Unicity of the Integrated Density of States for Relativistic Schrödinger Operators with Regular Magnetic Fields and Singular Electric Potentials

Viorel Iftimie[*], Marius Măntoiu[†‡] and Radu Purice[§¶]

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Abstract

We show coincidence of the two definitions of the integrated density of states (IDS) for a class of relativistic Schrödinger operators with magnetic fields and scalar potentials introduced in [21, 22], the first one relying on the eigenvalue counting function of operators induced on open bounded sets with Dirichlet boundary conditions, the other one involving the spectral projections of the operator defined on the entire space. In this way one generalizes the results of [10, 20] for non-relativistic operators. The proofs needs the magnetic pseudodifferential calculus developed in [21], as well as a Feynman-Kac-Itô formula for Lévy processes [19, 22]. In addition, in case when both the magnetic field and the scalar potential are periodic, one also proves the existence of the IDS.

1 Introduction

We specify first the class of operators we consider. For $d \geq 2$ we set

$$BC^\infty(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \mid \partial^\alpha f \in L^\infty(\mathbb{R}^d), \ \forall \alpha \in \mathbb{N}^d\},$$

and

$$C^\infty_{\text{pol}}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \mid \partial^\alpha f \text{ is polynomially bounded}, \ \forall \alpha \in \mathbb{N}^d\}.$$

The magnetic field $B = \frac{1}{2}\sum_{j,k=1}^d B_{jk} dx_j \wedge dx_k$ satisfies:

Hypothesis (i): $dB = 0$, $\quad B_{jk} = -B_{kj} \in BC^\infty(\mathbb{R}^d)$.

Using the transversal gauge one constructs a vector potential $A = \sum_{j=1}^d A_j dx_j$, with $A_j \in C^\infty_{\text{pol}}(\mathbb{R}^d)$, such that $dA = B$. The circulation of A through the segment $[x, y]$, $x, y \in \mathbb{R}^d$, can be written as

$$\int_{[x,y]} A = - < x - y, \Gamma^A(x,y) >, \quad \Gamma^A(x,y) := \int_0^1 ds A((1-s)x + sy). \tag{1.1}$$

In some papers [27, 25] one proposes the following quantization of a classical observable $a : T^*\mathbb{R}^d \to \mathbb{R}$:

$$\left[\mathfrak{Op}^A(a)u\right](x) := \iint_{\mathbb{R}^d \, \mathbb{R}^d} dy \, d\!\!\!{}^- \xi \, e^{i < x-y, \xi + \Gamma^A(x,y) >} \, a\left(\frac{x+y}{2}, \xi\right) u(y), \tag{1.2}$$

[*]Institute of Mathematics Simion Stoilow of the Romanian Academy, P.O. Box 1-764, Bucharest, Romania. Email: Viorel.Iftimie@imar.ro

[†]Universidad de Chile, Las Palmeras 3425, Casilla 653, Santiago Chile.

[‡]Institute of Mathematics Simion Stoilow of the Romanian Academy, P.O. Box 1-764, Bucharest, Romania. Email: Marius.Mantoiu@imar.ro

[§]Institute of Mathematics Simion Stoilow of the Romanian Academy, P.O. Box 1-764, Bucharest, Romania. Email: Radu.Purice@imar.ro

[¶]Laboratoire Européen Associé CNRS *Math-Mode*, Franco-Roumain

where $u \in \mathcal{S}(\mathbb{R}^d)$, $d\xi := (2\pi)^{-d} d\xi$ and the oscillatory integral makes sense if, for example, $a \in S^m(\mathbb{R}^d)$.

A symbolic calculus for the operators defined by (1.2), essential for the present work, has been developed in [21]. The quantization (1.2) has the important physical property of being gauge covariant: if $\varphi \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, then A and $A' = A + d\varphi$ define the same magnetic field B and

$$\mathfrak{Op}^{A'}(a) = e^{i\varphi} \mathfrak{Op}^A(a) e^{-i\varphi}.$$

There exists another approach for quantization in the presence of a magnetic field [13, 16, 17, 18, 19, 28, 29]. One defines $\mathfrak{Op}_A(a)$ by the Weyl quantization of the symbol

$$T^*\mathbb{R}^d \ni (x, \xi) \mapsto a(x, \xi - A(x)) \in \mathbb{R},$$

but in this way gauge covariance is lost, as shown in [21] for $a(\xi) = <\xi> := (1 + |\xi|^2)^{1/2}$. One notices, however, that both quantizations lead to the same magnetic non-relativistic Schrödinger operator.

We are concerned in the present paper with the case

$$a(\xi) = <\xi> - 1 \tag{1.3}$$

for which the two quantizations do not coincide.

As shown in [21, 22], the operator $\mathfrak{Op}^A(a)$ in $L^2(\mathbb{R}^d)$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^d)$. One denotes by H_A its closure; then $H_A \geq 0$ and its domain is the magnetic Sobolev space of order 1:

$$\mathcal{H}^1_A := \{ u \in L^2(\mathbb{R}^d) \mid (D_j - A_j)u \in L^2(\mathbb{R}^d),\ 1 \leq j \leq d \}.$$

We call H_A *the relativistic Schrödinger operator with magnetic field*. One should remark that another candidate exists for this concepts, the operator $[(D - A)^2 + 1]^{1/2} - 1$ (cf. [12] for instance), but this one cannot be deduced from a quantization which systematically applies to a whole class of symbols.

For the scalar potential V, let us first consider the following condition.

Hypothesis (ii): $V : \mathbb{R}^d \to \mathbb{R}$, $V = V_+ - V_-$, $V_\pm \geq 0$, $V_\pm \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, and the operator of multiplication by V_- is form-bounded with respect to H_0, with relative bound strictly less than 1.

In other words, there exist $\alpha \in [0, 1)$ and $\beta \geq 0$ such that

$$\int_{\mathbb{R}^d} V_- |u|^2 dx \leq \alpha \|H_0^{1/2} u\|^2 + \beta \|u\|^2, \quad u \in D(H_0^{1/2}) = \mathcal{H}^{1/2}(\mathbb{R}^d), \tag{1.4}$$

where $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^d)$ and $\mathcal{H}^s(\mathbb{R}^d)$ is the usual Sobolev space of order $s \in \mathbb{R}$.

We are going to show in Section 4 that under the assumptions (i) and (ii), the form sum

$$H \equiv H(A; V) := H_A \dot{+} V$$

is well-defined. The operator H will be self-adjoint and lower semi-bounded in $L^2(\mathbb{R}^d)$. In particular, $H_A = H(A; 0)$.

To use the Feynman-Kac-Itô formula from Section 4 we will need a stronger hypothesis, involving Kato's class $\mathcal{K}_d$ associated to the operator H_0, defined as follows: The semigroup generated by H_0 is given by convolution with a function p_t (defined in Section 3); a function $W \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $W \geq 0$, belongs to $\mathcal{K}_d$ if

$$\lim_{t \searrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t \left[\int_{\mathbb{R}^d} p_s(x - y) W(y) dy \right] ds = 0. \tag{1.5}$$

In particular, if $W \in L^\infty(\mathbb{R}^d)$, $W \geq 0$, then $W \in \mathcal{K}_d$. In [36, 6, 9] one shows that $W \in \mathcal{K}_d$ verifies (1.4) for any $\alpha > 0$.

For our main results we shall need a stronger assumption on V.

Hypothesis (ii′): $V : \mathbb{R}^d \to \mathbb{R}$, $V = V_+ - V_-$, $V_\pm \geq 0$, $V_\pm \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and $V_- \in \mathcal{K}_d$.

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To define the integrated density of states (IDS) we need a family $\mathcal{F}$ of bounded open subsets of $\mathbb{R}^d$, satisfying:

Hypothesis (iii): For any $m \in \mathbb{N}^*$, there exists $\Omega \in \mathcal{F}$ such that the ball $B(0; m)$ centered in the origin, of radius m, is contained in Ω.

Hypothesis (iv): For any $\epsilon > 0$, there exists $m_0 \in \mathbb{N}^*$ such that if $\Omega \in \mathcal{F}$ and $B(0, m_0) \subset \Omega$, we have

$$|\{x \in \mathbb{R}^d \mid \text{dist}(x, \partial\Omega) < 1\}| < \epsilon\,|\Omega|,$$

where we set $|\Omega|$ for the Lebesgue measure of Ω.

Let us mention some basic references concerning IDS [5, 8, 11, 14, 32] and [10] that is closer related to our work. There are two definitions of IDS. The first one [5, 8] uses the operator H_Ω induced by H on $\Omega \in \mathcal{F}$, with Dirichlet boundary conditions (it is defined in Section 6, where we prove that H_Ω has compact resolvent on $L^2(\Omega)$). IDS is the function

$$\rho : \mathbb{R} \to \mathbb{R}_+, \quad \rho(\lambda) := \lim_{\substack{\Omega \to \mathbb{R}^d \\ \Omega \in \mathcal{F}}} \frac{N_\Omega(\lambda)}{|\Omega|}, \tag{1.6}$$

where $N_\Omega(\lambda)$ is the number of eigenvalues of H_Ω smaller than λ.

The second definition [8, 14] uses the fact (proved in Section 5) that the operator $\mathbf{1}_\Omega E_\lambda(H)\mathbf{1}_\Omega$ belongs to $\mathcal{I}_1$, i.e. is trace-class. Here $\mathbf{1}_\Omega$ is the operator of multiplication by the characteristic function of Ω, and $E_\lambda(H)$ is the spectral projection of H corresponding to the interval $(-\infty, \lambda]$, $\lambda \in \mathbb{R}$. Then IDS is also defined by

$$\rho(\lambda) := \lim_{\substack{\Omega \to \mathbb{R}^d \\ \Omega \in \mathcal{F}}} \frac{\text{tr}[\mathbf{1}_\Omega E_\lambda(H)\mathbf{1}_\Omega]}{|\Omega|}. \tag{1.7}$$

The existence of the limits (1.6) and (1.7) and their equality are both non-trivial problems.

In order to solve them one uses the notion of density of states for H, for which we also have two different definitions. We are going to see in Sections 5 and 6 that for any $f \in C_0(\mathbb{R})$ (continuous function with compact support on $\mathbb{R}$) the operators $f(H_\Omega)$ and $\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega$ belong to $\mathcal{I}_1$. By the Riesz-Markov Theorem for any $\Omega \in \mathcal{F}$ there exist Borel measures μ_Ω^D and μ_Ω on $\mathbb{R}$, such that

$$|\Omega|^{-1}\text{tr}f(H_\Omega) = \int_\mathbb{R} f d\mu_\Omega^D, \qquad |\Omega|^{-1}\text{tr}\left[\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega\right] = \int_\mathbb{R} f d\mu_\Omega.$$

One notices that the two expressions in (1.6) and (1.7) are exactly the distribution functions of these two measures:

$$\mu_\Omega^D((-\infty, \lambda]) = |\Omega|^{-1}N_\Omega(\lambda), \qquad \mu_\Omega((-\infty, \lambda]) = |\Omega|^{-1}\text{tr}\left[\mathbf{1}_\Omega E_\lambda(H)\mathbf{1}_\Omega\right].$$

If Borel measures μ^D, μ on $\mathbb{R}$ exists such that

$$\lim_{\mathcal{F}\ni\Omega\to\mathbb{R}^d} \mu_\Omega^D = \mu^D, \qquad \lim_{\mathcal{F}\ni\Omega\to\mathbb{R}^d} \mu_\Omega = \mu,$$

meaning that for any $f \in C_0(\mathbb{R})$ and any $\epsilon > 0$ there exists $m_0 \in \mathbb{N}^*$ such that if $B(0; m_0) \subset \Omega$, then

$$\left|\int_\mathbb{R} f d\mu_\Omega^D - \int_\mathbb{R} f d\mu^D\right| < \epsilon, \qquad \left|\int_\mathbb{R} f d\mu_\Omega - \int_\mathbb{R} f d\mu\right| < \epsilon,$$

each of them is called *the density of states of H*. The main result of this article is the equivalence of these definitions:

Theorem 1.1. *Under assumptions (i), (ii'), (iii) and (iv), the density of states μ^D exists if and only if the density of states μ exists. In addition, if one of them exists, then $\mu^D = \mu$.*

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For the non-relativistic Schrödinger operator, such a result has been obtained in [10] for $V_- = 0$ and in [20] for $V_- \neq 0$. There are several results concerning the existence and unicity of IDS for non-relativistic Schrödinger operators without magnetic field (see [10] for references). The case of a constant magnetic field has been treated in [14] ($V \in C^\infty(\mathbb{R}^d)$ periodic) or in [15] (V random potential, eventually unbounded from below). We also mention the results in [2, 3, 4, 7, 14] and references therein.

The existence of IDS has been proved in [20] for periodic magnetic fields and scalar potentials. The second goal of this paper is to extend this last result to the relativistic case.

We consider a lattice Γ in $\mathbb{R}^d$, generated by a base $e_1, \ldots, e_d$:

$$\Gamma = \{\sum_{j=1}^d \alpha_j e_j \mid \alpha_j \in \mathbb{Z}, \ 1 \leq j \leq d\}.$$

Let us also denote by F a fundamental domain of $\mathbb{R}^d$ with respect to Γ; for instance

$$F = \{\sum_{j=1}^d t_j e_j \mid 0 \leq t_j < 1, 1 \leq j \leq d\}.$$

We make the following hypothesis:

(v) V and B_{jk}, $1 \leq j, k \leq d$ are Γ-periodic functions.

Theorem 1.2. *Under the hypothesis (i), (ii'), (iii), (iv) and (v), the integrated density of states of H exists and for each $f \in C_0(\mathbb{R})$ we have*

$$\lim_{\mathcal{F} \ni \Omega \to \mathbb{R}^d} |\Omega|^{-1} \mathrm{tr}\left[\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega\right] = |F|^{-1} \mathrm{tr}_\Gamma f(H), \tag{1.8}$$

where tr_Γ is the Γ-trace in the sense of Atiyah [1].

The plan of this paper is as follows:

In Section 2 we review first some properties of the magnetic pseudodifferential calculus, established in [21, 22]. Some refined result about commutators are obtained and one overlines approximation by regularisations (using the magnetic convolution) and cut-offs.

In Section 3 we present the Feller semi-group defined by the free Hamiltonian H_0 and the associated Lévy process. The diamagnetic inequality (3.10) will be a consequence of a Feynman-Kac-Itô formula for the relativistic Hamiltonian H_A.

Section 4 is devoted to the construction of the relativistic Schrödinger operator $H = H(A; V)$. Using the Feynman-Kac-Itô formula, representing the semi-group generated by H, we prove the important fact that $C_0^\infty(\mathbb{R}^d)$ is an essential domain for the form associated to H and we present some consequences regarding commutators and covariance under gauge transformations.

In Section 5 we estimate the trace norm of some operators of the form $\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega$, Ω being a bounded open subset of $\mathbb{R}^d$ and $f : \mathbb{R} \to \mathbb{C}$ a suitable function. The hypothesis $V \in \mathcal{K}_d$ is essential, allowing us to use some estimations for the integral kernel of the semi-group generated by $H(0; -V_-)$ (cf. [36, 9]).

In Section 6 one defines H_Ω as a pseudo-self-adjoint operator on $L^2(\mathbb{R}^d)$, using a result in [33] on monotone sequences of quadratic forms. One also estimates the $\mathcal{I}_1$-norm of operators of the form $f(H_\Omega)$.

Section 7 is dedicated to the proof of Theorem 1.1. The main difficulty is the $\mathcal{I}_1$-norm estimate of operators of the form $\mathbf{1}_\Omega(H + \lambda)^{-m}\mathbf{1}_\Omega - (H_\Omega + \lambda)^{-m}$. Then, using ideas from [10] finishes the proof.

In Section 8 we prove Theorem 1.2, on the lines of the proof of Theorem 1.6 from [20].

2 The magnetic pseudodifferential calculus

Let us recall first some properties of operators defined by (1.2), proved in [21]. We are going to assume everywhere that $B = dA$ fulfills hypothesis (i).

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Definition 2.1. *Let $m \in \mathbb{R}$.*

(a) A function $f \in C^\infty(\mathbb{R}^{2d})$ belongs to the symbol space $S^m(\mathbb{R}^d)$ if for any $\alpha, \beta \in \mathbb{N}^d$ there is a constant $C_{\alpha,\beta} > 0$ such that

$$\left| \left(\partial_x^\alpha \partial_\xi^\beta f \right)(X) \right| \leq C_{\alpha,\beta} <\xi>^{m-\beta}, \qquad X = (x,\xi) \in \mathbb{R}^{2d}.$$

The space $S^m(\mathbb{R}^d)$ is endowed with the usual Fréchet topology.

(b) $S^{-\infty}(\mathbb{R}^d) := \cap_{m \in \mathbb{R}} S^m(\mathbb{R}^d)$ is endowed with the projective limit topology.

(c) A symbol $f \in S^m(\mathbb{R}^d)$ is called elliptic *if for some positive constants C, R one has*

$$|f(X)| \geq C <\xi>^m, \qquad \forall X = (x,\xi) \in \mathbb{R}^{2d}, \quad |\xi| \geq R.$$

(d) We call principal symbol *of an operator of the form $\mathfrak{Op}^A(f)$, where $f \in S^m(\mathbb{R}^d)$, any element $f_0 \in S^m(\mathbb{R}^d)$ such that $f - f_0 \in S^{m-1}(\mathbb{R}^d)$.*

Proposition 2.2. *Let $f \in S^m(\mathbb{R}^d)$ and $g \in S^{m'}(\mathbb{R}^d)$.*

(a) $\mathfrak{Op}^A(f)$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^d)$ and on $\mathcal{S}'(\mathbb{R}^d)$.

(b) $\mathfrak{Op}^A(\overline{f})$ is the formal adjoint of $\mathfrak{Op}^A(f)$, i.e.

$$\left(\mathfrak{Op}^A(f)u, v \right)_{L^2(\mathbb{R}^d)} = \left(u, \mathfrak{Op}^A(\overline{f})v \right)_{L^2(\mathbb{R}^d)}, \qquad \forall u, v \in \mathcal{S}(\mathbb{R}^d).$$

(c) There exists a unique element $f \circ^B g \in S^{m+m'}(\mathbb{R}^d)$ such that

$$\mathfrak{Op}^A(f)\mathfrak{Op}^A(g) = \mathfrak{Op}^A(f \circ^B g).$$

Moreover, a principal symbol of $\mathfrak{Op}^A(f)\mathfrak{Op}^A(g)$ is fg.

Proposition 2.3. *Let $f \in S^0(\mathbb{R}^d)$. Then $\mathfrak{Op}^A(f) \in B(L^2(\mathbb{R}^d))$, and its norm in $B(L^2(\mathbb{R}^d))$ is dominated by a semi-norm of f in $S^0(\mathbb{R}^d)$.*

Definition 2.4. *Let $s \in \mathbb{R}_+$, $p_s(\xi) := <\xi>^s \in S^s(\mathbb{R}^d)$, $P_s := \mathfrak{Op}^A(p_s)$.*

a) An element $u \in L^2(\mathbb{R}^d)$ belongs to the magnetic Sobolev space $\mathcal{H}_A^s(\mathbb{R}^d)$ if $P_s u \in L^2(\mathbb{R}^d)$. $\mathcal{H}_A^s(\mathbb{R}^d)$ is a Hilbert space for the norm

$$\|u\|_{s,A} := \left(\|P_s u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2}, \tag{2.1}$$

and $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{H}_A^s(\mathbb{R}^d)$.

b) $\mathcal{H}_A^{-s}(\mathbb{R}^d)$ will be the dual of $\mathcal{H}_A^s(\mathbb{R}^d)$ endowed with the natural norm.

Remark 2.5. If $s \in \mathbb{N}$, we have

$$\mathcal{H}_A^s(\mathbb{R}^d) = \{ u \in L^2(\mathbb{R}^d) \mid (D-A)^\alpha u \in L^2(\mathbb{R}^d), \ \forall \alpha \in \mathbb{N}^d, |\alpha| \leq s \},$$

and a norm equivalent to (2.1) is given by

$$\|u\|_{s,A}' = \left(\sum_{|\alpha| \leq s} \|(D-A)^\alpha u\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2}.$$

We used the notation $(D-A)^\alpha = (D_1 - A_1)^{\alpha_1} \ldots (D_n - A_n)^{\alpha_n}$.

Proposition 2.6. *For each $s, m \in \mathbb{R}$ and each $f \in S^m(\mathbb{R}^d)$,*

$$\mathfrak{Op}^A(f) \in B\left(\mathcal{H}_A^s(\mathbb{R}^d), \mathcal{H}_A^{s-m}(\mathbb{R}^d) \right).$$

Proposition 2.7. *Let $p \in S^m(\mathbb{R}^d)$ be real and elliptic, $m \geq 0$. We assume $p(X) \geq 0$ for $|\xi| \geq R$ ($R > 0$ large enough). Then the operator $\mathfrak{Op}^A(p)$, defined on $S(\mathbb{R}^d)$, is essentially self-adjoint in $L^2(\mathbb{R}^d)$. Its closure P will be a lower semi-bounded self-adjoint operator on the domain $\mathcal{H}_A^m(\mathbb{R}^d)$.*

Remark 2.8. This proposition applies to the case $p(X) = <\xi> -1$. The corresponding operator, denoted by H_A, will be a lower semi-bounded self-adjoint operator on $L^2(\mathbb{R}^d)$ with domain

$$\mathcal{H}^1_A(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) \mid (D_j - A_j)u \in L^2(\mathbb{R}^d), 1 \le j \le d\}. \tag{2.2}$$

In fact, adapting arguments from [17] (where the quantification $\mathfrak{Op}_A$ is used), one can show that $H_A \ge 0$.

The next result has been proved in [21] for B admitting a vector potential with bounded derivatives and for the general case of hypothesis (i) in [23].

Proposition 2.9. *Let us consider verified the hypothesis of Proposition 2.7.*
a) If $\lambda \in \mathbb{R}$, $\lambda < \inf \sigma(P)$, then $(P - \lambda)^{-1}$ is the closure in $L^2(\mathbb{R}^d)$ of an operator $\mathfrak{Op}^A(p_{(\lambda)})$, with $p_{(\lambda)} \in S^{-m}(\mathbb{R}^d)$. If in addition $\lambda \le \inf p - 1$, then a principal symbol of $\mathfrak{Op}^A(p_{(\lambda)})$ is $(p - \lambda)^{-1}$.
b) If $m > 0$, $p \ge 1$ and $P \ge 1$, then for every $s \in \mathbb{R}$, P^s is the closure in $L^2(\mathbb{R}^d)$ of a operator $\mathfrak{Op}^A(q_{(s)})$, $q_{(s)} \in S^{sm}(\mathbb{R}^d)$, which admits p^s like principal symbol.

In the remaining part of this section we are going to prove two properties of commutators of magnetic pseudo-differential operators, as well as applications to approximation by regularization or cut-off.

Proposition 2.10. *Let $m \in \mathbb{R}$ or $m = -\infty$ and $g \in S^{m'}(\mathbb{R}^d)$, $m' \in \mathbb{R}$. Then*
a) $f \circ^B g - g \circ^B f \in S^{m+m'-1}(\mathbb{R}^d)$, $\forall f \in S^m(\mathbb{R}^d)$.
b) Let $\overline{m} \in \mathbb{R}$ and M be a subset of $S^m(\mathbb{R}^d)$ formed by symbols independents of $x \in \mathbb{R}^d$, such that the set $\{\partial_{\xi_1} f, \ldots, \partial_{\xi_d} f \mid f \in M\}$ is bounded in $S^{\overline{m}}(\mathbb{R}^d)$. Then the set $\{f \circ^B g - g \circ^B f \mid f \in M\}$ is bounded in $S^{\overline{m}+m'}(\mathbb{R}^d)$.

Proof. a) It follows from the Proposition 2.2 (c).
To verify b), one uses the composition formula from [21] for $f \in M$, in which the integral is oscillatory:

$$(f \circ^B g)(X) = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \mathit{d}Y \, \mathit{d}Z \, e^{-2i[Y,Z]} \, \omega^B(x,y,z) \, f(\xi - \eta) \, g(X - Z), \quad X \in \mathbb{R}^{2d}, \tag{2.3}$$

where $X = (x,\xi)$, $Y = (y,\eta)$, $Z = (z,\zeta)$ are points in $\mathbb{R}^{2d}$, $\mathit{d}Y = \pi^{-d}dY$, $[Y,Z] = <\eta,z> - <\zeta,y>$ ($<\cdot,\cdot>$ is the scalar product in $\mathbb{R}^d$) and $\omega^B(x,y,z) = e^{-4iF_B(x,y,z)}$, where $F_B \in C^\infty(\mathbb{R}^{3d})$, depending only on the magnetic field B and its first order derivatives, are of the form:

$$\sum_{1 \le j \le d} [D_j(x,y,z)y_j + E_j(x,y,z)z_j], \quad D_j, E_j \in BC^\infty(\mathbb{R}^{3d}). \tag{2.4}$$

Using the Leibnitz-Newton formula

$$f(\xi - \eta) = f(\xi) - \sum_{1 \le j \le d} \eta_j \int_0^1 (\partial_j f)(\xi - t\eta)dt$$

and the fact that $1 \circ^B g = g$, one writes (2.3) as

$$(f \circ^B g)(X) = f(\xi)g(X) + \rho_f(X), \tag{2.5}$$

where

$$\rho_f(X) = - \sum_{1 \le j \le d} \int_0^1 dt \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \mathit{d}Y \, \mathit{d}Z \, \eta_j \, e^{-2i[Y,Z]} \, \omega^B(x,y,z) \, (\partial_j f)(\xi - t\eta)g(X - Z). \tag{2.6}$$

We use the identity

$$\eta_j e^{-2i[Y,Z]} = -\frac{1}{2i}\partial_{z_j}\left(e^{-2i[Y,Z]}\right)$$

to integrate by parts with respect to z_j. We also use (2.4) as well as

$$y_k e^{-2i[Y,Z]} = \frac{1}{2i}\partial_{\zeta_k}\left(e^{-2i[Y,Z]}\right), \quad z_k e^{-2i[Y,Z]} = -\frac{1}{2i}\partial_{\eta_k}\left(e^{-2i[Y,Z]}\right)$$

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to integrate by parts with respect to ζ_k and η_k. This gives

$$\rho_f(X) = \sum_{1 \le j \le d} \int_0^1 dt \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} dY\,dZ\, e^{-2i[Y,Z]}\, \omega^B(x,y,z)\Big[\sum_{1 \le k \le d} D_{jk}(x,y,z)(\partial_{\xi_k} g)(X - Z)$$

$$(\partial_j f)(\xi - \eta) + t \sum_{1 \le k \le d} E_{jk}(x,y,z)(\partial_j \partial_k f)(\xi - t\eta)g(X - Z) -$$

$$-(\partial_j f)(\xi - t\eta)(\partial_{x_j} g)(X - Z)\Big], \quad D_{jk},\, E_{jk} \in BC^\infty(\mathbb{R}^{3d}).$$

By hypothesis, the sets $\{\partial_j f \mid f \in M\}$, $1 \le j \le d$ are bounded in $S^{\overline{m}}(\mathbb{R}^d)$.

Using the standard integration by parts procedure with respect to y, z, η, ζ, starting from the equality

$$< \eta >^{2N} e^{-2i[Y,Z]} = \left(1 - \frac{1}{4}\Delta_z\right)^N e^{-2i[Y,Z]}, \quad N \in \mathbb{N}$$

and its analogs, by eliminating the monomials in y and z, as above, one obtains the estimation

$$|\rho_f(X)| \le p(f)\,q(g) \int_0^1 dt \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} dY\,dZ\, < z >^{-2N_\eta} < \zeta >^{-2N_y} < \eta >^{-2N_z} < y >^{-2N_\zeta}\, \cdot$$

$$\cdot < \xi - t\eta >^{\overline{m}} < \xi - \zeta >^{m'},$$

where N_η, N_y, N_z, N_ζ are natural integers which must be chosen in a suitable way in order to have absolute convergence of the integrals, $p(f) = \sum_{1 \le j \le d} p_j(\partial_j f)$, p_j is a continuous semi-norm on $S^{\overline{m}}(\mathbb{R}^d)$ and q a continuous semi-norm on $S^{m'}(\mathbb{R}^d)$. Since

$$< \xi - t\eta >^{\overline{m}} \le C < \xi >^{\overline{m}} < \eta >^{|\overline{m}|}, \quad < \xi - \zeta >^{m'} \le C < \xi >^{m'} < \zeta >^{|m'|}, \quad C \in \mathbb{R}_+,$$

one can choose $N_\eta = N_\zeta = d$, $N_y = d + |m'|$, $N_z = d + |\overline{m}|$ and get

$$|\rho_f(X)| \le C_0\, p(f)\,q(g) < \xi >^{\overline{m}+m'}, \quad C_0 > 0 \text{ constant.}$$

Analogously one estimates the derivatives of ρ and obtains that the set $\{\rho_f \mid f \in M\}$ is bounded in $S^{\overline{m}+m'}(\mathbb{R}^d)$.

In the same way one can show that $g \circ^B f = fg + \rho'_f$ and $\{\rho'_f; f \in M\}$ is bounded in $S^{\overline{m}+m'}(\mathbb{R}^d)$. $\qquad\square$

Definition 2.11. *Let $u \in \mathcal{S}'(\mathbb{R}^d)$, $f \in \mathcal{S}(\mathbb{R}^d)$. One calls* magnetic convolution *of u with f, the function $u \star^A f \in C^\infty(\mathbb{R}^d)$ defined by*

$$(u \star^A f)(x) := < u(y),\, e^{i<x-y,\Gamma^A(x,y)>} f(x - y) >, \quad x \in \mathbb{R}^d, \tag{2.7}$$

where $< \cdot, \cdot >$ is the duality bracket between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.

Remark 2.12. Using the equality $f(x - y) = \int_{\mathbb{R}^d} e^{i<x-y,\xi>} \widehat{f}(\xi)d\xi$, one finds out that $u \star^A f = \mathfrak{Op}^A(\widehat{f})u$.

To regularize a distribution by means of the magnetic convolution, one uses a standard δ-sequence. Let us consider a function $\theta \in C_0^\infty(\mathbb{R}^d)$, $\theta \ge 0$, $\mathrm{supp}\,\theta \subset B(0;1)$, $\int_{\mathbb{R}^d} \theta(x)dx = 1$. For $j \ge 1$ one defines $\theta_j(x) := j^d \theta(jx)$, $x \in \mathbb{R}^d$. Then $\theta_j \in C_0^\infty(\mathbb{R}^d)$, $\mathrm{supp}\,\theta_j \subset B(0;1/j)$, $\widehat{\theta}_j \in \mathcal{S}(\mathbb{R}^d)$, $\widehat{\theta}_j(\xi) = \widehat{\theta}(j^{-1}\xi)$, $\xi \in \mathbb{R}^d$. For $u \in \mathcal{S}'(\mathbb{R}^d)$ we set $R_j u := u \star^A \theta_j$.

Proposition 2.13. *(a) If $u \in \mathcal{E}'(\mathbb{R}^d)$, $R_j u \in C_0^\infty(\mathbb{R}^d)$ and $\mathrm{supp}R_j u \subset \{x \in \mathbb{R}^d \mid \mathrm{dist}(x, \mathrm{supp}\,u) \le 1/j\}$.*

(b) If $u \in L^\infty(\mathbb{R}^d)$, $R_j u \in L^\infty(\mathbb{R}^d)$ and $\|R_j u\|_{L^\infty(\mathbb{R}^d)} \le \|u\|_{L^\infty(\mathbb{R}^d)}$.

(c) If $u \in L^2(\mathbb{R}^d)$, $R_j u \in L^2(\mathbb{R}^d)$ and $\lim_{j\to\infty} R_j u = u$ on $L^2(\mathbb{R}^d)$.

(d) Let $P = \mathfrak{Op}^A(p)$, $p \in S^{1/2}(\mathbb{R}^d)$. If $u \in L^2(\mathbb{R}^d)$ and $Pu \in L^2(\mathbb{R}^d)$, then $\lim_{j\to\infty} PR_j u = Pu$ in $L^2(\mathbb{R}^d)$.

Proof. Properties (a) and (b) are evident.

(c) Since $\widehat{\theta} \in \mathcal{S}(\mathbb{R}^d) \subset S^{-\infty}(\mathbb{R}^d) \subset S^0(\mathbb{R}^d)$ and $R_j = \mathfrak{Op}^A(\widehat{\theta}_j)$, Proposition 2.3 shows that $R_j u \in L^2(\mathbb{R}^d)$. But

$$(R_j)\, u(x) = \int_{\mathbb{R}^d} \theta(z)\, u\,(x - z/j)\, e^{-(i/j)<z,\Gamma^A(x,x-z/j)>} dz$$

and $u(x) = \int_{\mathbb{R}^d} \theta(z)\, u(x)\, dz$. Using twice the Dominated Convergence Theorem and the continuity of u in mean, one gets

$$\|R_j u - u\|_{L^2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} \theta(z)\big[\|u(\cdot - z/j) - u(\cdot)\|_{L^2(\mathbb{R}^d)}+$$

$$+ \|u(\cdot)\left(e^{-(i/j)<z,\Gamma^A(\cdot,\cdot-z/j)>} - 1\right)\|_{L^2(\mathbb{R}^d)}\big] dz \xrightarrow[j\to\infty]{} 0.$$

(d) The sequence $\{j^{1/2}\partial_k\widehat{\theta}_j\}_{j\geq 1}$ is bounded in $S^{-1/2}(\mathbb{R}^d)$ for $1 \leq k \leq d$. One applies Proposition 2.10 (b) with $M = \{j^{1/2}\widehat{\theta}_j;\ j \geq 1\}$, $m = -\infty$, $\overline{m} = -\frac{1}{2}$, $m' = \frac{1}{2}$, $f(\xi) = j^{1/2}\widehat{\theta}_j(\xi)$, $g = p$ and deduces that the set

$$\{j^{1/2}(\widehat{\theta}_j \circ^B p - p \circ^B \widehat{\theta}_j) \mid j \geq 1\}$$

is bounded in $S^0(\mathbb{R}^d)$. By Proposition 2.3, there is a constant $C > 0$ such that

$$\|R_j P - P R_j\|_{B(L^2(\mathbb{R}^d))} \leq C j^{-1/2},\ j \geq 1.$$

Thus $\lim_{j\to\infty}\left(P R_j u - R_j P u\right) = 0$ in $L^2(\mathbb{R}^d)$ and $\lim_{j\to\infty} R_j P u = P u$ in $L^2(\mathbb{R}^d)$ implies the conclusion. $\qquad\square$

Proposition 2.14. *Let* $P = \mathfrak{Op}^A(p)$, $p \in S^m(\mathbb{R}^d)$, $m \leq 1$ *and* $\varphi \in BC^\infty(\mathbb{R}^d)$. *Let us suppose that there exists* $N \in \mathbb{N}$, $N > m + d - 1$ *such that* $|M_N| < \infty$, *where*

$$M_N := \bigcup_{|\alpha|=N+1} \operatorname{supp} \partial^\alpha\varphi.$$

Then there is a constant $C > 0$ *independent on* φ, *operators* $S_\alpha, S'_\alpha \in B(L^2(\mathbb{R}^d))$, $1 \leq |\alpha| \leq N$, *independent on* φ *and operators* $T_N, T'_N \in \mathcal{I}_2$ *(the Hilbert-Schmidt space on* $L^2(\mathbb{R}^d)$*) such that*

$$\|T_N\|_{\mathcal{I}_2} + \|T'_N\|_{\mathcal{I}_2} \leq C \max_{|\alpha|=N+1}\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)}|M_N|^{1/2} \tag{2.8}$$

and

$$[\varphi, P] := \varphi P - P\varphi = \sum_{1\leq|\alpha|\leq N}(\partial^\alpha\varphi)S_\alpha + T_N = \sum_{1\leq|\alpha|\leq N}S'_\alpha(\partial^\alpha\varphi) + T'_N, \tag{2.9}$$

with the convention that the sums in (2.9) do not exist if $N = 0$.

Proof. Using (1.2) and Taylor's formula

$$\varphi(x) - \varphi(y) = -\sum_{1\leq|\alpha|\leq N}\frac{(y-x)^\alpha}{\alpha!}(\partial^\alpha\varphi)(x) + r_N(x,y),$$

where

$$r_N(x,y) = -\sum_{|\alpha|=N+1}\frac{(y-x)^\alpha}{N!}\int_0^1 (1-t)^N(\partial^\alpha\varphi)(x + t(y-x))dt,$$

one obtains the first equality from (2.9) with $S_\alpha = -\frac{1}{\alpha!}\mathfrak{Op}^A(D_\xi^\alpha p)$ and

$$T_N u(x) := -\frac{1}{N!}\sum_{|\alpha|=N+1}\int_0^1 dt(1-t)^N\int_{\mathbb{R}^d} dy\hspace{-0.3em}\text{-}d\xi\, e^{i<x-y,\xi+\Gamma^A(x,y)>} \times \tag{2.10}$$

$$\times (D_\xi^\alpha p)\left(\frac{x+y}{2},\xi\right)(\partial^\alpha \varphi)(x+t(y-x))u(y), \quad u \in \mathcal{S}(\mathbb{R}^d).$$

Since for $|\alpha| \geq 1$ one has $D_\xi^\alpha p \in S^0(\mathbb{R}^d)$, by Proposition 2.3 one has $S_\alpha \in B(L^2(\mathbb{R}^d))$. But, using in (2.10) the identity $<x-y>^{2s} e^{i<x-y,\xi>} = (1-\Delta_\xi)^s(e^{i<x-y,\xi>})$, for $s \in \mathbb{N}$, to integrate by parts, one sees that for any $s \in \mathbb{N}$, T_N can be written as an integral operator with kernel

$$K_N(x,y) := -\frac{1}{N!}\sum_{|\alpha|=N+1}\int_0^1 dt(1-t)^N\int_{\mathbb{R}^d} d\xi \, e^{i<x-y,\xi+\Gamma^A(x,y)>}\times$$

$$\times <x-y>^{-2s}\left[(1-\Delta_\xi)^s D_\xi^\alpha p\right]\left(\frac{x+y}{2},\xi\right)(\partial^\alpha\varphi)(x+t(y-x)).$$

Denoting by $\mathbf{1}_M$ the characteristic function of a set $M \subset \mathbb{R}^d$, one obtains for each $s \in \mathbb{N}$

$$|K_N(x,y)| \leq C_s <x-y>^{-2s}\max_{|\alpha|=N+1}\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)}\int_0^1 \mathbf{1}_{M_N}(x+t(y-x))dt,$$

with C_s a constant depending on s but not on φ. Choosing $s > d/4$ one gets

$$\|T_N\|_{\mathcal{I}_2} = \|K_N\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \leq$$

$$\leq C_s \max_{|\alpha|=N+1}\|\partial^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)}\sup_{t\in[0,1]}\left[\int_{\mathbb{R}^{2d}} <x-y>^{-4s}\mathbf{1}_{M_N}(x+t(y-x))\,dx\,dy\right]^{1/2}.$$

The integral in the previous formula equals

$$\int_{\mathbb{R}^{2d}} <z>^{-4s}\mathbf{1}_{M_N}(x+tz)dxdz = \int_{\mathbb{R}^d}\mathbf{1}_{M_N}(y)dy\int_{\mathbb{R}^d} <z>^{-4s}dz = |M_N|\int_{\mathbb{R}^d} <z>^{-4s}dz.$$

It follows that the norm $\|T_N\|_{\mathcal{I}_2}$ is bounded with the right member of the inequality (2.8).

In the same way one gets the second equality from (2.9) and the corresponding bound for $\|T_N'\|_{\mathcal{I}_2}$. $\quad\square$

A first application of the Proposition 2.14 concerns cut-off approximations. Let $\psi \in C_0^\infty(\mathbb{R}^d)$, $0 \leq \psi \leq 1$, $\text{supp}\,\psi \subset B(0;2)$, $\psi = 1$ on $B(0;1)$. For $j \geq 1$ one sets $\psi_j(x) := \psi(x/j)$, $x \in \mathbb{R}^d$.

Proposition 2.15. *Let $u \in L^2(\mathbb{R}^d)$ and $P = \mathfrak{Op}^A(p)$, $p \in S^m(\mathbb{R}^d)$, $m \leq 1$.*
a) $\psi_j u \in L^2(\mathbb{R}^d)$ and $\lim_{j\to\infty}\psi_j u = u$ in $L^2(\mathbb{R}^d)$.
b) If $Pu \in L^2(\mathbb{R}^d)$, then $P(\psi_j u) \in L^2(\mathbb{R}^d)$ and $\lim_{j\to\infty} P(\psi_j u) = Pu$ in $L^2(\mathbb{R}^d)$.

Proof. a) is trivial.
b) follows if for some constant $C > 0$ one obtains the inequality

$$\|[\psi_j, P]\|_{B(L^2(\mathbb{R}^d))} \leq Cj^{-1}, \quad j \leq 1. \tag{2.11}$$

For this one applies Proposition 2.14 with $\varphi = \psi_j$ and $N = d+1$. Since $\text{supp}\,\partial^\alpha\psi_j \subset B(0;2j)$, $\forall \alpha \in \mathbb{N}^d$, there is a constant $C_1 > 0$ such that $|M_N^{(j)}|^{1/2} \leq C_1 j^{d/2}$, $\forall j \geq 1$, where $M_N^{(j)} := \bigcup_{|\alpha|=N+1}\text{supp}\,\partial^\alpha\psi_j$. On the other hand $(\partial^\alpha\psi_j)(x) = j^{-|\alpha|}(\partial^\alpha\psi)(x/j)$, hence there is a constant $C_2 > 0$ such that $\|\partial^\alpha\psi_j\|_{L^\infty(\mathbb{R}^d)} \leq C_2 j^{-|\alpha|}$, for any $j \geq 1$ and for all $\alpha \in \mathbb{N}^d$ with $1 \leq |\alpha| \leq N+1$. Then (2.11) follows from (2.8) and (2.9). $\quad\square$

3 The Feller semigroup

In this section we are going to recall some well-known properties of the semi-group $\left\{e^{-tH_0}\right\}_{t\geq 0}$, where H_0 is the free relativistic Hamiltonian for $A = 0$. H_0 is self-adjoint on $L^2(\mathbb{R}^d)$, $H_0 \geq 0$ and its domain is the standard Sobolev space $\mathcal{H}^1(\mathbb{R}^d) \equiv \mathcal{H}_0^1(\mathbb{R}^d)$. Its restriction to $\mathcal{S}(\mathbb{R}^d)$ is the operator $\mathfrak{Op}^0(p)$, with $p(X) = <\xi> -1$. By the Lévy-Khincin formula (see for example [31]), there exists a measure n on $\mathbb{R}^d$ such that

$$<\xi> -1 = -\int_{\mathbb{R}^d} \left[e^{i<y,\xi>} - 1 - i < y,\xi > \mathbf{1}_{B(0;1)}(y)\right] dn(y), \quad \forall \xi \in \mathbb{R}^d. \tag{3.1}$$

Cf. [17] one has the explicit formula

$$dn(y) = 2(2\pi)^{-\frac{d+1}{2}} |y|^{-\frac{d+1}{2}} K_{\frac{d+1}{2}}(|y|)dy, \tag{3.2}$$

where K_ν is the modified Bessel function of the third kind and order ν, for which the next inequalities are verified for some positive constant C:

$$0 < K_\nu(r) \leq C \max\left(r^{-\nu}, r^{-1/2}\right) e^{-r}, \quad \forall r > 0, \ \forall \nu > 0. \tag{3.3}$$

By [19, 6], for $t > 0$, the operator e^{-tH_0} is given by the convolution with the function

$$p_t(x) := (2\pi)^{-d} \frac{t}{\sqrt{|x|^2 + t^2}} \int_{\mathbb{R}^d} e^{t-<\xi>\sqrt{|x|^2+t^2}} d\xi = \tag{3.4}$$

$$= 2^{-\frac{d-1}{2}} \pi^{-\frac{d+1}{2}} t\, e^t \left(|x|^2 + t^2\right)^{-\frac{d+1}{4}} K_{\frac{d+1}{2}}\left(\sqrt{|x|^2 + t^2}\right), \quad x \in \mathbb{R}^d.$$

One verifies the properties

$$p_t(x) > 0, \quad \int_{\mathbb{R}^d} p_t(x)dx = 1 \tag{3.5}$$

and the fact that e^{-tH_0} can be extended as an well-defined bounded operator on the Banach space

$$C_\infty(\mathbb{R}^d) := \left\{f \in C(\mathbb{R}^d) \mid \lim_{|x|\to\infty} f(x) = 0\right\},$$

equipped with the norm $\|\cdot\|_\infty$. One also checks easily the Feller semi-group properties for the family of these extensions.

By [6, 9], this Feller semi-group is generated by a Lévy process. More precisely, on the space Ω of the "càdlag" functions on $[0,\infty)$ ($\mathbb{R}^d$-valued, continuous to the right, having left limits), endowed with the smallest σ-algebra $\mathcal{F}$ for which all the coordinate functions

$$\Omega \ni \omega \mapsto X_t(\omega) := \omega(t) \in \mathbb{R}^d$$

are measurable, one can define for each $x \in \mathbb{R}^d$ a probabilistic measure P_x such that $P_x\{X_0 = x\} = 1$ and the random variables $X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent with distributions $p_{t_1-t_0}, \dots, p_{t_n-t_{n-1}}$ for each $0 = t_0 < t_1 < \cdots < t_n < \infty$. If we denote by E_x the expectation with respect to the probability P_x, then for any $f \in C_\infty(\mathbb{R}^d)$ and $t \geq 0$ we have

$$\left[e^{-tH_0}f\right](x) = E_x\left(f \circ X_t\right), \quad x \in \mathbb{R}^d. \tag{3.6}$$

By the Lévy-Itô Theorem [24, 19] one has

$$X_t = x + \int_0^{t_+} \int_{\mathbb{R}^d} y\, \widetilde{N}_X\left(ds\, dy\right), \tag{3.7}$$

where

$$\widetilde{N}_X\left(ds\, dy\right) := N_X(ds\, dy) - \widehat{N}_X(ds\, dy),$$

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$$\widehat{N}_X(ds\,dy) := E_x\left(N_X(ds\,dy)\right) = ds\,dn(y)$$

and N_X is a "counting measure" on $[0,\infty) \times \mathbb{R}^d$, defined by

$$N_X((t,t'] \times B) := \sharp\{s \in (t,t'] \mid X_s \neq X_{s-}, \; X_s - X_{s-} \in B\},$$

where $0 < t < t'$ and B is a Borel subset of $\mathbb{R}^d$. Using the procedure from [19] (where one works with the quantification $\mathfrak{Op}_A$), one gets a Feynman-Kac-Itô formula for the Hamiltonian H_A. For $u \in L^2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $t \geq 0$ one has

$$\left(e^{-tH_A}u\right)(x) = E_x\left((u \circ X_t)\,e^{-S(t,X)}\right), \tag{3.8}$$

where

$$S(t,X) := i\int_0^{t_+}\int_{\mathbb{R}^d} \widetilde{N}_X(dsdy)\left\langle \int_0^1 A(X_{s-} + ry)dy, y\right\rangle + \tag{3.9}$$

$$+i\int_0^t\int_{\mathbb{R}^d} \widehat{N}_X(dsdy)\left\langle \int_0^1 [A(X_s + ry) - A(X_s)]dr, y\right\rangle.$$

Let us remark that from (3.8) and (3.6) one obtains the diamagnetic inequality for the relativistic hamiltonian H_A:

$$\left|e^{-tH_A}u\right| \leq e^{-tH_0}|u|, \quad \forall\, u \in L^2(\mathbb{R}^d), \; \forall\, t \geq 0. \tag{3.10}$$

(3.10) implies another proof of the fact that $H_A \geq 0$: e^{-tH_0} is a contraction, thus e^{-tH_A} is a contraction too, which implies $H_A \geq 0$.

Once again from (3.10), it follows that for any $\lambda > 0$, $r > 0$ and $u \in L^2(\mathbb{R}^d)$ one has

$$\left|(H_A + \lambda)^{-r}u\right| \leq (H_0 + \lambda)^{-r}|u|. \tag{3.11}$$

This inequality is deduced using the fact that for any lower semi-bounded self-adjoint operator H in a complex Hilbert space $\mathcal{H}$, for any $r > 0$ and any $\lambda \in \mathbb{R}$ such that $\lambda + \inf \sigma(H) > 0$, one has

$$(H + \lambda)^{-r} = \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-\lambda t}e^{-tH}dt, \tag{3.12}$$

where Γ is the Euler function of the second kind.

4 The Hamiltonian $H(A;V)$

We denote by h_A the quadratic form associated to H_A:

$$h_A(u,v) := \left(H_A^{1/2}u, H_A^{1/2}v\right)_{L^2(\mathbb{R}^d)}, \quad u,v \in D(h_A) := D(H_A^{1/2}). \tag{4.1}$$

To a function $W \in L^1_{\text{loc}}(\mathbb{R}^d)$, $W \geq 0$, one assigns a quadratic form q_W:

$$q_W(u,v) := \int_{\mathbb{R}^d} W(x)u(x)\overline{v(x)}dx, \quad u,v \in D(q_W) := \{f \in L^2(\mathbb{R}^d) \mid W^{1/2}f \in L^2(\mathbb{R}^d)\}. \tag{4.2}$$

These forms are symmetric, closed and positive. We set

$$h_A(u) := h_A(u,u), \quad q_W(u) := q_W(u,u).$$

The next result is known [22], but for convenience we are going to include a proof.

Proposition 4.1. *We assume (i) and (ii). Then the sesquilinear form $h = h(A;V) := h_A + q_{V_+} - q_{V_-}$ is well-defined on $D(h_A) \cap D(q_{V_+})$, being symmetric, closed and lower semi-bounded. Thus it defines a lower semi-bounded self-adjoint operator on $L^2(\mathbb{R}^d)$, denoted by $H = H(A;V) := H_A \dot{+} V$ (in the sense of forms).*

Proof. The form $h_A + q_{V_+}$, defined on $D(h_A) \cap D(q_{V_+})$, is densely defined, symmetric, closed and positive. The conclusion of the Proposition would follow if we show that the form q_{V_-} is $(h_A + q_{V_+})$-bounded, with relative bound < 1.

We denote by $H_+ := H_A + V_+$ the unique self-adjoint operator ≥ 0 associated to the form $h_A + q_{V_+}$. Since $C_0^\infty(\mathbb{R}^d) \subset D(h_A) \cap D(q_{V_+})$ we can use the version from [26] of Kato-Trotter formula

$$e^{-tH_+} = s - \lim_{n \to \infty} \left[e^{-\frac{t}{n}H_A} e^{-\frac{t}{n}V_+} \right]^n, \quad \forall t \geq 0. \tag{4.3}$$

Combining with (3.10) and (3.12) we infer that for every $r > 0$, $\lambda > 0$ and $f \in L^2(\mathbb{R}^d)$ one has

$$\left| (H_+ + \lambda)^{-r} f \right| \leq (H_0 + \lambda)^{-r} |f|. \tag{4.4}$$

Let $g \in L^2(\mathbb{R}^d)$, $\lambda > 0$ large enough, $u := (H_0 + \lambda)^{-1/2} g$. By using the assumption (ii), there exists $\alpha \in (0,1)$, $\beta \geq 0$ and $\alpha' \in (0,1)$ such that

$$q_{V_-}(u) \leq \alpha \|H_0^{1/2} u\|_{L^2(\mathbb{R}^d)}^2 + \beta \|u\|_{L^2(\mathbb{R}^d)}^2 = \alpha \|H_0^{1/2}(H_0 + \lambda)^{-1/2} g\|_{L^2(\mathbb{R}^d)}^2 + \tag{4.5}$$

$$+ \beta \|(H_0 + \lambda)^{-1/2} g\|_{L^2(\mathbb{R}^d)}^2 \leq \left(\alpha + \frac{\beta}{\lambda} \right) \|g\|_{L^2(\mathbb{R}^d)}^2 \leq \alpha' \|g\|_{L^2(\mathbb{R}^d)}^2.$$

For $v \in D(h_A) \cap D(q_{V_+})$ we set $f := (H_+ + \lambda)^{1/2} v$ and $g := |f|$. Using (4.4) with $r = 1/2$, (4.5) and the explicit form of q_{V_-}, we get

$$q_{V_-}(v) = q_{V_-}[(H_+ + \lambda)^{-1/2} f] \leq q_{V_-}[(H_+ + \lambda)^{-1/2} g] \leq \alpha' \|g\|_{L^2(\mathbb{R}^d)}^2 =$$

$$= \alpha' \|(H_+ + \lambda)^{1/2} v\|_{L^2(\mathbb{R}^d)}^2 = \alpha'[h_A(v) + q_{V_+}(v) + \lambda \|v\|_{L^2(\mathbb{R}^d)}^2].$$

$\square$

The Feynman-Kac-Itô formula (3.8) can be extended to the Hamiltonian H (cf. [19]).

Proposition 4.2. *Under assumptions (i) and (ii), for any $u \in L^2(\mathbb{R}^d)$ and all $t \geq 0$, we have*

$$\left(e^{-tH} u \right)(x) = E_x \left[(u \circ X_t) e^{-S(t,X) - \int_0^t (V \circ X_s) ds} \right], \quad x \in \mathbb{R}^d. \tag{4.6}$$

By using ideas from [34] and Propositions 2.13, 2.15 and 4.2, we are going to prove

Proposition 4.3. *Under assumptions (i), (ii), $C_0^\infty(\mathbb{R}^d)$ is an essential domain for the form h.*

Proof. Due to Hypothesis (ii) the form h and the operator H are well-defined.

1. Let us first suppose that $V_- = 0$. We show that $D(h) \cap L_{\text{comp}}^\infty(\mathbb{R}^d)$ is an essential domain for h. It is known that the range $R(e^{-H})$ is an essential domain for h. By Proposition 4.2, for any $u \in L^2(\mathbb{R}^d)$

$$|e^{-H} u| \leq e^{-H(0,0)} |u|, \quad \text{a.e. on } \mathbb{R}^d \tag{4.7}$$

the function on the right hand side being of class $L^\infty(\mathbb{R}^d)$.

Let $u \in D(h) \cap L^\infty(\mathbb{R}^d)$, ψ and ψ_j as in Proposition 2.15 and $u_j := \psi_j u$, $j \geq 1$. Then

$$u_j \in L_{\text{comp}}^\infty(\mathbb{R}^d) \cap D(q_{V_+}), \quad \lim_{j \to \infty} u_j = u \text{ in } L^2(\mathbb{R}^d) \text{ and } \lim_{j \to \infty} q_{V_+}(u_j - u) = 0.$$

Let us notice that we have the equality

$$h_A(v, w) = \left((H_A + 1)^{1/2} v, (H_A + 1)^{1/2} w \right)_{L^2(\mathbb{R}^d)} - (v, w)_{L^2(\mathbb{R}^d)} \tag{4.8}$$

for any $v, w \in D(h_A) = D(H_A^{1/2}) = D[(H_A + 1)^{1/2}]$. The operator $H_A + 1$ is defined, by Proposition 2.8, by the magnetic pseudo-differential operator $\mathfrak{Op}^A(<\xi>)$, so by the point (b) of Proposition 2.9, $(H_A + 1)^{1/2}$ is defined by an operator $\mathfrak{Op}^A(q)$, where $q \in S^{1/2}(\mathbb{R}^d)$ and $q - <\xi>^{1/2} \in S^{-1/2}(\mathbb{R}^d)$. Since $u \in D[(H_A + 1)^{1/2}]$, we shall have $\mathfrak{Op}^A(q)u \in L^2(\mathbb{R}^d)$; using Proposition 2.15 (b) we infer that $\mathfrak{Op}^A(q)u_j$ belongs to $L^2(\mathbb{R}^d)$ and $\lim_{j\to\infty} \mathfrak{Op}^A(q)u_j = \mathfrak{Op}^A(q)u$ in $L^2(\mathbb{R}^d)$. Since $\mathfrak{Op}^A(q) - \mathfrak{Op}^A(<\xi>^{1/2}) \in B[L^2(\mathbb{R}^d)]$, by Proposition 2.3, it follows that $\mathfrak{Op}^A(<\xi>^{1/2})u_j \in L^2(\mathbb{R}^d)$, so $u_j \in \mathcal{H}_A^{1/2}(\mathbb{R}^d) = D(H_A^{1/2}) = D(h_A)$. Also using (4.8), we get

$$\lim_{j\to\infty} h_A(u_j - u) = 0, \quad \text{so} \lim_{j\to\infty} u_j = u \text{ in } D(h).$$

2. We prove that $C_0^\infty(\mathbb{R}^d)$ is an essential domain for $h(A; V_+)$. Obviously $C_0^\infty(\mathbb{R}^d) \subset D(h)$. Let $u \in D(h) \cap L_{\text{comp}}^\infty(\mathbb{R}^d)$ and $R_j u$, $j \geq 1$, defined as in Proposition 2.13. Then

$$R_j u \in C_0^\infty(\mathbb{R}^d), \quad \lim_{j\to\infty} R_j u = u \text{ in } L^2(\mathbb{R}^d), \quad \lim_{j\to\infty} \mathfrak{Op}^A(q)R_j u = \mathfrak{Op}^A(q)u \text{ in } L^2(\mathbb{R}^d),$$

where q has been defined above. It follows that $\lim_{j\to\infty} h_A(R_j u - u) = 0$.

On the other hand,

$$\text{supp}R_j u \subset \{x \in \mathbb{R}^d \mid \text{dist}(x, \text{supp}u) \leq 1\}, \ |(R_j u)(x) - u(x)| \leq 2\|u\|_{L^\infty(\mathbb{R}^d)}, \ x \in \mathbb{R}^d$$

and there is a subsequence $(R_{j_k}u)_{k\geq 1}$ such that $(R_{j_k}u)(x) \xrightarrow[k\to\infty]{} u(x)$ a.e. $x \in \mathbb{R}^d$. Using the Dominated Convergence Theorem we see that $\lim_{k\to\infty} q_{V_+}(R_{j_k}u - u) = 0$, thus $\lim_{k\to\infty} R_{j_k}u = u$ in $D(h)$.

3. In order to end the proof we have to notice that q_{V_-} is relatively bounded with respect to $h(A; V_+)$ and consequently any convergent sequence from $D(h(A; V_+))$ is also convergent in $D(q_{V_-})$. $\square$

Corollary 4.4. *Under hypothesis (i) and (ii), a vector $u \in D(h)$ belongs to $D(H)$ if and only if $\mathfrak{Op}^A(p)u + Vu \in L^2(\mathbb{R}^d)$, where $p(\xi) := <\xi> - 1$. Moreover $Hu = \mathfrak{Op}^A(p)u + Vu$ for any $u \in D(H)$.*

Proof. Let $u \in D(h)$. Since $C_0^\infty(\mathbb{R}^d)$ is an essential domain for h, $u \in D(H)$ if and only if there exists $f \in L^2(\mathbb{R}^d)$ such that

$$h(u, v) = (f, v)_{L^2(\mathbb{R}^d)}, \ \forall\, v \in C_0^\infty(\mathbb{R}^d);$$

if this is the case, then $Hu = f$. By Proposition 4.1, $V_\pm u \in L_{\text{loc}}^1(\mathbb{R}^d)$ and

$$q_{V_\pm}(u, v) = <V_\pm u, \overline{v}>, \quad \forall\, v \in C_0^\infty(\mathbb{R}^d), \tag{4.9}$$

where we denoted by $<\cdot, \cdot>$ the duality bracket between $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{D}'(\mathbb{R}^d)$. Let $\{u_j\}_{j\geq 1} \subset C_0^\infty(\mathbb{R}^d)$ such that $\lim_{j\to\infty} u_j = u$ in $D(h)$. Then $\lim_{j\to\infty} H_A^{1/2}u_j = H_A^{1/2}u$ in $L^2(\mathbb{R}^d)$.

Using Proposition 2.2 (a), we get

$$h_A(u, v) = \lim_{j\to\infty} (H_A^{1/2}u_j, H_A^{1/2}v)_{L^2(\mathbb{R}^d)} = \lim_{j\to\infty} (H_A u_j, v)_{L^2(\mathbb{R}^d)} = \tag{4.10}$$

$$= \lim_{j\to\infty} <\mathfrak{Op}^A(p)u_j, \overline{v}> = <\mathfrak{Op}^A(p)u, \overline{v}>, \quad \forall\, v \in C_0^\infty(\mathbb{R}^d).$$

The Proposition follows immediately from the equality

$$h(u, v) = <\mathfrak{Op}^A(p)u, \overline{v}> + <Vu, \overline{v}>, \quad v \in C_0^\infty(\mathbb{R}^d), \tag{4.11}$$

which is a consequence of (4.9) and (4.10). $\square$

Proposition 4.5. *We suppose that (i) and (ii) are true. Let $\varphi \in BC^\infty(\mathbb{R}^d)$ such that $|M| < \infty$, where $M := \bigcup_{|\alpha|=d+2} \operatorname{supp} \partial^\alpha \varphi$.*

(a) If $u \in D(H)$, then $\varphi u \in D(H)$. Moreover the commutator $[\varphi, H]$, which is well-defined on $D(H)$, can be extended to an element of $B[L^2(\mathbb{R}^d)]$.

(b) There exists a constant $C > 0$, independent of φ, operators

$$S_\alpha, \ S_\alpha' \in B[L^2(\mathbb{R}^d)], \ 1 \leq |\alpha| \leq d+1,$$

independent of φ and operators T, $T' \in \mathcal{I}_2$, such that

$$\|T\|_{\mathcal{I}_2} + \|T'\|_{\mathcal{I}_2} \leq C \max_{|\alpha|=d+2} \|\partial^\alpha \varphi\|_{L^2(\mathbb{R}^d)} |M|^{1/2} \tag{4.12}$$

and

$$[\varphi, H] = \sum_{1 \leq |\alpha| \leq d+1} (\partial^\alpha \varphi) S_\alpha + T = \sum_{1 \leq |\alpha| \leq d+1} S_\alpha'(\partial^\alpha \varphi) + T'. \tag{4.13}$$

(c) One has

$$[(H+\lambda)^{-1}, \varphi] = (H+\lambda)^{-1}[\varphi, H](H+\lambda)^{-1}, \quad \forall \lambda \in \mathbb{R}, \ \lambda > -\inf \sigma(H). \tag{4.14}$$

Proof. (a) Let $u \in D(H)$. Then $u \in D(q_{V_+}) \cap D(h_A)$. It follows that $\varphi u \in D(q_{V_+})$ and

$$u \in D(H_A^{1/2}) = D\left((H_A+1)^{1/2}\right) = \mathcal{H}_A^{1/2}(\mathbb{R}^d).$$

Since $\varphi \in S^0(\mathbb{R}^d)$ and $\mathfrak{Op}^A(\varphi)$ is the operator of multiplication by φ, by Proposition 2.6,

$$\varphi u \in \mathcal{H}_A^{1/2}(\mathbb{R}^d) = D\left(H_A^{1/2}\right);$$

thus $\varphi u \in D(h)$. By Proposition 2.10 (a) if follows that $[\mathfrak{Op}^A(p), \varphi] \in B[L^2(\mathbb{R}^d)]$, p being given by Corollary 4.4. Therefore, computing in $\mathcal{D}'(\mathbb{R}^d)$, we get

$$\mathfrak{Op}^A(p)(\varphi u) + V(\varphi u) = \varphi[\mathfrak{Op}^A(p)u + Vu] + [\mathfrak{Op}^A(p), \varphi]u \in L^2(\mathbb{R}^d). \tag{4.15}$$

From Corollary 4.4 we deduce that $\varphi u \in D(H)$. In addition, the equality (4.15) shows that

$$[\varphi, H] = [\varphi, \mathfrak{Op}^A(p)] \ \text{ on } \mathcal{D}(H), \tag{4.16}$$

which implies the last statement of point (a).

(b) follows from (4.16) and proposition 2.14 with $m = 1$ and $N = d+1$.

(c) is trivial. $\qquad \square$

We close this Section with a result on gauge covariance of the operator H.

Proposition 4.6. *We assume (i) and (ii). Let A be a vector potential for B with components in $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and let $\widetilde{A} = A - d\varphi$ for some real function $\varphi \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$. We denote by U the unitary operator of multiplication by $e^{-i\varphi}$ on $L^2(\mathbb{R}^d)$. Then*

$$U H(A; V) U^{-1} = H(\widetilde{A}; V). \tag{4.17}$$

Proof. We notice first that from the equality

$$\varphi(x) - \varphi(y) = <x - y, \int_0^1 (\nabla \varphi)((1-s)x + sy)ds>$$

and from Definition 1.2, one gets the relation

$$[e^{-i\varphi} \mathfrak{Op}^A(a)(e^{i\varphi} w)](x) = \left[\mathfrak{Op}^{\widetilde{A}}(a)w\right](x), \quad \forall x \in \mathbb{R}^d \tag{4.18}$$

for any $a \in S^m \mathbb{R}^d$ and any $w \in \mathcal{S}(\mathbb{R}^d)$.

Let $u \in D[H(\widetilde{A}; V)]$; cf. Corollary 4.4, $u \in D[h(\widetilde{A}; V)]$ and

$$\mathfrak{Op}^{\widetilde{A}}(p)u + Vu \in L^2(\mathbb{R}^d),$$

where $p(\xi) := <\xi> -1$. From (4.18) we deduce that

$$\mathfrak{Op}^A(p)(e^{i\varphi}u) + V(e^{i\varphi}u) = e^{i\varphi}[\mathfrak{Op}^{\widetilde{A}}(p)u + Vu] \in L^2(\mathbb{R}^d). \tag{4.19}$$

Let us show that $e^{i\varphi}u \in D[h(A; V)]$. Obviously $e^{i\varphi}u \in D(q_{V_+})$. We notice now that (2.2) implies that $w \in D(H_{\widetilde{A}})$ if and only if $e^{i\varphi}w \in D(H_A)$. From (4.18) we get $U H_A U^{-1} = H_{\widetilde{A}}$, so $U H_A^{1/2} U^{-1} = H_{\widetilde{A}}^{1/2}$ and then $U^{-1}[D(h_{\widetilde{A}})] = D(h_A)$. It follows that $e^{i\varphi}u \in D(h_A)$, so $e^{i\varphi}u \in D[h(A; V)]$. Using Corollary 4.4 and equality (4.19), we deduce $U^{-1}u \in D[H(A; V)]$ as well as (4.17). $\qquad\square$

5 Trace estimations

Proposition 5.1. *Let us suppose that (i) and (ii) are verified. There exists $\mu \geq 1$, only depending on V_-, such that for all*

$$\lambda \geq \lambda_0 := \max\{-\inf\sigma(H) + 1, \mu\}, \quad r \geq r_0 := d+1,$$

there exists $C > 0$ such that for every bounded open subset Ω of $\mathbb{R}^d$ we have $\mathbf{1}_\Omega (H+\lambda)^{-r} \in \mathcal{I}_2$ and

$$\|\mathbf{1}_\Omega (H+\lambda)^{-r}\|_{\mathcal{I}_2} \leq C|\Omega|^{1/2}. \tag{5.1}$$

We denoted by $\mathbf{1}_\Omega$ both the characteristic function of Ω and the associated multiplication operator on $L^2(\mathbb{R}^d)$.

Proof. We use (3.12) and Proposition 4.2 to obtain that for any $f \in L^2(\mathbb{R}^d)$, $\lambda \geq \lambda_0$ and $r \geq r_0$ one has

$$|(H+\lambda)^{-r}f| \leq \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-\lambda t} e^{-tH(0; -V_-)}|f| dt. \tag{5.2}$$

Since $V_- \in \mathcal{K}_d$, by Theorem 1.5 from [36] (or Theorem 2.9 from [9]), for any $t > 0$ the operator $e^{-tH(0, -V_-)}$ has an integral kernel satisfying: For any $\rho, \rho' > 1$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$, one can choose positive constants M, b such that

$$0 \leq e^{-tH(0, V_-)}(x, y) \leq Me^{bt} \sup_{z \in \mathbb{R}^d} [p_{t/2}(z)]^{1/\rho'} [p_t(x-y)]^{1/\rho}, \quad \forall t > 0, \; x, y \in \mathbb{R}^d, \tag{5.3}$$

where p_t is defined by (3.4). Using (3.4) and (3.3) it follows that there exists an absolute constant $C > 0$ such that

$$p_t(x) \leq Cte^t \left[(|x|^2 + t^2)^{-\frac{d+1}{2}} + (|x|^2 + t^2)^{-\frac{d+2}{4}} \right] e^{-(|x|^2+t^2)^{1/2}}, \quad \forall t > 0, \; x \in \mathbb{R}^d. \tag{5.4}$$

We choose $\rho = 4$ and $\rho' = \frac{4}{3}$ in (5.3). From (5.4) it follows that for some $C_1 > 0$:

$$\sup_{z \in \mathbb{R}^d} [p_{t/2}(z)]^{3/4} \leq C_1 \left(t^{-\frac{3d}{4}} + t^{-\frac{3d}{8}} \right), \quad \forall t > 0. \tag{5.5}$$

Using (5.2), (5.5), (5.3) and (5.4), we get the inequality

$$|[(H+\lambda)^{-r}f](x)| \leq \int_{\mathbb{R}^d} L(x-y)|f(y)|dy =: (T|f|)(x), \quad f \in L^2(\mathbb{R}^d), \; x \in \mathbb{R}^d, \tag{5.6}$$

where

$$L(x) := C_2 \int_0^\infty e^{-(\lambda - b - 1/4)t} \, t^{r-\frac{3}{4}} \left(t^{-\frac{3d}{4}} + t^{-\frac{3d}{8}} \right) \times \tag{5.7}$$

$$\times \left[(|x|^2 + t^2)^{-\frac{d+1}{8}} + (|x|^2 + t^2)^{-\frac{d+2}{16}} \right] e^{-\frac{1}{4}(|x|^2+t^2)^{1/2}} dt \leq$$

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$$\leq C_2 \left[|x|^{-\frac{d+1}{4}} + |x|^{-\frac{d+2}{8}} \right] e^{-\frac{|x|}{4}} \int_0^\infty e^{-(\lambda - b - \frac{1}{4})t} \, t^{r - \frac{3}{4}} \left(t^{-\frac{3d}{4}} + t^{-\frac{3d}{8}} \right) dt,$$

where C_2 is a positive constant. Choosing $\mu = b + \frac{1}{2}$, the assumptions insure the convergence of the last integral. It follows that $L \in L^2(\mathbb{R}^d)$, $\forall d \geq 2$.

The integral operator $\mathbf{1}_\Omega T$ is Hilbert-Schmidt, since

$$\|\mathbf{1}_\Omega T\|_{\mathcal{I}_2} = \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathbf{1}_\Omega(x) L(x-y)|^2 dx dy \right]^{1/2} = \|L\|_{L^2(\mathbb{R}^d)} |\Omega|^{1/2}. \tag{5.8}$$

The conclusion of the Proposition follows from (5.6), (5.8) and Theorem 2.13 from [53].

$\square$

Corollary 5.2. *Under the assumptions of Proposition 5.1, for any $m \geq 2r_0$ there exists $C > 0$ such that for any $\Omega \subset \mathbb{R}^d$ open and bounded, we have $\mathbf{1}_\Omega(H + \lambda)^{-m}\mathbf{1}_\Omega \in \mathcal{I}_1$ and*

$$\|\mathbf{1}_\Omega(H + \lambda)^{-m}\mathbf{1}_\Omega\|_{\mathcal{I}_1} \leq C|\Omega|. \tag{5.9}$$

Proof. We choose $r \geq r_0$, $s \geq r_0$, $r + s = m$. Then

$$\|\mathbf{1}_\Omega(H + \lambda)^{-m}\mathbf{1}_\Omega\|_{\mathcal{I}_1} \leq \|\mathbf{1}_\Omega(H + \lambda)^{-r}\|_{\mathcal{I}_2} \|(H + \lambda)^{-s}\mathbf{1}_\Omega\|_{\mathcal{I}_2},$$

and we use Proposition 5.1 to conclude.

$\square$

Corollary 5.3. *Let $f \in L^\infty(\mathbb{R})$, $\mathrm{supp} f \subset (-\infty, a]$, $a \in \mathbb{R}$. Under the assumptions of Proposition 5.1, $\exists C > 0$ such that for any $\Omega \subset \mathbb{R}^d$ open and bounded we have $\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega \in \mathcal{I}_1$ and*

$$\|\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega\|_{\mathcal{I}_1} \leq C|\Omega|. \tag{5.10}$$

Proof. We use the equality

$$\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega = \mathbf{1}_\Omega(H + \lambda)^{-r}(H + \lambda)^{2r} f(H)(H + \lambda)^{-r}\mathbf{1}_\Omega, \quad r \geq r_0$$

and Proposition 5.1, taking into account the fact that H, being lower semi-bounded, satisfies

$$(H + \lambda)^{2r} f(H) \in B[L^2(\mathbb{R}^d)].$$

$\square$

6 The operator H_Ω

We assume (i) and (ii) for a while; let $H = H(A; V)$ be the operator constructed in Section 4. Let Ω be an open subset of $\mathbb{R}^n$ and Ω^c its complement. For $n \in \mathbb{N}$, $n \geq 1$, we set $H_n := H + n\mathbf{1}_{\Omega^c}$, which is a self-adjoint operator on $L^2(\mathbb{R}^d)$ with domain $D(H_n) = D(H)$. The associated quadratic form

$$h_n(u, v) := h(u, v) + n(\mathbf{1}_{\Omega^c} u, \mathbf{1}_{\Omega^c} v), \qquad u, v \in D(h_n) = D(h) \tag{6.1}$$

is symmetric, lower semi-bounded and closed. We also have $h \leq h_n \leq h_{n+1}$, $\forall n \geq 1$.

We are going to identify $L^2(\Omega)$ with the closed subspace of $L^2(\mathbb{R}^d)$ whose elements are null on Ω^c. The operator $\mathbf{1}_\Omega$ will be the orthogonal projection of $L^2(\mathbb{R}^d)$ on $L^2(\Omega)$.

To the monotone sequence of forms $\{h_n\}_{n \geq 1}$ defined by (6.1) one assigns the form h_Ω defined on

$$D(h_\Omega) := \{u \in \cap_{n \geq 1} D(h_n) \mid \sup_{n \geq 1} h_n(u, u) < \infty\} = D(h) \cap L^2(\Omega) \tag{6.2}$$

by the equality

$$h_\Omega(u, v) = \lim_{n \to \infty} h_n(u, v) = h(u, v), \qquad u, v \in D(h_\Omega). \tag{6.3}$$

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The form h_Ω is not densely defined but, by Theorem 4.1 from [33], it is lower bounded and closed, defining a unique pseudo-self-adjoint operator H_Ω on $L^2(\mathbb{R}^d)$; we have $D(H_\Omega) \subset L^2(\Omega)$, $HD(H_\Omega) \subset L^2(\mathbb{R}^d)$ and H_Ω, considered as an operator in $L^2(\Omega)$, is self-adjoint. In addition, $\lim_{n\to\infty} H_n = H_\Omega$ in strong resolvent sense. We denote by $\mathcal{C}_H(\mathbb{R})$ the set of functions $f : [m_f, \infty) \to \mathbb{R}$, where $m_f < \inf \sigma(H)$ (maybe depending on f), f continuous and $\lim_{t\to\infty} f(t) = 0$. Since $\inf \sigma(H_n)$ and $\inf \sigma(H_\Omega)$ are smaller or equal than $\inf \sigma(H)$, one can define for any $f \in \mathcal{C}_H(\mathbb{R})$ the operators

$$f(H_n), f(H_\Omega) \in B\left[L^2(\mathbb{R}^d)\right].$$

The second one is defined as follows: $f(H_\Omega)|_{L^2(\Omega)}$ is the operator from $B\left[L^2(\Omega)\right]$ associated to H_Ω (seen as a self-adjoint operator in $L^2(\Omega)$) by the usual functional calculus, while $f(H_\Omega) = 0$ on $L^2(\Omega)^\perp$. Then we have $\lim_{n\to\infty} f(H_n) = f(H_\Omega)$ for the strong convergence in $B\left[L^2(\mathbb{R}^d)\right]$. We have

$$f(H_\Omega) = \mathbf{1}_\Omega f(H_\Omega) = f(H_\Omega)\mathbf{1}_\Omega. \tag{6.4}$$

In particular, the properties above are checked for the function

$$f(t) = (t + \lambda)^{-1}, \quad \lambda > -\inf \sigma(H),$$

defined on a neighborhood of $\sigma(H)$. Then

$$f(H) = (H + \lambda)^{-1}, \quad f(H_\Omega) = (H_\Omega + \lambda)^{-1}.$$

Lemma 6.1. *If we assume (i) and (ii), for any $\Omega \subset \mathbb{R}^d$ open bounded set, the operator H_Ω has compact resolvent.*

Proof. It will be enough to show that any $M \subset D(H_\Omega)$, bounded for the graph norm defined by H_Ω, is relatively compact in $L^2(\Omega)$. The set M will be bounded in $D(h_\Omega)$, thus also bounded in $L^2(\Omega)$ and $D(h)$. Hence the set $M_A := (H_A + 1)^{1/2}M$ is bounded in $L^2(\mathbb{R}^d)$ and $M = (H_A + 1)^{-1/2}M_A$.

Let

$$\chi \in C_0^\infty(\mathbb{R}^d), \quad 0 \le \chi \le 1, \quad \chi = 1 \text{ in a neighborhood of } \overline{\Omega}.$$

By (3.11) one has

$$|\chi(H_A + 1)^{-1/2}f| \le \chi(H_0 + 1)^{-1/2}|f|, \quad \forall f \in L^2(\mathbb{R}^d). \tag{6.5}$$

Since

$$R(H_0 + 1)^{-1/2} = \mathcal{H}^{1/2}(\mathbb{R}^d),$$

the operator $\chi(H_0 + 1)^{-1/2}$ is compact on $L^2(\mathbb{R}^d)$. By Pitt's Theorem [30] and by (6.5), the operator $\chi(H_A + 1)^{-1/2}$ is also compact on $L^2(\mathbb{R}^d)$. Since $\chi M = M$, the set M is relatively compact in $L^2(\Omega)$. $\quad\square$

Proposition 6.2. *We assume that (i) and (ii') are verified. For any $\lambda \ge \lambda_0$, $r \ge r_0$ (λ_0 and r_0 as in Proposition 5.1), there is a constant $C > 0$ such that for any open subsets U, Ω of $\mathbb{R}^d$ such that $U \cap \Omega$ is bounded we have $\mathbf{1}_U(H_\Omega + \lambda)^{-r} \in \mathcal{I}_2$ and the next inequality holds:*

$$\|\mathbf{1}_U(H_\Omega + \lambda)^{-r}\|_{\mathcal{I}_2} \le C|U \cap \Omega|^{1/2}. \tag{6.6}$$

Proof. By using the inequality (5.2) for H_n and the fact that

$$s - \lim_{n\to\infty}(H_n + \lambda)^{-r} = (H_\Omega + \lambda)^{-r},$$

one obtains that

$$|(H_\Omega + \lambda)^{-r}f| \le \frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-\lambda t}e^{-tH(0;-V_-)}|f|dt, \quad \forall f \in L^2(\mathbb{R}^d).$$

The proof is completed in the same way as for Proposition 5.1, since

$$\mathbf{1}_U(H_\Omega + \lambda)^{-r} = \mathbf{1}_U\mathbf{1}_\Omega(H_\Omega + \lambda)^{-r} = \mathbf{1}_{U\cap\Omega}(H_\Omega + \lambda)^{-r}.$$

$\square$

Corollary 6.3. *Under the assumptions of Proposition 6.2, for any $m \geq 2r_0$, $\exists C > 0$ such that for any $\Omega \subset \mathbb{R}^d$ bounded and open one has $(H_\Omega + \lambda)^{-m} \in \mathcal{I}_1$ and*

$$\| (H_\Omega + \lambda)^{-m} \|_{\mathcal{I}_1} \leq C|\Omega|. \tag{6.7}$$

Proof. We use the identity

$$(H_\Omega + \lambda)^{-m} = \mathbf{1}_\Omega (H_\Omega + \lambda)^{-r} (H_\Omega + \lambda)^{-s} \mathbf{1}_\Omega,$$

where

$$r \geq r_0, \quad s \geq r_0, \quad r + s = m,$$

as well as Proposition 6.2 with $U = \Omega$. $\qquad\square$

Corollary 6.4. *Let $f \in C_0(\mathbb{R})$. Under the assumptions of Proposition 6.2, there exists a constant $C > 0$ such that for any $\Omega \subset \mathbb{R}^d$ open and bounded, we have $f(H_\Omega) \in \mathcal{I}_1$ and*

$$\| f(H_\Omega) \|_{\mathcal{I}_1} \leq C|\Omega|. \tag{6.8}$$

Proof. For any $g \in \mathcal{C}_H(\mathbb{R})$, since $s - \lim_{n\to\infty} g(H_n) = g(H_\Omega)$, one obtains for each $\Omega \subset \mathbb{R}^d$ open set, the inequality

$$\| g(H_\Omega) \|_{B[L^2(\mathbb{R}^d)]} \leq \sup_{\mathbb{R}} |g|. \tag{6.9}$$

We choose

$$g(t) := (t + \lambda)^m f(t), \quad \text{where} \quad m \geq 2r_0, \ \lambda \geq \lambda_0, \ -\lambda \notin \mathrm{supp} f.$$

Then

$$g \in \mathcal{C}_H(\mathbb{R}), \quad -\lambda \notin \mathrm{supp} g \quad \text{and} \quad f(t) = (t + \lambda)^{-m} g(t), \ \forall t \in \mathbb{R}.$$

It follows that

$$f(H_\Omega) = (H_\Omega + \lambda)^{-m} g(H_\Omega),$$

so (6.8) is a consequence of (6.7) and (6.9). $\qquad\square$

7 Proof of Theorem 1.1

Lemma 7.1. *We assume (i) and (ii'). Let $\lambda > -\inf \sigma(H)$, $\Omega \subset \mathbb{R}^d$ an open bounded set and $\varphi \in BC^\infty(\mathbb{R}^d)$, $\varphi = 1$ on Ω^c. Then one has*

$$(H + \lambda)^{-1} - (H_\Omega + \lambda)^{-1} = \left[(H + \lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] \left[\varphi + [H, \varphi](H_\Omega + \lambda)^{-1} \right] = \tag{7.1}$$

$$= \left[\varphi - (H + \lambda)^{-1}[H, \varphi] \right] \left[(H + \lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right].$$

Proof. The function φ verifies the assumptions of Proposition 4.5, so the operator of multiplication by φ leaves $D(H)$ invariant and $[H, \varphi] \in B[L^2(\mathbb{R}^d)]$. Using (4.14) for H_n and the equality

$$[H_n, \varphi] = [H, \varphi], \quad \forall n \geq 1,$$

where $H_n := H + n\mathbf{1}_{\Omega^c}$, we deduce that

$$(H + \lambda)^{-1} - (H_n + \lambda)^{-1} = (H + \lambda)^{-1} n\mathbf{1}_{\Omega^c} \cdot \varphi(H_n + \lambda)^{-1} =$$

$$= (H + \lambda)^{-1} n\mathbf{1}_{\Omega^c}(H_n + \lambda)^{-1}\varphi + (H + \lambda)^{-1} n\mathbf{1}_{\Omega^c}(H_n + \lambda)^{-1}[H, \varphi](H_n + \lambda)^{-1} =$$

$$= \left[(H + \lambda)^{-1} - (H_n + \lambda)^{-1} \right] \left[\varphi + [H, \varphi](H_n + \lambda)^{-1} \right].$$

The first equality in (7.1) follows from the formula above in the limit $n \to \infty$, taking into account the relation

$$s - \lim_{n\to\infty} (H_n + \lambda)^{-1} = (H_\Omega + \lambda)^{-1}$$

and the fact that the sequence $\{(H_n + \lambda)^{-1}\}_{n \geq 1}$ is bounded in $B[L^2(\mathbb{R}^d)]$.

The second equality in (7.1) follows in the same way. $\qquad\square$

The next Proposition is basic for proving Theorem 1.1

Proposition 7.2. *We assume again (i) and (ii'). For any $\lambda \geq \lambda_0$ and $m \in \mathbb{N}$, $m \geq 4r_0$ (λ_0 and r_0 as in Proposition 5.1), there exists $C > 0$ such that for any bounded open subset Ω of $\mathbb{R}^d$*

$$\|\mathbf{1}_\Omega (H + \lambda)^{-m} \mathbf{1}_\Omega - (H_\Omega + \lambda)^{-m}\|_{\mathcal{I}_1} \leq C|\Omega|^{1/2}|\widetilde{\Omega}|^{1/2}, \tag{7.2}$$

where $\widetilde{\Omega} := \{x \in \mathbb{R}^d \mid \operatorname{dist}(x, \partial\Omega) < 1\}$.

Proof. We use (6.4) and infer that

$$\mathbf{1}_\Omega (H+\lambda)^{-m} \mathbf{1}_\Omega - (H_\Omega + \lambda)^{-m} = \sum_{0 \leq j \leq m-1} \mathbf{1}_\Omega (H+\lambda)^{j-m+1} \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] (H_\Omega + \lambda)^{-j} \mathbf{1}_\Omega. \tag{7.3}$$

We denote by E_j the general term of the sum. Let $\Omega \subset \mathbb{R}^d$ bounded and open. By taking the convolution of the characteristic function of a neighborhood of $\overline{\Omega^c}$ by a function from $C_0^\infty(\mathbb{R}^d)$ with the support included in a small neighborhood of the origin, one constructs a real function $\varphi \in BC^\infty(\mathbb{R}^d)$ such that $0 \leq \varphi \leq 1$, $\varphi = 1$ on Ω^c, $\varphi = 0$ on $\Omega \setminus \widetilde{\Omega}$ and such that $\|\partial^\alpha \varphi\|_{L^2(\mathbb{R}^d)} \leq C_\alpha$, $\forall \alpha \in \mathbb{N}^d$, with C_α independent of Ω.

We estimate first the $\mathcal{I}_1$-norm of E_j for $2r_0 \leq j \leq m-1$. We use the first equality form (7.1) and write $E_j = E_j' + E_j''$, where E_j' and E_j'' correspond to the two terms of the sum $\varphi + [H, \varphi]$. We have

$$\|E_j'\|_{\mathcal{I}_1} = \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] \varphi (H_\Omega + \lambda)^{-j} \mathbf{1}_\Omega\|_{\mathcal{I}_1} \leq$$

$$\leq \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right]\|_{B[L^2(\Omega)]} \|\mathbf{1}_U (H+\lambda)^{-j/2}\|_{\mathcal{I}_2} \|(H_\Omega + \lambda)^{-j/2} \mathbf{1}_\Omega\|_{\mathcal{I}_2},$$

where $U := \Omega^c \cup \widetilde{\Omega}$. Using (6.9) and Proposition 6.2 we get

$$\|E_j'\|_{\mathcal{I}_1} \leq C_1 |\widetilde{\Omega}|^{1/2} |\Omega|^{1/2} \tag{7.4}$$

for some positive constant C_1, independent of Ω. To estimate the $\mathcal{I}_1$ norm of E_j'', we write it as

$$E_j'' = \sum_{1 \leq |\alpha| \leq d+1} E_{j,\alpha}'' + E_{j,0}'',$$

where the terms $E_{j,\alpha}''$ and $E_{j,0}''$ correspond to the decomposition of $[H, \varphi]$ in the second of the inequalities (4.13). Using Propositions 4.5 and 6.2 we obtain inequalities, in which the constants are independent of Ω:

$$\|E_{j,\alpha}''\|_{\mathcal{I}_1} = \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] S_\alpha'(\partial^\alpha \varphi)(H_\Omega + \lambda)^{-j-1} \mathbf{1}_\Omega\|_{\mathcal{I}_1} \leq$$

$$\leq \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] S_\alpha'(\partial^\alpha \varphi)\|_{B[L^2(\mathbb{R}^d)]} \cdot$$

$$\cdot \|\mathbf{1}_{\widetilde{\Omega}} (H_\Omega + \lambda)^{-j/2}\|_{\mathcal{I}_2} \|(H_\Omega + \lambda)^{-j/2-1} \mathbf{1}_\Omega\|_{\mathcal{I}_2} \leq$$

$$\leq C' |\widetilde{\Omega}|^{1/2} |\Omega|^{1/2}, \qquad 1 \leq |\alpha| \leq d+1$$

and

$$\|E_{j,0}''\|_{\mathcal{I}_1} = \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] T'(H_\Omega + \lambda)^{-j-1} \mathbf{1}_\Omega\|_{\mathcal{I}_1} \leq$$

$$\leq \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right]\|_{B[L^2(\mathbb{R}^d)]} \|T'\|_{\mathcal{I}_2} \|(H_\Omega + \lambda)^{-j-1} \mathbf{1}_\Omega\|_{\mathcal{I}_2} \leq$$

$$\leq C'' |\widetilde{\Omega}|^{1/2} |\Omega|^{1/2}.$$

Thus we have

$$\|E_j''\|_{\mathcal{I}_1} \leq C_2 |\widetilde{\Omega}|^{1/2} |\Omega|^{1/2}. \tag{7.5}$$

Taking (7.4) into account we get

$$\|E_j\|_{\mathcal{I}_1} \leq C|\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}, \quad 2r_0 \leq j \leq m-1, \tag{7.6}$$

for some constant $C > 0$ independent of Ω.

Let us assume now that $0 \le j \le 2r_0 - 1$; then $m - j - 1 \ge 2r_0$. We use now the second equality in (7.1) to write $E_j = \widetilde{E}'_j + \widetilde{E}''_j$ where, as before, $\widetilde{E}'_j$ and $\widetilde{E}''_j$ correspond to the two terms in the sum $\varphi + [H, \varphi]$. The $\mathcal{I}_1$-norm of $\widetilde{E}''_j$ is estimated as above, writing

$$\widetilde{E}''_j = \sum_{1 \le |\alpha| \le d+1} \widetilde{E}''_{j,\alpha} + \widetilde{E}''_{j,0},$$

where the terms $\widetilde{E}''_{j,\alpha}$ and $\widetilde{E}''_{j,0}$ correspond to the decomposition of $[H, \varphi]$ in the first equality in (4.13). By (6.9) and Propositions 4.5 and 5.1, we get for Ω- independent constants

$$\|\widetilde{E}''_{j,\alpha}\|_{\mathcal{I}_1} = \|\mathbf{1}_\Omega (H+\lambda)^{j-m} (\partial^\alpha \varphi) S_\alpha \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] (H_\Omega + \lambda)^{-j} \mathbf{1}_\Omega\|_{\mathcal{I}_1} \le$$

$$B[L^2(\mathbb{R}^d)] \le \|\mathbf{1}_\Omega (H+\lambda)^{\frac{j-m}{2}}\|_{\mathcal{I}_2} \|(H+\lambda)^{\frac{j-m}{2}} \mathbf{1}_{\widetilde{\Omega}}\|_{\mathcal{I}_2} \times$$

$$\times \|(\partial^\alpha \varphi) S_\alpha \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] (H_\Omega + \lambda)^{-j} \mathbf{1}_\Omega\|_{B[L^2(\mathbb{R}^d)]} \le$$

$$\le C' |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}, \quad 1 \le |\alpha| \le d+1$$

and

$$\|\widetilde{E}''_{j,0}\|_{\mathcal{I}_1} = \|\mathbf{1}_\Omega (H+\lambda)^{j-m} T \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] (H_\Omega + \lambda)^{-j} \mathbf{1}_\Omega\|_{\mathcal{I}_1} \le$$

$$\le \|\mathbf{1}_\Omega (H+\lambda)^{j-m}\|_{\mathcal{I}_2} \|T\|_{\mathcal{I}_2} \| \left[(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1} \right] (H_\Omega + \lambda)^{-j} \mathbf{1}_\Omega\|_{B[L^2(\mathbb{R}^d)]} \le$$

$$\le C'' |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}.$$

So we have

$$\|\widetilde{E}''_j\|_{\mathcal{I}_1} \le C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}. \tag{7.7}$$

To estimate the $\mathcal{I}_1$-norm of $\widetilde{E}'_j$, we introduce another auxiliary real function

$$\psi \in C_0^\infty(\mathbb{R}^d), \ 0 \le \psi \le 1, \ \operatorname{supp} \psi \subset \Omega \cup \widetilde{\Omega}, \ \psi = 1 \text{ in a neighborhood of } \overline{\Omega},$$

such that for any $\alpha \in \mathbb{N}^d$ one has $\|\partial^\alpha \psi\|_{L^\infty(\mathbb{R}^d)} \le C_\alpha$, with C_α independent of Ω. We have

$$\|\widetilde{E}'_j\|_{\mathcal{I}_1} = \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \varphi [(H+\lambda)^{-1} - (H_\Omega + \lambda)^{-1}] (H_\Omega + \lambda)^{-j} \mathbf{1}_\Omega\|_{\mathcal{I}_1} \le$$

$$\le C \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \varphi (H+\lambda)^{-1} \psi\|_{\mathcal{I}_1} + C \|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \varphi \psi\|_{\mathcal{I}_1},$$

where we used the fact that $\psi \mathbf{1}_\Omega = \mathbf{1}_\Omega$ and we denoted by C various constants independent of Ω.

Since $\operatorname{supp}(\varphi \psi) \subset \widetilde{\Omega}$, using Proposition 5.1 as above, we get

$$\|\mathbf{1}_\Omega (H+\lambda)^{j-m+1} \varphi \psi\|_{\mathcal{I}_1} \le C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}. \tag{7.8}$$

For the last term that has to be estimated we use Proposition 4.5 and write

$$\varphi (H+\lambda)^{-1} = (H+\lambda)^{-1} \varphi - \sum_{1 \le |\alpha| \le d+1} (H+\lambda)^{-1} (\partial^\alpha \varphi) S_\alpha (H+\lambda)^{-1} - (H+\lambda)^{-1} T (H+\lambda)^{-1}.$$

One gets immediately the inequalities

$$\|\mathbf{1}_\Omega (H+\lambda)^{j-m} \varphi \psi\|_{\mathcal{I}_1} \le C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2},$$

$$\|\mathbf{1}_\Omega (H+\lambda)^{j-m} (\partial^\alpha \varphi) S_\alpha (H+\lambda)^{-1} \psi\|_{\mathcal{I}_1} \le C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}$$

and

$$\|\mathbf{1}_\Omega (H+\lambda)^{j-m} T (H+\lambda)^{-1} \psi\|_{\mathcal{I}_1} \le C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2},$$

which gives

$$\|\mathbf{1}_\Omega (H + \lambda)^{j-m+1} \varphi (H + \lambda)^{-1} \psi\|_{\mathcal{I}_1} \leq C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}. \tag{7.9}$$

From (7.8) and (7.9) we obtain

$$\|\widetilde{E}_j'\|_{\mathcal{I}_1} \leq C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2} \tag{7.10}$$

which, together with (7.7), implies the inequality

$$\|E_j\|_{\mathcal{I}_1} \leq C |\Omega|^{1/2} |\widetilde{\Omega}|^{1/2}, \quad 0 \leq j \leq 2r_0 + 1. \tag{7.11}$$

The relation (7.2) follows from (7.3), (7.6) and (7.11). $\qquad \square$

Theorem 1.1 is a direct consequence of the next Proposition:

Proposition 7.3. *Now we assume that the hypothesis (i), (ii'), (iii) and (iv) are fulfilled. Then for any* $f \in C_0(\mathbb{R})$ *and* $\epsilon > 0$, *there exists* $m_0 \in \mathbb{N}^*$ *such that*

$$|\mathrm{tr}[\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega] - \mathrm{tr} f(H_\Omega)| \leq \epsilon |\Omega| \tag{7.12}$$

for any $\Omega \in \mathcal{F}$ *with* $B(0, m_0) \subset \Omega$.

Proof. One uses ideas of [10] (see also [20]). Let λ_0 and r_0 the constants from Proposition 5.1. We set $a := \lambda_0 + 1$, $m_0 := 4r_0$. It will be enough to prove (7.12) for the real functions $f \in C_0(\mathbb{R})$ such that $\mathrm{supp}\, f \subset [-a + \frac{1}{2}, \infty)$. The functions

$$\left[-a + \frac{1}{2}, \infty\right) \ni t \mapsto (a + t)^{m_0} f(t) \in \mathbb{R}$$

and

$$[0, 2] \ni \tau \mapsto \tau^{-m_0} f(\tau^{-1} - a) \in \mathbb{R}$$

are continuous. For any $\epsilon > 0$ there is a polynomial P_ϵ with real coefficients such that

$$\left| \tau^{-m_0} f(\tau^{-1} - a) - P_\epsilon(\tau) \right| \leq \epsilon, \quad \forall \tau \in [0, 2].$$

Therefore

$$\left| (a + t)^{m_0} f(t) - P_\epsilon \left(\frac{1}{a + t} \right) \right| \leq \epsilon, \quad \forall t \geq -a + \frac{1}{2}.$$

Let

$$Q_\epsilon(t) := (a + t)^{-m_0} P_\epsilon \left(\frac{1}{a + t} \right).$$

Then in form-sense

$$-\epsilon (a + H)^{-m_0} \leq f(H) - Q_\epsilon(H) \leq \epsilon (a + H)^{-m_0},$$

so

$$-\epsilon \mathbf{1}_\Omega (a + H)^{-m_0} \mathbf{1}_\Omega \leq \mathbf{1}_\Omega f(H)\mathbf{1}_\Omega - \mathbf{1}_\Omega Q_\epsilon(H)\mathbf{1}_\Omega \leq \epsilon \mathbf{1}_\Omega (a + H)^{-m_0} \mathbf{1}_\Omega, \quad \Omega \in \mathcal{F}.$$

Using Corollaries 5.2 and 5.3 we obtain

$$|\mathrm{tr}[\mathbf{1}_\Omega f(H)\mathbf{1}_\Omega] - \mathrm{tr}[\mathbf{1}_\Omega Q_\epsilon(H)\mathbf{1}_\Omega]| \leq \epsilon \,\mathrm{tr}[\mathbf{1}_\Omega (a + H)^{-m_0} \mathbf{1}_\Omega] \leq C_1 \,\epsilon \,|\Omega|, \tag{7.13}$$

where C_1 is a constant independent on ϵ and $\Omega \in \mathcal{F}$.

In the same way, using Corollaries 6.3 and 6.4, one shows that for some constant C_2, independent on ϵ and $\Omega \in \mathcal{F}$, one has

$$|\mathrm{tr} f(\Omega) - \mathrm{tr} Q_\epsilon(H_\Omega)| \leq \epsilon \,\mathrm{tr}(a + H_\Omega)^{-m_0} \leq C_2 \,\epsilon \,|\Omega|. \tag{7.14}$$

Inequality (7.12) follows from (7.13), (7.14), (7.2) and hypothesis (iv).

$\qquad \square$

8 Proof of Theorem 1.2

Let us suppose that (i), (ii'), (iii), (iv) and (v) are verified. Since the proof of Theorem 1.2 is very close to that of Theorem 1.6 from [20], we shall not indicate all the details.

Let us notice first that the hypothesis (i) and (v), the proofs of Proposition 5.1 and Corollary 5.2 from [20] show that there exists a constant magnetic field

$$B^0 = \frac{1}{2} \sum_{1 \le j,k \le d} B_{jk}^0 dx_j \wedge dx_k, \quad B_{jk}^0 = -B_{kj}^0 \in \mathbb{R}$$

and a vector potential

$$A^{(p)} = \sum_{1 \le j \le d} A_j^{(p)} dx_j,$$

where the components $A_j^{(p)}$ belong to $C_{\text{pol}}^\infty(\mathbb{R}^d)$ and are Γ-periodic, such that $B - B^0 = dA^{(p)}$. Since $B^0 = dA^0$ with

$$A^0 = \sum_{1 \le j \le d} A_j^0 dx_j, \quad A_j^0(x) = \frac{1}{2} \sum_{1 \le k \le d} B_{kj}^0 x_k,$$

we have $B = d(A^{(p)} + A^0)$, so, by Proposition 4.6, we will assume in the sequel that the vector potential defining the magnetic field B is $A := A^{(p)} + A^0$.

For $\gamma \in \Gamma$ we define the function $\varphi_\gamma : \mathbb{R}^d \to \mathbb{R}$,

$$\varphi_\gamma(x) := \sum_{1 \le j \le d} A_j^0(\gamma) x_j = \frac{1}{2} \sum_{1 \le j,k \le d} B_{kj}^0 \gamma_k x_j$$

(so $d\varphi_\gamma = A^0(\gamma)$) and the unitary operators of multiplication with $e^{i\varphi_\gamma}$ on $L^2(\mathbb{R}^d)$ denoted by U_γ. Let us put $(L_\gamma u)(x) := u(x - \gamma)$ and $T_\gamma := U_\gamma L_\gamma$. The operators T_γ are the magnetic translations [37].

Lemma 8.1. *Let us suppose that (i), (ii') and (v) are verified. Then the operator $H = H(A;V)$ constructed in Section 4 commutes with T_γ, i.e.*

$$HT_\gamma = T_\gamma H, \quad \forall \gamma \in \Gamma. \tag{8.1}$$

Proof. By Proposition 4.6, we have $H(A;V)U_\gamma = U_\gamma H(A - d\varphi_\gamma; V)$. So we only need to show that

$$L_\gamma H(A;V) = H(A - d\varphi_\gamma; V)L_\gamma, \quad \forall \gamma \in \Gamma. \tag{8.2}$$

Since

$$\Gamma^A(x - \gamma, y - \gamma) = \Gamma^A(x,y) - A^0(\gamma) = \Gamma^{A - d\varphi_\gamma}(x,y), \ x,y \in \mathbb{R}^d,$$

it follows that for any $a \in S^m(\mathbb{R}^d)$, Γ-periodic in x, and for any $w \in \mathcal{S}(\mathbb{R}^d)$, we have

$$L_\gamma \mathfrak{Op}^A(a)w = \mathfrak{Op}^{A - d\varphi_\gamma}(a)(L_\gamma w). \tag{8.3}$$

Let $u \in D(H(A;V))$, so $u \in D(h(A;V))$ and $\mathfrak{Op}^A(p)u + Vu \in L^2(\mathbb{R}^d)$ with $p(\xi) := < \xi > -1$. From (8.3) we have

$$\left[\mathfrak{Op}^{A - d\varphi_\gamma}(p)\right](L_\gamma u) + V(L_\gamma u) = L_\gamma \left[\mathfrak{Op}^A(p)u + Vu\right] \in L^2(\mathbb{R}^d). \tag{8.4}$$

Let us show that $L_\gamma u \in D(h(A - d\varphi_\gamma; V))$. Obviously $L_\gamma u \in D(q_{V_+})$. From (2.2) it follows that

$$w \in D(H_A) \text{ if and only if } L_\gamma w \in D(H_A) = D(H_{A - d\varphi_\gamma}).$$

From (8.3) we deduce that $L_\gamma H_A L_\gamma^{-1} = H_{A - d\varphi_\gamma}$, so $L_\gamma H_A^{1/2} L_\gamma^{-1} = H_{A - d\varphi_\gamma}^{1/2}$. It follows that $L_\gamma [D(h_A)] = D(h_{A - d\varphi_\gamma})$, so $L_\gamma u \in D(h_{A - d\varphi_\gamma})$ and then

$$L_\gamma u \in D[h(A - d\varphi_\gamma; V)].$$

From Corollary 4.4 and equality (8.4) we get $L_\gamma u \in D[H(A - d\varphi_\gamma; V)]$ as well as (8.2). $\qquad\square$

The family $\{T_\gamma\}_{\gamma \in \Gamma}$ satisfies

$$T_\alpha T_\beta = e^{-i\varphi_\beta(\alpha)} T_{\alpha+\beta}, \quad \alpha, \beta \in \Gamma,$$

so it doesn't form a group. However, using [1] as a model (cf. also [20]), one can define a Γ-trace for a class of operators on $B(L^2(\mathbb{R}^d))$ commuting with the magnetic translations T_γ.

Definition 8.2. *An operator $S \in B(L^2(\mathbb{R}^d))$ belongs to $\mathcal{I}_1^\Gamma$ if $T_\gamma S = S T_\gamma$, $\forall \gamma \in \Gamma$ and if for every $\varphi, \psi \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ one has $\varphi S \psi \in \mathcal{I}_1$.*

One can show that for all $\varphi, \varphi', \psi, \psi' \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ such that

$$\sum_{\gamma \in \Gamma} L_\gamma(\varphi\psi) = \sum_{\gamma \in \Gamma} L_\gamma(\varphi'\psi') = 1, \quad \forall S \in \mathcal{I}_1,$$

we have the equality $\text{tr}(\varphi S \psi) = \text{tr}(\varphi' S \psi')$. This justifies

Definition 8.3. *Let $S \in \mathcal{I}_1^\Gamma$. We call Γ-trace of S the expression*

$$\text{tr}_\Gamma S := \text{tr}(\varphi S \psi),$$

where $\varphi, \psi \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ and $\sum\limits_{\gamma \in \Gamma} L_\gamma(\varphi\psi) = 1$.

One can prove (see [20])

Lemma 8.4. *Let $S = S^* \in \mathcal{I}_1^\Gamma$. Then K_S, the integral kernel of S, is a locally integrable function on $\mathbb{R}^d \times \mathbb{R}^d$, its restriction to the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$ is well-defined and locally integrable and one has*

$$\text{tr}_\Gamma S = \int_F K_S(x,x) dx, \tag{8.5}$$

where F is a fundamental domain of $\mathbb{R}^d$ with respect to Γ.

From Corollary 5.3 and Lemma 8.1 it follows that for any $f \in C_0(\mathbb{R})$ one has $f(H) \in \mathcal{I}_1^\Gamma$. From Lemma 8.4 we know that the restriction to the diagonal of integral kernel $K_{f(H)}$ exists as a locally integrable function. Then for any $\Omega \in \mathcal{F}$ one has

$$\text{tr}(\mathbf{1}_\Omega f(H) \mathbf{1}_\Omega) = \int_\Omega K_{f(H)}(x,x) dx. \tag{8.6}$$

By the proof of Theorem 1.6 in [20] we get

$$\lim_{\Omega \to \mathbb{R}^d, \Omega \in \mathcal{F}} \frac{1}{|\Omega|} \int_\Omega K_{f(H)}(x,x) dx = \frac{1}{|F|} \int_F K_{f(H)}(x,x) dx. \tag{8.7}$$

Then (1.9) follows from (8.6), (8.7) and (8.5).

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