Phase Transitions for the Growth Rate of Linear Stochastic Evolutions[1]

Nobuo YOSHIDA[2]

Abstract

We consider a simple discrete-time Markov chain with values in $[0,\infty)^{\mathbb{Z}^d}$. The Markov chain describes various interesting examples such as oriented percolation, directed polymers in random environment, time discretizations of binary contact path process and the voter model. We study the phase transition for the growth rate of the "total number of particles" in this framework. The main results are roughly as follows: If $d \geq 3$ and the Markov chain is "not too random", then, with positive probability, the growth rate of the total number of particles is of the same order as its expectation. If on the other hand, $d = 1, 2$, or the Markov chain is "random enough", then the growth rate is slower than its expectation. We also discuss the above phase transition for the dual processes and its connection to the structure of invariant measures for the Markov chain with proper normalization.

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Contents

1 Introduction

In this paper, we discuss phase transitions for the large time behavior of a Markov chain with values in $[0,\infty)^{\mathbb{Z}^d}$, where $\mathbb{Z}^d$ is the d dimensional integer lattice. We consider the Markov chain $N_t = (N_{t,y})_{y\in\mathbb{Z}^d}$, $t = 1, 2, ...$ obtained by:

$$N_t = N_0 A_1 A_2 \cdots A_t, \quad t = 1, 2, ...,$$

where $N_0 \in [0,\infty)^{\mathbb{Z}^d}$ is the initial state (regarded as a row vector), and $A_t = (A_{t,x,y})_{x,y\in\mathbb{Z}^d}$, $t = 1, 2, ...$ are i.i.d. random matrices (cf. (1.1)–(1.7) below for more details). This framework includes various interesting examples such as (generalized) oriented percolation, directed polymers in random environment, time discretizations of binary contact path process and

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[2] Division of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan. email: nobuo@math.kyoto-u.ac.jp URL: http://www.math.kyoto-u.ac.jp/~nobuo/. Supported in part by JSPS Grant-in-Aid for Scientific Research, Kiban (C) 17540112

the voter model. We interpret $N_{t,y} \geq 0$ as the "number of particles" at time-space (t, y), though we do not assume in general that it is an integer. In the oriented percolation, $N_{t,y}$ is the number of open oriented paths from $(0, x)$ for some $x \in \mathbb{Z}^d$ to (t, y), and in the directed polymers in random environment, $N_{t,y}$ is the partition function for the polymer chain $S_0, S_1, .., S_{t-1}, S_t$ with $S_t = y$.

We look at the growth rate of the "total number" of particles:

$$|N_t| = \sum_{y \in \mathbb{Z}^d} N_{t,y} \quad t = 1, 2, ...$$

which will be kept finite for all t by our assumptions. We first show that $|N_t|$ has the expected value $|N_0||a|^t$, where $|a|$ is a positive number (cf. (1.8) and Lemma 1.2.1), so that $|a|^t$ can be considered as the mean growth rate of $|N_t|$. The main purpose of this paper is to investigate whether the limit:

$$|\overline{N}_\infty| \stackrel{\text{def}}{=} \lim_{t \to \infty} |N_t|/|a|^t$$

vanishes almost surely or not. Our results can be summarized as follows:

i) If $d \geq 3$ and the matrix A_t is not "too random", then, $|\overline{N}_\infty| > 0$ with positive probability (Lemma 2.1.1).

ii) In any dimension d, if the matrix A_t is "random enough", then, $|\overline{N}_\infty| = 0$, almost surely (Theorem 3.1.1). Moreover, the convergence is exponentially fast.

iii) For $d = 1, 2$, $|\overline{N}_\infty| = 0$, almost surely, under mild assumptions on A_t (Theorem 3.2.1. The assumptions are so mild that, for many examples, they merely amount to saying that A_t is "random at all". Moreover, the convergence is exponentially fast for $d = 1$.

We will refer i) as *regular growth phase*, and ii)—iii) as *slow growth phase*. In the regular growth phase, $|N_t|$ grows as fast as its mean growth rate with positive probability, whereas in the slow growth phase, the growth of $|N_t|$ is slower than its mean growth rate almost surely. We remark that the exponential decay of $|N_t|/|a|^t$, mentioned in ii)–iii) above are interpreted as the positivity of the Lyapunov exponents.

The phenomena i)–iii) mentioned above have been widely observed for various models; for continuous-time linear interacting particle systems [8, Chapter IX], for directed polymers in random environment [1, 2, 3], and for branching random walks in random environment [7, 12]. Here, we capture phenomena i)–iii) above by a simple discrete-time Markov chain, which however includes various, old and new examples. Here, "old examples" means that some of our results are known for them, such as directed polymers in random environment, whereas "new examples" means that our results are new for them, such as (generalized) oriented percolation. For oriented percolation, it is traditional to discuss the presence/absence of the open oriented paths to certain time-space location. On the other hand, our results show that the model exhibits a new type of phase transition, if we look at not only the presence/absence of the open oriented paths, but also their number.

In section 4, we discuss the phase transition i)–iii) for the dual processes and its connection to the structure of invariant measures for the Markov chain $(|N_t|/|a|^t)$.

1.1 The linear stochastic evolution

Here are remarks on the usage of notation in this paper. We write $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^* = \{1, 2, ...\}$ and $\mathbb{Z} = \{\pm x \; ; \; x \in \mathbb{N}\}$. For $x = (x_1, .., x_d) \in \mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x| = \sum_{i=1}^d |x_i|$. For $\xi = (\xi_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, $|\xi| = \sum_{x \in \mathbb{Z}^d} |\xi_x|$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We write $P[X] = \int X \, dP$ and $P[X : A] = \int_A X \, dP$ for a r.v.(random variable) X and an event A.

Let $A_t = (A_{t,x,y})_{x,y\in\mathbb{Z}^d}$, $t \in \mathbb{N}^*$ be a sequence of i.i.d. random matrices with non-negative entries. We denote by $(\Omega, \mathcal{F}, P)$ the probability space on which these random matrices are defined. Here are the set of assumptions we assume throughout this article:

$$\text{For fixed } t \in \mathbb{N}^*, \text{ column vectors } (A_{t,x,y})_{x\in\mathbb{Z}^d},\ y \in \mathbb{Z}^d \text{ are independent,} \tag{1.1}$$
$$P[A_{1,x,y}^2] < \infty \quad \text{for all } x, y \in \mathbb{Z}^d, \tag{1.2}$$
$$A_{t,x,y} = 0 \ \text{ a.s. if } |x-y| > r_A \text{ for some non-random } r_A \in \mathbb{N}, \tag{1.3}$$
$$A_{1,x,y} \text{ is not a constant a.s. for some } x, y \in \mathbb{Z}^d, \tag{1.4}$$
$$(A_t \circ \theta_{s,z})_{t\in\mathbb{N}^*} \stackrel{\text{law}}{=} (A_t)_{t\in\mathbb{N}^*} \quad \text{for all } (s,z) \in \mathbb{N} \times \mathbb{Z}^d, \tag{1.5}$$

where $A_t \circ \theta_{s,z} = (A_{t+s,x+z,y+z})_{x,y\in\mathbb{Z}^d}$ for $(s,z) \in \mathbb{N} \times \mathbb{Z}^d$. Depending on the results we prove in the sequel, some of these conditions can be relaxed. However, we choose not to bother ourselves with the pursuit of the minimum assumptions for each result.

We define a Markov chain $(N_t)_{t\in\mathbb{N}}$ with values in $[0,\infty)^{\mathbb{Z}^d}$ by

$$\sum_{x\in\mathbb{Z}^d} N_{t-1,x} A_{t,x,y} = N_{t,y}, \quad t \in \mathbb{N}^*. \tag{1.6}$$

Here and in the sequel (with only exception in Theorem 4.1.3 below), we suppose that the initial state N_0 is non-random and *finite* in the sense that

$$\text{the set } \{x \in \mathbb{Z}^d \ ; \ N_{0,x} > 0\} \text{ is finite and non-empty.} \tag{1.7}$$

The Markov chain defined above can be thought of as the time discretization of the particle system considered in the last Chapter in T. Liggett's book [8, Chapter IX]. Thanks to the time discretization, the definition is considerably simpler here. Though we *do not* assume in general that $(N_t)_{t\in\mathbb{N}}$ takes values in $\mathbb{N}^{\mathbb{Z}^d}$, we refer $N_{t,y}$ as the "number of particles" at time-space (t,y), and $|N_t|$ as "total number of particles" at time t.

We write:

$$a_y = P[A_{t,0,y}], \quad |a| = \sum_{y\in\mathbb{Z}^d} |a_y|. \tag{1.8}$$

We now see that various interesting examples are included in this simple framework.

• **Oriented percolation (OP):** Let $\eta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d.r.v.'s with $P(\eta_{t,y} = 1) = p \in (0,1)$ and $N_0 = (N_{0,x})_{x\in\mathbb{Z}^d} \in \{0,1\}^{\mathbb{Z}^d}$. An *open oriented path* with the end-point $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N}^* \times \mathbb{Z}^d$ such that $N_{0,x_0} = 1$, $x_t = y$, $|x_s - x_{s-1}| = 1$, $\eta_{s,x_s} = 1$ for all $s = 1,..,t$. Then, the number of open oriented paths with the end-point (t,y) is given by (1.6) with

$$A_{t,x,y} = \mathbf{1}_{|x-y|=1}\eta_{t,y}.$$

• **Generalized oriented percolation (GOP):** We generalize OP as follows. Let $\eta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d.r.v.'s with $P(\eta_{t,y} = 1) = p \in (0,1]$ and let $\zeta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0,1\}$-valued i.i.d.r.v.'s with $P(\zeta_{t,y} = 1) = q \in [0,1]$, which are independent of $\eta_{t,y}$'s. We refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.6) with

$$A_{t,x,y} = \mathbf{1}_{|x-y|=1}\eta_{t,y} + \delta_{x,y}\zeta_{t,y}$$

as the *generalized oriented percolation* (GOP). Thus, the OP is the special case ($q = 0$) of GOP. The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{|y|=1} + q\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},\widetilde{y}}] = \begin{cases} q & \text{if } y = \widetilde{y} = x = \widetilde{x}, \\ p & \text{if } y = \widetilde{y},\ |x-y| = |\widetilde{x}-y| = 1, \\ a_{y-x}a_{\widetilde{y}-\widetilde{x}} & \text{if otherwise.} \end{cases} \tag{1.9}$$

In particular, we have $|a| = 2dp + q$.

• **Directed polymers in random environment (DPRE):** Let $\{\eta_{t,y}\ ;\ (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$ be i.i.d. with $\exp(\lambda(\beta)) \stackrel{\text{def.}}{=} P[\exp(\beta\eta_{t,y})] < \infty$ for any $\beta \in (0,\infty)$. The following expectation is called the partition function of the *directed polymers in random environment*:

$$N_{t,y} = P_S^0\left[\exp\left(\beta\sum_{u=1}^t \eta_{u,S_u}\right) : S_t = y\right], \quad (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d,$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ is the simple random walk on $\mathbb{Z}^d$. We refer the reader to a review paper [3] and the references therein for more information. Starting from $N_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d}$, the above expectation can be obtained inductively by (1.6) with

$$A_{t,x,y} = \frac{\mathbf{1}_{|x-y|=1}}{2d}\exp(\beta\eta_{t,y}).$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{e^{\lambda(\beta)}\mathbf{1}_{|y|=1}}{2d}, \quad P[A_{t,x,y}A_{t,\widetilde{x},\widetilde{y}}] = \begin{cases} a_{y-x}a_{\widetilde{y}-\widetilde{x}} & \text{if } y \neq \widetilde{y}, \\ e^{\lambda(2\beta)-2\lambda(\beta)}a_{y-x}a_{y-\widetilde{x}} & \text{if } y = \widetilde{y}. \end{cases} \tag{1.10}$$

In particular, we have $|a| = e^{\lambda(\beta)}$.

• **The binary contact path process (BCPP):** The binary contact path process is a continuous-time Markov process with values in $\mathbb{N}^{\mathbb{Z}^d}$, originally introduced by D. Griffeath [6]. In this article, we consider a discrete-time variant as follows. Let

$$\{\eta_{t,y} = 0,1\ ;\ (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}, \quad \{\zeta_{t,y} = 0,1\ ;\ (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\},$$
$$\{e_{t,y}\ ;\ (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$$

be families of i.i.d.r.v.'s with $P(\eta_{t,y} = 1) = p \in (0,1]$, $P(\zeta_{t,y} = 1) = q \in [0,1]$, and $P(e_{t,y} = e) = \frac{1}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We suppose that these three families are independent of each other. Starting from an $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$, we define a Markov chain $(N_t)_{t\in\mathbb{N}}$ with values in $\mathbb{N}^{\mathbb{Z}^d}$ by

$$N_{t+1,y} = \eta_{t+1,y}N_{t,y-e_{t+1,y}} + \zeta_{t+1,y}N_{t,y}, \quad t \in \mathbb{N}.$$

We interpret the process as the spread of an infection, with $N_{t,y}$ infected individuals at time t at the site y. The $\zeta_{t+1,y}N_{t,y}$ term above means that these individuals remain infected at time $t+1$ with probability q, and they recover with probability $1-q$. On the other hand, the $\eta_{t+1,y}N_{t,y-e_{t+1,y}}$ term means that, with probability p, a neighboring site $y - e_{t+1,y}$ is picked at random (say, the wind blows from that direction), and $N_{t,y-e_{t+1,y}}$ individuals at site y are infected anew at time $t+1$. This Markov chain is obtained by (1.6) with

$$A_{t,x,y} = \eta_{t,y}\mathbf{1}_{e_{t,y}=y-x} + \zeta_{t,y}\delta_{x,y}.$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$
\begin{aligned}
a_y &= \frac{p\mathbf{1}_{|y|=1}}{2d} + q\delta_{0,y}, \qquad (1.11)\\
P[A_{t,x,y}A_{t,\widetilde{x},\widetilde{y}}] &= \begin{cases} a_{y-x}a_{\widetilde{y}-\widetilde{x}} & \text{if } y \neq \widetilde{y},\\ a_{y-x} & \text{if } x = \widetilde{x} \text{ and } y = \widetilde{y},\\ q\delta_{x,y}a_{y-\widetilde{x}} + q\delta_{\widetilde{x},y}a_{y-x} & \text{if } x \neq \widetilde{x} \text{ and } y = \widetilde{y}.\end{cases} \qquad (1.12)
\end{aligned}
$$

In particular, we have $|a| = p + q$.

• **Voter model (VM):** Let $e_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\mathbb{Z}^d$-valued i.i.d.r.v.'s with $P(e_{t,y} = 0) = 1 - p$ $(p \in (0, 1])$ and $P(e_{t,y} = e) = \frac{p}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We then refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.6) with

$$A_{t,x,y} = \delta_{x,y+e_{t,y}}$$

as the *voter model* (VM). Let us suppose that $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$ for simplicity. This process describes the behavior of voters in a certain election. At time 0, a voter at $y \in \mathbb{Z}^d$ supports the candidate $N_{0,y}$. Then, at time $t = 1$, the voter makes a decision in a random way. With probability $1 - p$, the voter still supports the same candidate, and with probability $p/(2d)$, he/she finds the candidate supported by his/her neighbor at $y + e_{1,y}$ $(|e_{1,y}| = 1)$ more attractive, and starts to support $N_{0,y+e_{1,y}}$, instead of $N_{0,y}$. The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\frac{\mathbf{1}_{|y|=1}}{2d} + (1-p)\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},\widetilde{y}}] = \begin{cases} a_{y-x}a_{\widetilde{y}-\widetilde{x}} & \text{if } y \neq \widetilde{y},\\ \delta_{x,\widetilde{x}}a_{y-x} & \text{if } y = \widetilde{y}.\end{cases} \qquad (1.13)$$

In particular, we have $|a| = 1$.

Remark: The branching random walk in random environment considered in [7, 12] can also be considered as a "close relative" to the models considered here, although it does not exactly fall into our framework.

1.2 Some basic properties

In this subsection, we lay basis to study the growth of $|N_t|$ as $t \nearrow \infty$. We denote by $\mathcal{F}_t$, $t \in \mathbb{N}^*$ the σ-field generated by $A_1, ..., A_t$.

First of all, we identify the mean growth rate of $|N_t|$ with $|a|^t$.

Lemma 1.2.1

$$P[N_{t,y}] = |a|^t \sum_{x\in\mathbb{Z}^d} N_{0,x}P_S^x(S_t = y),$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ *is the random walk on* $\mathbb{Z}^d$ *such that*

$$P_S^x(S_0 = x) = 1 \text{ and } P_S^x(S_1 = y) = \overline{a}_{y-x} \stackrel{\text{def.}}{=} a_{y-x}/|a|.$$

Moreover, $(|\overline{N}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ *is a martingale, where we have defined* $\overline{N}_t = \left(\overline{N}_{t,x}\right)_{x\in\mathbb{Z}^d}$ *by*

$$\overline{N}_{t,x} = |a|^{-t}N_{t,x}. \qquad (1.14)$$

Proof: The first equality is obtained by averaging the identity:

$$N_{t,y} = \sum_{x_0,,..,x_{t-1}} N_{0,x_0} A_{1,x_0,x_1} A_{2,x_1,x_2} \cdots A_{t,x_{t-1},y}. \tag{1.15}$$

It is also easy to see from the above identity that $(|\overline{N}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ is a martingale. $\square$

We next compare $|N_t|$ and its mean growth rate $|a|^t$.

Lemma 1.2.2 *Referring to Lemma 1.2.1, the limit*

$$|\overline{N}_\infty| = \lim_{t\to\infty} |\overline{N}_t| \tag{1.16}$$

exists a.s. and

$$P[|\overline{N}_\infty|] = |N_0| \ \ or \ \ 0. \tag{1.17}$$

Moreover, $P[|\overline{N}_\infty|] = |N_0|$ if and only if the limit (1.16) is convergent in $\mathbb{L}^1(P)$.

Before we prove Lemma 1.2.2, we introduce some notation and definitions. For $(s, z) \in \mathbb{N}\times\mathbb{Z}^d$, we define $N_t^{s,z} = (N_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$ and $\overline{N}_t^{s,z} = (\overline{N}_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$, $t \in \mathbb{N}$ respectively by

$$\begin{aligned} &N_{0,y}^{s,z} = \delta_{z,y}, \quad N_{t+1,y}^{s,z} = \sum_{x\in\mathbb{Z}^d} N_{t,x}^{s,z} A_{s+t+1,x,y}, \\ &\text{and} \quad \overline{N}_{t,y}^{s,z} = |a|^{-t} N_{t,y}^{s,z}. \end{aligned} \tag{1.18}$$

In particular, $(N_t^{0,z})_{t\in\mathbb{N}}$ is the Markov chain (1.6) with the initial state $N_0^{0,z} = (\delta_{z,y})_{y\in\mathbb{Z}^d}$. Moreover, we have

$$N_{t,y} = \sum_{z\in\mathbb{Z}^d} N_{0,z} N_{t,y}^{0,z} \quad \text{for any initial state } N_0. \tag{1.19}$$

Now, it follows from Lemma 1.2.2 that

$$P[|\overline{N}_\infty^{0,0}|] = 1, \text{ or } = 0.$$

We will refer to the former case as *regular growth phase* and the latter as *slow growth phase.* By (1.19) and the shift invariance, $P[|\overline{N}_\infty|] = |N_0|$ for all N_0 in the regular growth phase and $P[|\overline{N}_\infty|] = 0$ for all N_0 in the slow growth phase. The regular growth means that, at least with positive probability, the growth of the "total number" $|N_t|$ of the particles is of the same order as its expectation $|a|^t|N_0|$. On the other hand, the slow growth means that, almost surely, the growth of $|N_t|$ is slower than its expectation.

Proof of Lemma 1.2.2: By multiplying N_t by $|N_0|^{-1}$, we may assume that $|N_0| = 1$. The limit (1.16) exists by the martingale convergence theorem, and $\ell \stackrel{\text{def.}}{=} P[|\overline{N}_\infty|] \le 1$ by Fatou's lemma. To show (1.17), we will prove that $\ell = \ell^2$, using the argument in [8, page 433, Theorem 2.4(a)]. Using the notation (1.18), we write

(1) $|\overline{N}_{s+t}| = \sum_y \overline{N}_{s,y} |\overline{N}_t^{s,y}|.$

Since $|\overline{N}_t^{s,y}| \stackrel{\text{law}}{=} |\overline{N}_t|$, the limit

$$|\overline{N}_\infty^{s,y}| = \lim_{t\to\infty} |\overline{N}_t^{s,y}|$$

exists a.s. and is equally distributed as $|\overline{N}_\infty|$. Moreover, by letting $t \nearrow \infty$ in (1), we have that

$$|\overline{N}_\infty| = \sum_y \overline{N}_{s,y}|\overline{N}^{s,y}_\infty|.$$

and hence by Jensen's inequality that

$$P[\exp(-|\overline{N}_\infty|)|\mathcal{F}_s] \geq \exp\left(-P[|\overline{N}_\infty||\mathcal{F}_s]\right) = \exp\left(-|\overline{N}_s|\ell\right) \geq \exp\left(-|\overline{N}_s|\right).$$

By letting $s \nearrow \infty$ in the above inequality, we obtain

$$\exp(-|\overline{N}_\infty|) \stackrel{\text{a.s.}}{\geq} \exp\left(-|\overline{N}_\infty|\ell\right) \geq \exp\left(-|\overline{N}_\infty|\right),$$

and thus, $|\overline{N}_\infty| \stackrel{a.s.}{=} |\overline{N}_\infty|\ell$. By taking expectation, we get $\ell = \ell^2$. Once we know (1.17), the final statement of the lemma is standard([5, page 257–258, (5.2)], for example). □

Let us now take a brief look at the condition for the extinction: $\lim_{t\to\infty} |N_t| = 0$ a.s., although our main objective in this article is to study $|\overline{N}_\infty| = \lim_{t\to\infty} |\overline{N}_t|$.

If $|a| < 1$, we have

$$\lim_{t\to\infty} |N_t| = \lim_{t\to\infty} |a|^t |\overline{N}_t| = 0.$$

For $|a| = 1$, we will present an argument below (Lemma 1.2.3), which applies when $(N_t)_{t\in\mathbb{N}}$ is $\mathbb{N}^{\mathbb{Z}^d}$-valued. Consequently, we will see that $\lim_{t\to\infty} |N_t| = 0$ for GOP with $(1-p)(1-q) \neq 0$ and for VM with $p \in (0, 1]$. For GOP, we apply Lemma 1.2.3 directly. For VM, we slightly modify the argument (See the remark after the lemma).

It follows from the observations above that

$$\lim_{t\to\infty} |N_t| = 0, \text{ a.s. if} \begin{cases} 2dp + q \leq 1 \text{ and } (1-p)(1-q) \neq 0 & \text{for GOP,} \\ \lambda(\beta) < 0 & \text{for DPRE,} \\ p + q \leq 1 \text{ and } (1-p)(1-q) \neq 0 & \text{for BCPP,} \\ p \in (0, 1] & \text{for VM.} \end{cases} \tag{1.20}$$

Lemma 1.2.3 *Let $\mathcal{O}_t$ be the set of occupied sites at time t,*

$$\mathcal{O}_t = \{x \in \mathbb{Z}^d \ ; \ N_{t,x} > 0\}$$

and $|\mathcal{O}_t|$ be its cardinality. Suppose that

$$\delta \stackrel{\text{def.}}{=} P\left(\bigcap_{x\in\mathbb{Z}^d} \{A_{1,x,0} = 0\}\right) > 0. \tag{1.21}$$

Then,

$$P(\lim_{t\to\infty} |\mathcal{O}_t| \in \{0, \infty\}) = 1.$$

Proof: We will see that

(1) $\{|\mathcal{O}_t| \leq m \ \text{ i.o.}\} \stackrel{\text{a.s.}}{=} \{|\mathcal{O}_t| = 0 \ \text{ i.o.}\}$ for any $m \in \mathbb{N}$,

which immediately implies the lemma:

$$\{|\mathcal{O}_t| \not\longrightarrow \infty\} = \bigcup_{m\in\mathbb{N}} \{|\mathcal{O}_t| \leq m \ \text{ i.o.}\} \stackrel{\text{a.s.}}{=} \{|\mathcal{O}_t| = 0 \ \text{ i.o.}\}.$$

For (1), we have only to show the $\stackrel{\text{a.s.}}{\subset}$ part. We write $\widetilde{\mathcal{O}}_{t-1} = \bigcup_{x \in \mathcal{O}_{t-1}} \{y \in \mathbb{Z}^d \; ; \; |x-y| \le r_A\}$ (cf. (1.3)). Since

$$|\mathcal{O}_t| = 0 \iff |N_t| = \sum_{x,y \in \mathbb{Z}^d} N_{t-1,x} A_{t,x,y} = 0,$$

we have

$$\begin{aligned} P(|\mathcal{O}_t| = 0 | \mathcal{F}_{t-1}) &= P\left(\bigcap_{y \in \widetilde{\mathcal{O}}_{t-1}} \{ \sum_{x \in \mathbb{Z}^d} N_{t-1,x} A_{t,x,y} = 0 \} \middle| \mathcal{F}_{t-1} \right) \\ &\ge P\left(\bigcap_{y \in \widetilde{\mathcal{O}}_{t-1}} \bigcap_{x \in \mathbb{Z}^d} \{A_{1,x,y} = 0\} \middle| \mathcal{F}_{t-1} \right) \\ &= \prod_{y \in \widetilde{\mathcal{O}}_{t-1}} P\left(\bigcap_{x \in \mathbb{Z}^d} \{A_{1,x,y} = 0\} \right) = \delta^{|\widetilde{\mathcal{O}}_{t-1}|}. \end{aligned}$$

This, together with the generalized second Borel-Cantelli lemma ([5, page 237]) implies that

$$\{|\mathcal{O}_t| \le m \text{ i.o.}\} \subset \left\{ \sum_{t=1}^{\infty} P(|\mathcal{O}_t| = 0 | \mathcal{F}_{t-1}) = \infty \right\} \stackrel{\text{a.s.}}{=} \{|\mathcal{O}_t| = 0 \text{ i.o.}\}.$$

□

Remark: For VM, we argue as follows. Since $|a| = 1$, $|N_t|$ is a martingale and hence converges a.s. Since $|N_t|$ is $\mathbb{N}$-valued, we have $|N_{t-1}| = |N_t|$ for large t, a.s. On the other hand, for some $c = c(p,d) > 0$, we have

$$\{1 \le |\mathcal{O}_{t-1}| \le m\} \subset \{P\left(|N_{t-1}| > |N_t| | \mathcal{F}_{t-1}\right) \ge c^m\} \quad \text{for all } m \in \mathbb{N}^*.$$

(Replace $N_{t-1,y}$ on all y on the interior boundaries of $\mathcal{O}_{t-1}$ with 0, while keeping all the other $N_{t-1,y}$ unchanged.) This implies that $\lim_{t \to \infty} |N_t| = 0$, via a similar argument as in Lemma 1.2.3.

2 Regular growth phase

2.1 Regular growth via second moments

The purpose of this subsection is to give a sufficient condition for the regular growth phase. This can be done by expressing the two-point function

$$P[N_{t,y} N_{t,\widetilde{y}}]$$

in terms of a Feynman-Kac type expectation with respect to the independent product of the random walks in Lemma 1.2.1. It is convenient to introduce the following notation:

$$w(x,\widetilde{x},y,\widetilde{y}) = \begin{cases} \dfrac{P[A_{1,x,y} A_{1,\widetilde{x},\widetilde{y}}]}{a_{y-x} a_{\widetilde{y}-\widetilde{x}}} = \left(\dfrac{P[A_{1,x-y,0} A_{1,\widetilde{x}-y,0}]}{a_{y-x} a_{y-\widetilde{x}}} \right)^{\delta_{y,\widetilde{y}}}, & \text{if } a_{y-x} a_{\widetilde{y}-\widetilde{x}} \ne 0, \\ 0, & \text{if } a_{y-x} a_{\widetilde{y}-\widetilde{x}} = 0. \end{cases} \tag{2.1}$$

Remark: For OP and DPRE, we see from (1.9) and (1.10) that

$$P[A_{t,x,y} A_{t,\widetilde{x},\widetilde{y}}] = \gamma^{\delta_{y,\widetilde{y}}} a_{y-x} a_{\widetilde{y}-\widetilde{x}}, \tag{2.2}$$

where

$$\gamma = 1/p \text{ and } \exp(\lambda(2\beta) - 2\lambda(\beta))$$

respectively for OP and DPRE. If (2.2) is satisfied, then,

$$w(x, \widetilde{x}, y, \widetilde{y}) = \begin{cases} \gamma^{\delta_{y,\widetilde{y}}} & \text{if } a_{y-x} a_{\widetilde{y}-\widetilde{x}} \neq 0, \\ 0, & \text{if } a_{y-x} a_{\widetilde{y}-\widetilde{x}} = 0. \end{cases} \tag{2.3}$$

For GOP, we have

$$w(x, \widetilde{x}, y, \widetilde{y}) = \begin{cases} 1/q & \text{if } y = \widetilde{y} = x = \widetilde{x}, \\ 1/p & \text{if } y = \widetilde{y},\ |x - y| = |\widetilde{x} - y| = 1, \\ 1 & \text{if neither of the above and } a_{y-x} a_{\widetilde{y}-\widetilde{x}} \neq 0, \\ 0 & \text{if } a_{y-x} a_{\widetilde{y}-\widetilde{x}} = 0, \end{cases} \tag{2.4}$$

where $1/q$ on the first line is replaced by 0 if $q = 0$. For BCPP, we have

$$w(x, \widetilde{x}, y, \widetilde{y}) = \begin{cases} 1/q & \text{if } x = \widetilde{x} = y = \widetilde{y}, \\ 1/p & \text{if } x = \widetilde{x},\ y = \widetilde{y},\ |x - y| = 1, \\ 1 & \text{if } y = \widetilde{y} = x, |\widetilde{x} - y| = 1, \\ 1 & \text{if } y = \widetilde{y} = \widetilde{x}, |x - y| = 1, \\ 1 & \text{if } y \neq \widetilde{y} \text{ and } a_{y-x} a_{\widetilde{y}-\widetilde{x}} \neq 0, \\ 0 & \text{if otherwise,} \end{cases} \tag{2.5}$$

where $1/q$ on the first line is replaced by 0 if $q = 0$.

We let $(S, \widetilde{S}) = ((S_t, \widetilde{S}_t)_{t \in \mathbb{N}}, P^{x,\widetilde{x}}_{S,\widetilde{S}})$ denote the independent product of the random walks in Lemma 1.2.1. We have the following Feynman-Kac formula.

Lemma 2.1.1

$$P[N_{t,y} N_{t,\widetilde{y}}] = |a|^{2t} \sum_{x, \widetilde{x} \in \mathbb{Z}^d} N_{0,x} N_{0,\widetilde{x}} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_t : (S_t, \widetilde{S}_t) = (y, \widetilde{y})] \tag{2.6}$$

for all $t \in \mathbb{N}$, $y, \widetilde{y} \in \mathbb{Z}^d$, *where*

$$e_t = \prod_{u=1}^{t} w(S_{u-1}, \widetilde{S}_{u-1}, S_u, \widetilde{S}_u). \tag{2.7}$$

Consequently,

$$P[|\overline{N}_t|^2] = \sum_{x, \widetilde{x} \in \mathbb{Z}^d} N_{0,x} N_{0,\widetilde{x}} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_t], \tag{2.8}$$

and

$$\sup_{t \in \mathbb{N}} P[|\overline{N}_t|^2] < \infty \iff \sup_{t \in \mathbb{N}} P^{0,0}_{S,\widetilde{S}}[e_t] < \infty \tag{2.9}$$

$$\implies P[|\overline{N}_\infty|] = |N_0|. \tag{2.10}$$

Proof: By (1.15) and the independence, we have

(1) $$P[N_{t,y} N_{t,\widetilde{y}}] = \sum_{x_0,,..,x_{t-1}} \sum_{\widetilde{x}_0,,..,\widetilde{x}_{t-1}} N_{0,x_0} N_{0,\widetilde{x}_0} \prod_{s=1}^{t} P[A_{1,x_{s-1},x_s} A_{1,\widetilde{x}_{s-1},\widetilde{x}_s}],$$

with the convention that $x_t = y$, $\widetilde{x}_t = \widetilde{y}$. We have on the other hand that

$$P[A_{1,x_{s-1},x_s} A_{1,\widetilde{x}_{s-1},\widetilde{x}_s}] = |a|^2 w(x_{s-1}, \widetilde{x}_{s-1}, x_s, \widetilde{x}_s) \overline{a}_{x_s - x_{s-1}} \overline{a}_{\widetilde{x}_s - \widetilde{x}_{s-1}}.$$

Plugging this into (1), we get (2.6). (2.8) is an immediate consequence of (2.6). We now recall (1.19) and that $|N_t^{0,z}| \stackrel{\text{law}}{=} |N_t^{0,0}|$ for all $t \in \mathbb{N}$ and $z \in \mathbb{Z}^d$. Therefore, it is enough to prove (2.9) for $N_t = N_t^{0,0}$. But this follows immediately from (2.8). (2.10) is a consequence of Lemma 1.2.2. □

Remarks: 1) When (2.3) holds, we have

$$e_t = \gamma^{\sum_{u=1}^t \delta_{S_u, \widetilde{S}_u}}. \tag{2.11}$$

Let us assume (2.11) and set

$$\pi_x = P^{x,0}_{S,\widetilde{S}}(S_t = \widetilde{S}_t \text{ for some } t \in \mathbb{N}^*).$$

We then have for all $x \in \mathbb{Z}^d$ that

$$P^{x,0}_{S,\widetilde{S}}\left(\sum_{u=1}^{\infty} \delta_{S_u,\widetilde{S}_u} = k\right) = \begin{cases} 1 - \pi_x & \text{for } k = 0, \\ \pi_x \pi_0^{k-1}(1-\pi_0) & \text{for } k = 1, 2, .., \end{cases}$$

and hence that

$$\sup_{t \in \mathbb{N}} P^{0,0}_{S,\widetilde{S}}[e_t] < \infty \iff \pi_0 \gamma < 1 \tag{2.12}$$

$$\implies \lim_{t\to\infty} P^{x,0}_{S,\widetilde{S}}[e_t] = \begin{cases} 1 + \pi_x \dfrac{\pi_0(\gamma-1)}{1-\pi_0\gamma} & \text{if } x \neq 0, \\ 1 + \dfrac{\pi_0(\gamma-1)}{1-\pi_0\gamma} & \text{if } x = 0. \end{cases} \tag{2.13}$$

On the other hand, it can be seen from (2.6) that

$$P[|\overline{N}^{0,x}_\infty||\overline{N}^{0,\widetilde{x}}_\infty|] = \lim_{t\to\infty} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_t],$$

using the notation (1.18). Thus, (2.13) provides us with the formula for covariances of $(|\overline{N}^x_\infty|)_{x\in\mathbb{Z}^d}$. This observation was made by D. Griffeath for the binary contact path process in continuous time [6, 9] and by F. Comets for DPRE (private communications). Also, it follows from (2.9) and (2.12) that

$$\sup_{t\in\mathbb{N}} P[|\overline{N}_t|^2] < \infty \iff d \geq 3 \text{ and } \begin{cases} p > \pi_0 & \text{for OP}, \\ \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_0) & \text{for DPRE}. \end{cases} \tag{2.14}$$

Similar arguments using (2.4) and (2.5) show that

$$\sup_{t\in\mathbb{N}} P[|\overline{N}_t|^2] < \infty \impliedby d \geq 3 \text{ and } \begin{cases} p \wedge q > \pi_0 & \text{for GOP with } q \neq 0, \\ p > \pi_0 & \text{for BCPP with } q = 0, \\ p \wedge q > \pi_0 & \text{for BCPP with } q \neq 0. \end{cases} \tag{2.15}$$

For OP, DPRE and BCPP with $q = 0$, $(S_t)_{t\in\mathbb{N}}$ is the simple random walks. In this case, π_0 is the same as the return probability of the simple random walk itself, for which we have $1/(2d) < \pi_0 \leq 0.3405...$ for $d \geq 3$ [11, page 103]. (2.14) for DPRE case can be found in [10].

2) By the second moment method discussed here, it is possible to get the central limit theorem for the spacial distribution of the particles. In fact, (2.9) implies that

$$\lim_{t\to\infty} \sum_{x\in\mathbb{Z}^d} f\left((x-mt)/\sqrt{t}\right) \overline{N}_{t,x} = |\overline{N}_\infty| \int_{\mathbb{R}^d} f d\nu \;\text{ in } \mathbb{L}^2(P) \text{ for all } f \in C_{\rm b}(\mathbb{R}^d),$$

where $m = \sum_{x\in\mathbb{Z}^d} x\overline{a}_x$ and ν is the Gaussian measure with

$$\int_{\mathbb{R}^d} x_i d\nu(x) = 0, \quad \int_{\mathbb{R}^d} x_i x_j d\nu(x) = \sum_{x\in\mathbb{Z}^d} (x_i - m_i)(x_j - m_j)\overline{a}_x, \quad i,j = 1,..,d.$$

This will be carried out in a work in preparation by M.Nakashima, together with the investigations of related topics.

3 Slow growth phase

3.1 Slow growth in any dimension

We give the following sufficient condition for the slow growth phase in any dimension. The condition is typically applies to the limited regions of parameters, which makes particles "hard to survive" (Remark 1 after Theorem 3.1.1).

Theorem 3.1.1 *Suppose that*

$$\sum_{y\in\mathbb{Z}^d} P\left[A_{1,0,y} \ln A_{1,0,y}\right] > |a| \ln |a|. \tag{3.1}$$

Then, there exists a non-random $c > 0$ *such that*

$$|\overline{N}_t| = O(e^{-ct}), \quad \textit{as } t \to \infty, \textit{ a.s.}$$

Remarks: 1) It is easy to see that

$$(3.1) \iff \begin{cases} 2dp + q < 1 & \text{for GOP,} \\ \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d) & \text{for DPRE,} \\ p + q < 1 & \text{for BCPP.} \end{cases}$$

2) Theorem 3.1.1 generalizes [2, Theorem 1.3(a)], which is obtained in the setting of DPRE. Theorem 3.1.1 can also be thought of as the discrete-time analogue of [8, page 455, Theorem 5.1].

Proof of Theorem 3.1.1: By (1.19) and the shift invariance, it is enough to prove the result for $N_t = N_t^{0,0}$. We write

$$|N_t| = \sum_y A_{1,0,y} |N_{t-1}^{2,y}|.$$

Thus, for $h \in (0,1]$,

$$|N_t|^h \le \sum_y A_{1,0,y}^h |N_{t-1}^{2,y}|^h.$$

Since $|N_{t-1}^{2,y}| \stackrel{\rm law}{=} |N_{t-1}|$, we have

$$P[|N_t|^h] \le \sum_y P[A_{1,0,y}^h] P[|N_{t-1}|^h],$$

and hence

$$P[|\overline{N}_t|^h] \le \varphi(h) P[|\overline{N}_{t-1}|^h], \quad \text{with } \varphi(h) = \sum_y P\left[\left(\tfrac{A_{1,0,y}}{|a|}\right)^h\right].$$

Note that $\varphi(1) = 1$ and that

$$\varphi'(1-) = \sum_{y \in \mathbb{Z}^d} P\left[\frac{A_{1,0,y}}{|a|} \ln\left(\frac{A_{1,0,y}}{|a|}\right)\right] > 0.$$

(For the differentiability, note that $x^h |\ln x| \le (he)^{-1}$ for $x \in [0,1]$, and $x^h |\ln x| \le x \ln x$ for $x \ge 1$.) These imply that there exists $h_0 \in (0,1)$ such that $\varphi(h_0) < 1$, and hence that $P[|\overline{N}_t|^{h_0}] \le \varphi(h_0)^t$, $t \in \mathbb{N}$. Finally the theorem follows from the Borel-Cantelli lemma. □

3.2 Slow growth in dimensions one and two

We now state a result (Theorem 3.2.1) for slow growth phase in dimensions one and two. Unlike Theorem 3.1.1, Theorem 3.2.1 is typically applies to the entire region of the parameters in various models (cf. Remarks after Theorem 3.2.1).

For $f, g \in [0,\infty)^{\mathbb{Z}^d}$ with $|f|, |g| < \infty$, we define their convolution $f * g \in [0,\infty)^{\mathbb{Z}^d}$ by

$$(f * g)_x = \sum_{y \in \mathbb{Z}^d} f_{x-y} g_y.$$

The identity : $|(f * g)| = |f||g|$ will often be used in the sequel.

Theorem 3.2.1 *Let $d = 1, 2$. Suppose that $P[A_{1,0,y}^3] < \infty$ for all $y \in \mathbb{Z}^d$ and that there is a constant $c \in (0,\infty)$ such that*

$$\sum_{x,\widetilde{x},y \in \mathbb{Z}^d} \xi_x \xi_{\widetilde{x}} P[A_{1,x,y} A_{1,\widetilde{x},y}] \ge (1+c)|(a * \xi)^2| \tag{3.2}$$

for all $\xi \in [0,\infty)^{\mathbb{Z}^d}$ such that $|\xi| < \infty$. Then, almost surely,

$$|\overline{N}_t| = \begin{cases} O(\exp(-ct)) & \text{if } d = 1, \\ \longrightarrow 0 & \text{if } d = 2 \end{cases} \quad \text{as } t \longrightarrow \infty, \tag{3.3}$$

where c is a non-random constant.

Remarks: 1) By (2.2), (3.2) holds for DPRE for all $\beta \in (0,\infty)$. For (3.2), it is sufficient that there exists $c > 0$ such that

$$b_x^A \ge b_x + c\delta_{0,x} \quad \text{for all } x \in \mathbb{Z}^d, \tag{3.4}$$

where b and b^A are defined by

$$b_x = \sum_{y \in \mathbb{Z}^d} a_y a_{y-x} \quad \text{and} \quad b_x^A = \sum_{y \in \mathbb{Z}^d} P[A_{1,0,y} A_{1,x,y}] \tag{3.5}$$

For GOP, we have by (1.9) that

$$b_x \begin{cases} = 2dp^2 + q^2, & \text{if } x = 0, \\ = 2pq & \text{if } |x| = 1, \\ > 0 & \text{if } |x| = 2, \end{cases} \qquad b_x^A = \begin{cases} 2dp + q, & \text{if } x = 0, \\ 2pq & \text{if } |x| = 1, \\ p^{-1} b_x & \text{if } |x| = 2, \end{cases} \qquad b_x = b_x^A = 0 \text{ if } |x| \ge 3.$$

Thus, (3.2) holds for GOP whenever p or q is in $(0,1)$.

2) For $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ with $|\xi| < \infty$, we denote its Fourier transform by $\widehat{\xi}(\theta) = \sum_{x \in \mathbb{Z}^d} \xi_x \exp(\mathbf{i} x \cdot \theta)$, $\theta \in [-\pi,\pi]^d$. Then, (3.2) follows from that

$$c_1 \stackrel{\text{def.}}{=} \min_{\theta \in [-\pi,\pi]^d} \left(\widehat{b^A}(\theta) - |\widehat{a}(\theta)|^2 \right) > 0. \tag{3.6}$$

This can be seen as follows. Note that (3.2) can be written as:

$$\sum_{x,\widetilde{x} \in \mathbb{Z}^d} \xi_x \xi_{\widetilde{x}} b^A_{x-\widetilde{x}} \geq (1+c)|(a*\xi)^2|.$$

Then, by Plancherel's identity and the fact that $|(a*\xi)^2| \leq |a|^2|\xi^2|$, we have that

$$\begin{aligned}\sum_{x,\widetilde{x} \in \mathbb{Z}^d} \xi_x \xi_{\widetilde{x}} b^A_{x-\widetilde{x}} &= (2\pi)^{-d} \int_{[-\pi,\pi]^d} \widehat{b^A}(\theta)|\widehat{\xi}(\theta)|^2 d\theta \geq (2\pi)^{-d} \int_{[-\pi,\pi]^d} (|\widehat{a}(\theta)|^2 + c_1)|\widehat{\xi}(\theta)|^2 d\theta \\ &= |(a*\xi)^2| + c_1|\xi^2| \geq (1 + c_1/|a|^2)|(a*\xi)^2|.\end{aligned}$$

The criterion (3.6) can effectively be used to check (3.2) for BCPP if $d \geq 2$ or $1-p+q(1-q) \neq 0$. In fact, we have by (1.11) and (1.12) that

$$b_x \begin{cases} = \frac{p^2}{2d} + q^2, & \text{if } x = 0, \\ = \frac{pq}{d} & \text{if } |x| = 1, \\ > 0 & \text{if } |x| = 2, \\ = 0 & \text{if } |x| \geq 3 \end{cases} \qquad b^A_x = \begin{cases} p+q, & \text{if } x = 0, \\ \frac{pq}{d} & \text{if } |x| = 1, \\ 0 & \text{if } |x| \geq 2, \end{cases}$$

Hence, (3.4) fails in this case. On the other hand,

$$\widehat{a}(\theta) = \frac{p}{d}\sum_{j=1}^d \cos\theta_j + q, \quad \widehat{b^A}(\theta) = p + q + \frac{2pq}{d}\sum_{j=1}^d \cos\theta_j.$$

Thus, (3.6) can be verified as follows:

$$\widehat{b^A}(\theta) - |\widehat{a}(\theta)|^2 = p + q - q^2 - \left(\frac{p}{d}\sum_{j=1}^d \cos\theta_j\right)^2 \geq p\left(1 - \frac{p}{d^2}\right) + q(1-q) > 0.$$

3) Suppose that $\sup_{x,y \in \mathbb{Z}^d} A_{1,x,y}$ is a bounded r.v. Then, instead of (3.2), it is enough to assume that there is a $\gamma \in [0,\infty)^{\mathbb{Z}^d}$, $\gamma_x \not\equiv 0$ such that

$$\sum_{x,\widetilde{x},y \in \mathbb{Z}^d} \xi_x \xi_y P[A_{1,x,y} A_{1,\widetilde{x},y}] \geq |(a*\xi)^2| + |(\gamma*\xi)^2| \tag{3.7}$$

for all $\xi \in [0,\infty)^{\mathbb{Z}^d}$ such that $|\xi| < \infty$. (In this case we check (3.10) in the proof of Lemma 3.2.3 by the fact that $X_{t,y}$ is bounded. Then, we replace $\overline{a}$ by γ in Lemma 3.2.4). However, we do not know examples to which this observation can effectively be applied.

4) Theorem 3.2.1 is a generalization of [1, Theorem 1.1], [2, Theorem 1.3(b)] and [4, Theorem 1.1], which are obtained in the setting of DPRE. The proof of Theorem 3.2.1 will be built on ideas and techniques developed there. Theorem 3.2.1 can also be thought of as a discrete-time analogue of [8, page 451, Theorem 4.5].

We first prepare a general lemma.

Lemma 3.2.2 *Suppose that $(X_n)_{n\in\mathbb{N}}$ be non-negative independent r.v.'s such that*

$$\sum_{n\geq 0} m_n = 1,\ \ \sum_{n\geq 0} P[X_n^3] < \infty,$$
$$\sum_{n\geq 0} P[(X_n - m_n)^3] \leq c_0 \sum_{n\geq 0} \mathrm{var}(X_n),$$

where $m_n = P[X_n]$ and c_0 is a constant. Then, for $h \in (0,1)$, there is a constant $c_1 \in (0,\infty)$ such that

$$\frac{1}{2+c_0}\sum_{n\geq 0}\mathrm{var}(X_n) \leq P\left[\frac{(U-1)^2}{U+1}\right] \leq c_1 P\left[1-U^h\right] \quad \textit{where}\ U = \sum_{n\geq 0} X_n.$$

Proof: We have

$$\begin{aligned}\sum_{n\geq 0}\mathrm{var}(X_n) &= P[(U-1)^2] = P\left[\frac{U-1}{\sqrt{U+1}}\sqrt{U+1}\right] \\ &\leq P\left[\frac{(U-1)^2}{U+1}\right]^{1/2} P\left[(U-1)^2(U+1)\right]^{1/2}.\end{aligned}$$

On the other hand,

$$P[(U-1)^3] = \sum_{n\geq 0} P[(X_n-m_n)^3] \leq c_0 \sum_{n\geq 0}\mathrm{var}(X_n).$$

Therefore,

$$P\left[(U-1)^2(U+1)\right] = P\left[(U-1)^3 + 2(U-1)^2\right] \leq (c_0+2)\sum_{n\geq 0}\mathrm{var}(X_n).$$

Combining these, we get the first inequality. To get the second, we define a function:

$$f(u) = 1 + h(u-1) - u^h, \quad u \in [0,\infty).$$

Note that $P[U] = 1$ and that there is a constant $c_2 \in (0,\infty)$ such that

$$f(u) \geq c_2 \frac{(u-1)^2}{u+1} \quad \text{for all } u \in [0,\infty).$$

We then see that

$$P\left[1-U^h\right] = P[f(U)] \geq c_2 P\left[\frac{(U-1)^2}{U+1}\right].$$

□

We denote the density of the particles by:

$$\rho_{t,x} = \mathbf{1}_{\{|N_t|>0\}}\frac{N_{t,x}}{|N_t|}, \ \ t\in\mathbb{N}, x\in\mathbb{Z}^d. \tag{3.8}$$

Lemma 3.2.3 *For $h \in (0,1)$, there is a constant $c \in (0,\infty)$ such that*

$$P\left[1-U_t^h|\mathcal{F}_{t-1}\right] \geq c|(a*\rho_{t-1})^2| \quad \textit{for all } t\in\mathbb{N}^*,$$

where $U_t = \frac{1}{|a|}\sum_{x,y\in\mathbb{Z}^d}\rho_{t-1,x}A_{t,x,y}$.

Proof: We may focus on the event $\{|N_{t-1}|>0\}$, since the inequality to prove is trivially true on $\{|N_{t-1}|=0\}$. We write

$$U_t=\sum_{y\in\mathbb{Z}^d}X_{t,y} \text{ with } X_{t,y}=\frac{1}{|a|}\sum_{x\in\mathbb{Z}^d}\rho_{t-1,x}A_{t,x,y}.$$

For fixed $t\in\mathbb{N}^*$, $\{X_{t,y}\}_{y\in\mathbb{Z}^d}$ are non-negative r.v.'s, which are conditionally independent given $\mathcal{F}_{t-1}$. We will prove the lemma by applying Lemma 3.2.2 to these r.v.'s under the conditional probability. The (conditional) expectations and the variances of $\{X_{t,y}\}_{y\in\mathbb{Z}^d}$ are computed as follows:

$$\begin{aligned}
m_{t,y} &\stackrel{\text{def.}}{=} P[X_{t,y}|\mathcal{F}_{t-1}]=(\rho_{t-1}*\overline{a})_y,\\
v_{t,y} &\stackrel{\text{def.}}{=} P[(X_{t,y}-m_{t,y})^2|\mathcal{F}_{t-1}]\\
&= \frac{1}{|a|^2}\sum_{x_1,x_2\in\mathbb{Z}^d}\rho_{t-1,x_1}\rho_{t-1,x_2}\mathrm{cov}(A_{t,x_1,y},A_{t,x_2,y}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{y\in\mathbb{Z}^d}m_{t,y} &= |\rho_{t-1}*\overline{a}|=1,\\
\sum_{y\in\mathbb{Z}^d}v_{t,y} &= \frac{1}{|a|^2}\sum_{x_1,x_2,y\in\mathbb{Z}^d}\rho_{t-1,x_1}\rho_{t-1,x_2}\sum_{y\in\mathbb{Z}^d}\mathrm{cov}(A_{t,x_1,y},A_{t,x_2,y})\\
&\stackrel{(3.2)}{\geq} c_0\sum_{y\in\mathbb{Z}^d}(\rho_{t-1}*\overline{a})_y^2=c_0|(\rho_{t-1}*\overline{a})^2|. \qquad (3.9)
\end{aligned}$$

We will check that there exists $c_1\in(0,\infty)$ such that

$$\sum_{y\in\mathbb{Z}^d}P[(X_{t,y}-m_{t,y})^3|\mathcal{F}_{t-1}]\leq c_1\sum_{y\in\mathbb{Z}^d}v_{t,y} \quad \text{for all } t\in\mathbb{N}^*. \qquad (3.10)$$

Then, the lemma follows from Lemma 3.2.2 and (3.9). There exists $c_2\in(0,\infty)$ such that

(1) $P[A^3_{1,0,y}]\leq c_2a_y^3$ for all $y\in\mathbb{Z}^d$.

This can be seen as follows: Note that $a_y=0\Leftrightarrow A_{1,0,y}=0$, a.s. This implies that, for each $y\in\mathbb{R}^d$, there is $c_y\in[0,\infty)$ such that $P[A^3_{1,0,y}]=c_ya_y^3$. Therefore, we have (1) with $c_2=\sup_{|y|\leq r_A}c_y$ (cf. (1.3)). By (1), we get

$$\begin{aligned}
P[X^3_{t,y}|\mathcal{F}_{t-1}] &= \frac{1}{|a|^3}\sum_{x_1,x_2,x_3\in\mathbb{Z}^d}\left(\prod_{j=1}^3\rho_{t-1,x_j}\right)P\left[\prod_{j=1}^3A_{t,x_j,y}\right]\\
&\stackrel{\text{Hölder}}{\leq} c_2\sum_{x_1,x_2,x_3\in\mathbb{Z}^d}\left(\prod_{j=1}^3\rho_{t-1,x_j}\overline{a}_{y-x_j}\right)=c_2(\rho_{t-1}*\overline{a})_y^3. \qquad (3.11)
\end{aligned}$$

Consequently, (3.10) can be verified as follows:

$$\begin{aligned}
\sum_{y\in\mathbb{Z}^d}P[(X_{t,y}-m_{t,y})^3|\mathcal{F}_{t-1}] &\leq 3\sum_{y\in\mathbb{Z}^d}(P[X^3_{t,y}|\mathcal{F}_{t-1}]+m^3_{t,y})\\
&\stackrel{(3.11)}{\leq} c_3\sum_{y\in\mathbb{Z}^d}(\rho_{t-1}*\overline{a})_y^3\stackrel{(3.9)}{\leq}\frac{c_3}{c_0}\sum_{y\in\mathbb{Z}^d}v_{t,y}.
\end{aligned}$$

□

Lemma 3.2.4 *For $h \in (0,1)$ and $\Lambda \subset \mathbb{Z}^d$,*

$$P\left[|\overline{N}_{t-1}|^h|(\overline{a}*\rho_{t-1})^2|\right] \geq \frac{1}{|\Lambda|}P\left[|\overline{N}_{t-1}|^h\right] - \frac{2}{|\Lambda|}P_S^0(S_t \notin \Lambda)^h, \tag{3.12}$$

for all $t \in \mathbb{N}^*$, where $((S_t)_{t\in\mathbb{N}}, P_S^0)$ is the random walk in Lemma 1.2.1.

Proof: We have on the event $\{|N_t| > 0\}$ that

$$\begin{aligned}
|\Lambda||(\rho_{t-1}*\overline{a})^2| &\geq |\Lambda|\sum_{z\in\Lambda}(\rho_{t-1}*\overline{a})_y^2 \geq \left(\sum_{y\in\Lambda}(\rho_{t-1}*\overline{a})_y\right)^2 \\
&= \left(1-\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^2 \geq 1-2\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y \\
&\geq 1-2\left(\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^h. \qquad (3.13)
\end{aligned}$$

Note also that

$$\begin{aligned}
P\left[\left(|\overline{N}_{t-1}|\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^h\right] &\leq P\left[|\overline{N}_{t-1}|\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right]^h \\
&= P\left[\sum_{y\notin\Lambda}(\overline{N}_{t-1}*\overline{a})_y\right]^h = P(S_t\notin\Lambda)^h, \qquad (3.14)
\end{aligned}$$

where the last equality comes from Lemma 1.2.1. We therefore see that

$$\begin{aligned}
|\Lambda|P\left[|\overline{N}_{t-1}|^h|(\rho_{t-1}*\overline{a})^2|\right] &\overset{(3.13)}{\geq} P\left[|\overline{N}_{t-1}|^h\right] - 2P\left[\left(|\overline{N}_{t-1}|\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^h\right] \\
&\overset{(3.14)}{\geq} P\left[|\overline{N}_{t-1}|^h\right] - 2P_S^0(S_t\notin\Lambda)^h.
\end{aligned}$$

□

Proof of (3.3) for $d = 2$: We will prove that for $h \in (0,1)$,

$$P[|\overline{N}_t|^h] = \begin{cases} O(\exp(-ct^{1/3})) & \text{if } d=1, \\ O(\exp(-c\sqrt{\ln t})) & \text{if } d=2 \end{cases} \quad \text{as } t \longrightarrow \infty, \tag{3.15}$$

where $c \in (0,\infty)$ is a constant. We have

$$|\overline{N}_t| = \frac{1}{|a|}\sum_{x,y\in\mathbb{Z}^d}\overline{N}_{t-1,x}A_{t,x,y} = |\overline{N}_{t-1}|U_t, \tag{3.16}$$

where U_t is from Lemma 3.2.3. We then see from Lemma 3.2.3 that for $h \in (0,1)$

$$P[|\overline{N}_t|^h|\mathcal{F}_{t-1}] - |\overline{N}_{t-1}|^h = |\overline{N}_{t-1}|^h P\left[U_t^h - 1|\mathcal{F}_{t-1}\right] \leq -c|\overline{N}_{t-1}|^h|(\rho_{t-1}*\overline{a})^2|.$$

We therefore have by Lemma 3.2.4 that

(1) $$P[|\overline{N}_t|^h] \le \left(1 - \frac{c}{|\Lambda|}\right) P[|\overline{N}_{t-1}|^h] + \frac{2c}{|\Lambda|} P_S^0(S_t \notin \Lambda)^h.$$

We set $\Lambda = (-\sqrt{t\ell_t}/2, \sqrt{t\ell_t}/2]^d \cap \mathbb{Z}^d$, where for $\ell_t = t^{1/3}$ for $d = 1$, and $\ell_t = \sqrt{\ln t}$ for $d = 2$. Then,

$$P_S^0(S_t \notin \Lambda) = P_S^0\left(\left|S_t/\sqrt{t}\right| \ge \sqrt{\ell_t}/2\right) \le c_1 \exp(-c_2 \ell_t),$$

so that (1) reads,

$$P[|\overline{N}_t|^h] \le \left(1 - \frac{c}{(t\ell_t)^{d/2}}\right) P[|\overline{N}_{t-1}|^h] + c_3 \exp(-c_2 \ell_t).$$

By iteration, we conclude (3.15). □

Proof of (3.3) for $d = 1$: Suppose that $d = 1$. We will prove that for $h \in (0, 1)$,

$$P[|\overline{N}_t|^h] = O(\exp(-ct)), \quad t \longrightarrow \infty,$$

where $c \in (0, \infty)$ is a constant. Then, (3.3) for $d = 1$ follows from the Borel-Cantelli lemma. Since the left-hand-side is non-increasing in t, it is enough to show that for some $s \in \mathbb{N}^*$,

(1) $$P[|\overline{N}_{ns}|^h] = O(\exp(-cn)), \quad n \longrightarrow \infty.$$

We write

$$|N_{s+t}| = \sum_y N_{s,y} |N_t^{s,y}| \ \text{ with } \ |N_t^{s,y}| = \sum_{x_1,..,x_t} A_{s+1,y,x_1} A_{s+2,x_1,x_2} \cdots A_{s+t,x_{t-1},x_t}.$$

Thus, for $h \in (0, 1)$,

$$|N_{s+t}|^h \le \sum_y N_{s,y}^h |N_t^{s,y}|^h.$$

Since $|N_t^{s,y}| \overset{\text{law}}{=} |N_t|$, we have by (3.15) that

(2) $$P[|N_{s+t}|^h] \le \sum_y P[N_{s,y}^h] P[|N_t|^h] \le c_1 s \exp(-c_2 s^{1/3}) P[|N_t|^h] \text{ for all } t \in \mathbb{N}^*.$$

We now take $s \in \mathbb{N}^*$ such that $c_1 s \exp(-c_2 s^{1/3}) < 1$. Then, (1) follows from (2). □

4 Dual processes

In this section, we associate a dual object to the process $(N_t)_{t\in\mathbb{N}}$ and thereby investigate invariant measures for $(\overline{N}_t)_{t\in\mathbb{N}}$. This can be considered as a discrete analogue of the duality theory for the continuous-time linear systems in the book by T. Liggett [8, Chapter IX].

4.1 Dual processes and invariant measures

We define a Markov chain $(M_t)_{t\in\mathbb{N}}$ with values in $[0, \infty)^{\mathbb{Z}^d}$ by

$$\sum_{x\in\mathbb{Z}^d} A_{t,y,x} M_{t-1,x} = M_{t,y}, \quad t \in \mathbb{N}, \tag{4.1}$$

where the initial state $M_0 \in [0, \infty)^{\mathbb{Z}^d}$ is a non-random and finite (cf. (1.7)). We refer $(M_t)_{t\in\mathbb{N}}$ as the *dual process* of $(N_t)_{t\in\mathbb{N}}$ defined by (1.6). Regarding (M_t) as column vectors, we can interpret (4.1) as:

$$M_t = A_t A_{t-1} \cdots A_1 M_0.$$

The dual process can also be understood as being defined in the same way as (1.6), except that the matrix A_t is replaced by its transpose: $A^*_t = (A_{t,y,x})_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d}$.

By the same proof as Lemma 1.2.1, we have:

Lemma 4.1.1
$$P[M_{t,y}] = |a|^t \sum_{x\in\mathbb{Z}^d} M_{0,x} P^x_S(-S_t = y),$$
where $((S_t)_{t\in\mathbb{N}}, P^x_S)$ *is the random walk in Lemma 1.2.1. Moreover,* $(|\overline{M}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ *is a martingale, where we have defined* $\overline{M}_t = \left(\overline{M}_{t,x}\right)_{x\in\mathbb{Z}^d}$ *by*

$$\overline{M}_{t,x} = |a|^{-t} M_{t,x}. \tag{4.2}$$

Also, Lemma 1.2.2 holds true with $\overline{N}_t$ replaced by $\overline{M}_t$. Accordingly, we have the definition of *regular/slow growth phase* for the dual process in the same way as for the (N_t)-process. For $(s,z)\in\mathbb{N}\times\mathbb{Z}^d$, we define $M^{s,z}_t = (M^{s,z}_{t,y})_{y\in\mathbb{Z}^d}$ and $\overline{M}^{s,z}_t = (\overline{M}^{s,z}_{t,y})_{y\in\mathbb{Z}^d}$, $t\in\mathbb{N}$ respectively by

$$\begin{aligned} &M^{s,z}_{0,y} = \delta_{z,y}, \quad N^{s,z}_{t+1,y} = \sum_{x\in\mathbb{Z}^d} M^{s,z}_{t,x} A_{s+t+1,y,x}, \\ &\text{and} \quad \overline{M}^{s,z}_{t,y} = |a|^{-t} M^{s,z}_{t,y}. \end{aligned} \tag{4.3}$$

$(N_t)_{t\in\mathbb{N}}$ and $(M_t)_{t\in\mathbb{N}}$ are dual to each other in the following sense:

Lemma 4.1.2 *For each fixed* $t\in\mathbb{N}^*$,

$$\left(N^{0,x}_{t,y}\right)_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d} \stackrel{\text{law}}{=} \left(M^{0,y}_{t,x}\right)_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d}. \tag{4.4}$$

Proof: We have

$$\begin{aligned} M^{0,y}_{t,x} &= \sum_{x_1,\ldots,x_{t-1}\in\mathbb{Z}^d} A_{t,y,x_1} A_{t-1,x_1,x_2}\cdots A_{2,x_{t-2},x_{t-1}} A_{1,x_{t-1},x} \\ &\stackrel{\text{law}}{=} \sum_{x_1,\ldots,x_{t-1}\in\mathbb{Z}^d} A_{1,y,x_1} A_{2,x_1,x_2}\cdots A_{t-1,x_{t-2},x_{t-1}} A_{t,x_{t-1},x} = N^{0,x}_{t,y}. \end{aligned}$$

This shows that the left-hand-side of (4.4) is obtained from the right-hand-side by the measure-preserving transform $(A_1, A_2, .., A_t) \mapsto (A_t, A_{t-1}, .., A_1)$. □

The following result show that the structure of invariant measures of $(\overline{N}_t)$ depends on whether the dual process (M_t) is in the regular or slow growth phase. To state the theorem, it is convenient to introduce the following notation: Let $\mathcal{P}([0,\infty)^{\mathbb{Z}^d})$ be the set of Borel probability meaures on $[0,\infty)^{\mathbb{Z}^d}$, and

$$\begin{aligned} \mathcal{I} &= \{\mu\in\mathcal{P}([0,\infty)^{\mathbb{Z}^d})\ ;\ \mu \text{ is invariant for the Markov chain } (\overline{N}_t)\}, \\ \mathcal{S} &= \{\mu\in\mathcal{P}([0,\infty)^{\mathbb{Z}^d})\ ;\ \mu \text{ is invariant with respect to the shift of } \mathbb{Z}^d\}. \end{aligned}$$

Theorem 4.1.3 a) *Suppose that* $P[|\overline{M}^{0,0}_\infty|] = 1$. *Then, for each* $\alpha\in[0,\infty)$, *there is a* $\nu_\alpha\in\mathcal{I}\cap\mathcal{S}$ *such that*

$$\int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\nu_\alpha(\eta) = \alpha. \tag{4.5}$$

Moreover, ν_α *is extremal in* $\mathcal{I}\cap\mathcal{S}$.

b) *Suppose on the contrary that* $P[|\overline{M}^{0,0}_\infty|] = 0$. *Then,*

$$\left\{\mu \in \mathcal{I} \cap \mathcal{S}, \; ; \int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\mu(\eta) < \infty \right\} = \{\delta_{\mathbf{0}}\},$$

where $\delta_{\mathbf{0}}$ *is the unit point mass on* $\mathbf{0} = (0)_{x\in\mathbb{Z}^d}$.

Proof: a): Let $(N^{\mathbf{1}}_t)_{t\in\mathbb{N}}$ be the (N_t)-process such that $N^{\mathbf{1}}_{0,x} \equiv 1$ for all $x \in \mathbb{Z}^d$. We have by Lemma 4.1.2 that

$$\alpha\overline{N}^{\mathbf{1}}_t \stackrel{\text{law}}{=} (\alpha|\overline{M}^{0,y}_t|)_{y\in\mathbb{Z}^d},$$

where $\alpha\overline{N}^{\mathbf{1}}_t = \left(\alpha\overline{N}^{\mathbf{1}}_{t,y}\right)_{y\in\mathbb{Z}^d}$. Since the right-hand-side converges a.s. to $(\alpha|\overline{M}^{0,y}_\infty|)_{y\in\mathbb{Z}^d}$ as $t \to \infty$, we see that the weak limit

$$\nu_\alpha \stackrel{\text{def.}}{=} \lim_{t\to\infty} P\left(\alpha\overline{N}^{\mathbf{1}}_t \in \cdot\right),$$

exists and that

(1) $\nu_\alpha = P\left((\alpha|\overline{M}^{0,y}_\infty|)_{y\in\mathbb{Z}^d} \in \cdot\right)$.

We see $\nu_\alpha \in \mathcal{I}$ from the way ν_α is defined. Also, $\nu_\alpha \in \mathcal{S}$, since $P\left(\alpha\overline{N}^{\mathbf{1}}_t \in \cdot\right) \in \mathcal{S}$ for any $t \in \mathbb{N}^*$ by (1.5). Moreover, (1) implies (4.5). The extremality of ν_α follows from the same argument as in [8, page 437, Corollary 2.1.5].
b): This follows from the same argument as in [8, page 435, Theorem 2.7]. □

4.2 Regular/slow growth for the dual process

In this subsection, we adapt arguments from sections 2 and 3 to obtain sufficient conditions for regular/slow growth phases the dual process. A motivation to investigate these sufficient conditions is explained by Theorem 4.1.3.

We let $(S, \widetilde{S}) = ((S_t, \widetilde{S}_t)_{t\in\mathbb{N}}, P^{x,\widetilde{x}}_{S,\widetilde{S}})$ denote the independent product of the random walks in Lemma 1.2.1. We have the following Feynman-Kac formula for the two-point functions of the dual process. The proof is the same as that of Lemma 2.1.1.

Lemma 4.2.1

$$P[M_{t,y}M_{t,\widetilde{y}}] = |a|^{2t} \sum_{x,\widetilde{x}\in\mathbb{Z}^d} M_{0,x}M_{0,\widetilde{x}}P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^*_t : (-S_t, -\widetilde{S}_t) = (y, \widetilde{y})] \quad \textit{for all } y, \widetilde{y} \in \mathbb{Z}^d, \tag{4.6}$$

where

$$e^*_t = \prod_{u=1}^{t} w(-S_u, -\widetilde{S}_u, -S_{u-1}, -\widetilde{S}_{u-1}), \quad \textit{(cf. (2.1))}. \tag{4.7}$$

Consequently,

$$P[|\overline{N}_t|^2] = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} M_{0,x}M_{0,\widetilde{x}}P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^*_t], \tag{4.8}$$

and

$$\sup_{t\in\mathbb{N}} P[|\overline{M}_t|^2] < \infty \iff \sup_{t\in\mathbb{N}} P^{0,0}_{S,\widetilde{S}}[e^*_t] < \infty \tag{4.9}$$

$$\implies P[|\overline{M}_\infty|] = |M_0|. \tag{4.10}$$

We see from Lemma 4.2.1, as in (2.14) and (2.15) that

$$\sup_{t\in\mathbb{N}} P[|\overline{M}_t|^2] < \infty \quad \Longleftrightarrow \quad d \geq 3 \text{ and } \begin{cases} p > \pi_0 & \text{for OP,} \\ \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_0) & \text{for DPRE.} \end{cases} \tag{4.11}$$

$$\sup_{t\in\mathbb{N}} P[|\overline{M}_t|^2] < \infty \quad \Longleftarrow \quad d \geq 3 \text{ and } \begin{cases} p \wedge q > \pi_0 & \text{for GOP with } q \neq 0, \\ p > \pi_0 & \text{for BCPP with } q = 0, \\ p \wedge q > \pi_0 & \text{for BCPP with } q \neq 0. \end{cases} \tag{4.12}$$

Let us now turn to sufficient conditions for the dual process to be in the slow growth phase. We first note that exactly the same statement as Theorem 3.1.1 holds true with $\overline{N}_t$ replaced by $\overline{M}_t$, since the proof works for the dual process without change. In particular,

$$|\overline{M}_t| = o(e^{-ct}), \quad \text{as } t \to \infty, \text{ a.s. if} \begin{cases} 2dp + q < 1 & \text{for GOP,} \\ \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d) & \text{for DPRE,} \\ p + q < 1 & \text{for BCPP.} \end{cases}$$

In analogy with Theorem 3.2.1, we have:

Theorem 4.2.2 *Let $d = 1, 2$. Suppose that $P[A^3_{1,0,y}] < \infty$ for all $y \in \mathbb{Z}^d$ and that*

$$\textit{the r.v. } \sum_{x\in\mathbb{Z}^d} A_{1,x,0} \textit{ is not a constant a.s.} \tag{4.13}$$

Then, almost surely,

$$|\overline{M}_t| = \begin{cases} O(\exp(-ct)) & \textit{if } d = 1, \\ \longrightarrow 0 & \textit{if } d = 2 \end{cases} \quad \textit{as } t \longrightarrow \infty, \tag{4.14}$$

where $c > 0$ is a non-random constant.

To explain the proof of Theorem 4.2.2, we introduce

$$V_t = \frac{1}{|a|} \sum_{x,y\in\mathbb{Z}^d} \rho^*_{t-1,y} A_{t,x,y}, \quad t \in \mathbb{N}^*, \qquad \text{where} \quad \rho^*_{t-1,x} = \mathbf{1}_{\{|M_{t-1}|>0\}} M_{t-1,x}/|M_{t-1}|. \tag{4.15}$$

We then have $|\overline{M}_t| = V_t|\overline{M}_{t-1}|$, $t \in \mathbb{N}^*$. Using this relation instead of (3.16), we can show Theorem 4.2.2 in the same way as Theorem 3.2.1, except that we replace Lemma 3.2.3 by Lemma 4.2.3 below.

Lemma 4.2.3 *For $h \in (0, 1)$, there is a constant $c \in (0, \infty)$ such that*

$$P\left[1 - V_t^h|\mathcal{F}_{t-1}\right] \geq c|(\rho^*_{t-1})^2| \quad \textit{for all } t \in \mathbb{N}^*.$$

Proof: We may focus on the event $\{|M_{t-1}| > 0\}$, since the inequality to prove is trivially true on $\{|M_{t-1}| = 0\}$. By the last part of the proof of Lemma 3.2.2, we see that there exists a constant $c_1 \in (0, \infty)$ such that

(1) $P\left[1 - V_t^h|\mathcal{F}_{t-1}\right] \geq c_1 P\left[\frac{(V_t - 1)^2}{V_t + 1}|\mathcal{F}_{t-1}\right]$ for all $t \in \mathbb{N}^*$.

We write

$$V_t = \sum_{x\in\mathbb{Z}^d} \rho^*_{t-1,y} X_{t,y} \text{ with } X_{t,y} = \frac{1}{|a|}\sum_{x\in\mathbb{Z}^d} A_{t,x,y}.$$

For fixed $t \in \mathbb{N}^*$, $\{X_{t,y}\}_{y\in\mathbb{Z}^d}$ are non-negative r.v.'s, which are i.i.d. with mean one, given $\mathcal{F}_{t-1}$. Furthermore, $X_{t,y}$ is not a constant a.s., because of (4.13). We therefore see from [2, Lemma 2.1] that there exists a constant $c_2 \in (0,\infty)$ such that

$$P\left[\frac{(V_t-1)^2}{V_t+1}|\mathcal{F}_{t-1}\right] \geq c_2|(\rho^*_{t-1})^2| \quad \text{for all } t \in \mathbb{N}^*,$$

which, together with (1), proves the lemma. □

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