Global unique solvability of 3D MHD equations in a thin periodic domain

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Abstract

We study magnetohydrodynamic equations for a viscous incompressible resistive fluid in a thin 3D domain. We prove the global existence and uniqueness of solutions corresponding to a large set of initial data from Sobolev type space of the order 1/2 and forcing terms from L_2 type space. We also show that the solutions constructed become smoother for positive time and prove the global existence of (unique) strong solutions.

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1 Introduction

Let $\mathcal{O}_\varepsilon = (0, l_1) \times (0, l_2) \times (0, \varepsilon)$, where $l_1, l_2 > 0$ and $\varepsilon \in (0, 1]$. We consider solutions $u, b : \mathbb{R}^3 \times \mathbb{R}_+ \mapsto \mathbb{R}^3$ of magnetohydrodynamic (MHD) equations which have the form (see, e.g., [12] or [16]):

$$\partial_t u - \nu_1 \triangle u + \sum_{j=1}^{3} u_j \partial_j u = -\nabla \left(p + \frac{s}{2}|b|^2 \right) + s \sum_{j=1}^{3} b_j \partial_j b + f, \quad (1.1)$$

$$\partial_t b - \nu_2 \triangle b + \sum_{j=1}^{3} u_j \partial_j b = \sum_{j=1}^{3} b_j \partial_j u + g, \quad (1.2)$$

$$\operatorname{div} u = 0, \quad \operatorname{div} b = 0 \qquad \text{and} \qquad \int_{\mathcal{O}_\varepsilon} u dx = 0, \quad \int_{\mathcal{O}_\varepsilon} b dx = 0, \tag{1.3}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $b = (b_1(x,t), b_2(x,t), b_3(x,t))$ denote velocity and magnetic field, $p(x,t)$ is a scalar pressure, $x = (x', x_3) = (x_1, x_2, x_3) \in \mathbb{R}^3$. For the sake of definiteness and for simplicity we consider the case of periodic boundary conditions, i.e. we assume that u, b and p are periodic with periods l_1, l_2 and ε in x_1, x_2 and x_3 directions. In equations above ν_1 is the kinematic viscosity, ν_2 is the magnetic diffusivity (which is determined from magnetic permeability and conductivity of the fluid), the parameter s is defined by the relation $s = Ha^2\nu_1\nu_2$, where Ha is the so-called Hartman number. The given periodic (in x) functions $f = f(x,t)$ and $g = g(x,t)$ represent external volume forces and the curl of external current applied to the fluid.

We equip system (1.1) -(1.3) with the initial data

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x) \quad \text{in} \quad \mathcal{O}_\varepsilon\,. \tag{1.4}$$

Well-posedness questions for MHD equations with different type of boundary conditions are discussed in [13] (see also [5, 19]). Basically, the results available are similar to those known for Navier-Stokes equations. In 2D case there exists a unique global weak solution. In 3D case it was proved (i) the global existence of weak solutions, (ii) the uniqueness of strong solutions and their local existence, (iii) global existence of strong solutions with small initial data and external forces.

Our main results (see Theorem 3.2) states that if the thickness ε of the domain $\mathcal{O}_\varepsilon$ is small enough, then for initial data and force terms from a large (in the sense of thin domain problems, see, e.g., [17, 18]) set in the corresponding phase/parameter space there exists a solution which is unique in the class of weak Leray solutions. Moreover we show that these solutions become smoother for $t > 0$ and prove the global existence of (unique) strong solutions.

In our considerations we rely on some ideas and methods which have been developed for 3D Navier-Stokes equations on thin domains in recent years.

The study of global existence of strong solutions of the Navier-Stokes equations on thin three-dimensional domains began with the papers of Raugel and Sell ([17], [18]), who proved global existence of strong solutions for large initial data and forcing terms in the case of periodic conditions or mixed periodic-Dirichlet conditions. After these publications a number of papers by various authors followed, where the results were sharpened and/or extended to other boundary conditions [1, 7, 8, 9, 10, 11, 14, 15, 21]. See also extensions to thin spherical domains in [22], to thin two-layer domains in [4], and to stochastic problems in [2, 3].

In this paper, we show how to extend the global existence and uniqueness results available for 3D Navier-Stokes equations to the case of the MHD equations in thin three-dimensional domains. We rely mainly on the same method as Iftimie and Raulel [9] did in the case of Navier-Stokes equations with periodic and periodic-Dirichlet boundary conditions. Our hypotheses concerning dependence of initial data and force terms on ε are compatible with the corresponding requirements for the Navier-Stokes equations (see Theorem 1.1 and Remark 1.1 in [9]). We also note that recently Kukavica and Ziane [11] have suggested an approach to 3D periodic Navier-Stokes equations in a thin domain which leads to a weaker requirements (in contrast with [9]) concerning initial data and volume forces. However we cannot apply the method from [11] in our case because the paper [11] relies substantially on the fact that the corresponding 2D problem satisfies the so-called enstrophy conservation property. Due to the presence of magnetic field the corresponding property is not true for the MHD case. See Remark 3.4 for more details.

The paper is organized as follows. In Sect. 2 we rewrite the problem in symmetric form, introduce appropriate functional spaces and recall several auxiliary facts borrowed from [9]. In Sect. 3 we first state and discuss uniqueness theorem which is an MHD analog of Theorem 3.1[9] and which, as we hope, has independent interest. Then we formulate and comment our main results which are collected in Theorem 3.2. Sect 4 is devoted to the proofs.

We also note that our results provide an answer to the question suggested by Professor John D Gibbon during author's short visit to Imperial College London in May 2007. The author is thankful to him for his hospitality and stimulating discussions.

2 Preliminaries

In this section we rewrite problem (1.1)–(1.4) in symmetric form, introduce the main functional spaces and collect several auxiliary results.

Due to the fact that the both velocity and magnetic fields have the same type of boundary conditions it is convenient to use Elsasser's variables (see, e.g., [16]):

$$z^+ = u + \sqrt{s}\cdot b \quad \text{and} \quad z^- = u - \sqrt{s}\cdot b, \tag{2.1}$$

and write MHD equations (1.1) and (1.2) in the symmetric form

$$\partial_t z^+ - \nu^+ \triangle z^+ - \nu^- \triangle z^- + (z^-, \nabla) z^+ = -\nabla P + f^+, \tag{2.2}$$

$$\partial_t z^- - \nu^+ \triangle z^- - \nu^- \triangle z^+ + (z^+, \nabla) z^- = -\nabla P + f^-, \tag{2.3}$$

where $P = p + \frac{1}{8}|z^+ - z^-|^2$ is the total pressure, $\nu^\pm = \frac{\nu_1 \pm \nu_2}{2}$ and $f^\pm = f \pm \sqrt{s}\cdot g$. Relations (1.3) give us the conditions

$$\operatorname{div} z^\pm = 0 \quad \text{and} \quad \int_{\mathcal{O}_\varepsilon} z^\pm dx = 0. \tag{2.4}$$

We also need to supply equations (2.2)–(2.4) with (l_1, l_2, ε)-periodic boundary conditions and with the initial data

$$z^\pm(x, 0) = z_0^\pm(x) \equiv u_0(x) \pm \sqrt{s} b_0(x) \quad \text{in} \quad \mathcal{O}_\varepsilon\,. \tag{2.5}$$

This remakable symmetry of the equations for z^+ and z^- enables us to expect that the results for the MHD equations and Navier-Stokes equations would be quite similar. This is partially true only because the pure hydrodinamics (the case $z^+ = z^-$) provides *additional* symmetry in the model in comparison with the MHD (the case $z^+ \neq z^-$), see Remark 3.4.

Let $H^s_{per}(\mathcal{O}_\varepsilon)$ be the Sobolev space of order s consisting of real functions on $\mathbb{R}^3$ which are periodic with periods l_1, l_2 and ε in x_1, x_2 and x_3 directions. This space can be described in terms of Fourier series as a set of functions of the form

$$v(x) = \sum_{k\in\mathbb{Z}^3} v_k e_k(x) \quad \text{with} \quad e_k(x) = \frac{1}{\sqrt{l_1 l_2 \varepsilon}} \exp\left\{2\pi i\left(\frac{k_1 x_1}{l_1} + \frac{k_2 x_2}{l_1} + \frac{k_3 x_3}{\varepsilon}\right)\right\}, \tag{2.6}$$

where the complex Fourier coefficients $v_k = \int_{\mathcal{O}_\varepsilon} v(x) e_k(x) dx$ satisfy relation $\bar{v}_k = v_{-k}$ and also

$$\|v\|^2_s \equiv |v_0|^2 + \sum_{k\in\mathbb{Z}^3, k\neq 0} \left(\left|\frac{k_1}{l_1}\right|^2 + \left|\frac{k_2}{l_2}\right|^2 + \left|\frac{k_3}{\varepsilon}\right|^2 \right)^s |v_k|^2 < \infty. \tag{2.7}$$

We also use the notations

$$\dot{H}^s_{per}(\mathcal{O}_\varepsilon) = \left\{ v \in H^s_{per}(\mathcal{O}_\varepsilon) \, : v_0 \equiv \frac{1}{\sqrt{l_1 l_2 \varepsilon}} \int_{\mathcal{O}_\varepsilon} v dx = 0 \right\}$$

and

$$V^s_\varepsilon = \left\{ w \in \left[\dot{H}^s_{per}(\mathcal{O}_\varepsilon)\right]^3 \, : \, \operatorname{div} w = 0 \right\}$$

for the space of periodic divergence-free vector fields on $\mathbb{R}^3$.

We consider the bilinear form

$$a_\varepsilon(u, v) = \sum_{j=1}^{3} \int_{\mathcal{O}_\varepsilon} \nabla u_j \cdot \nabla v_j \, dx, \quad u, v \in V^1_\varepsilon,$$

and denote by A_ε the Stokes operator, defined as an isomorphism from V^1_ε onto its dual V^{-1}_ε by the relation

$$(A_\varepsilon u, v)_{V^1_\varepsilon, V^{-1}_\varepsilon} = a_\varepsilon(u, v), \quad u, v \in V_\varepsilon \, .$$

This operator can be extended to $H_\varepsilon \equiv V^0_\varepsilon$ as a linear self-adjoint positive operator with the domain $\mathcal{D}(A_\varepsilon) = V^2_\varepsilon$ such that

$$(A_\varepsilon u)(x) = -\Delta u(x), \quad x \in \mathcal{O}_\varepsilon, \quad \text{for every } u \in \mathcal{D}(A_\varepsilon) = V^2_\varepsilon.$$

Moreover (see [9] and also [6]), there exits a constant $c_0 > 1$ independent of ε such that

$$c_0^{-2} \sum_{i=1}^{3} \|(-\Delta)^s v_i\|^2_{L_2} \leq \|A^s_\varepsilon v\|^2_{L_2} \leq c_0^2 \sum_{i=1}^{3} \|(-\Delta)^s v_i\|^2_{L_2}, \quad 0 \leq s \leq 1, \tag{2.8}$$

for any function $v = (v_1, v_2, v_3) \in V_\varepsilon^s$. We note that for $v \in \dot H^s_{per}(\mathcal{O}_\varepsilon)$ of the form (2.6) we have

$$(-\Delta)^s v = \sum_{k\in\mathbb{Z}^3, k\neq 0} \left(\left|\frac{k_1}{l_1}\right|^2 + \left|\frac{k_2}{l_2}\right|^2 + \left|\frac{k_3}{\varepsilon}\right|^2 \right)^s v_k e_k(x), \quad s \in \mathbb{R}.$$

In the calculations below we shall use the norm

$$|v|_s \equiv |A_\varepsilon^{s/2} v| \equiv \|A_\varepsilon^{s/2} v\|_{L_2(\mathcal{O}_\varepsilon)}, \quad s \in \mathbb{R}, \tag{2.9}$$

and the corresponding inner product on the space V_ε^s (for $s = 0$ we drop the subscript). Relation (2.8) shows that the norm $|\cdot|_s$ and the norm induced by (2.7) are equivalent on V_ε^s for $0 \le s \le 1$ with the constants independent of ε.

Now we consider the trilinear form

$$b_\varepsilon(u, v, w) = \sum_{j,l=1}^{3} \int_{\mathcal{O}_\varepsilon} u_j\, \partial_j v_l\, w_l\, dx, \quad u, v \in \mathcal{D}(A_\varepsilon),\ w \in (L^2(\mathcal{O}_\varepsilon))^3.$$

It defines a bilinear operator $B_\varepsilon : V_\varepsilon^1 \mapsto V_\varepsilon^{-1}$ by the formula

$$(B_\varepsilon(u, v), w)_{V_\varepsilon^1, V_\varepsilon^{-1}} = b_\varepsilon(u, v, w), \quad u, v, w \in V_\varepsilon^1.$$

For smooth functions we have that $B(u, v) = \Pi_\varepsilon(u, \nabla)v$, where Π_ε is the Leray projector on H_ε in $\left[L^2_{per}(\mathcal{O}_\varepsilon)\right]^3$.

Now problem (2.2)–(2.5) in the space H_ε can be written in the form

$$\partial_t z^+ + \nu^+ A_\varepsilon z^+ + \nu^- A_\varepsilon z^- + B_\varepsilon(z^-, z^+) = \tilde f^+, \quad z^+(0) = z_0^+, \tag{2.10a}$$

$$\partial_t z^- + \nu^+ A_\varepsilon z^- + \nu^- A_\varepsilon z^+ + B_\varepsilon(z^+, z^-) = \tilde f^-, \quad z^-(0) = z_0^-, \tag{2.10b}$$

where $\tilde f^\pm = \Pi_\varepsilon f^\pm$. Below we also use the notation $\tilde f = \{\tilde f^+; \tilde f^-\}$.

Definition 2.1 A couple $z = \{z^+; z^-\}$ is said to be a *weak Leray solution* for MHD problem (2.10) if

$$z(t) \in L_\infty(\mathbb{R}_+; H_\varepsilon \times H_\varepsilon) \cap L_2^{loc}(\mathbb{R}_+; V_\varepsilon^1 \times V_\varepsilon^1), \tag{2.11}$$

it satisfies the energy inequality

$$\frac{1}{2}|z(t)|^2 + \int_0^t \left[\nu^+ |A_\varepsilon^{1/2} z|^2 + 2\nu^-(A_\varepsilon z^-, z^+)\right] d\tau \le \frac{1}{2}|z(0)|^2 + \int_0^t (\tilde f, z) d\tau \tag{2.12}$$

and relations (2.10) hold in the sense of distributions. Here and below we use the following notations:

$$|y|_s^2 = |y^+|_s^2 + |y^-|_s^2,\ A_\varepsilon^s y = \{A_\varepsilon^s y^+; A_\varepsilon^s y^-\},\ (y, z)_s = (y^+, z^+)_s + (y^-, z^-)_s \tag{2.13}$$

for every $s \in \mathbb{R}$ and for any couples of the form $y = \{y^+; y^-\}$ and $z = \{z^+; z^-\}$.

We note that for $z = \{z^+; z^-\}$ with z^+ and z^- defined by (2.1) we have that

$$\nu^+|A_\varepsilon^{1/2}z|^2 + 2\nu^-(A_\varepsilon z^-, z^+) = 2\left(\nu_1|A_\varepsilon^{1/2}u|^2 + \nu_2 s|A_\varepsilon^{1/2}b|^2\right) \geq \frac{\nu_0}{2}|A_\varepsilon^{1/2}z|^2, \tag{2.14}$$

where $\nu_0 = \min\{\nu_1, \nu_2\}$. In particular, the bilinear form under the integral in the left hand side of (2.12) is positively defined.

Definition 2.2 A weak Leray solution $z = \{z^+; z^-\}$ to MHD problem (2.10) is said to be *strong* if

$$z(t) \in C(\mathbb{R}_+; V_\varepsilon^1 \times V_\varepsilon^1) \cap L_2^{loc}(\mathbb{R}_+; V_\varepsilon^2 \times V_\varepsilon^2). \tag{2.15}$$

In the case when $f^\pm \in L_\infty(\mathbb{R}_+, H_\varepsilon)$ and $z_0^\pm \in H_\varepsilon$ the existence of weak Leray solutions was established in [13] (see also [5, 19]). Their uniqueness is unknown. As it was mentioned in the Introduction the results on the existence and unqueness of weak and strong solutions for the MHD equations are similar to those known for Navier-Stokes problem (see [13, 5, 19]).

For any function from $L_2(\mathcal{O}_\varepsilon)$ we define its averaging in the thin direction x_3 by the formula

$$(Mu)(x) = \frac{1}{\varepsilon}\int_0^\varepsilon u(x', \eta)\, d\eta, \ u \in L_2(\mathcal{O}_\varepsilon), \tag{2.16}$$

where $x = (x', x_3) \in \mathcal{O}_\varepsilon$. The operator M is an orthogonal projector in each space $\dot{H}^s_{per}(\mathcal{O}_\varepsilon)$ and thus the operator $N = I - M$ is also an orthoprojector in $\dot{H}^s_{per}(\mathcal{O}_\varepsilon)$. We need these operators M and N to formulate the following assertion which was established in [9].

Proposition 2.3 ([9]) *There are constants K_1, K_2, K_3 independent of $\varepsilon \in (0,1)$ such that*

- *if $w_i \in \dot{H}^{s_i}_{per}(\mathcal{O}_\varepsilon)$ are three functions satisfying $Mw_i = 0$, $0 \leq s_i < 3/2$ for $i = 1, 2, 3$, and $s_1 + s_2 + s_3 = 3/2$, then*

$$\left|\int_{\mathcal{O}_\varepsilon} w_1(x)w_2(x)w_3(x)dx\right| \leq K_1\|w_1\|_{s_1}\|w_2\|_{s_2}\|w_3\|_{s_3}; \tag{2.17}$$

- *if $v_1 \in \dot{H}^{s_1}_{per}(\mathcal{O}_\varepsilon)$ and $w_i \in \dot{H}^{s_i}_{per}(\mathcal{O}_\varepsilon)$, $i = 2, 3$, are functions satisfying $Nv_1 = 0$, $Mw_2 = Mw_3 = 0$, $0 \leq s_i < 1$ for $i = 1, 2, 3$, and $s_1 + s_2 + s_3 = 1$, then*

$$\left|\int_{\mathcal{O}_\varepsilon} v_1(x)w_2(x)w_3(x)dx\right| \leq K_2\varepsilon^{-1/2}\|v_1\|_{s_1}\|w_2\|_{s_2}\|w_3\|_{s_3}; \tag{2.18}$$

- *for any $v \in V_\varepsilon^1$ and $w \in \left[\dot{H}^{3/2}_{per}(\mathcal{O}_\varepsilon)\right]^3$ satisfying $Nv = 0$ and $Mw = 0$ we have*

$$\left|\int_{\mathcal{O}_\varepsilon} v(x)\nabla w(x)(-\Delta)^{1/2}w(x)dx\right| \leq K_3\varepsilon^{-1/2}|v|_1|w|_{1/2}|w|_{3/2}. \tag{2.19}$$

We recall that in (2.17) and (2.18) we use the norms defined by (2.7) and in (2.19) the norms $|\cdot|_s$ are given by (2.9). We also note that the first two inequalities in Proposition 2.3 are proved in Lemma 2.1[9]. For the third one we refer to Lemma 2.4[9].

Remark 2.4 According Remark 2.1 in [9] relations of the form (2.17) and (2.18) remain true without the assumption $Mw_i = 0$. However in this case the constant K_1 and K_2 *depend* on ε. In particular, with an appropriate choice of the exponents s_i in the corresponding relations one can show that there exists a constant C_ε such that for the trilinear form b_ε the following inequalities holds:

$$|b_\varepsilon(w_1, w_2, w_3)| \le C_\varepsilon |w_1|_{1/2} |w_2|_{3/2} |w_3|_{1/2}, \quad w_1, w_3 \in V_\varepsilon^{1/2}, w_2 \in V_\varepsilon^{3/2}; \tag{2.20}$$

$$|b_\varepsilon(w_1, M\psi, w_2)| \le C_\varepsilon |w_1|_{1/2} |w_2|_{1/2} |M\psi|_1, \quad w_1, w_2 \in V_\varepsilon^{1/2}, \psi \in V_\varepsilon^1; \tag{2.21}$$

and

$$|b_\varepsilon(w_1, w_2, M\psi)| \le C_\varepsilon |w_1|_{1/2} |w_2|_{3/2} |M\psi| \tag{2.22}$$

for $w_1 \in V_\varepsilon^{1/2}, w_2 \in V_\varepsilon^{3/2}, \psi \in H_\varepsilon$.

Below we also use extensions of the operators M and N on the space H_e of vector fields. We define these extensions by the formulas

$$Mu = (Mu_1, Mu_2, Mu_3), \; N = u - Mu, \;\; u = (u_1, u_2, u_3) \in H_\varepsilon, \tag{2.23}$$

where Mu_j are defined by (2.16). One can see that the operators M and N are orthogonal projectors in V_ε^s. They commute with spatial derivatives ∂_j, $j = 1, 2, 3$, and also with any power A_ε^s of the Stokes operator A_ε. Moreover, we obviously have that $b_\varepsilon(w_1, w_2, w_3) = 0$ provided that one of the vectors w_j lies in NV_ε^1 and two others belong to MV_ε^1. We shall use this observation in further considerations.

3 Main results

We start with the following MHD analog of the uniqueness theorem which was proved in [9] for 3D Navier-Stokes equations.

Theorem 3.1 (Uniqueness) *Let $z = \{z^+; z^-\}$ be a weak Leray solution for MHD problem* (2.10) *such that*

$$(I - M)z(t) \in L_\infty(\mathbb{R}_+; V_\varepsilon^{1/2} \times V_\varepsilon^{1/2}) \cap L_2^{loc}(\mathbb{R}_+; V_\varepsilon^{3/2} \times V_\varepsilon^{3/2}), \tag{3.1}$$

where $Mz = \{Mz^+; Mz^-\}$ and M is defined in (2.23). *Then z is unique in the class of the weak Leray solutions.*

To prove this theorem we apply the same argument as in [9] for the case Navier-Stokes equations. Therefore in Sect. 4 we give a sketch of the proof only. We also note that, as in the case of Navier-Stokes equations (cf. [8] and [9]) Theorem 3.1

implies the uniqueness of 2D solutions in the class of 3D weak Leray solutions. The point is that any 2D solution can be characterized as a 3D solution $z(t)$ with the property $(I-M)z(t)=0$.

Our main result stated in Theorem 3.2 below provides conditions on initial data and forcing terms which guarantee the global existence of solutions of the 3D MHD problem possessing property (3.1). We also include in Theorem 3.2 some other properties of the solution constucted.

Theorem 3.2 *Assume that there exist $c,\delta>0$ and $\sigma<1$ such that we have that*

$$|Nz_0|_{1/2}^2 \le c\varepsilon^\delta, \quad |Mz_0|^2 \le c\varepsilon\left[\log\frac{1}{\varepsilon}\right]^\sigma \tag{3.2}$$

and

$$\sup_{\mathbb{R}_+}|N\tilde{f}|^2 \le \frac{c}{\varepsilon}\left[\log\frac{1}{\varepsilon}\right]^\sigma, \quad \sup_{\mathbb{R}_+}|M\tilde{f}|^2 \le c\varepsilon\left[\log\frac{1}{\varepsilon}\right]^\sigma, \tag{3.3}$$

for every $\varepsilon\in(0,1)$, where $z_0=\{z_0^+;z_0^-\}$ and $\tilde{f}=\{\tilde{f}^+;\tilde{f}^-\}$ are the data in (2.10)*. Then there exists $\varepsilon_*\in(0,1]$ such that for every $\varepsilon\in(0,\varepsilon_*)$ MHD problem* (2.10) *has a weak Leray solution $z=\{z^+;z^-\}$ satisfying* (3.1)*. Moreover, this solution z possesses properties:*

- *we have that*

$$z(t)\in C(\mathbb{R}_+;H_\varepsilon\times H_\varepsilon), \quad \partial_t z(t)\in L_2(\mathbb{R}_+;V_\varepsilon^{-1}\times V_\varepsilon^{-1}) \tag{3.4}$$

  and also

$$z(t)\in L_\infty(\mathbb{R}_+;V_\varepsilon^{1/2}\times V_\varepsilon^{1/2})\cap L_2^{loc}(\mathbb{R}_+;V_\varepsilon^{3/2}\times V_\varepsilon^{3/2}); \tag{3.5}$$

- *there exist C_1, C_2 and $0<\eta<\delta$ (independent of ε) such that*

$$|Nz(t)|_{1/2}^2 \equiv |Nz^+(t)|_{1/2}^2+|Nz^-(t)|_{1/2}^2 \le C_1\varepsilon^\eta,\ t\ge 0, \tag{3.6}$$

  and

$$\limsup_{t\to\infty}|Nz(t)|_{1/2}^2 \le C_2\varepsilon^{2+\eta}\max_{\tau\in\mathbb{R}_+}|N\tilde{f}(\tau)|^2; \tag{3.7}$$

- *(regularizing effect): for any $T>0$ there exists a positive constant B_T (also depending on ε, $|z_0|_{1/2}$ and $|\tilde{f}|$) such that*

$$|z(t)|_1^2+\frac{1}{t}\int_0^t \tau|z(\tau)|_2^2 d\tau \le \frac{B_T}{t}, \quad 0<t<T, \tag{3.8}$$

  which implies that

$$z(t)\in L_\infty([T_1,T_2];V_\varepsilon^1\times V_\varepsilon^1)\cap L_2^{loc}([T_1,T_2];;V_\varepsilon^2\times V_\varepsilon^2)$$

  for all $0<T_1<T_2<\infty$;

- *if in addition we have that $z_0^\pm \in V_\varepsilon^1$, then the solution $z(t)$ satisfies* (2.15) *and thus it is strong.*

Remark 3.3 It is easy to see from the definition of the norm $|\cdot|_s$ in (2.9) that there is $k_0 > 0$ independent of ε such that $|Nz_0|_s \le k_0\varepsilon^\sigma |Nz_0|_{s+\sigma}$ for any $s \in \mathbb{R}$ and $\sigma \ge 0$. Therefore the first condition in (3.2) holds if $Nz_0 \in V_\varepsilon^1 \times V_\varepsilon^1$ and $|Nz_0|_1^2 \le c\varepsilon^{-1+\delta}$. By (2.8) we can state the conditions on Nz_0 in the same form by using the Sobolev norms (2.7). One can also see that relations (3.2) and (3.3) are valid when, for instance, we have that

$$\frac{1}{\varepsilon}\int_{\mathcal{O}_\varepsilon}\left(|\nabla z_0^\pm(x)|^2 + |\tilde{f}^\pm(x,t)|^2\right)dx \le C\left(\ln\frac{1}{\varepsilon}\right)^\sigma, \quad \sigma > 0.$$

This means that large values of the data in (2.10) are allowed for small ε.

Remark 3.4 In the case when $z_0^+ = z_0^-$ and $\tilde{f}^+ = z_0^-$ one can see that there is a solution $z = \{z^+; z^-\}$ of MHD problem (2.10) with $z^+(t) \equiv z^-(t)$. Moreover, in this case the function $u = z^+(t) = z^-(t)$ (see (2.1)) solves 3D Navier-Stokes problem. Thus as a consequence of Theorem 3.2 we obtain a result for Navier-Stokes equations on a thin 3D periodic domain. Partially this result is a slight reformulation of Theorem 1.1[9] for the purely periodic case. However, it seems that the properties stated in (3.5) and (3.8) are new even for the corresponding Navier-Stokes models. We also note that, as it was shown in [9, Theorems 1.3 and 5.1] and in [11], the set of admissible initial data and forcing terms given by Theorem 1.1[9] (and our Theorem 3.2) can be substantially extended in the Navier-Stokes case. However, we cannot do the same in the MHD case. The main obstacle is related to the enstrophy conservation property for the corresponding 2D problem. The point is that 2D analog of the value

$$b_\varepsilon(z^-, z^+, \Delta z^+) + b_\varepsilon(z^+, z^-, \Delta z^-)$$

is zero in the purely hydrodynamic case ($z^+(t) \equiv z^-(t)$) and does not generically vanish in the MHD case ($z^+(t) \ne z^-(t)$). This property is quite important in the arguments given in [9, Theorems 1.3 and 5.1] and in [11].

Remark 3.5 We also note that results similar to Theorem 3.2 hold true for other boundary conditions imposed on z^+ and z^-. For instance, we can consider free-periodic boundary conditions, i.e., free in the thin direction:

$$\partial_3 z_1^\pm = \partial_3 z_2^\pm = 0,\ z_3^\pm = 0 \ \text{ for } \ x_3 = 0 \text{ and } x_3 = \varepsilon,$$

and periodic on the lateral boundary. Physically this type of boundary conditions corresponds to the case when the boundaries $x_3 = 0$ and $x_3 = \varepsilon$ are perfectly conducting free surfaces (see, e.g., [16] and also [5, 13, 19]). In this case mean value operator M has the structure $M(u_1, u_2, u_3) = (Mu_1, Mu_2, 0)$. A description of a set of admissible data z_0 and $\tilde{f}$ is also changed. The main reason is that estimate (2.19) is known in the purely periodic case only (see a discussion in [9]).

In a similar way one can also consider different boundary conditions for the velocity u and for the magnetic field b. As a physically reasonable case we point out the situation when u satisfies Dirichlet-periodic and b free-periodic boundary conditions (see discussions in [13, 19]). However in this case of mixed boundary conditions it is more convenient to base all considerations on the original form (1.1)–(1.4) of the MHD model. We do not give the corresponding arguments because they are quite similar to those in Section 4. The modifications needed are minor and the same as one changes boundary conditions in the corresponding Navier-Stokes model (see, e.g., [9] and [21]).

4 Proofs

4.1 Sketch of the proof of Theorem 3.1

We first note that by the same argument as in [9] for the case of Navier-Stokes equations we can show that property (3.1) implies the regularity in (3.4) of the solution considered. This additional regularity makes it possible by the same method as in [9] to obtain the relation

$$\frac{1}{2}|w(t)|^2 + \int_0^t \left[\nu^+|A_\varepsilon^{1/2}w|^2 + 2\nu^-(A_\varepsilon w^-, w^+)\right] d\tau \\ \leq -\int_0^t \left[b_\varepsilon(w^-, z^+, w^+) + b_\varepsilon(w^+, z^-, w^-)\right] d\tau, \qquad (4.1)$$

where $w(t) = z(t) - \tilde{z}(t)$ and $\tilde{z}(t)$ is another week Leray solution of the same problem. We note that the idea behind the method applied in [9] is well known (see, e.g., [20] and the references therein). Following this idea in our case we first sum energy inequalities (see (2.12)) for $z(t)$ and $\tilde{z}(t)$ and then subtract from this sum the result of calculation

$$(z, \tilde{z})\Big|_0^t = \int_0^t \partial_t(z, \tilde{z})d\tau = \int_0^t \left[(\partial_t z, \tilde{z}) + (z, \partial_t \tilde{z})\right] d\tau$$

on the solutions considered. The additional smoothness of z in (3.4) is enough to perform all calculations (we refer to [9] for details in the Navier-Stokes case).

Now we apply estimates (2.20) and (2.21) to obtain the relation

$$\begin{aligned} |b_\varepsilon(w^-, z^+, w^+)| &\leq |b_\varepsilon(w^-, Nz^+, w^+)| + |b_\varepsilon(w^-, Mz^+, w^+)| \\ &\leq C_\varepsilon |w^-|_{1/2}|w^+|_{1/2}\left[|Nz^+|_{3/2} + |Mz^+|_1\right]. \end{aligned}$$

Therefore using interpolation we have that

$$|b_\varepsilon(w^-, z^+, w^+)| \leq \eta|w|_1^2 + C_{\varepsilon,\eta}|w|^2\left[|Nz^+|_{3/2}^2 + |Mz^+|_1^2\right]$$

for every $\eta > 0$. Using this estimate and the same estimate for $b_\varepsilon(w^+, z^-, w^-)$ with $\eta > 0$ small enough, from the ellipticity property in (2.14) and from (4.1) we get that

$$|w(t)|^2 \le C_\varepsilon \int_0^t |w(\tau)|^2 \left[|Nz(\tau)|^2_{3/2} + |Mz(\tau)|^2_1\right] d\tau.$$

Since $|Nz|^2_{3/2} + |Mz|^2_1 \in L_1(\mathbb{R}_+)$, Gronwall's lemma yields $w(t) = 0$ for almost all $t \ge 0$. This completes the proof of Theorem 3.1.

4.2 Proof of Theorem 3.2

Our arguments below are formal. To make them rigorous we can follow the standard idea (see, e.g., [20]) and use Galerkin approximations based on the eigenfunction basis of the Stokes operator with periodic boundary conditions. The point is that under the conditions imposed on the initial data z_0 and the forcing term $\tilde{f}$ problem (2.10) has a weak Leray solution $z(t)$ which can be constucted as a limit of sequence $\{z_k(t)\}$ of Galerkin approximations based on on the eigenfunction basis (see [13, 19]). Therefore the main point in the proof is to obtain appropriate additional *a priori* estimates for this solution. This can be done by proving the corresponding uniform bounds for the Galerkin approximate solutions z_k. This is exaclty what we have in mind in algebraic manipulations below.

For vector fields z^+ and z^- we define their projections

$$m^\pm = Mz^\pm, \quad n^\pm = Nz^\pm \equiv (I - M)z^\pm.$$

We also use notations similar to (2.13) for $m = \{m^+; m^-\}$ and $n = \{n^+; n^-\}$.

Step 1: the existence of solutions with property (3.1)**.** If we multiply equation (2.10a) in H_ε by $A_\varepsilon^{1/2} n^+$, then using properties of the projectors N and M we obtain that

$$\frac{1}{2}\frac{d}{dt}|n^+|^2_{1/2} + \nu^+|n^+|^2_{3/2} + \nu^-(n^-, n^+)_{3/2} + b_\varepsilon(n^-, n^+, A_\varepsilon^{1/2} n^+)$$
$$+ b_\varepsilon(n^-, m^+, A_\varepsilon^{1/2} n^+) + b_\varepsilon(m^-, n^+, A_\varepsilon^{1/2} n^+) = (N\tilde{f}^+, A_\varepsilon^{1/2} n^+). \quad (4.2)$$

Proposition 2.3 and interpolation arguments make it possible to estimate trilinear terms in (4.2). Indeed, (2.17) with $s_1 = 1$, $s_2 = 0$, $s_3 = 1/2$ implies that

$$\begin{aligned} |b_\varepsilon(n^-, n^+, A_\varepsilon^{1/2} n^+)| &\le C|n^-|_1 |n^+|_1 |n^+|_{3/2} \le C|n|_{1/2}|n|^2_{3/2} \\ &\le C\delta^{-1}|n|^2_{1/2}|n|^2_{3/2} + \delta|n|^2_{3/2} \end{aligned} \quad (4.3)$$

for every $\delta > 0$. By (2.18) with $s_1 = 0$, $s_2 = s_3 = 1/2$ we have that

$$|b_\varepsilon(n^-, m^+, A_\varepsilon^{1/2} n^+)| \le C\varepsilon^{-1/2}|n^-|_{1/2}|m^+|_1|n^+|_{3/2}$$
$$\le C\varepsilon^{-1/2}|m|_1|n|_{1/2}|n|_{3/2} \le C\delta^{-1}\varepsilon^{-1}|m|^2_1|n|^2_{1/2} + \delta|n|^2_{3/2} \quad (4.4)$$

for any $\delta > 0$. By (2.19) we also have that

$$
\begin{aligned}
|b_\varepsilon(m^-, n^+, A_\varepsilon^{1/2} n^+)| &\le C\varepsilon^{-1/2}|m^-|_1|n^+|_{1/2}|n^+|_{3/2} \\
&\le C\varepsilon^{-1/2}|m|_1|n|_{1/2}|n|_{3/2} \le C\delta^{-1}\varepsilon^{-1}|m|_1^2|n|_{1/2}^2 + \delta|n|_{3/2}^2 \qquad (4.5)
\end{aligned}
$$

for any $\delta > 0$. Now we use a relation similar to (4.2) for n^- along with the corresponding inequalities (4.3)–(4.5), where the superscripts "+" and "−" are interchanged, and also the inequality

$$
\begin{aligned}
|(N\tilde{f}^\pm, A_\varepsilon^{1/2} n^\pm)| &\le |N\tilde{f}^\pm||n^\pm|_1 \le C\varepsilon^{1/2}|N\tilde{f}^\pm||n^\pm|_{3/2} \\
&\le C\delta^{-1}\varepsilon|N\tilde{f}|^2 + \delta|n|_{3/2}^2
\end{aligned}
$$

for any $\delta > 0$ to obtain (after rescaling δ) the relation

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|n|_{1/2}^2 + \nu^+|n|_{3/2}^2 + 2\nu^-(n^-, n^+)_{3/2} \\
&\quad \le \frac{c_1}{\delta}|n|_{1/2}^2\left[\varepsilon^{-1}|m|_1^2 + |n|_{3/2}^2\right] + \frac{c_2}{\delta}\varepsilon|N\tilde{f}|^2 + \delta|n|_{3/2}^2.
\end{aligned}
$$

Similar to (2.14) for any $\sigma \ge 0$ we have that

$$
\nu^+|n|_\sigma^2 + 2\nu^-(n^-, n^+)_\sigma = 2\left(\nu_1|Nu|_\sigma^2 + \nu_2 s|Nb|_\sigma^2\right) \ge \frac{\nu_0}{2}|n|_\sigma^2,
$$

where $\nu_0 = \min\{\nu_1, \nu_2\}$. Therefore choosing $\delta = \nu_0/4$ yields

$$
\frac{d}{dt}|n|_{1/2}^2 + \frac{\nu_0}{2}|n|_{3/2}^2 \le \frac{c_1}{\nu_0}|n|_{1/2}^2\left[\varepsilon^{-1}|m|_1^2 + |n|_{3/2}^2\right] + \frac{c_2}{\nu_0}\varepsilon|N\tilde{f}|^2. \qquad (4.6)
$$

Assume that

$$
|n(0)|_{1/2}^2 = |(I-M)z^+(0)|_{1/2}^2 + |(I-M)z^-(0)|_{1/2}^2 \le K_*^2. \qquad (4.7)
$$

for some K_* such as $K_*^2 < \nu_0^2(4c_1)^{-1}$. By continuity we have that there exists $0 < T_* \le +\infty$ such that

$$
|n(t)|_{1/2}^2 \le \nu_0^2(4c_1)^{-1} \quad \text{for any} \quad 0 \le t < T_*.
$$

If $T_* < +\infty$, then we can assume $|n(T_*)|_{1/2}^2 = \nu_0^2(4c_1)^{-1}$. We show that this relation leads to a contradiction.

We have from (4.6) that

$$
\frac{d}{dt}|n|_{1/2}^2 + \frac{\nu_0}{4}|n|_{3/2}^2 \le \frac{c_1}{\nu_0\varepsilon}|n|_{1/2}^2|m|_1^2 + \frac{c_2}{\nu_0}\varepsilon|Nf|^2, \quad 0 \le t \le T_*. \qquad (4.8)
$$

Since $|n|_{3/2}^2 \ge k_0^{-2}\varepsilon^{-2}|n|_{1/2}^2$, in the same way as in [9] by Gronwall's type argument we obtain that

$$
|n(t)|_{1/2}^2 \le |n(0)|_{1/2}^2 e^{h^*(t)} + \frac{c_2 k_0^2}{\nu_0^2}\varepsilon^3 \max_{\tau\in\mathbb{R}_+}|N\tilde{f}(\tau)|^2 \max_{0<\tau<t} e^{h(t)-h(\tau)} \qquad (4.9)
$$

for every $0 \le t \le T_*$, where

$$h(t) = -\frac{\nu_0}{8k_0^2\varepsilon^2}t + \frac{c_1}{\nu_0\varepsilon}\int_0^t |m(\tau)|_1^2 d\tau, \quad h^*(t) = -\frac{\nu_0}{8k_0^2\varepsilon^2}t + h(t)$$

It follows from (2.10) that

$$\frac{1}{2}\frac{d}{dt}|z^\pm|_0^2 + \nu^+|z^\pm|_1^2 + \nu^-(z^-, z^+)_1 = (\tilde{f}^\pm, z^\pm).$$

Therefore using (2.14) we obtain that

$$\frac{1}{2}\frac{d}{dt}|z|^2 + \frac{\nu_0}{2}|z|_1^2 \le (\tilde{f}^+, z^+) + (\tilde{f}^-, z^-).$$

Since

$$\begin{aligned}(\tilde{f}^\pm, z^\pm) &\le |N\tilde{f}^\pm||n^\pm| + |M\tilde{f}^\pm||m^\pm| \\ &\le \frac{c}{\nu_0}\left(\varepsilon^2|N\tilde{f}|^2 + |M\tilde{f}|^2\right) + \frac{\nu_0}{8}|z|_1^2,\end{aligned}$$

we obtain

$$\frac{d}{dt}|z|^2 + \frac{\nu_0}{2}|z|_1^2 \le \frac{c}{\nu_0}\left(\varepsilon^2 \sup_{\mathbb{R}_+}|N\tilde{f}|^2 + \sup_{\mathbb{R}_+}|M\tilde{f}|^2\right) \equiv \frac{B}{\nu_0}.$$

Since $|z|_1^2 \ge \mu_0^{-2}|z|^2$ for some $\mu_0 > 0$, this implies that

$$|z(t)|^2 \le e^{-\nu_0 t/(2\mu_0^2)}|z(0)|^2 + \frac{2B}{\nu_0^2}\mu_0^2$$

and

$$|z(t)|^2 + \frac{\nu_0}{2}\int_\tau^t |z|_1^2 \le |z(\tau)|^2 + (t-\tau)\frac{B}{\nu_0}.$$

Consequently

$$h(t) - h(\tau) \le -\frac{\nu_0(t-\tau)}{8k_0^2\varepsilon^2} + \frac{c_1}{\nu_0^2\varepsilon}\left(|z(0)|^2 + \frac{B}{\nu_0}\left[(t-\tau) + \min\{\nu_0^{-1}, \tau\}\right]\right).$$

If

$$\varepsilon^2 \sup_{\mathbb{R}_+}|N\tilde{f}|^2 + \sup_{\mathbb{R}_+}|M\tilde{f}|^2 \equiv \frac{B}{c} \le \frac{c_*\nu_0^4}{\varepsilon} \tag{4.10}$$

with an appropriate $c_* > 0$, then we have that

$$h(t) - h(\tau) \le \frac{c_1}{\nu_0^2\varepsilon}\left(|z(0)|^2 + \frac{B}{\nu_0^2}\min\{1, \nu_0\tau\}\right).$$

Therefore, since $|z(0)|^2 \le \varepsilon |n(0)|^2_{1/2} + |m(0)|^2$, (4.9) yields

$$\begin{aligned} |n(t)|^2_{1/2} \quad &\le \quad \exp\left\{ \frac{c_1}{\nu_0^2} \left(|n(0)|^2_{1/2} + \frac{1}{\varepsilon}|m(0)|^2 \right) \right\} \Bigg[|n(0)|^2_{1/2} \qquad (4.11) \\ &+ \frac{c_2}{\nu_0^2}\varepsilon^3 \max_{\mathbb{R}_+} |N\tilde{f}|^2 \exp\left\{ \frac{c_1}{\nu_0^4} \left(\varepsilon \sup_{\mathbb{R}_+} |N\tilde{f}|^2 + \frac{1}{\varepsilon} \sup_{\mathbb{R}_+} |M\tilde{f}|^2 \right) \right\} \Bigg] \end{aligned}$$

for every $0 \le t \le T_*$. Thus, if

$$\begin{aligned} &\exp\left\{ \frac{c_1}{\nu_0^2} \left(|n(0)|^2_{1/2} + \frac{1}{\varepsilon}|m(0)|^2 \right) \right\} \Bigg[|n(0)|^2_{1/2} \qquad (4.12) \\ &+ \frac{c_2}{\nu_0^2}\varepsilon^3 \max_{\mathbb{R}_+} |N\tilde{f}|^2 \exp\left\{ \frac{c_1}{\nu_0^4} \left(\varepsilon \sup_{\mathbb{R}_+} |N\tilde{f}|^2 + \frac{1}{\varepsilon} \sup_{\mathbb{R}_+} |M\tilde{f}|^2 \right) \right\} \Bigg] < \nu_0^2 (4c_1)^{-1} \end{aligned}$$

and relations (4.7) with K^* small enough and (4.10) hold, then (4.11) at time T_* will contradict to relation $|n(T_*)|^2_{1/2} = \nu_0^2(4c_1)^{-1}$. Thus in this case $T_* = \infty$. It is easy to see that the latter conditions follow from the hypotheses in (3.2) and (3.3) provided ε is small enough. Thus it follows from (4.8) and (4.11) that there is a solution $z(t)$ satisfying (3.1).

Remark 4.1 The solvability conditions in Theorem 3.2 can be formulated in the form similar to [9]. Namely we can state that there exists K_* such that under the conditions in (4.7), (4.10) and (4.12) problem (2.10) has a weak Leray solution satisfying (3.1).

Step 2: relations (3.4)–(3.7)**.** As it was mentioned in the sketch of the proof of Theorem 3.1 property (3.4) follows from (3.1).

The properties in (3.6) and (3.7) are direct consequences of (4.11).

Thus we need only to establish relation (3.5). It is sufficient to prove this property for m component. As above we have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|m^+|^2_{1/2} + \nu^+|m^+|^2_{3/2} + \nu^-(m^-, m^+)_{3/2} \\ &\qquad + b_\varepsilon(n^-, n^+, A_\varepsilon^{1/2} m^+) + b_\varepsilon(m^-, m^+, A_\varepsilon^{1/2} m^+) = (M\tilde{f}^+, A_\varepsilon^{1/2} m^+). \end{aligned}$$

By (2.18) we have that

$$|b_\varepsilon(n^-, n^+, A_\varepsilon^{1/2} m^+)| \le C_\varepsilon |n^-|_1 |n^+|_1 |m^+|_{3/2} \le C_{\varepsilon,\delta} |n|^2_{1/2} |n|^2_1 + \delta |m|^2_{3/2}$$

for every $\delta > 0$. Similarly (2.21) yields

$$|b_\varepsilon(m^-, m^+, A_\varepsilon^{1/2} m^+)| \le C_{\varepsilon,\delta} |m|^2_{1/2} |m|^2_1 + \delta |m|^2_{3/2}.$$

Thus with an appropriate choice of $\delta > 0$ we obtain that

$$\frac{d}{dt}|m|^2_{1/2} + \frac{\nu_0}{2}|m|^2_{3/2} \le c_1 |m|^2_{1/2} |m|^2_1 + c_2 \left[|n|^2_{1/2} |n|^2_1 + |M\tilde{f}|^2 \right].$$

Since $\int_0^t \left[|m|_1^2 + |n|_1^2\right] < \infty$ and $\sup_{\mathbb{R}_+} |n(t)|_{1/2} \le C_\varepsilon$, Gronwall's type argument yields the conclusion desired.

Step 3: regularization property and strong solutions. If we multiply equation (2.10a) in H_ε by $A_\varepsilon z^+$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|z^+|_1^2 + \nu^+|z^+|_2^2 + \nu^-(z^-, z^+)_2 + b_\varepsilon(z^-, z^+, A_\varepsilon m^+)$$
$$+ b_\varepsilon(n^-, n^+, A_\varepsilon n^+) + b_\varepsilon(n^-, m^+, A_\varepsilon n^+) + b_\varepsilon(m^-, n^+, A_\varepsilon n^+) = (\tilde{f}^+, A_\varepsilon z^+).$$

By (2.22) we have

$$|b_\varepsilon(z^-, z^+, A_\varepsilon m^+)| \le C|z|_{1/2}|z|_{3/2}|A_\varepsilon m^+|.$$

Here and in the next relation the constant C may depend on ε. One can also see from (2.18) that

$$|b_\varepsilon(n^-, m^+, A_\varepsilon n^+) + b_\varepsilon(m^-, n^+, A_\varepsilon n^+)| \le C|z|_{1/2}|z|_{3/2}|A_\varepsilon n^+|.$$

Now we estimate the term $b_\varepsilon(n^-, n^+, A_\varepsilon n^+)$. By the Hölder inequality

$$|b_\varepsilon(n^-, n^+, A_\varepsilon n^+)| \le c_0 \|n^-\|_{L_3(\mathcal{O}_\varepsilon)} \|\nabla n^+\|_{L_6(\mathcal{O}_\varepsilon)} |A_\varepsilon n^+|,$$

where c_0 is independent of ε. Since $H^{1/2}(\mathcal{O}_\varepsilon) \subset L_3(\mathcal{O}_\varepsilon)$ and $H^1(\mathcal{O}_\varepsilon) \subset L_6(\mathcal{O}_\varepsilon)$, we have estimates

$$\|n^-\|_{L_3(\mathcal{O}_\varepsilon)} \le c_1 |n^-|_{1/2} \quad \text{and} \quad \|\nabla n^+\|_{L_6(\mathcal{O}_\varepsilon)} \le c_2 |n^+|_2$$

We emphasize that in these relations the constants c_1 and c_2 do not depend on ε (it follows from the interpolation and from the fact that $\|n^+\|_{L_6(\mathcal{O}_\varepsilon)} \le C|\nabla n^+|$ with the constant independent of ε, see, e.g., [21]). Thus we obtain

$$|b_\varepsilon(n^-, n^+, A_\varepsilon n^+)| \le c|n^-|_{1/2}|A_\varepsilon n^+|^2,$$

where c is independent of ε. After applying the same procedure to equation (2.10b) we arrive to the inequality

$$\frac{d}{dt}|z|_1^2 + \frac{\nu_0}{2}|z|_2^2 \le c_0|n|_{1/2}|A_\varepsilon z|^2 + C_\varepsilon \left[|z|_{1/2}^2|z|_{3/2}^2 + |\tilde{f}|^2\right].$$

where c_0 does not depend on ε. Therefore by (3.6) we have that

$$\frac{d}{dt}|z|_1^2 + \left(\frac{\nu_0}{2} - c_0 C_1^{1/2}\varepsilon^{\eta/2}\right)|z|_2^2 \le C_\varepsilon \left[|z|_{1/2}^2|z|_{3/2}^2 + |\tilde{f}|^2\right].$$

for all $t \ge 0$. Thus taking ε_* small enough yields

$$\frac{d}{dt}|z|_1^2 + \frac{\nu_0}{4}|z|_2^2 \le C_\varepsilon \left[|z|_{1/2}^2|z|_{3/2}^2 + |\tilde{f}|^2\right], \quad t \ge 0. \tag{4.13}$$

Multiplying this relation by t, after integration we obtain

$$t|z(t)|_1^2 + \frac{\nu_0}{4}\int_0^t \tau|z|_2^2 d\tau \le \int_0^t |z|_1^2 d\tau + C_\varepsilon \int_0^t \tau|z|_{1/2}^2|z|_{3/2}^2 d\tau + C_\varepsilon \int_0^t \tau|\tilde{f}|^2 d\tau.$$

By (3.5) this implies (3.8).

If $z_0 \in V_\varepsilon^1$, then it follows from (4.13) that

$$|z(t)|_1^2 + \int_0^t |z(\tau)|_2^2 d\tau \le |z(0)|_1^2 + C_T(\varepsilon, |z_0|_{1/2}, |\tilde{f}|), \quad t \in [0, T].$$

This implies that the solution $z(t)$ is strong. Thus the proof of Theorem 3.2 is complete.

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