EXACT DERIVATIVE FORMULA OF THE POTENTIAL FUNCTION OF THE SRB MEASURE

MIAOHUA JIANG

ABSTRACT. For a C^r-diffeomorphism ($r \geq 3$) f on a smooth compact Riemannian manifold possessing a hyperbolic attractor, the potential function for the SRB measure $-\log J^u f(h_f(x))$ is differentiable with respect to f in a C^r-neighborhood of f. We show that if we calculate the unstable Jacobian $J^u f$ with respect to a Hölder continuous metric ω_0 under which the stable and unstable subspaces are orthogonal, the derivative formula in a given direction δf, a vector field on M evaluated at $f(x)$, is given exactly by

$$\delta(\log J_0^u f(h_f(x))) = \mathrm{div}_\rho^u X^u(f(x))$$

where X^u, X^s are the projections of the vector field $\delta f \circ f^{-1}$ onto unstable and stable subbundles, $\mathrm{div}_\rho^u X^u$ is the divergence of X^u with respect to the volume form induced by the SRB measure ρ of f, and $J_0^u f$ is the unstable Jacobian with respect to the metric ω_0 on the unstable manifold of f. This result complements Ruelle's formula by identifying a metric under which the coboundary term can be determined exactly and also gives an alternative proof of the derivative formula of the SRB measure.

1. Introduction

Let f be a $C^{1+\alpha}$-diffeomorphism of a smooth (C^∞) compact manifold M possessing a hyperbolic attractor Δ_f, assuming that it is also topologically mixing on Δ_f. The SRB measure of f, ρ_f is the unique equilibrium state for the potential function $\varphi(x) = -\log J^u f(x)$, where $J^u f(x)$ is the Jacobian of f along the unstable manifold, i.e., $\mu = \rho_f$ is the unique invariant measure satisfying the equation (*variational principle*)

$$h_\mu + \int \varphi(x) d\mu = 0,$$

where h_μ is the metric entropy of f with respect to μ.

In [7], Ruelle proved that the map $f \to \rho_f$ is differentiable and the derivative, or the *linear response function* is given by

$$\delta\rho_f(\Phi) = \sum_{n=0}^{\infty} \rho_f < \mathrm{grad}(\Phi \circ f^n), X >, \tag{1}$$

in a suitable functional setup.

To be precise, assume that f_0 is a C^r-diffeomorphism ($r \geq 3$) of a smooth compact Riemannian manifold possessing a topologically mixing hyperbolic attractor Δ_0. Let $U(f_0)$ denote the C^r-neighborhood of f_0 such that any map $f \in U(f_0)$ possesses a hyperbolic attractor Δ_f and is topologically conjugate to f_0 via a Hölder

Date: August 24, 2009.
2000 MSC 37D20 37D35 37C40 .

continuous map h_f: $\Delta_0 \to \Delta_f$:

$$f \circ h_f = h_f \circ f_0.$$

Then, the SRB measure ρ_f pulled back onto Δ_0 by the conjugating map h_f is the unique equilibrium state for the potential function $-\log J^u f(h_f(x))$:

$$H_{h_f^* \rho_f} + \int_{\Delta_0} -\log J^u f(h_f(x)) dh_f^* \rho_f = 0,$$

where $H_{h_f^*\rho_f}$ is the measure-theoretic entropy of f with respect to the measure $h_f^*\rho_f$. Given any smooth function Φ on the manifold, the functional

$$f \to \int_{\Delta_f} \Phi d\rho_f = \int_{\Delta_0} \Phi \circ h_f dh_f^* \rho_f$$

is differentiable in $f \in U(f_0)$ and its derivative formula evaluated at f is given by (1).

The derivation of this formula [7] relies mainly on the calculation of the derivative of the potential function $-\log J^u f(h_f(x))$ with respect to f. Ruelle showed that the derivative of $\log J^u f(h_f(x))$, in the direction δf (a vector field on M evaluated at $f(x)$) is coboundary with the divergence of the vector field $X = \delta f \circ f^{-1}$ restricted to the unstable manifold with respect to the volume form induced by the SRB measure's density, i.e. there exists a Borel measurable function $\phi(x)$ on Δ_f such that

$$\delta(\log J^u f(h_f(x))) = \mathrm{div}_\rho^u X^u(h_f(x)) + \phi(f(h_f(x))) - \phi(h_f(x)).$$

Note that there is no essential difference in evaluating the derivative formula at the point f_0 or any other point $f = h_f \circ f_0 \circ h_f^{-1} \in U(f_0)$. The expression of the formula becomes simpler if it is evaluated at f_0:

$$\delta(\log J^u f(h_f(x)))\big|_{f_0} = \mathrm{div}_\rho^u X^u(x) + \phi(f_0(x)) - \phi(x).$$

In this article, we show that if we calculate the unstable Jacobian with respect to a Hölder continuous metric ω_0, under which the stable and unstable subspaces of f_0 are orthogonal (for example, the Lyapunov metric), the coboundary term ϕ can be determined explicitly. Our calculation is also more elementary, although not necessary shorter, and thus, provides an alternative proof of the derivative formula.

Let $f \in U(f_0)$ and h_f be the conjugating map: $f \circ h_f = h_f \circ f_0$. Let $J_0^u f(x)$ denote the Jacobian of the map f restricted to the unstable manifold with respect to the volume form induced by the metric ω_0 whose precise definition will be given in the next section. Let δf be a C^r-vector field evaluated at $f_0(x)$. Thus, $X = \delta f \circ f_0^{-1}$ is a C^r-vector field on M. Let X^u, X^s be the projections of X onto stable and unstable invariant subspaces of Df_0. We have the following theorem.

Theorem 1. *The map $f \to \log J_0^u f(h_f(x))$ is differentiable in terms of $f \in U(f_0)$. Its derivative at the point f_0, in a given direction specified by a vector field evaluated at $f_0(x)$, δf, is given by*

$$\delta(\log J_0^u f(h_f(x)))\big|_{f_0} = \mathrm{div}_\rho^u X^u(f_0(x)),$$

where $\mathrm{div}_\rho^u X^u(x)$ is the divergences of X^u on the unstable manifold with respect to the volume form induced by the density of the SRB measure ρ_{f_0}.

In next section, we will give definitions of the terms in this theorem and recall some related results. They include the topology of the spaces of the potential functions, the differentiability of the conjugating map h_f in terms of f and stable and unstable subspaces, the divergence of a non-smooth vector fields, the volume form induced by the density of the SRB measure on the unstable manifold which is only defined up to a constant factor, the Hölder continuous metric ω_0, as well as the Lyapunov metric.

2. The preliminaries

For standard definitions of commonly used terms such as uniformly hyperbolic maps, attractors, topological mixing, C^r-topology, conjugating map, exponential splitting etc., we refer to the book [5].

2.1. Spaces of subbunbles of the tangent bundle and Hölder continuous maps.

Much of the setup is the same as that in [7]. We include here a brief description.

2.1.1. *The Banach manifold structure of* $\mathrm{Diff}^{C^r}(M)$. What we need is a description of its tangent space. Let $\mathcal{A}_x$ be the exponential map from the tangent space T_xM to M at the point x. Given a diffeomorphism $f \in \mathrm{Diff}^{C^r}(M)$, for any map g in its small C^r neighborhood, The map $\tilde{g}(x) : x \to \mathcal{A}^{-1}_{f(x)} g(x) \in T_{f(x)}M$ defines a C^r vector field on M. Note that this vector field is evaluated at the point $f(x)$. With the linear structure of the tangent space, we have the following Banach space

$$\mathcal{B} = \{\tilde{g}(x) : x \to \xi(x) = \mathcal{A}^{-1}_{f(x)} g(x) \in T_{f(x)}M, C^r\}$$

equipped with the norm

$$\|\tilde{g}(x)\|_{C^r} = \sup_{x\in M} \|\mathcal{A}^{-1}_{f(x)} g(x)\|_{C^r}$$

where the C^r-norm in $T_{f(x)}M$ is induced by the Riemannian metric on M. Clearly, the map $\tilde{g} \to g$ is a C^r diffeomorphism in a small neighborhood of the zero section.

In a similar way, we can define the Banach manifold structure of Hölder continuous maps on M. If h is a Hölder continuous map of M with a Hölder exponent $0 < \alpha \le 1$, its C^α neighborhood, as a Banach manifold, is identified with the C^α neighborhood of the zero vector field in the space of all C^α vector fields on M. The C^α norm of a vector field $v(x)$ is defined by

$$\|v(x)\|_{C^\alpha} = \|v(x)\|_{C^0} + \sup_{x\in M} \sup_{y\in M, y\neq x} \frac{\|v(x) - v(y)\|}{d(x,y)^\alpha},$$

where $\|v(x) - v(y)\|$ is the norm in the tangent space $T_{h(x)}M$ and $d(x, y)$ is the distance on M. When h is the identity, we denote its C^α neighborhood by $C^\alpha(M, M)$. If we replace the norm by the absolute value, we have the Banach space of Hölder continuous functions on M and we denote it by $C^\alpha(M, \mathbb{R})$.

2.1.2. *The Banach manifold structure of the space of Hölder continuous Grassmannian bundles of the tangent bundle.* Let $G_k(M)$ denote the space of Hölder continuous Grassmannian bundles of order k of the tangent bundle. Each point of $G_k(M)$, $\mathcal{J}(x)$ is a k-dimensional linear subspace bundle of the tangent bundle TM. $\mathcal{J}(x)$ is Hölder continuous with respect to x in the Grassmannian metric on the linear subspaces. Such Grassmannian bundles are also defined on any subset of M, in particular, on the hyperbolic attractor of f. Since the tangent space has a linear structure with a Riemannian metric, $G_k(M)$ forms naturally a Banach manifold with the Grassmannian distance defined between k-dimensional linear subspaces of the tangent space T_xM induced by this metric.

Let $E^u_f(x)$ and $E^s_f(x)$ denote the unstable and stable invariant subspaces of f. They are Hölder continuous in x over the hyperbolic attractor Δ_f. Let u and s denote also the dimensions of the unstable and stable subspaces of Df. For any given $f \in U(f_0)$, since $h_f(x)$ is close to x, $E^u_f(h_f(x)), E^s_f(h_f(x))$ can be identified with subspaces of the tangent space T_xM using local coordinate charts. They can thus, be considered as Hölder continuous subbundles of the tangent bundle over the hyperbolic attractor Δ_{f_0}. We have that $E^u_f(h_f(x))$ and $E^s_f(h_f(x))$ are elements of $G_u(\Delta_{f_0})$ and $G_s(\Delta_{f_0})$ for each $f \in U(f_0)$.

The following proposition summarizes the preliminary results we need concerning the differentiability of the conjugating map, the exponential splittings, and the potential function.

Proposition 1. [7] *If f_0 is $C^r, r \geq 3$, then,*

(1) *the map $f \in U(f_0) \to h_f \in C^\alpha(M, M)$ is C^{r-1} for some $0 < \alpha < 1$. The derivative in the direction of δf is given by*
$$\delta h_f(x) = \sum_{n=0}^{\infty} Df^n X^s - \sum_{n=1}^{\infty} Df^{-n} X^u.$$
where $X = \delta f \circ f^{-1}$ and $X = X^s + X^u$.

(2) *the maps $f \to E^u_f(h_f(x)), E^s_f(h_f(x))$ are C^{r-2} from $U(f_0)$ to $G_u(\Delta_{f_0})$, $G_s(\Delta_{f_0})$, respectively.*

(3) *the map $f \to -\log J^u f(h_f(x))$ is C^{r-2} from $U(f_0)$ to $C^\alpha(M, \mathbb{R})$.*

2.2. **The divergence of X^u.** Let $\delta f \in T_f U(f_0)$ be an element in the tangent space of $U(f_0)$ at f. It is a C^r vector field evaluated at the point $f(x)$. Thus, the composition $X = \delta f \circ f^{-1}$ is a C^r vector field on M. Since $T_x(M) = E^u_f(x) \oplus E^s_f(x)$ for $x \in \Delta_f$, the vector field X can be projected onto these invariant subspaces of the differential operator Df: $X = X^u + X^s$. Since Δ_f is an attractor, the global and local unstable manifolds are smooth submanifolds equipped with induced Riemannian metric from M. Thus, X^u is a well-defined vector field on both global and local unstable manifolds. Since $E^s_f(x)$ is only Hölder continuous along W^u_f, the vector field X^u is only Hölder continuous, in general. However, using a weaker definition of divergence (a distribution), given any smooth volume form on the unstable manifold, the divergence of X^u respect to the given volume form exists due to the absolute continuity of the holonomy map defined by the stable foliation.

Definition 1. [4] *Let X be a continuous vector field defined in an open set U on a Riemannian manifold with a volume form dw. Let $\mathcal{G}$ denote the family of C^∞ functions in U with compact support. Assume that there is an integrable function $h(x)$ such that*
$$\int < \mathrm{grad} g, X > dw = -\int g(x)h(x)dw,$$
for all $g \in \mathcal{G}$. Then, we call $h(x)$ the divergence of X and denote it by $\mathrm{div}X$.

The next proposition gives a sufficient condition for the existence of the divergence in our context. Since the definition of the divergence concerns only the vector field in a small open neighborhood, we state the proposition in the case when the unstable manifold is a linear subspace $\mathbb{R}^u$ in a n-dimensional linear space $\mathbb{R}^n$ equipped with Riemannian metric.

Proposition 2. [4] *Let X be a C^1-vector field on $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ equipped with a smooth Riemannian metric with the property that the metric is independent of the coordinate in $\mathbb{R}^s$. Let W be an absolutely continuous foliation with smooth leaves transversal to $\mathbb{R}^u$. Let $E^s(x)$ be the distribution tangent to W at $x \in \mathbb{R}^u$ and $X = X^u + X^s$ denote the projections of X onto $\mathbb{R}^u$ and $E^s(x)$, respectively. Then, $\mathrm{div}^u X^u$, the divergence of X^u on the manifold $\mathbb{R}^u$ with respect to the volume form restricted to the manifold is well defined. Moreover, when the Jacobian $J\xi(x^u, t)$ of the the holonomy map $\xi(x^u, t)$ from $\mathbb{R}^u$ to $\mathbb{R}^u + tX$ defined by the foliation W is differentiable in t and continuous in x^u, we have*

$$\mathrm{div}^u X^u = \mathrm{div}^u X_c^u - \frac{d}{dt} J\xi(x^u, t)|_{t=0}, x^u \in \mathbb{R}^u \tag{2}$$

where X_c^u is the projection of the vector field X onto the coordinate subspaces $\mathbb{R}^u$ and $\mathbb{R}^s$: $X = X_c^u + X_c^s$.

For a uniformly hyperbolic attractor, the stable foliation is absolutely continuous. The Jacobian of the holonomy map $\xi(x^u, t)$ defined by this foliation from the unstable manifold $\mathbb{R}^u$ to $\mathbb{R}^u + tX$ is given by [6, p. 255]

$$J\xi(x^u, t) = \prod_{n=0}^{\infty} \frac{|\det(D_{p_n} f\Big|_{E_n})|}{|\det(D_{p'_n} f\Big|_{E'_n})|},$$

where $p_n = f^n(x), p'_n = f^n(\xi(x, t)), E_n = D_{p_n} f^n E^u(x)$, and $E'_n = D_{p'_n} f^n E_t^u$, where $E_t^u = E^u(x) + tX(x)$. It is Hölder continuous in x^u and differentiable in t. Thus, $\mathrm{div}^u X^u$ is a well-defined Hölder continuous function on the unstable manifold. In the stable direction, since the unstable subspace is Hölder continuous, $\mathrm{div}^u X^u$ is Hölder continuous as well. Thus, $\mathrm{div}^u X^u$ is Hölder continuous on the hyperbolic attractor.

The divergence, by its definition, can be taken with respect to any smooth volume form on the manifold. We denote $\mathrm{div}_\omega^u X^u$ the divergence taken with respect to a volume form ω. For two smooth volume forms ω_1 and ω_2, let $p(x) = \frac{d\omega_1}{d\omega_2}$ be the relative density function. We have the relation

$$\mathrm{div}_{\omega_1}^u X^u = \mathrm{div}_{\omega_2}^u X^u + < \mathrm{grad} \log p(x), X^u > . \tag{3}$$

When volume forms differ only by a constant factor, the divergence is clearly the same. Thus, the divergence can be taken with respect to the volume form given by the density function of the SRB measure ρ on the unstable manifold. We denote this divergence by $\mathrm{div}_\rho^u X^u$. We use $\mathrm{div}^u X^u$ to denote the divergence taken with respect to the volume form induced by the original Riemannian metric on M.

Note that the volume form given by the SRB measure is independent of the original choice of the Riemannian metric as long as the volume forms induced are equivalent. The density function $p(x)$ of the SRB measure is given by

$$\frac{p(x)}{p(y)} = \prod_{i=1}^{\infty} \frac{|\det Df_{y_i}|_{E_{y_i}^u}|}{|\det Df_{x_i}|_{E_{x_i}^u}|} = \prod_{i=0}^{\infty} \frac{|\det Df_{x_i}^{-1}|_{E_{x_i}^u}|}{|\det Df_{y_i}^{-1}|_{E_{y_i}^u}|}, \tag{4}$$

where $x_i = f^{-i}(x)$, $y_i = f^{-i}(y)$, and y is a given point on the unstable manifold $W^u(x)$. We have

(5)

$$\mathrm{div}^u_\rho X^u = \mathrm{div}^u X^u + < \mathrm{grad} \log p(x), X^u > \; = \mathrm{div}^u X^u + \sum_{n=1}^{\infty} < \mathrm{grad} p_f \circ f^{-n}, X^u >,$$

where p_f is the potential function $-\log |\det Df|_{E^u} = -\log J^u f(x)$.

2.3. **Lyapunov metric.** Lyapunov metric is defined only on the attractor Δ_f for a given $f \in U(f_0)$. Given any point $x \in \Delta_f$, let v_1, v_2 be two vectors in the unstable subspace $E^u(x)$, the new inner product $< \cdot, \cdot >'_x$ is defined by

$$< v_1, v_2 >'_x = \sum_{k=0}^{\infty} < Df^{-k} v_1, Df^{-k} v_2 > |_{f^{-k}(x)} = < v_1, v_2 >_x +$$

$$\sum_{k=1}^{\infty} < Df^{-1}_{f^{-k+1}(x)} Df^{-1}_{f^{-k+2}(x)} \cdots Df^{-1}_x v_1, Df^{-1}_{f^{-k+1}(x)} Df^{-1}_{f^{-k+2}(x)} \cdots Df^{-1}_x v_2 > |_{f^{-k}(x)},$$

where $< v_1, v_2 >_x$ denotes the inner product of v_1, v_2 in the tangent space $T_x M$.

For vectors in the stable subspace, Lyapunov metric's inner product is defined by

$$< v_1, v_2 >'_x = \sum_{k=0}^{\infty} < Df^k v_1, Df^k v_2 > |_{f^k(x)}.$$

For vectors $v \in E^u_x$ and $w \in E^s_x$, the inner product is defined to be zero. For general vectors, the inner product is defined by using their projections on the stable and unstable subspaces and the bi-linearity of the inner product. Clearly, under this metric, stable and unstable subspaces are orthogonal. However, Lyapunov metric is only a Hölder continuous metric since stable and unstable subspaces are Hölder continuous with respect to the base point x.

Proposition 3. *The volume form induced by the Lyapunov metric on the unstable manifold is smooth and equivalent to the volume form induced by the original Riemannian metric. Its density function $r(x)$ is a smooth function on the unstable manifold and Hölder continuous on Δ_f. The composition function $r(h_f(x))$ depends on f differentiably as a Hölder continuous function on Δ_0.*

Proof. We only need to show that $r(h_f(x))$ is differentiable in terms of f. Since the metric is defined by

$$< v_1, v_2 >'_x = \sum_{k=0}^{\infty} < Df^{-k} v_1, Df^{-k} v_2 > |_{f^{-k}(x)},$$

we have that the density function at the point x induced by the Lyapunov metric

$$r = 1 + \sum_{k=1}^{\infty} \det Df^{-k}_{E^u_x} = 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \det Df^{-1}_{E^u_{f^{-i+1}(x)}}$$

$$= 1 + \sum_{k=1}^{\infty} \left[\prod_{i=1}^{k} \det Df_{E^u_{f^{-i}(x)}} \right]^{-1} = 1 + \sum_{k=1}^{\infty} \left[\prod_{i=1}^{k} J^u f_{E^u_{f^{-i}(x)}} \right]^{-1}.$$

Thus,

$$r(h_f(x)) = 1 + \sum_{k=1}^{\infty} \left[\prod_{i=1}^{k} J^u f_{E^u_{f^{-i}(h_f(x))}} \right]^{-1} = 1 + \sum_{k=1}^{\infty} \left[\prod_{i=1}^{k} J^u f(h_f(f_0^{-i})) \right]^{-1}.$$

Since $J^u f(h_f(x))$ depends on f differentiably by Proposition 1, we can differentiate each term in the series with respect to f. Note that the series is essentially a power series in f, both the series and its termwise derivative series converge uniformly due to the uniform hyperbolicity. The function $r(h_f(x))$ is thus, differentiable in f. □

We will see that this proposition allows us to calculate the unstable Jacobian $J^u f(h_f(x))$ with respect to the volume form given by Lyapunov metric and it is still differentiable in terms of f, i.e., $f \to J^u f(h_f(x))$ is differentiable in f. But unfortunately, the dependence of the metric on f makes it difficult to estimate the first order term in the difference between $J^u f(h_f(x))$ and $J^u f_0(x)$. The metric we use in the calculation comes from f_0 only. Since the Lyapunov metric induced by f_0 is only defined on the hyperbolic attractor Δ_0, in the next section, we explain how it can be extended into the neighborhood of Δ_0 so that $J^u f(h_f(x))$ can be calculated with this metric.

2.4. **Choice of volume forms on the unstable manifolds.** Assume that ω is the given smooth Riemannian metric on M. The unstable Jacobian $J^u f(h_f(x))$ is calculated in terms of the induced volume form (also denoted by ω) on unstable manifolds at $h_f(x)$ and $f(h_f(x)) = h_f(f_0(x))$. Note that the computing of this function $J^u f(h_f(x))$ depends on choices of volume forms at points $h_f(x)$ and $f(h_f(x)) = h_f(f_0(x))$. Theoretically, any volume forms defined on the unstable manifolds of the hyperbolic attractor Δ_f can be used to calculate the potential functions. The volume forms can be dependent on the map f as f is fixed when we calculated the value of the function. Indeed, even the induced volume form ω depends on the map f since the unstable manifold depends on f. For any other volume form equivalent to ω, assume the density function is given by $g(z), z \in \Delta_f$.

With respect to the volume form $w_g = g(z)\omega$, we have the following coboundary relation:

$$\log J^u_\omega f(h_f(x)) = \log J^u_{\omega_g} f(h_f(x)) + \log g(h_f(f_0(x))) - \log g(h_f(x)). \tag{6}$$

As long as $g(h_f(x))$ is a differentiable function in f, the new unstable Jacobian is again differentiable in f. In particular, it means we can use the volume forms on the unstable manifold induced by either the Lyapunov metric or the SRB measure. In the case of SRB measures, $g(z)$ is piecewise defined on Δ_f and the derivative of $g(h_f(x))$ with respect to f will be a piecewise continuous function in x. However, neither metric is convenient: one needs to estimate the changes of the metric in terms of f.

Note that the point x is fixed when we consider the differentiability of $\log J^u f(h_f(x))$ with respect to f, the continuity of $g(h_f(x))$ in x is not needed: its measurability is sufficient. The metric can be defined locally depending on x.

Given any local smooth metric $\omega_0(x)$ in a neighborhood of x which is independent of f, its induced volume form on the local unstable manifold of f passing the point $h_f(x)$ is always differentiably dependent on f. This follows from the fact that the induced volume form depends on f through the unstable subspace $E^u(h_f(x))$ which depends on f differentiably.

Definition of the local metric $\omega_0(x)$

Take any smooth metric defined on the local unstable manifold of f_0 at x, for example, the metric ω induced by the original Riemannian metric, the Lyapunov metric of f_0, or the induced metric ω times the density function of the SRB measure. We extend it into an open neighborhood of x in the following way. Identify an open neighborhood of x with a neighborhood of the origin in $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ so that that the local unstable manifold of f_0 at x, $W^u(x)$, is contained in the subspace $(\mathbb{R}^u, 0)$. The leaf of the local stable foliation passing the point $x_0^u = (x^u, 0) \in W^u(x)$ can be expressed as $(F(x_0^u, x^s), x^s), x^s \in \mathbb{R}^s$, where $F(x_0^u, x^s)$ is a C^r map from $\mathbb{R}^s$ to $\mathbb{R}^u$. Given any point $z = (x^u, x^s) \in \mathbb{R}^u \oplus \mathbb{R}^s$ near the origin, there exists $x_0^u \in W^u(x)$ such that $x^u = F(x_0^u, x^s)$. For a fixed x^s, the map $x^u \to x_0^u$ so defined is the holonomy map induced by the stable foliation from $(\mathbb{R}^u, x^s)$ to $W^u(x)$. Indeed, (x_0^u, x^s) gives another coordinate for each point in this small neighborhood of x. In this coordinate system, the local stable manifolds are simply the coordinate hypersurfaces $x_0^u = C$. The metric ω_0 at point $z = (x^u, x^s)$ is defined by the following rules:

(1) If the vectors of T_zM are in the subspace $\mathbb{R}^u$, then, the metric is the same as the metric in $\mathbb{R}^u$ at the point $(x_0^u, 0)$.

(2) If the vectors are in the tangent space of the stable leaf, they inherit the original Riemannian metric.

(3) The metric ω_0 is defined so that $\mathbb{R}^u$ and the tangent space of the stable leaf are orthogonal.

At each point (x^u, x^s) in the local stable submanifold passing through the point $(x_0^u, 0)$, the metric ω_0, restricted to the subspace $\mathbb{R}^u$ of the tangent space, is independent of the coordinate x^s. When the metric is restricted to any stable leaf, it is smooth. With this definition, we have a Hölder continuous metric in an open neighborhood of every $x \in \Delta_0$. We can then, partition an appropriately small closed neighborhood of Δ_0 into finite compact subsets so that each subset belongs to at least one of such open neighborhoods. For each element of the partition, we fix a choice of the metric ω_0. The metric ω_0 is thus only measurable over this closed neighborhood of Δ_0. It is Hölder continuous in x when restricted to one open neighborhood. The potential function calculated using this metric is denoted by $\log J_0^u f(h_f(x))$.

Because of the lack of the differentiability of the metric ω_0 with respect to the base point x, the differentiability of $\log J_0^u f(h_f(x))$ becomes a question in the computation of the derivative, we need a sequence of smooth metrics w_n that approximate the metric ω_0. This sequence of smooth (local) metrics are obtained by approximating the stable foliation by smooth foliations that are locally invariant under f_0.

Definition of the local smooth metric $\omega_n(x)$

For each $n = 1, 2, \cdots$, we define maps $F_n(x_0^u, x^s)$ so that it is C^r $(r > 2)$ in both x_0^u and x^s and both $F_n(x_0^u, x^s)$ and $D_{x^s} F_n(x_0^u, x^s)$ converge to $F(x_0^u, x^s)$ and $D_{x^s} F(x_0^u, x^s)$ uniformly in the neighborhood of x. These maps surely exist, e.g., see [1] Pages 137-138. The metric w_n is then defined in the same way as ω_0 is defined by replacing $F(x_0^u, x^s)$ with $F_n(x_0^u, x^s)$. The metric ω_n is differentiable in the base point x. When restricted to $\mathbb{R}^u$, ω_n is the same as ω_0. We also require that $F_n(x_0^u, x^s)$ satisfy the local invariance under f_0: the leaves of foliations satisfy the relation

$$f_0(F_n(x_0^u, x^s)) \subset F_n(f_0(x_0^u), x^s).$$

The local smooth foliation defined by $F_n(x_0^u, x^s)$ is denoted by W_n^s. For each fixed n, we have a local coordinate system (x_0^u, x^s) defined by this smooth foliation.

3. Computation of the derivative of the map $f \to -\log J^u f(h_f(x))$

The computation of the derivative is divided into three steps: reduction of the number of variables from 3 to 2 and calculation of the two resulting partial derivatives. Since the metric ω_0 is not differentiable, we calculate the derivative formula with respect to the metric ω_n. We show that for each n the derivative is bounded and converges to a Hölder continuous function. Following a standard real analysis argument (see Lemma 6.6 in [3]), we obtain both the differentiability and the derivative formula of the map $f \to -\log J_0^u f(h_f(x))$, where the unstable Jacobian is calculated with respect to the metric ω_0. We use $J^u f$ to denote the unstable Jacobian under a generic smooth metric.

3.1. **Reduction of the Number of Variables.** Even though the function $-\log J^u f(y)$ is differentiable in y along the unstable manifold of f, it is, in general, not differentiable over the entire hyperbolic attractor of f, Δ_f [11]. We can not use the chain rule to obtain the derivative of the composition $p_f \circ h_f = -\log J^u f(h_f(x))$. In the potential function $p_f \circ h_f = -\log J^u f(h_f(x))$, there are three terms that vary with f: the unstable Jacobian $J^u f$ in terms of f, the unstable direction E_f^u in which $J^u f$ is calculated, and the base point $h_f(x)$. We describe the dependence of the potential function on f in a neighborhood of f_0 with the help of local coordinate systems on the manifold. Note that we need only to calculate the derivative formula at the given map f_0.

We first extend the domain of the map $f \to p_f \circ h_f = -\log J^u f(h_f(x))$ to a cross-product of two infinite dimensional manifolds: $U(f_0)$ and the space of Grassmannian bundle over M.

We consider the space of Grassmannian bundle over M of order k, where k is the dimension of the unstable subspace. The space consisting of all continuous Grassmannian bundles forms a smooth Banach manifold $\mathcal{G}$ with the usual supremum metric provided by the Grassmannian distance. For any diffeomorphism $f \in U(f_0)$, the small C^r-neighborhood of f_0, we define a map in the form of

$$(g, f) \in \mathcal{G} \times U(f_0) \to Df(x)|_{g(x)} = Df(g(x)) \in \mathcal{G}.$$

Note that for a given point x, $Df(x)|_{g(x)}$ is a linear map on the subspace $g(x)$ of the tangent space T_xM. When $g(x) = E_f^u(x), x \in \Delta_f$, the invariant unstable subspace, we have $Df(x)|_{E_f^u(x)} = Df(E_f^u(x)) = E_f^u(f(x))$. Once the metric is chosen, we have

$$J^u f|_{E_f^u}(h_f(x)) = |\det Df_{E_f^u(h_f(x))}|. \tag{7}$$

Since $h_f(x)$ is close to x when f is close to f_0, we identify the tangent spaces at these two points. Thus, $E_f^u(h_f(x))$, the unstable subspace of f at the point $h_f(x)$ can be identified with a subspace in the tangent space T_xM close to the unstable subspace of Df_0, $E_{f_0}^u(x)$, and vice versa. We now compare the determinants of two linear maps $Df_{E^u_f(h_f(x))}$ and $Df_{E^u_{f_0}(x)}$, where $E^u_{f_0}(x)$ is considered as a subspace of the tangent space at the point $h_f(x)$. Note that the map

$$f \to J^u f|_{E^u_{f_0}(x)}(h_f(x)) = |\det Df_{E^u_{f_0}(x)}| \tag{8}$$

is differentiable in terms of f when the subspace $E^u_{f_0}(x)$ is held independent of f. The determinant depends on f and h_f.

What we shall prove next is that these two determinants (7) and (8), when calculated with the metric ω_0, differ by a higher order term in terms of the C^r-distance of f and f_0.

Lemma 1. *Let $\epsilon = \mathrm{dist}^{C^3}(f, f_0)$ be the C^3 distance between f and f_0. Then, we have*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}|\log J^u_0 f_{E^u_f}(h_f(x)) - \log J^u_0 f_{E^u_{f_0}(x)}(h_f(x))| = 0,$$

The lemma implies that the functionals $f \to \log J^u_0 f_{E^u_f}(h_f(x))$ and $f \to \log J^u_0 f_{E^u_{f_0}(x)}(h_f(x))$ will have the same differentiability with respect to f and thus, the same derivative formula at the point f_0 when differentiable.

Lemma 1 is a direct consequence of the following linear algebra result.

Proposition 4. *Let E^k, E^s be two invariant subspaces of a linear transformation T on an n-dimensional inner product space $\mathbb{R}^n$ and $E^k \oplus E^s = \mathbb{R}^n$. Assume that E^k, E^s are almost orthogonal: there exists a small number $\delta > 0$ such that for any unit vectors $e \in E^k$ and $w \in E^s$, $| < e, w > | < \delta$. Assume that E'^k is another linear space which is ϵ-close to E^k, i.e., E'^k is a linear span of vectors $(e_1 + w_1, e_2 + w_2, \cdots, e_k + w_k)$, where $w_i \in \mathbb{R}^n$ and $\|w_i\| < \epsilon$. Then, there exists a constant C such that*

$$|\det T|_{E^k} - \det T|_{E'^k}| < C(\delta\epsilon + \epsilon^2).$$

Proof. Let $(e_1, e_2, \cdots, e_k)$ be an orthonormal basis of E^k. Let $(v_1, v_2, \cdots, v_s)$ be an orthonormal basis of E^s. Since $\delta > 0$ is a small number and $E^k \oplus E^s = \mathbb{R}^n$, $(e_1, e_2, \cdots, e_k, v_1, v_2, \cdots, v_s)$ is a basis of $\mathbb{R}^n$. Under this basis, the matrix representation of the linear transformation T takes a block diagonal form since E^k and E^s are invariant under T:

$$T = \begin{pmatrix} A & 0 \\ 0 & B. \end{pmatrix}$$

Using the Gram-Schmidt orthogonalization procedure, we can obtain an orthonormal basis of $\mathbb{R}^n$ $(e_1, e_2, \cdots, e_k, v'_1, v'_2, \cdots, v'_s)$, where

$$v'_1 = (1 - \beta_1)(v_1 + \alpha_{11}e_1 + \alpha_{12}e_2 + \cdots + \alpha_{1k}e_k),$$

$\alpha_{1i} = - < e_i, v_1 >, i = 1, 2, \cdots, k$, $\beta_1 = 1 - \sqrt{1 - \alpha_{11}^2 - \alpha_{12}^2 - \cdots - \alpha_{1k}^2}$. Note that

$$< v_2, v'_1 >= (1 - \beta_1)(\alpha_{11} < v_2, e_1 > + \cdots + \alpha_{1k} < v_2, e_k >).$$

We have

$$v'_2 = (1 - \beta_2)(v_2 + \eta_1 v'_1 + \alpha_{21}e_1 + \alpha_{22}e_2 + \cdots + \alpha_{2k}e_k),$$

where $\alpha_{2i} = - < e_i, v_2 >$, $\eta_1 = - < v_2, v'_1 >$, and

$$\beta_2 = 1 - \sqrt{1 - \eta_1^2 - \alpha_{21}^2 - \alpha_{22}^2 - \cdots - \alpha_{2k}^2}.$$

Inductively, we conclude

$$(v_1', v_2', \cdots, v_s') = (e_1, e_2, \cdots, e_k, v_1, v_2, \cdots, v_s) \begin{bmatrix} P \\ I + Q \end{bmatrix},$$

where the absolute value of the entries of P is bounded by δ and the absolute value of the entries of Q is bounded by $C\delta^2$ for some constant C. Under this new orthonormal basis $(e_1, \cdots, e_k, v_1', \cdots, v_s')$, the transformation's matrix representation becomes

$$\begin{bmatrix} I & P \\ 0 & I+Q \end{bmatrix}^{-1} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I+Q \end{bmatrix} = \begin{bmatrix} I & P^* \\ 0 & I+Q^* \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I+Q \end{bmatrix}$$
$$= \begin{bmatrix} A & AP + P^*B + P^*BQ \\ 0 & B + Q^*B + BQ + Q^*BQ \end{bmatrix},$$

where $Q^* = -\sum_{i=1}^{\infty} Q^n$ and $P^* = -P - PQ^*$. The matrices Q^* and P^* has entries on the order of δ^2 and δ, respectively. We denote $\tilde{P} = AP + P^*B + P^*BQ$ and $\tilde{Q} = Q^*B + BQ + Q^*BQ$. Clearly, $\det T|_{E^k} = \det(A)$. To calculate $\det T|_{E'^k}$, we take an orthonormal basis $(e_1', \cdots, e_k')$ of E'^k. Under the basis of $(e_1, \cdots, e_k, v_1', \cdots, v_s')$, the k column vectors $(Te_1', \cdots, Te_k')$ form a matrix H of size $n \times k$. Then, $|\det T|_{E'^k}| = \sqrt{\det(H'H)}$. We now compute H.

Since E'^k is ϵ-close to E_k, we can take a basis of E'^k in the form of

$$(e_1, \cdots, e_k, v_1', \cdots, v_s') \begin{bmatrix} I \\ E \end{bmatrix}$$

with entries of E bounded by ϵ. We apply again the Gram-Schmidt orthogonalization procedure to this basis of E'^k to obtain an orthonormal basis $(e_1', \cdots, e_k')$. By the same argument, we have

$$(e_1', \cdots, e_k') = (e_1, \cdots, e_k, v_1', \cdots, v_s') \begin{bmatrix} I \\ E \end{bmatrix} (I + R),$$

where R is a matrix of size $k \times k$ whose entries are on the order of ϵ^2. Then

$$H = \begin{bmatrix} A & \tilde{P} \\ 0 & B + \tilde{Q} \end{bmatrix} \begin{bmatrix} I \\ E \end{bmatrix} (I + R) = \begin{bmatrix} A + AR + \tilde{P}E(I+R) \\ (B + \tilde{Q})E(I+R) \end{bmatrix}$$

Finally, we obtain the leading terms of $H'H$ up to the second order of ϵ, δ.

$$H'H = A'A + A'AR + A'PE + (AR)'A + (PE)'A + (BE)'BE.$$

Thus, we have

$$|\det T|_{E'^k}| = \sqrt{\det(H'H)} = \sqrt{\det(A'A)} + C(\delta\epsilon + \epsilon^2),$$

for some constant C. □

Proof of Lemma 1. Let f be sufficiently C^3-close to f_0. The unstable and stable subspaces $E_f^u(h_f(x))$ and $E_f^s(h_f(x))$ are invariant under Df and ϵ-almost orthogonal since they are within the ϵ-distance of $E_{f_0}^u(x)$ and $E_{f_0}^s(x)$, respectively. Thus, we have

$$\log J_0^u f_{E_f^u}(h_f(x))) - \log J_0^u f_{E_{f_0}^u}(h_f(x))) = O(\epsilon^2),$$

where the determinants are calculated with the metric ω_0 at $h_f(x)$ and $h_f(f_0(x))$. □

Now we consider the differentiability and the derivative of the function $-\log J_0^u f_{E^u_{f_0}}(h_f(x))$ with respect to f. We denote by $-\log J_n^u f_{E^u_{f_0}}(h_f(x))$ the same Jacobian but calculated with the local smooth metric ω_n, $n = 1, 2, 3, \cdots$. Since both the subspace $E^u_{f_0}(x)$ and the metric ω_n are fixed, for each $n \geq 1$, $-\log J_n^u f_{E^u_{f_0}}(y)$ is a continuously differentiable function in terms of $f \in \mathrm{Diff}^{C^2}(f_0)$ and it is continuously differentiable in $y \in M$. Thus, we can calculate the derivative of $-\log J_n^u f_{E^u_{f_0}}(h_f(x))$ with respect to f at the point f_0 by calculating two partial derivatives.

Take a fixed perturbation δf of f_0 and consider a small interval of $[-r, r]$, $r \ll 1$. We show that the sequence of functions $g_n(\epsilon) = -\log J_n^u f^\epsilon_{E^u_{f_0}}(h_{f_\epsilon}(x))$, $\epsilon \in (-r, r)$ is differentiable in ϵ, where $f^\epsilon = f_0 + \epsilon\delta f$ and their derivatives are bounded uniformly in n and the sequence of derivatives converge to a function continuous in ϵ. Thus, we conclude that the derivative of the limit function $g_0(\epsilon) = -\log J_0^u f^\epsilon_{E^u_{f_0}}(h_f(x))$ is differentiable in ϵ and its derivative is given by the limit of the derivatives of $g_n(\epsilon)$. We drop the negative sign in front of the function in our calculation for simplicity.

We can now determine the derivative formula of $\log J_n^u f^\epsilon_{E^u_{f_0}}(h_{f^\epsilon}(x))$ with respect to ϵ using the chain rule. The derivative at any given point $\epsilon_0 \in (-r, r)$ is

$$\frac{\partial}{\partial \epsilon} \log J_n^u f^\epsilon_{E^u_{f_0}}(h_{f^\epsilon}(x))|_{\epsilon_0}$$
$$= \frac{\partial}{\partial \epsilon} \log J_n^u f^\epsilon_{E^u_{f_0}}(h_{f^{\epsilon_0}}(x))|_{\epsilon_0} + \frac{\partial}{\partial y} \log J_n^u f^{\epsilon_0}_{E^u_{f_0}}(y)|_{h_{f^{\epsilon_0}}(x)} \ \frac{\partial}{\partial \epsilon} h_{f^\epsilon}|_{\epsilon_0}.$$

3.2. Calculating the first partial derivative $\frac{\partial}{\partial \epsilon} \log J_n^u f^\epsilon_{E^u_{f_0}}(h_{f^{\epsilon_0}}(x))|_{\epsilon_0}$.

Since the calculation of the function $\log J_n^u f^\epsilon_{E^u_{f_0}}(h_{f^{\epsilon_0}}(x))$ does not depend on any specific coordinate system, we identify the neighborhood of the given point x with $\mathbb{R}^u \times R^s$ where the local unstable manifold of f_0 at x is identified with an open set in $(\mathbb{R}^u, 0) \subset \mathbb{R}^u \times R^s$. For each $n \geq 1$, the local smooth foliation W_n^s gives us a coordinate system whose leaves and $(\mathbb{R}^u, x^s), x^s \in R^s$ form the coordinate hyper-surfaces. Denote the tangent space of W_n^s by E_n^s. Under the metric ω_n, the subspaces $\mathbb{R}^u$ and E_n^s are orthogonal by definition.

Let X be the vector field defined by the composition $\delta f \circ (f^{\epsilon_0})^{-1}$. Let X_n^u and X_n^s denote the projections of X onto $\mathbb{R}^u$ and E_n^s, respectively. We have

Theorem 2.
$$\frac{\partial}{\partial \epsilon} \log J_n^u f^\epsilon_{E^u_{f_0}}(h_{f^{\epsilon_0}}(x))|_{\epsilon_0} = \mathrm{div}^u_{\omega_n} X_n^u(f^{\epsilon_0}(x)).$$

Proof. We first carry out the calculation in the special case $\epsilon_0 = 0$. We need a more detailed description of the space where f is taking values. Let $\mathcal{F}^{C^r}$ denote the collection of the following vector fields

$$\mathcal{F}^{C^r} = \{\delta f : x \to \delta f(x) \in T_{f_0(x)} \text{ is } C^r\}.$$

When ϵ is small, $f^\epsilon(x) = f_0(x) + \epsilon\delta f(x)$ is identified with a point in the neighborhood of $f_0(x)$ by using the exponential map at this point x. Since our calculation is restricted in a small open neighborhood, $\delta f(x)$ can be considered as a map from an open neighborhood of x to $\mathbb{R}^n$ and $\|\delta f\|_{C^r}$ is small. We thus, have $Df^\epsilon(x) = Df_0(x) + \epsilon D\delta f(x)$. Furthermore,

$$\det(Df_0(x)+\epsilon D\delta f(x))\big|_{E^u_{f_0}} = \det\left(I_{f_0(x)}+\epsilon D\delta f(x)\cdot Df_0^{-1}(f_0(x))\right)|_{E^u_{f_0}} \det Df_0|_{E^u_{f_0}}(x),$$

where $I_{f_0(x)}$ denotes the identity map of the tangent space $T_{f_0(x)}$. Let $X(x)$ denote the vector field $\delta f \circ f_0^{-1}(x)$. We have

$$D\delta f(x) \cdot Df_0^{-1}(f_0(x)) = DX(f_0(x)),$$

the differential of the vector field X evaluated at the point $f_0(x)$.

By Lemma 2 that we shall prove next,

$$\frac{d}{d\epsilon} \log \det(I + \epsilon DX(x))|_{E^u_{f_0}} = \mathrm{div} X^u(x).$$

In general, given any $\epsilon_0 \in (-r, r)$, $I + \epsilon DX = (I + \epsilon_0 DX) + (\epsilon - \epsilon_0)DX = \big(I + (\epsilon - \epsilon_0)DX(I + \epsilon_0 DX)^{-1}\big)(I + \epsilon_0 DX)$ and $DX(I + \epsilon_0 DX)^{-1} = D\big[\delta f \circ (f^{\epsilon_0})^{-1}\big]$. Thus, we have

$$\frac{\partial}{\partial \epsilon} \log J^u_n f^\epsilon_{E^u_{f_0}}(h_{f^{\epsilon_0}}(x))|_{\epsilon_0} = \mathrm{div}^u_{\omega_n} X^u_n(f^{\epsilon_0}(x)).$$

□

It leaves us to prove the following lemma.

Lemma 2. *Let X be a smooth vector field defined in an open neighborhood U of x on a Riemannian manifold. Assume that the smooth metric w on U satisfies the following conditions:*

(1) there exists a split of tangent space $\mathbb{R}^n = \mathbb{R}^u \oplus \mathbb{R}^s$ such that the volume form $dw = \rho(x)dx_1 \wedge \cdots \wedge dx_n$ induced by the metric at the point $x = (x^u, x^s) \in U$ has the property that $\rho(x^u, x^s) = \rho^u(x^u)$ for some smooth function ρ^u defined on $\mathbb{R}^u$.

(2) The subspaces $\mathbb{R}^u$ and $\mathbb{R}^s$ are orthogonal under the metric w.

Then,

$$\frac{d}{d\epsilon} \log \det(I + \epsilon DX(x))|_{\mathbb{R}^u} = \mathrm{div} X^u(x),$$

where the derivative is evaluated at $\epsilon = 0$, $X^u + X^s = X$ is the projection of the vector field X onto $\mathbb{R}^u$ and $\mathbb{R}^s$, and the divergence is taken with respect to the induced volume form $dw = \rho^u(x^u)dx_1 \wedge \cdots \wedge dx_u$ on $\mathbb{R}^u$.

Proof. Let $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ denote the matrix representation of the operator DX under the orthonormal bases provided by the subspaces $\mathbb{R}^u$ and $\mathbb{R}^s$ at the corresponding points x and $x + \epsilon X(x)$. We have

$$\det(I + \epsilon DX(x))|_{\mathbb{R}^u} = \sqrt{\det\left[(I + \epsilon A_{11} \;\; \epsilon A_{21})^T \begin{pmatrix} I + \epsilon A_{11} \\ A_{21} \end{pmatrix}\right]} \cdot \frac{\rho(x + \epsilon X(x))}{\rho(x)}$$

$$= \sqrt{\det(I + 2\epsilon A_{11} + \epsilon^2 A_{11}^T A_{11} + \epsilon^2 A_{21}^T A_{21})} \cdot \frac{\rho(x + \epsilon X(x))}{\rho(x)}.$$

This gives

$$\frac{d}{d\epsilon} \log \det(I + \epsilon DX(x))|_{\mathbb{R}^u} = \mathrm{trace}(A_{11}) + \frac{d}{d\epsilon} \log \rho(x + \epsilon X(x)).$$

We write $x + \epsilon X(x) = (x^u + \epsilon X^u(x), x^s + \epsilon X^s(x)$, where $X(x) = X^u(x) + X^s(x)$ is the decomposition over $\mathbb{R}^u \oplus \mathbb{R}^s$. Since $\rho(x + \epsilon X(x)) = \rho^u(x^u + \epsilon X^u(x))$, we have

$$\frac{d}{d\epsilon} \log \rho(x + \epsilon X(x)) =< \mathrm{grad} \log \rho^u(x^u), X^u(x) > .$$

Thus,

$$\det(I+\epsilon DX(x))|_{\mathbb{R}^u} = \mathrm{trace}(A_{11}) + < \mathrm{grad}\log\rho^u(x^u), X^u(x) >,$$

i.e., the divergence of the vector field X^u on $\mathbb{R}^u$ with respect to induced volume form $\rho^u dx_1 \wedge \cdots \wedge dx_u$.

□

As we take the limit $n \to \infty$, the metric ω_n converges to ω_0 uniformly. So do the leaves of the smooth foliation $W_n^s(x)$ to the stable foliation. As a linear operator, DX can still be represented as a matrix in four blocks in the limiting metric ω_0. Both terms trace(A_{11}) and $< \mathrm{grad}\log\rho^u(x^u), X^u(x) >$ are well-defined for the limiting metric and are Hölder continuous. This implies the uniform boundedness of $\mathrm{div}^u_{\omega_n} X^u_n$ in n when $X = \delta f \circ (f^{\epsilon_0})^{-1}$. We conclude that $\log J^u_0 f^{\epsilon}_{E^u_{f_0}}(x)$ is differentiable in ϵ and

$$\frac{\partial}{\partial\epsilon}\log J^u_0 f^{\epsilon}_{E^u_{f_0}}(x)\big|_{\epsilon=0} = \mathrm{div}X^u(f_0(x)),$$

where the determinant and the divergence are taken with respect to the metric ω_0.

Remark. Even though the value of the derivative $\frac{\partial}{\partial\epsilon}\log J^u_n f^{\epsilon}_{E^u_{f_0}}(x)\big|_{\epsilon=0}$ does not depend on the choice of distributions complementary to $E^u_{f_0}$. Its calculation can be made to depend on the complementary subspaces $E^s(x)$ to represent the value in terms of a divergence of a vector field on the unstable submanifold.

3.3. Calculating the second partial derivative $\dfrac{\partial}{\partial y}\log J^u_n f^{\epsilon_0}_{E^u_{f_0}}(y)|_{h_{f^{\epsilon_0}}(x)}\ \dfrac{\partial}{\partial\epsilon}h_{f^\epsilon}|_{\epsilon_0}$.

Let $p_{f^{\epsilon_0}}(y) = -\log J^u_n f^{\epsilon_0}_{E^u_{f_0}}(y)$. Since both f^{ϵ_0} and $E^u_{f_0}$ are now fixed and ω_n is a smooth metric. The function $p_{f^{\epsilon_0}}(y)$ is smooth in y within an open neighborhood of x. By Proposition 1, we have

$$\frac{\partial}{\partial y}\log J^u_n f^{\epsilon_0}_{E^u_{f_0}}(y)|_{h_{f^{\epsilon_0}}(x)}\ \frac{\partial}{\partial\epsilon}h_{f^\epsilon}|_{\epsilon_0}$$

$$= \sum_{k=1}^{\infty} < \mathrm{grad}(p_{f^{\epsilon_0}} \circ (f^{\epsilon_0})^{-k}), X^u > - \sum_{k=0}^{\infty} < \mathrm{grad}(p_{f^{\epsilon_0}} \circ (f^{\epsilon_0})^{k}), X^s >,$$

where $X^u + X^s = X$, $X = \delta f \circ (f^{\epsilon_0})^{-1}$, X^u and X^s are projections of X onto the unstable and stable subspaces of f^{ϵ_0}.

This sequence is uniformly bounded in n and continuous in $\epsilon_0 \in (-r, r)$ due to the uniform hyperbolicity. Its limit as $n \to \infty$ is

$$\sum_{k=1}^{\infty} < \mathrm{grad}(p_{f_0} \circ {f_0}^{-k}), X^u > - \sum_{k=0}^{\infty} < \mathrm{grad}(p_{f_0} \circ {f_0}^{k}), X^s >,$$

where $X^u + X^s = X$ and $X = \delta f \circ f_0^{-1}$.

Lemma 3. *Let X^s be any vector in the tangent space of the stable submanifold W^s at x.*

$$< \mathrm{grad}(\log J^u_0 f_0\big|_{E^u_{f_0}}(x), X^s >= 0$$

Proof. Assume that X^s_n is a vector in the tangent space of the smooth foliation W^s_n and $\lim_{n\to\infty} X^s_n = X^s$.

We show that

$$< \mathrm{grad}(\log J^u_n f_0\big|_{E^u_{f_0}}(x), X^s_n >= 0$$

We use the coordinate system given by $E^u_{f_0}$ and the subspace tangent to the smooth foliation. Since the leaves are invariant under f_0, we have

$$Df_0(y) = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

for all y near x. Note that $E^u_{f_0}$ in invariant under $Df_0(x)$. Thus, $C = 0$ when $y = x$ and C is of order ϵ when $|y - x| = \epsilon$. The submatrix A is independent of y as y moves along the leaves of W^s_n. By the orthogonality of the metric, we have

$$|\det Df_0(y)| = \sqrt{\det[A^T A + C^T C]}.$$

Since $C^T C$ is of second order in ϵ, we have

$$< \operatorname{grad} \log |\det Df_{0E^u_{f_0}}|(x), X^s >= 0$$

for any given direction X^s tangent to the leaves of W^s_n. The result of the lemma follows as we take the limit $n \to \infty$.

□

Combining the two partial derivatives, we obtain the derivative formula

$$\delta(\log J^u_0 f(h_f(x)))\big|_{f_0} = \operatorname{div}^u X^u(f_0(x)) + \sum_{k=1}^{\infty} < \operatorname{grad}(p_{f_0} \circ {f_0}^{-k}), X^u > .$$

Since

$$\operatorname{div}^u_\rho X^u = \operatorname{div}^u X^u + \sum_{k=1}^{\infty} < \operatorname{grad}(p_{f_0} \circ {f_0}^{-k}), X^u >,$$

we have

$$\delta(\log J^u_0 f(h_f(x)))\big|_{f_0} = \operatorname{div}^u_\rho X^u(x) + \operatorname{div}^u X^u(f_0(x)) - \operatorname{div}^u X^u(x).$$

If we choose ω_0 so that its density function coincides with that induced by the SRB measure ρ, we have

$$\delta(\log J^u_0 f(h_f(x)))\big|_{f_0} = \operatorname{div}^u_\rho X^u(f_0(x)),$$

as claimed in Theorem 1.

References

[1] D. Anosov, Geodesic Flows on Closed Riemann Manifolds with Negative Curvature, Proc. of the Stecklov Inst. of Math., 90 (1967). English Translation, AMS, Providence R.I. (1969)

[2] D. Dolgopyat, On differentiability of SRB states for partially hyperbolic systems, *Invent. Math.* 155 (2004), no. 2, 389–449.

[3] M. Jiang and R. de la Llave Smooth dependence of thermodynamic limits of SRB measures *Comm. Math. Physics* 211, (2000), 303-333

[4] M. Jiang and R. de la Llave, Linear response function for coupled hyperbolic attractors, *Comm. Math. Phys.* 261 (2006), no. 2, 379–404

[5] A. Katok and B. Hasselblatt, Introduction to the Modern Theorey of Dynamical Systems, Cambridge University Press, 1996.

[6] L. Barreira and Y. Pesin, Nonuniform hyperbolicity, Dynamics of systems with nonzero Lyapunov exponents. Encyclopedia of Mathematics and its Applications, 115. Cambridge University Press, Cambridge, 2007

[7] D. Ruelle, Differentiation of SRB States, *Commun. Math. Phys.* 187, (1997), 227-241, Correction and complements: "Differentiation of SRB states" *Comm. Math. Phys.* 234 (2003), no. 1, 185–190.

[8] D. Ruelle, Smooth dynamics and new theoretical ideas in nonequilibrium statistical mechanics. *J. Statist. Phys.* 95, no. 1-2, (1999), 393–468.
[9] D. Ruelle, Thermodynamic Formalism. *Encyclopedia of Mathematics and Its Applications* No.5 Addison Wesley New York (1978)
[10] I. G. Sinai Topics in ergodic theory Princeton University Press Princeton, N.J. (1994)
[11] J. Schmeling, Hölder continuity of the holonomy maps for hyperbolic basic sets. II. Math. Nachr. 170 (1994), 211–225.

Department of Mathematics, Wake Forest University, Winston Salem, NC 27109

Fax: 1 336 758 7190, Phone: 1 336 758 5951
E-mail address: jiangm@wfu.edu