ is a twist map having an invariant curve Γ, homotopic to the circle $\mathbb{T} \times \{0\}$, of rotation number $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Concretely, there exists an embedding $\gamma : \mathbb{R} \to A$, such that $\Gamma = \gamma(\mathbb{R})$, satisfying $\gamma(x + 1) = \gamma(x) + (1, 0)$ for all $x \in \mathbb{R}$, and making the following diagram commute

$$
\begin{array}{ccc}
\Gamma \subset A & \xrightarrow{\;F\;} & \Gamma \subset A \\
\uparrow{\scriptstyle\gamma} & & \uparrow{\scriptstyle\gamma} \\
\mathbb{R} & \xrightarrow{\;R_\theta\;} & \mathbb{R}
\end{array}
\qquad\qquad F(\gamma(x)) = \gamma(x + \theta). \tag{27}
$$

Since F is a twist map, the Birkhoff Graph Theorem (see [11]) ensures that Γ is a Lipschitz graph over its projection on the circle $\mathbb{T} \times \{0\}$, and hence the dynamics on Γ induces a circle homeomorphism f_Γ simply by projecting the iterates, i.e., $f_\Gamma(X(\gamma(x))) = X(F(\gamma(x))$. We observe that, if F and γ are $\mathcal{C}^r$-diffeomorphisms, then $f_\Gamma \in \mathrm{Diff}_+^r(\mathbb{T})$.

From now on, we fix an angle $x_0 \in \mathbb{T}$ and identify invariant curves with points $y_0 \in I$. Then, if (x_0, y_0) belongs to an invariant curve Γ, we also denote the previous circle map as f_{y_0} instead of f_Γ. Of course, the parameterization γ is unknown in general, so we do not have an expression for f_{y_0}. But we can evaluate the orbit $(x_n, y_n) = F^n(x_0, y_0)$ and consider $x_n = f_{y_0}^n(x_0)$. We recall that this is the only that we need to compute numerically the rotation number θ using the method of [27] (reviewed in subsection 2.2).

Remark 5.1. *If the map F does not satisfy the twist condition, their invariant curves are not necessarily graphs over the circle $\mathbb{T} \times \{0\}$. Of course, if Γ is an invariant curve of F, its dynamics still induces a circle diffeomorphism, even though its construction is not so obvious. Since the non-twist case presents another kind of difficulties and has its own interest, we plan to adapt the method to consider the general situation in a subsequent work [22].*

If F is a $\mathcal{C}^r$-integrable twist map, then there is a $\mathcal{C}^r$-family of invariant curves of F satisfying (27), and $y_0 \mapsto f_{y_0}$ is a one-parameter family in $\mathrm{Diff}_+^r(\mathbb{T})$. In this case, we obtain a $\mathcal{C}^r$-function $y_0 \in I \mapsto \rho(f_{y_0})$. Of course, this is not the general situation and, actually, we do not expect this function to be defined for every $y_0 \in I$. Nevertheless, in many problems we have a family of invariant curves defined on a Cantor subset $J \subset I$ having positive Lebesgue measure and we still have differentiability of $\rho(f_{y_0})$ in the sense of Whitney. For example, if the map F is a perturbation of an integrable twist map that is symplectic or satisfies the intersection condition, KAM theory establishes (under other general assumptions) the existence of such a Cantor family of invariant curves (we refer to [6, 23]).

For practical purposes, even if a point $(x_0, y_0) \in A$ does not belong to a quasi-periodic invariant curve, we can compute the orbit $x_n = f_{y_0}^n(x_0) = X(F^n(x_0, y_0))$, even though f_{y_0} is not a circle diffeomorphism. Then, we can also compute the averaged sums $S_N^p(f_{y_0})$ of these iterates but we cannot guarantee in general that $\Theta_{q,p}(f_{y_0})$ converges when $q \to \infty$. Nevertheless, if (x_0, y_0) is an initial condition close enough to an invariant curve of Diophantine rotation number θ, we expect $\Theta_{q,p}(f_{y_0})$ to converge to a number close to θ, due to the existence of neighboring invariant curves for a set of big relative measure (that is called *condensation* phenomena in KAM theory). On the other hand, if (x_0, y_0) belongs to a periodic island, then we expect $\Theta_{q,p}(f_{y_0})$ to converge to the winding number of the "central" periodic orbit. Finally, we recall that the Aubry-Mather theorem (we refer to [11]) states that F has orbits of all rotation numbers, so it can occur that the method converges if (x_0, y_0) corresponds to a periodic orbit or to a ghost curve (Cantori).

On the other hand, in order to approximate the derivatives of the rotation number by means of $\Theta_{q,p,p-d}^d(f_{y_0})$, we have to compute the derivatives of the iterates x_n. However, as we do not have an explicit formula for the induced map f_{y_0}, the scheme for computing the derivatives of the iterates is slightly different from the one presented in subsection 3.3. Modified recurrences are detailed in the moment that they are required.

5.2 Numerical continuation of invariant curves

Let us consider $\alpha : \Lambda \subset \mathbb{R} \mapsto F_\alpha$ a one-parameter family of twist maps on $\mathcal{A}$, that induces a function $(\alpha, y_0) \in U \subset \Lambda \times I \mapsto \rho(f_{\alpha,y_0})$ differentiable in the sense of Whitney. In this situation, we can compute the derivatives of this function (at the points where they exist) by using the method of section 3. Our goal now is to use these derivatives to compute numerically invariant curves of F_α by means of the Newton method, similarly as we did in subsection 4.2 for computing the Arnold Tongues.

Concretely, let Γ_{α_0} be an invariant curve of rotation number $\theta \in \mathcal{D}$ for the map F_{α_0}. Then, given any α close to α_0, we want to compute the curve Γ_α, invariant under F_α, having the same rotation number. Once we have fixed an angle $x_0 \in \mathbb{T}$, we identify the invariant curve Γ_α by the point $(x_0, y(\alpha)) \in \Gamma_\alpha$. Then, our purpose is to solve, with respect to y, the equation $\rho(f_{\alpha,y}) = \theta$ by continuing the known solution $(\alpha_0, y(\alpha_0)) \in \Lambda \times I$. We just remark that, when solving this equation by means of the Newton method, we have to prevent us from falling into a resonant island, where the rotation number is locally constant around this point.

Now, in order to approximate numerically $D_\alpha \rho$ and $D_{y_0}\rho$, we have to discuss the computation of the derivatives of the iterates, i.e. $D_\alpha(x_n)$ and $D_{y_0}(x_n)$, where $x_n = f^n_{\alpha,y_0}(x_0)$. Omitting the dependence on the parameter α in the family of twist maps, we denote $F_1 = X \circ F$ and $F_2 = Y \circ F$, and we obtain the recurrent expression

$$D_{y_0}(x_n) = \partial_x F_1(z_{n-1}) D_{y_0}(x_{n-1}) + \partial_y F_1(z_{n-1}) D_{y_0}(y_{n-1}), \tag{28}$$

where $z_n := (x_n, y_n)$. Furthermore, $D_{y_0}(y_n)$ follows from a similar expression replacing F_1 by F_2. According to our convention of fixing $x_0 \in \mathbb{T}$, the computations have to be initialized by $D_{y_0}(x_0) = 0$ and $D_{y_0}(y_0) = 1$. Analogous formulas hold for $D_\alpha(x_n)$:

$$D_\alpha(x_n) = \partial_\alpha F_1(z_{n-1}) + \partial_x F_1(z_{n-1}) D_\alpha(x_{n-1}) + \partial_y F_1(z_{n-1}) D_\alpha(y_{n-1}),$$

and similarly for $D_\alpha(y_n)$ using F_2. The recursive computations are now initialized by $D_\alpha(x_0) = 0$ and $D_\alpha(y_0) = 0$.

Let us illustrate the above ideas studying the well known Hénon family, that is a paradigmatic example since it appears generically in the study of a saddle-node bifurcation. In Cartesian coordinates, the family can be written as

$$H_\alpha : \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} \cos(2\pi\alpha) & -\sin(2\pi\alpha) \\ \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix} \begin{pmatrix} u \\ v - u^2 \end{pmatrix}. \tag{29}$$

Note that the origin is an elliptic fixed point that corresponds to a “singular” invariant curve. We can blow-up the origin if, for example, we bring the map to the annulus by means of polar coordinates $x = \arctan(v/u)$ and $y = \sqrt{u^2 + v^2}$, thus obtaining a family $\alpha \in \Lambda = [0, 1) \mapsto F_\alpha$ of maps $F_\alpha : \mathbb{S} \times I \mapsto \mathbb{S} \times I$, given by

$$X \circ F_\alpha = \arctan \frac{\sin(x + 2\pi\alpha) - \cos(2\pi\alpha) y (\cos(x))^2}{\cos(x + 2\pi\alpha) + \sin(2\pi\alpha) y (\cos(x))^2}, \tag{30}$$

$$Y \circ F_\alpha = y\sqrt{1 - 2y(\cos(x))^2 \sin(x) + y^2(\cos(x))^4}. \tag{31}$$

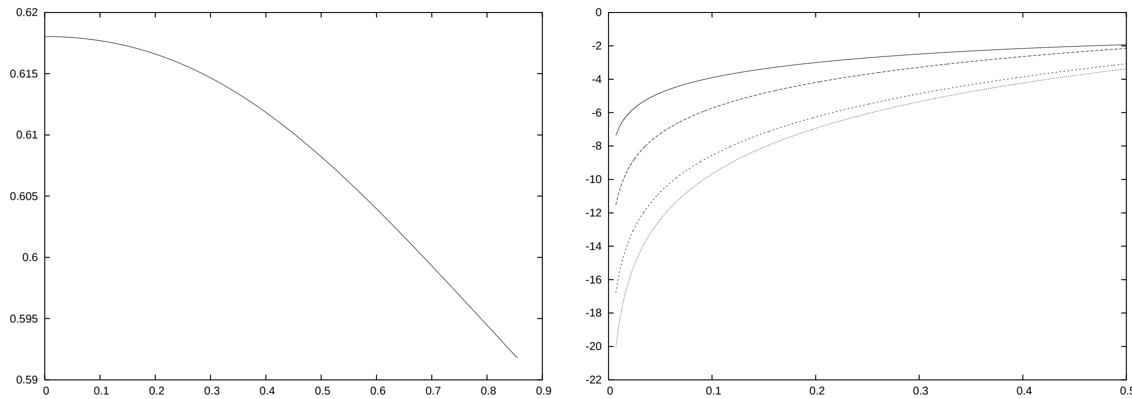

Figure 5: Left: Numerical continuation of y_0 (horizontal axis) with respect to α (vertical axis) of the invariant curve of rotation number $\theta = (\sqrt{5}-1)/2$ for the Hénon map (29). Right: Difference in $\log_{10}$ scale between $\alpha(y_0)$ in the left plot and its truncated Taylor expansion (32) up to order d (see table 3). The curves from top to bottom correspond, respectively, to $d = 2, 4, 6$ and 8.

We remark that, analogously as we did in section 4, in this application we consider angles in $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$ in order to avoid factors 2π that would appear in the derivatives (specially in subsection 5.3 when we consider higher order ones).

Albeit it is not difficult to check that the twist condition $\partial(X \circ F)/\partial y \neq 0$ is not fulfilled in these polar coordinates, we can perform a close to the identity change of variables to guarantee the twist condition except for the values $\alpha = 1/3, 2/3$. Then, it turns out that there exist invariant curves of F_α in a neighborhood of $\mathbb{S} \times \{0\}$, whose rotation number tend to α and are "close to the identity" to graphs over $\mathbb{S} \times \{0\}$. However, for values of α close to $1/3$ and $2/3$, meandering phenomena arises (we refer to [10, 29]), i.e., there are folded invariant curves (see remark 5.1).

As an example, we study the invariant curves of rotation number $\theta = (\sqrt{5}-1)/2$ by continuing the initial values $\alpha_0 = \theta$ and $y_0 = 0$, i.e, the curve $\mathbb{S} \times \{0\}$. The computations have been performed by using the *double-double* data type, a fixed averaging order $p = 8$ and up to 2^{23} iterates of the map, at most. As usual, we estimate the error in the rotation number by using (11), and we validate the computation when the error is smaller than 10^{-26}. For the Newton method, we require a tolerance smaller than 10^{-23} when comparing two successive computations. Finally, we do not require a prescribed tolerance in the computation of the derivatives $D_\alpha \rho$ and $D_{y_0}\rho$, but the biggest error in their computation is less that 10^{-21}.

The resulting curve in the space $I \times \Lambda$ is shown in figure 5 (left). During the continuation, the step in α is typically taken between 10^{-4} and 10^{-3}, but falls to 10^{-10} when we compute the last point $(\alpha, y(\alpha)) = (0.5917905628, 0.8545569509)$. In figure 6 we plot the graph corresponding to this invariant curve and its derivative. We observe that, even though the curve is still a graph, this parameterization is close to have a vertical tangency, so our approach is not suitable for continuing the curve. However, since the fractalization of the curve has not occurred, we expect that it still exists beyond this point. To continue the family of curves in this situation it is convenient to use another approach (see remark 5.1).

5.3 Computing expansions with respect to parameters

In the same situation of subsection 5.2, our aim now is to use the variational information of the rotation number to compute the Taylor expansion at the origin of figure 5 (left). Notice that in the selected example $\alpha'(0) = 0$, so we work with the expansion of the function $\alpha(y_0)$ rather than $y_0(\alpha)$.

In general, if (x_0, y_0^*) is a point on an invariant curve of rotation number θ for a twist map F_{α^*}, then we consider the expansion

$$\alpha(y_0) = \alpha^* + \alpha'(y_0^*)(y_0 - y_0^*) + \frac{\alpha''(y_0^*)}{2!}(y_0 - y_0^*)^2 + \cdots , \tag{32}$$

that corresponds to the value of the parameter for which (x_0, y_0) is contained in an invariant curve of $F_{\alpha(y_0)}$ having the same rotation number. We know that if $\theta \in \mathcal{D}$ and the family F_α is analytic, then (32) is an analytic function around y_0^*. Once again, during the rest of the section, we omit the dependence on the parameter α in the family of twist maps, and we denote $F_1 = X \circ F$ and $F_2 = Y \circ F$.

Like in subsection 4.3, we use that the family $y_0 \mapsto f_{\alpha(y_0),y_0} \in \mathrm{Diff}_+^\omega(\mathbb{S})$ induced by $y_0 \mapsto F_{\alpha(y_0)}$ has constant rotation number, together with remark 3.2. Concretely, for any integer $d \geq 1$ we have

$$0 = \Theta^d_{q,p,p}(f_{\alpha(y_0),y_0}) + \mathcal{O}(2^{-(p+1)q}), \tag{33}$$

where $\Theta^d_{q,p,p}$ is the extrapolation operator (16). We observe that the value of $\Theta^d_{q,p,p}(f_{y_0})$ at the point y_0^* only depends on the derivatives $\alpha^{(r)}(y_0^*)$ up to $r \leq d$. We use this fact to compute inductively these derivatives from equation (33). To achieve this, we have to isolate them from $D^d_{y_0}(x_n)|_{y_0=y_0^*}$ for any $d \geq 1$, as we discuss through the next paragraphs.

The following formula generalizes (28):

$$\begin{aligned} D^d_{y_0}(x_n) \;=\; & \sum_{j=0}^{d-1} \binom{d-1}{j} \Big\{ D^j_{y_0}(\partial_\alpha F_1(z_{n-1}))\, \alpha^{(d-j)}(y_0) \\ & + D^j_{y_0}(\partial_x F_1(z_{n-1})) D^{d-j}_{y_0}(x_{n-1}) + D^j_{y_0}(\partial_y F_1(z_{n-1})) D^{d-j}_{y_0}(y_{n-1}) \Big\}, \end{aligned} \tag{34}$$

while a similar equation holds for $D^d_{y_0}(y_n)$ replacing F_1 by F_2. Moreover, as in subsection 3.3, we compute the derivatives $D^r_{y_0}$ of $\partial_\alpha F_1(z_{n-1})$, $\partial_x F_1(z_{n-1})$ and $\partial_y F_1(z_{n-1})$ by means of the

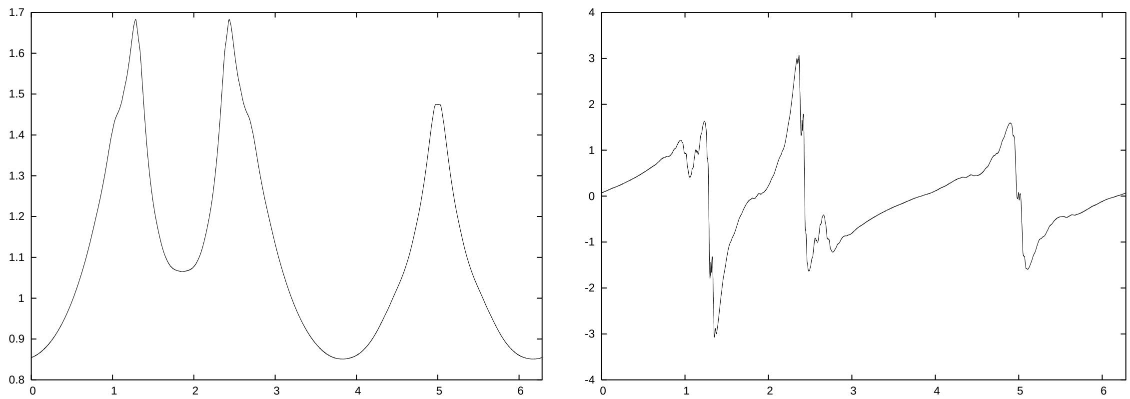

Figure 6: Left: Invariant curve of (29) of rotation number $\theta = (\sqrt{5}-1)/2$ corresponding to the last computed point in figure 5 (see text) expressed as a graph $x \mapsto y$ on the annulus $\mathbb{S} \times I$. Right: Derivative of the left plot computed using finite differences.

following recurrent expression

$$
\begin{aligned}
D^r_{y_0}(\partial^{k,l,m}_{\alpha,x,y} F_i(z_{n-1})) \;=\; & \sum_{j=0}^{r-1} \binom{r-1}{j} \Big\{ D^j_{y_0}(\partial^{k+1,l,m}_{\alpha,x,y} F_i(z_{n-1}))\, \alpha^{(r-j)}(y_0) \\
& + D^j_{y_0}(\partial^{k,l+1,m}_{\alpha,x,y} F_i(z_{n-1})) D^{r-j}_{y_0}(x_{n-1}) \\
& + D^j_{y_0}(\partial^{k,l,m+1}_{\alpha,x,y} F_i(z_{n-1})) D^{r-j}_{y_0}(y_{n-1}) \Big\},
\end{aligned}
$$

which only requires to evaluate the partial derivatives of F_1 and F_2 with respect to α, x and y.

Using the above expressions, we reproduce the inductive argument of subsection 4.3. Let us assume that the values $\alpha'(y_0^*), \ldots, \alpha^{(d-1)}(y_0^*)$ are known. Then, we observe that if we set $y_0 = y_0^*$ and $\alpha = \alpha^*$ in equation (34), the only term containing the derivative $\alpha^{(d)}(y_0^*)$ is the one corresponding to $j = 0$. By induction, it is easy to find that

$$
D^d_{y_0}(x_n)|_{y_0=y_0^*} = \mathcal{X}^d_n\, \alpha^{(d)}(y_0^*) + \hat{\mathcal{X}}^d_n, \qquad D^d_{y_0}(y_n)|_{y_0=y_0^*} = \mathcal{Y}^d_n\, \alpha^{(d)}(y_0^*) + \hat{\mathcal{Y}}^d_n,
$$

where the coefficients $\mathcal{X}^d_n$, $\hat{\mathcal{X}}^d_n$, $\mathcal{Y}^d_n$ and $\hat{\mathcal{Y}}^d_n$ are obtained recursively and only depend on the derivatives $\alpha^{(r)}(y_0^*)$, with $r < d$. Concretely, $\mathcal{X}^d_n$ and $\hat{\mathcal{X}}^d_n$ satisfy

$$
\begin{aligned}
\mathcal{X}^d_n \;=\; & \big(\partial_\alpha F_1(z_{n-1}) + \partial_x F_1(z_{n-1}) \mathcal{X}^d_{n-1} + \partial_y F_1(z_{n-1}) \mathcal{Y}^d_{n-1}\big)\,|_{y_0=y_0^*}, \\
\hat{\mathcal{X}}^d_n \;=\; & \Bigg(\partial_x F_1(z_{n-1}) \hat{\mathcal{X}}^d_{n-1} + \partial_y F_1(z_{n-1}) \hat{\mathcal{Y}}^d_{n-1} + \sum_{j=1}^{d-1} \binom{d-1}{j} \Big\{ D^j_{y_0}(\partial_\alpha F_1(z_{n-1})) \alpha^{(d-j)}(y_0) \\
& + D^j_{y_0}(\partial_x F_1(z_{n-1})) D^{d-j}_{y_0}(x_{n-1}) + D^j_{y_0}(\partial_y F_1(z_{n-1})) D^{d-j}_{y_0}(y_{n-1}) \Big\}\Bigg)\Bigg|_{y_0=y_0^*},
\end{aligned}
$$

d	$2\pi\alpha^{(d)}(0)$	e_1
0	3.8832220774509331546937312599254	-
1	$2.9215929940647956972904287221575 \cdot 10^{-29}$	$6\cdot 10^{-27}$
2	$-3.9914536995187621201317645570286\cdot 10^{-1}$	$6\cdot 10^{-28}$
3	$-7.2312013917244657534375078612123\cdot 10^{-1}$	$5\cdot 10^{-27}$
4	$-1.4570409862191278806067261207843\cdot 10^{0}$	$9\cdot 10^{-27}$
5	$2.0167847130561842764416032369501 \cdot 10^{1}$	$4\cdot 10^{-26}$
6	$1.2357011948811946999538300791232 \cdot 10^{2}$	$1\cdot 10^{-25}$
7	$-9.1717201199029959021691212417954\cdot 10^{1}$	$2\cdot 10^{-25}$
8	$-3.0832824868383111456060167381447\cdot 10^{3}$	$4\cdot 10^{-24}$
9	$-7.2541251340271326844826925983923\cdot 10^{4}$	$2\cdot 10^{-22}$

Table 3: Derivatives of $2\pi\alpha(y_0)$ at the origin for $\theta = (\sqrt{5}-1)/2$. The column e_1 corresponds to the estimated error using (11).

and similar equations hold for $\mathcal{Y}_n^d$ and $\hat{\mathcal{Y}}_n^d$ replacing F_1 by F_2. These sequences are initialized as

$$\mathcal{X}_0^1 := \hat{\mathcal{X}}_0^1 := \mathcal{Y}_0^1 := 0,\, \hat{\mathcal{Y}}_0^1 := 1, \quad \text{and} \quad \mathcal{X}_0^d := \hat{\mathcal{X}}_0^d := \mathcal{Y}_0^d := \hat{\mathcal{Y}}_0^d := 0, \quad \text{for } d > 1.$$

Finally, if we evaluate the extrapolation operator $\Theta_{q,p}$ for the sequences $\mathcal{X}^d = \{\mathcal{X}_n^d\}_{n=1,\ldots,N}$ and $\hat{\mathcal{X}}^d = \{\hat{\mathcal{X}}_n^d\}_{n=1,\ldots,N}$, then we obtain from (33) the following expression

$$\alpha^{(d)}(y_0^*) = -\frac{\Theta_{q,p}(\hat{\mathcal{X}}^d)}{\Theta_{q,p}(\mathcal{X}^d)} + \mathcal{O}(2^{-(p+1)q}).$$

Now, we apply this methodology to the Hénon family $\alpha \in \Lambda = [0,1) \mapsto F_\alpha$ given by (30) and (31). In particular, we fix $x_0 = 0$ and compute the expansion (32) at $y_0^* = 0$ corresponding to invariant curves of rotation number $\alpha^* = \theta = (\sqrt{5}-1)/2$.

Observe that the derivatives of this map are hard to compute explicitly, so we have to introduce another recursive scheme for them. Moreover, in order reduce the amount of computations, we use that the iterates of $(0,0)$ are $x_n = 2\pi n\theta$ and $y_n = 0$.

We detail the computations of $\partial_{\alpha,x,y}^{k,l,m}(Y \circ F_\alpha)$ at the point $(\alpha, x, y) = (\theta, x_n, 0)$, while the derivatives of $X \circ F_\alpha$ satisfy completely analogous expressions. Let us introduce the function

$$g(x,y) = 1 - 2y(\cos(x))^2 \sin(x) + y^2(\cos(x))^4,$$

so we can write $Y \circ F_\alpha(x,y) = y\sqrt{g(x,y)}$. First, we observe that for any $s \in \mathbb{Q}$ we have $\partial_{\alpha,x,y}^{k,l,m}(yg^s)|_{y=0} = 0$ provided $k \neq 0$ or $m = 0$. Otherwise, the required derivatives can be computed by means of the following recurrent expressions

$$\partial_{x,y}^{l,m}(yg^s) = \partial_{x,y}^{l,m-1}(g^s) + s\sum_{i=0}^{l}\sum_{j=0}^{m-1}\binom{l}{i}\binom{m-1}{j}\partial_{x,y}^{i,j}(yg^{s-1})\partial_{x,y}^{l-i,m-j}(g)$$

and

$$\partial_{x,y}^{l,m}(g^s) = s\sum_{i=0}^{l}\sum_{j=0}^{m-1}\binom{l}{i}\binom{m-1}{j}\partial_{x,y}^{i,j}(yg^{s-1})\partial_{x,y}^{l-i,m-j}(g).$$

Finally, we observe that the derivatives $\partial_{x,y}^{l-i,m-j}(g)$ can be computed easily by expanding the function as a trigonometric polynomial

$$g(x,y) = 1 - \frac{y}{2}\left(\sin(3x)+\sin(x)\right) + \frac{y^2}{2}\left(\frac{3}{4}+\cos(2x)+\frac{1}{4}\cos(4x)\right).$$

The computations are performed by using *double-double* data type, $p=7$ and 2^{21} iterates, at most. We stop the computations if the estimated error is less than 10^{-25}. The derivatives of the expansion (32) and their estimated error, are given in table 3. Finally, in order to verify the results, we compare the truncated expansions of the curve with the numerical approximation computed in section 5.2. The deviation is plotted in $\log_{10}$ scale in figure 5 (right).

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