THE QUINTIC NLS AS THE MEAN FIELD LIMIT OF A BOSON GAS WITH THREE-BODY INTERACTIONS

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ABSTRACT. We investigate the dynamics of a boson gas with three-body interactions in dimensions $d = 1, 2$. We prove that in the limit where the particle number N tends to infinity, the BBGKY hierarchy of k-particle marginals converges to a limiting (Gross-Pitaevskii (GP)) hierarchy for which we prove existence and uniqueness of solutions. The solutions of the GP hierarchy are shown to be determined by solutions of a quintic nonlinear Schrödinger equation. Our proof is based on, and extends, methods of Erdös-Schlein-Yau, Klainerman-Machedon, and Kirkpatrick-Schlein-Staffilani.

1. Introduction

In the work at hand, we study the dynamical mean field limit of a nonrelativistic Bose gas with 3-particle interactions in space dimensions $d = 1, 2$. We prove that the BBGKY hierarchy of marginal density matrices converges to an infinite hierarchy whose solutions are determined by solutions of a *quintic* nonlinear Schrödinger equation (NLS), provided that the initial conditions have product form. Our proof is based on adapting methods of Erdös-Schlein-Yau, [6], Klainerman-Machedon, [12], and Kirkpatrick-Schlein-Staffilani, [13], to this problem, and parts of our exposition follow quite closely [13] and [12]. In a companion paper, we discuss the Cauchy problem for the GP hierarchy in more generality, [4].

We consider a system of N bosons whose dynamics is generated by the Hamiltonian

$$H_N := \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N^2}\sum_{1\leq i<j<k\leq N} N^{2d\beta}V(N^\beta(x_i-x_j), N^\beta(x_i-x_k))\,, \quad (1.1)$$

on the Hilbert space $\mathcal{H}_N = L^2_{sym}(\mathbb{R}^{dN})$. The elements $\Psi(x_1,\dots,x_N) \in \mathcal{H}_N$ are fully symmetric with respect to permutations of the arguments x_j. We assume that the translation-invariant three-body potential V has the properties

$$V \geq 0 \quad , \quad V(x,y) = V(y,x) \quad , \quad V \in W^{3,p}(\mathbb{R}^{2d}) \quad (1.2)$$

for $2d < p \leq \infty$. We note that, since evidently,

$$\begin{aligned} U(x_1-x_2, x_2-x_3, x_1-x_3) &= U(x_1-x_2, -(x_1-x_2)+(x_1-x_3), x_1-x_3) \\ &\equiv V(x_1-x_2, x_1-x_3)\,, \end{aligned} \quad (1.3)$$

every translation invariant three-body interaction potential U can be written in the above form.

The solutions of the Schrödinger equation

$$i\partial_t \Psi_{N,t} \;=\; H_N\, \Psi_{N,t} \tag{1.4}$$

with initial condition $\Psi_N \in \mathcal{H}_N$ determine the N-particle density matrix

$$\gamma_N(t;\underline{x}_N;\underline{x}_N') \;=\; \overline{\Psi_{N,t}(\underline{x}_N)}\Psi_{N,t}(\underline{x}_N') \tag{1.5}$$

and its k-particle marginals

$$\gamma^{(k)}_{N,t}(t;\underline{x}_k;\underline{x}_k') \;=\; \int d\underline{x}_{N-k}\gamma_N(t;\underline{x}_k,\underline{x}_{N-k};\underline{x}_k',\underline{x}_{N-k})\,, \tag{1.6}$$

for $k=1,\dots,N$, where $\underline{x}_k=(x_1,\dots,x_k)$, $\underline{x}_{N-k}=(x_{k+1},\dots,x_N)$, etc.

The BBGKY hierarchy is given by

$$\begin{aligned} i\partial_t\gamma^{(k)}_{N,t} \;=\; & \sum_{j=1}^k[-\Delta_{x_j},\gamma^{(k)}_{N,t}] + \frac{1}{N^2}\sum_{1\leq i<j<\ell\leq k}[V_N(x_i-x_j,x_i-x_\ell),\gamma^{(k)}_{N,t}] \\ & + \frac{(N-k)}{N^2}\sum_{1\leq i<j\leq k}\mathrm{Tr}_{k+1}[V_N(x_i-x_j,x_i-x_{k+1}),\gamma^{(k+1)}_{N,t}] \\ & + \frac{(N-k)(N-k-1)}{N^2}\sum_{j=1}^k \mathrm{Tr}_{k+1}\mathrm{Tr}_{k+2}[V_N(x_j-x_{k+1},x_j-x_{k+2}),\gamma^{(k+2)}_{N,t}] \end{aligned} \tag{1.7}$$

where

$$V_N(x,y) \;:=\; N^{2d\beta}V(N^\beta x,N^\beta y)\,. \tag{1.8}$$

We note that in the limit $N\to\infty$, the sums weighted by combinatorial factors have the following size. In the first interaction term on the rhs, we have $\frac{k^2}{N^2}\to 0$ for every fixed k and sufficiently small β, and for the second term $\frac{(N-k)k}{N^2}\approx\frac{k}{N}\to 0$. For the third interaction term on the rhs, we note that $\frac{(N-k)(N-k-1)}{N^2}\to 1$ for every fixed k. Accordingly, a rigorous argument outlined in Section 3 shows that in the limit $N\to\infty$, one obtains the infinite hierarchy

$$i\partial_t\gamma^{(k)}_{\infty,t} \quad = \quad \sum_{j=1}^k[-\Delta_{x_j},\gamma^{(k)}_{\infty,t}] \;+\; b_0\sum_{j=1}^k B_{j;k+1,k+2}\gamma^{(k+2)}_{\infty,t} \tag{1.9}$$

where

$$b_0 \;=\; \int dx_1\,dx_2\,V(x_1,x_2) \tag{1.10}$$

is the coupling constant, and where we will sometimes refer to

$$\begin{aligned} & B_{j;k+1,k+2}\gamma^{(k+2)}_{\infty,t}(x_1,\dots,x_k;x_1',\dots,x_k') \\ & \quad := \int dx_{k+1}dx_{k+1}'dx_{k+2}dx_{k+2}' \\ & \qquad\quad \big[\,\delta(x_j-x_{k+1})\delta(x_j-x_{k+1}')\delta(x_j-x_{k+2})\delta(x_j-x_{k+2}') \\ & \qquad\qquad -\delta(x_j'-x_{k+1})\delta(x_j'-x_{k+1}')\delta(x_j'-x_{k+2})\delta(x_j'-x_{k+2}')\,\big] \\ & \qquad\quad \gamma^{(k+2)}_{\infty,t}(x_1,\dots,x_{k+2};x_1',\dots,x_{k+2}') \end{aligned} \tag{1.11}$$

as the "contraction operator". The topology in which this convergence holds is described in Section 3 below, and is here adopted from [6, 13].

Written in integral form,

$$\gamma^{(k)}_{\infty,t} = \mathcal{U}^{(k)}(t)\,\gamma^{(k)}_{\infty,0} - i\,b_0 \sum_{j=1}^{k} \int_0^t ds\,\mathcal{U}^{(k)}(t-s) B_{j;k+1,k+2}\gamma^{(k+2)}_{\infty,s} \tag{1.12}$$

where

$$\mathcal{U}^{(k)}(t)\,\gamma^{(k)}_{\infty,s} := e^{it\Delta^{(k)}_{\pm}}\,\gamma^{(k)}_{\infty,s}\,, \tag{1.13}$$

and

$$\Delta^{(k)}_{\pm} = \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k} \tag{1.14}$$

with

$$\Delta_{\pm,x_j} = \Delta_{x_j} - \Delta_{x'_j}\quad,\quad \Delta_{\underline{x}_k} = \sum_{j=1}^{k} \Delta_{x_j}\,. \tag{1.15}$$

Accordingly, it is easy to see that

$$\gamma^{(k)}_{\infty,t} = \big|\,\phi_t\,\big\rangle\big\langle\,\phi_t\,\big|^{\otimes k} \tag{1.16}$$

is a solution of (1.12) if ϕ_t satisfies the quintic NLS

$$i\partial_t\phi_t + \Delta_x\phi_t - b_0\,|\phi_t|^4\,\phi_t = 0 \tag{1.17}$$

with $\phi_0 \in L^2(\mathbb{R}^d)$.

Theorem 1.1. *Assume that $d \in \{1,2\}$, and that $V \in W^{3,p}$ for $p > 2d$, $V(x,x') = V(x',x)$, $V \geq 0$, and $0 < \beta < \frac{1}{2d+3}$. Let $\{\Psi_N\}_N$ denote a family such that $\frac{1}{N}\big\langle\Psi_N, H_N\Psi_N\big\rangle < \infty$, and which exhibits asymptotic factorization; that is, there exists $\phi \in L^2(\mathbb{R}^d)$ such that $\mathrm{Tr}\big|\,\gamma^{(1)}_N - |\phi\rangle\langle\phi|\,\big| \to 0$ as $N \to \infty$. Then, it follows for the k-particle marginals $\gamma^{(k)}_{N,t}$ associated to $\Psi_{N,t} = e^{-itH_N}\Psi_N$ that*

$$\mathrm{Tr}\Big|\,\gamma^{(k)}_{N,t} - |\,\phi_t\rangle\langle\phi_t\,|^{\otimes k}\,\Big| \to 0 \qquad (N\to\infty) \tag{1.18}$$

where ϕ_t solves the defocusing quintic nonlinear Schrödinger equation

$$i\partial_t\phi_t + \Delta\phi_t - b_0\,|\phi_t|^4\phi_t = 0 \tag{1.19}$$

with initial condition $\phi_0 = \phi$, and with $b_0 = \int_{\mathbb{R}^{2d}} dx\,dx'\,V(x,x')$.

The mathematical study of systems of interacting Bose gases is a central research area in mathematical physics which is currently experiencing remarkable progress. A problem of fundamental importance is to prove, in mathematically rigorous terms, that Bose-Einstein condensation occurs for such systems. Fundamental progress in the understanding of this problem and its solution in crucial cases, is achieved by Lieb, Seiringer, Yngvason, et al., in a landmark body of work; see for instance [2, 14, 15, 16] and the references therein.

A related, very active line of research addresses the derivation of the mean field dynamics for a dilute Bose gas, in a scaling regime where the interparticle interactions and the kinetic energy are comparable in magnitude. Some important early results were obtained in [11, 19].

In a highly influential series of works, Erdös, Schlein and Yau have proved for a Bose gas in $\mathbb{R}^3$, with a pair interaction potential that scales to a delta distribution for particle number $N \to \infty$, that the limiting dynamics is governed by a cubic NLS, see [5, 6, 7, 18] and the references therein. In their proof, the BBGKY hierarchy of k-particle marginal density matrices is proven to converge to an infinite limiting hierarchy (the *Gross-Pitaevskii (GP) hierarchy*) in the limit $N \to \infty$, and the existence and uniqueness of solutions is established for the infinite hierarchy. Their uniqueness proof uses sophisticated Feynman diagram expansions which are closely related to renormalization methods in quantum field theory, and represents the most involved part of their analysis.

Recently, Klainerman and Machedon have developed an alternative method to prove the uniqueness of solutions of the GP hierarchy, based on the recursive verification of certain spacetime bounds satisfied by the k-particle marginals, for the model in $\mathbb{R}^3$, [12]. Their result assumes an a priori spacetime bound which is not proven in [12]. Subsequently, Kirkpatrick, Schlein and Staffilani have proven a variant of this a priori bound for the model on $\mathbb{R}^2$ and on the torus $\mathbb{T}^2$, and derived the corresponding mean-field limits, [13]. In dimension 1, the cubic NLS is derived in [1]. Control of the rate of convergence of the quantum evolution towards a mean-field limit of Hartree type as $N \to \infty$ has recently been obtained by Rodnianski and Schlein, [17]. The derivation of mean-field limits based on operator-theoretic methods is developed in work of Fröhlich et al., [8, 9, 10].

All of the works cited above investigate properties of Bose gases with pair interactions, which is natural in the absence of interactions with any external fields. However, once the interaction of the Bose gas with a background field of matter is included in the model (for instance with phonons, photons, or other kinds of matter), averaging over the latter will typically lead to a linear combination of effective (renormalized, in the sense of quantum field theory) n-particle interactions, $n = 2, 3, \dots$. For systems exhibiting effective interactions of this general structure, it remains a key problem to determine the mean field dynamics. For n-particle interactions with $n = 2, 3$, where the microscopic Hamiltonian would have the form

$$\begin{aligned} H_N \ &:= \ \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\le i<j\le N} N^{2d\beta}V_2(N^\beta(x_i-x_j)) \\ &\quad + \frac{1}{N^2}\sum_{1\le i<j<k\le N} N^{2d\beta}V_3(N^\beta(x_i-x_j),N^\beta(x_i-x_k))\,, \end{aligned} \tag{1.20}$$

a combination of the analysis given in [13] with the one presented here will straightforwardly produce a mean field limit described by the defocusing NLS

$$i\partial_t\phi_t + \Delta\phi_t - \lambda_2|\phi_t|^2\phi_t - \lambda_3|\phi_t|^4\phi_t \ = \ 0 \tag{1.21}$$

in $d = 1, 2$, where $\lambda_2 = \int dx V_2(x) \geq 0$ and $\lambda_3 = \int dx dx' V_3(x, x') \geq 0$ account for the mean-field strength of the 2- and 3-body interactions.

Now we briefly describe the approach that we follow in this paper. We prove Theorem 1.1 by modifying the strategy of Erdös-Schlein-Yau [6], Klainerman-Machedon [12] and Kirkpatrick-Schlein-Staffilani [13]. More precisely, we prove the convergence of the BBGKY hierarchy to the GP hierarchy by straightforwardly adapting the arguments from the work [6] (the details are given in sections 2 and 3 where we follow the exposition of these arguments as presented in [13]). In order to prove the uniqueness of the limiting hierarchy, we expand the approach introduced in

[12] and subsequently used in [13]. Roughly speaking this approach consists of two ingredients:

(1) Expressing the solution $\gamma^{(k)}$ to the infinite hierarchy (1.9) in terms of the subsequent iterates $\gamma^{(k+2)}$, ... , $\gamma^{(k+2n)}$ using Duhamel's formula. However since the second term on the rhs of (1.9) involves the sum, the iterated Duhamel's formula has $k(k+2)...(k+2n-2)$ terms. In [12], Klainerman and Machedon introduced an elegant way to group these terms in much fewer $O(C^n)$ sets of terms, by introducing a certain "board game" strategy, by use of which they kept track of the relevant combinatorics. Inspired by the board game of [12], in this paper we define a different board game to suit the new operators $B_{j;k+1,k+2}$ that appear in our limiting hierarchy. This new board game helps us organize the Duhamel's expansions in a similar manner as in [12].

(2) Establishing two types of bounds:
   (a) Space-time $L^2_t L^2_x$ bounds for the freely evolving limiting hierarchy (please, see Theorem 5.1), which shall be used iteratively along the nested Duhamel's expansions.
   (b) Spatial a-priori L^2_x bound for the full limiting hierarchy (please, see Theorem 4.2).

   When $d = 2$, we prove both types of bounds, in a similar way as the authors of [13] do in the context of 2-body limiting hierarchy. On the other hand, when $d = 1$, the argument used to produce $L^2_t L^2_x$ bound of the type (a) for the freely evolving limiting hierarchy would produce a divergent bound, so instead we establish another spatial bound (stated in Theorem 4.3) for the full limiting hierarchy. We use this bound iteratively and at the end combine it with the spatial bound of the type (b).

Organization of the paper. In section 2 we derive a-priori energy bounds for solutions to the BBGKY hierarchy. In section 3 we summarize main steps in the proof of compactness of the sequence of k-particle marginals and their convergence to the infinite hierarchy. In Section 4 we present three types of spatial bounds on the limiting hierarchy, while in section 5 we give spacetime bounds on the freely evolving infinite hierarchy. The sections 6 - 8 are devoted to the proof of uniqueness of the limiting hierarchy. In particular, in section 6 we state the theorem that guaranties uniqueness of the infinite hierarchy, while section 7 concentrates on combinatorial arguments that will be used (together with results of sections 4 and 5) in section 8, where the uniqueness result is proved.

2. A priori energy bounds

To begin with, we derive a priori bounds of the form

$$\mathrm{Tr}(1-\Delta_{x_1})\cdots(1-\Delta_{x_k})\gamma^{(k)}_{N,t} \, < \, C^k \tag{2.1}$$

which are obtained from energy conservation, following [7, 6] and [13].

Proposition 2.1. *There exists a constant C, and for every k, there exists $N_0(k)$ such that for all $N \geq N_0(k)$,*

$$\big\langle\, \psi\,,\, (H_N + N)^k\, \psi\,\big\rangle \;\geq\; C^k N^k \big\langle\, \psi\,,\, (1-\Delta_{x_1})\,\cdots\,(1-\Delta_{x_k})\,\psi\,\big\rangle \tag{2.2}$$

for all $\psi \in L^2_s(\mathbb{R}^{dN})$.

Proof. We adapt the proof in [13] to the current case, which is based on induction in k. We first note that for $k = 0$, the statement is trivial, and that for $k = 1$, it follows from $V_N \geq 0$. For the induction step, we assume that for all $k \leq n$, the statement is correct. We then prove its validity for $n + 2$. Following [13], we write $S_i = (1 - \Delta_{x_i})^{1/2}$ and $H_N + N = h_1 + h_2$ with

$$\begin{aligned} h_1 &= \sum_{j=n+1}^{N} S_j^2 \\ h_2 &= \sum_{j=1}^{n} S_j^2 + \sum_{1\leq i<j<\ell\leq N} N^{-2} V_N(x_i - x_j, x_i - x_\ell)\,. \end{aligned} \tag{2.3}$$

Using the induction assumption, we infer that

$$\begin{aligned} &\big\langle\, \psi\,,\, (H_N + N)^{n+2}\, \psi\,\big\rangle \\ &\quad\geq\; C^n N^n \big\langle\, \psi\,,\, (H_N + N)\, S_1^2 \cdots\, S_n^2\, (H_N + N)\,\psi\,\big\rangle \\ &\quad\geq\; C^n N^n \big\langle\, \psi\,,\, h_1\, S_1^2 \cdots\, S_n^2\, h_1\,\psi\,\big\rangle \,+\, C^n N^n \big(\big\langle\, \psi\,,\, h_1\, S_1^2 \cdots\, S_n^2\, h_2\,\psi\,\big\rangle \,+\, c.c.\big) \\ &\quad\geq\; C^n N^n (N-n)(N-n-1) \big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+2}^2\,\psi\,\big\rangle \\ &\qquad +C^n N^n (N-n) n \big\langle\, \psi\,,\, h_1\, S_1^4 \cdots\, S_{n+1}^2\,\psi\,\big\rangle \\ &\qquad +C^n N^n \frac{(N-n)}{N^2} \,\cdot \\ &\qquad \cdot \sum_{1\leq i<j<\ell\leq N} \big(\big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+1}^2\, V_N(x_i - x_j, x_i - x_\ell)\,\psi\,\big\rangle \,+\, c.c.\big)\,. \end{aligned} \tag{2.4}$$

Making use of the permutation symmetry of ψ with respect to its arguments, there exists $N_0(n)$ such that for all $N > N_0(n)$,

$$\begin{aligned} &\big\langle\, \psi\,,\, (H_N + N)^{n+2}\, \psi\,\big\rangle \\ &\quad\geq\; C^{n+2} N^{n+2} \big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+2}^2\,\psi\,\big\rangle \,+\, C^{n+1} N^{n+1} \big\langle\, \psi\,,\, S_1^4 \cdots\, S_{n+1}^2\,\psi\,\big\rangle \\ &\qquad +\Big[C^n N^{n-2} (N-n)(N-n-1)(N-n-2)(N-n-3) \\ &\qquad\qquad\qquad \big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+1}^2\, V_N(x_{n+2} - x_{n+3}, x_{n+2} - x_{n+4})\,\psi\,\big\rangle \\ &\qquad +C^n N^{n-2} (N-n)(N-n-1)(N-n-2)(n+1) \\ &\qquad\qquad\qquad \big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+1}^2\, V_N(x_1 - x_{n+2}, x_1 - x_{n+3})\,\psi\,\big\rangle \\ &\qquad +C^n N^{n-2} (N-n)(N-n-1) n (n-1) \\ &\qquad\qquad\qquad \big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+1}^2\, V_N(x_1 - x_2, x_1 - x_{n+2})\,\psi\,\big\rangle \\ &\qquad +C^n N^{n-2} (N-n)(n+1) n (n-1) \big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+1}^2\, V_N(x_1 - x_2, x_1 - x_3)\,\psi\,\big\rangle \\ &\qquad + c.c. \Big]\,. \end{aligned} \tag{2.5}$$

Now we shall control the terms in $[\cdots]$ of (2.5). As in [13], we note that the first term in $[\cdots]$ can be neglected because it is positive. More precisely, since all

$S_1,\dots,S_{n+1}$ commute with $V_N(x_{n+2}-x_{n+3},x_{n+2}-x_{n+4})$ we have:

$$\begin{aligned}&\big\langle\,\psi\,,\,S_1^2\cdots\,S_{n+1}^2\,V_N(x_{n+2}-x_{n+3},x_{n+2}-x_{n+4})\,\psi\,\big\rangle\\&\quad=\int d\underline{x}_N V_N(x_{n+2}-x_{n+3},x_{n+2}-x_{n+4})\,|(S_1\cdots\,S_{n+1}\psi)(\underline{x}_N)|^2\,\geq\,0\end{aligned}\tag{2.6}$$

which follows from the positivity of V_N. Hence, in order to obtain a lower bound on (2.5), the first term in $[\cdots]$ can be discarded.

For the second term in $[\cdots]$ in (2.5), we use that

$$\begin{aligned}&\big\langle\,\psi\,,\,S_1^2\cdots\,S_{n+1}^2\,V_N(x_1-x_{n+2},x_1-x_{n+3})\,\psi\,\big\rangle\\
&\quad\geq\;-\,|\big\langle\,\psi\,,\,S_{n+1}\cdots S_2S_1[S_1,V_N(x_1-x_{n+2},x_1-x_{n+3})]S_2\cdots\,S_{n+1}\,\psi\,\big\rangle|\\
&\quad\geq\;-\,|\big\langle\,\psi\,,\,S_{n+1}\cdots S_2S_1|\nabla_{x_1}V_N(x_1-x_{n+2},x_1-x_{n+3})|S_2\cdots\,S_{n+1}\,\psi\,\big\rangle|\\
&\quad\geq\;-\mu\,|\big\langle\,\psi\,,\,S_{n+1}^2\cdots S_1^2\,\psi\,\big\rangle|\\
&\qquad\;-\mu^{-1}\,|\big\langle\,\psi\,,\,S_{n+1}\cdots S_2|\nabla_{x_1}V_N(x_1-x_{n+2},x_1-x_{n+3})|^2S_2\cdots\,S_{n+1}\,\psi\,\big\rangle|\\
&\quad\geq\;-\,\mu\,|\big\langle\,\psi\,,\,S_{n+1}^2\cdots S_1^2\,\psi\,\big\rangle|\\
&\qquad\;-C\,\mu^{-1}\,\|\nabla V_N\|_{L^{2p}(\mathbb{R}^{2d})}^2\big\langle\,\psi\,,\;S_1^2\cdots\,S_{n+2}^2\,\psi\,\big\rangle\,,\end{aligned}\tag{2.7}$$

for $p>d$, where in order to obtain (2.7) we used the Schwarz inequality and Lemma 2.2. As a consequence of

$$\|\nabla V_N\|_{L^{2p}(\mathbb{R}^{2d})}\,\leq\,CN^{2\beta(d+\frac{1}{2}-\frac{d}{2p})}\|\nabla V\|_{L^{2p}(\mathbb{R}^{2d})}\,,\tag{2.8}$$

the expression (2.7) implies

$$\begin{aligned}&\big\langle\,\psi\,,\,S_1^2\cdots\,S_{n+1}^2\,V_N(x_1-x_{n+2},x_1-x_{n+3})\,\psi\,\big\rangle\\&\qquad\geq\,-\,C\,N^{\beta(2d+1)}\,\big\langle\,\psi\,,\,S_1^2\cdots\,S_{n+2}^2\,\psi\,\big\rangle\end{aligned}\tag{2.9}$$

for a constant $C>0$.

For the third term in $[\cdots]$ in (2.5), we use

$$\begin{aligned}&\big\langle\,\psi\,,\,S_1^2\cdots\,S_{n+1}^2\,V_N(x_1-x_2,x_1-x_{n+2})\,\psi\,\big\rangle\\
&\quad\geq\;-\,|\big\langle\,\psi\,,\,S_{n+1}\cdots S_3S_2S_1[S_1S_2,V_N(x_1-x_2,x_1-x_{n+2})]S_3\cdots\,S_{n+1}\,\psi\,\big\rangle|\\
&\quad\geq\;-\,|\big\langle\,\psi\,,\,S_{n+1}\cdots S_2S_1|\nabla_{x_1}\nabla_{x_2}V_N(x_1-x_2,x_1-x_{n+2})|S_3\cdots\,S_{n+1}\,\psi\,\big\rangle|\\
&\quad\geq\;-\nu\,|\big\langle\,\psi\,,\,S_{n+1}^2\cdots S_1^2\,\psi\,\big\rangle|\\
&\qquad\;-\nu^{-1}\,|\big\langle\,\psi\,,\,S_{n+1}\cdots S_3|\nabla_{x_1}\nabla_{x_2}V_N(x_1-x_2,x_1-x_{n+2})|^2S_3\cdots\,S_{n+1}\,\psi\,\big\rangle|\\
&\quad\geq\;-\,\nu\,|\big\langle\,\psi\,,\,S_{n+1}^2\cdots S_1^2\,\psi\,\big\rangle|\\
&\qquad\;-C\,\nu^{-1}\,\|V_N\|_{W^{2,2p}(\mathbb{R}^{2d})}^2\big\langle\,\psi\,,\;S_1^2\cdots\,S_{n+1}^2\,\psi\,\big\rangle\\
&\quad\geq\;-\,\nu\,|\big\langle\,\psi\,,\,S_{n+1}^2\cdots S_1^2\,\psi\,\big\rangle|\\
&\qquad\;-C\,\nu^{-1}N^{4\beta(d+1)}\|V\|_{W^{2,2p}(\mathbb{R}^{2d})}^2\big\langle\,\psi\,,\;S_1^2\cdots\,S_{n+1}^2\,\psi\,\big\rangle\\
&\quad\geq\;-CN^{2d\beta+2}\|V\|_{W^{2,2p}(\mathbb{R}^{2d})}^2\big\langle\,\psi\,,\;S_1^2\cdots\,S_{n+2}^2\,\psi\,\big\rangle\,,\end{aligned}\tag{2.10}$$

where we used the Schwarz inequality, Lemma 2.2 with $p>d$, and optimized the constant ν.

Finally, for the last term inside $[\cdots]$ in (2.5), we use

$$\begin{aligned}
&\big\langle\, \psi\,,\, S_1^2 \cdots\, S_{n+1}^2\, V_N(x_1-x_2,x_1-x_3)\,\psi\,\big\rangle \\
&\quad\geq\; -\,\big|\big\langle\, \psi\,,\, S_{n+1}\cdots S_3S_2S_1[S_1S_2S_3, V_N(x_1-x_2,x_1-x_3)]S_4\cdots\, S_{n+1}\,\psi\,\big\rangle\big| \\
&\quad\geq\; -\,\big|\big\langle\, \psi\,,\, S_{n+1}\cdots S_1|\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}V_N(x_1-x_2,x_1-x_3)|S_4\cdots\, S_{n+1}\,\psi\,\big\rangle\big| \qquad (2.11)\\
&\quad\geq\; -\,\big|\big\langle\, S_1\cdots S_{n+1}\,\psi\,,\, |\nabla_{x_1}\nabla_{x_2}\nabla_{x_3}V_N(x_1-x_2,x_1-x_3)|S_4\cdots\, S_{n+1}\,\psi\,\big\rangle\big| \\
&\quad\geq\; -\,\|\, S_1\cdots S_{n+1}\,\psi\,\|_{L^2} \qquad (2.12)\\
&\qquad\qquad \|\,|((\nabla_{x_2}+\nabla_{x_3})\nabla_{x_2}\nabla_{x_3}V_N)(-x_2,-x_3)|S_4\cdots\, S_{n+1}\,\psi\,\|_{L^2} \\
&\quad\geq\; -C\|\, S_1\cdots S_{n+1}\,\psi\,\|_{L^2}\,\|V_N\|_{W^{3,2p}(\mathbb{R}^{2d})}\|\, S_2\cdots S_{n+1}\,\psi\,\|_{L^2} \qquad (2.13)\\
&\quad\geq\; -CN^{2\beta(d+\frac{3}{2}-\frac{d}{2p})}\|V\|_{W^{3,2p}(\mathbb{R}^{2d})}\big\langle\, \psi\,,\; S_1^2\cdots\, S_{n+1}^2\,\psi\,\big\rangle\,, \qquad (2.14)
\end{aligned}$$

where we have translated $x_2 \to x_2 + x_1$ and $x_3 \to x_3 + x_1$ to get (2.12). In order to obtain (2.13) we used the Hölder estimate combined with Sobolev embedding. To pass from (2.13) to the last line, we applied $\|\, S_2\cdots S_{n+1}\,\psi\,\| \leq \|\, S_1S_2\cdots S_{n+1}\,\psi\,\|$, using that $S_1 \geq 1$.

In conclusion, the sum of terms inside $[\cdots]$ in (2.5) is bounded below by

$$[\cdots]\;\geq\; -C(n)N^{n-1+2\beta(d+\frac{3}{2})}\big\langle\, \psi\,,\; S_1^2\cdots\, S_{n+1}^2\,\psi\,\big\rangle\,, \qquad (2.15)$$

which is dominated by the first term on the rhs of (2.5), for $\beta < \frac{1}{2d+3}$. Moreover, the second term on the rhs of (2.5) is positive. This immediately establishes the induction step $n \to n+2$. ☐

Lemma 2.2. *For dimension d, the estimate*

$$\begin{aligned}
&\big\langle\, \psi_1\,,\, V(x_1,x_2)\,\psi_2\,\big\rangle \qquad (2.16)\\
&\qquad\leq\; C_{p,d}\,\|V\|_{L^p_{x_1,x_2}}\|\,\langle\nabla_{x_1}\rangle\langle\nabla_{x_2}\rangle\,\psi_1\,\|_{L^2_{x_1,x_2}}\,\|\,\langle\nabla_{x_1}\rangle\langle\nabla_{x_2}\rangle\,\psi_2\,\|_{L^2_{x_1,x_2}}
\end{aligned}$$

holds for any $p \geq d$ if $d \geq 2$, and for any $p > 1$ if $d = 1$.

Proof. Clearly, using the Hölder (for $1 = \frac{1}{p} + \frac{1}{2p'} + \frac{1}{2p'}$) and Sobolev inequalities,

$$\begin{aligned}
|\big\langle\, \psi_1\,,\, V(x_1,x_2)\,\psi_2\,\big\rangle| &\leq C_{p,d}\,\|V\|_{L^p_{x_1,x_2}}\|\psi_1\|_{L^{2p'}_{x_1,x_2}}\|\psi_2\|_{L^{2p'}_{x_1,x_2}} \\
&\leq C_{p,d}\,\|V\|_{L^{2p}_{x_1,x_2}}\|\psi_1\|_{H^1_{x_1,x_2}}\|\psi_2\|_{H^1_{x_1,x_2}} \qquad (2.17)
\end{aligned}$$

provided that $2 \leq 2p' \leq \frac{4d}{2d-2}$ if $d \geq 2$ (interpreting (x_1,x_2) as a point in $\mathbb{R}^{2d}$), and $2 \leq 2p' < \infty$ if $d = 1$. This immediately implies that $d \leq p < \infty$ for $d \geq 2$, and $1 < p < \infty$ for $d = 1$. Moreover, it is clear that

$$\|\psi\|^2_{H^1_{x_1,x_2}} \;=\; \big\langle\, \psi\,,\,(1-\Delta_{x_1}-\Delta_{x_2})\,\psi\,\big\rangle \;\leq\; \big\langle\, \psi\,,\,(1-\Delta_{x_1})(1-\Delta_{x_2})\,\psi\,\big\rangle\,, \qquad (2.18)$$

from $\big\langle\, \psi\,,\,\Delta_{x_1}\Delta_{x_2}\,\psi\,\big\rangle = \|\nabla_{x_1}\nabla_{x_2}\psi\|^2_{L^2} \geq 0$. The claim follows immediately. ☐

In conclusion, we have found the following a priori estimate.

Corollary 2.3. *Define*

$$\widetilde{\psi}_N \;:=\; \frac{\chi(\frac{\kappa}{N}H_N)\psi_N}{\|\chi(\frac{\kappa}{N}H_N)\psi_N\|} \qquad (2.19)$$

where χ is a bump function supported on $[0,1]$, and $\kappa > 0$ is a real parameter. Let $\widetilde{\psi}_{N,t} = e^{-itH_N}\widetilde{\psi}_N$, and let $\widetilde{\gamma}^{(k)}_{N,t}$ be the corresponding k-particle marginal. Then, *there exists a constant C independent of k and there exists an integer $N_0(k)$ for every $k \geq 1$, such that for all $N > N_0(k)$,*

$$\mathrm{Tr}(1-\Delta_{x_1})\cdots(1-\Delta_{x_k})\widetilde{\gamma}^{(k)}_{N,t} \leq C^k . \tag{2.20}$$

3. Compactness and convergence to the infinite hierarchy

In this section, we summarize from [6, 13] the main steps of the proof of compactness of the sequence of k-particle marginals and the convergence to the infinite hierarchy as $N \to \infty$, see also [13]. In the present context, these arguments can be adopted almost verbatim from these works. We outline them here for the convenience of the reader, closely following [13], and without claiming any originality from our part.

The appropriate topology on the space of density matrices is defined as follows in [6]. Letting $\mathcal{K}_k = \mathcal{K}(L^2((\mathbb{R}^d)^k))$ denote the space of compact operators on $L^2(\mathbb{R}^d)$ equipped with the operator norm topology, and $\mathcal{L}^1_k := \mathcal{L}^1(L^2((\mathbb{R}^d)^k))$ the space of trace class operators on $L^2((\mathbb{R}^d)^k)$ equipped with the trace class norm, it is a standard fact that $\mathcal{L}^1_k = \mathcal{K}^*_k$. $\mathcal{K}_k$ is separable, so there is a dense subset $\{J^{(k)}_j\}$, with $\|J^{(k)}_j\| \leq 1$, of the unit ball of $\mathcal{K}_k$. On $\mathcal{L}^1_k$, the metric

$$\eta_k(\gamma^{(k)},\widetilde{\gamma}^{(k)}) = \sum_{j\in\mathbb{N}} 2^{-j}\Big| \mathrm{Tr}J^{(k)}_j\big(\gamma^{(k)}-\widetilde{\gamma}^{(k)}\big)\Big| \tag{3.1}$$

is defined in [6]. A uniformly bounded sequence $\gamma^{(k)}_N \in \mathcal{L}^1_k$ converges to $\gamma^{(k)} \in \mathcal{L}^1_k$ with respect to the weak* topology if and only if $\eta_k(\gamma^{(k)}_N,\gamma^{(k)}) \to 0$ as $N \to \infty$.

Moreover, $C([0,T],\mathcal{L}^1_k)$ shall denote the space of $\mathcal{L}^1_k$-valued functions of $t \in [0,T]$ that are continuous with respect to the metric η_k. On $C([0,T],\mathcal{L}^1_k)$, the metric $\widehat{\eta}_k(\gamma^{(k)}(\cdot),\widetilde{\gamma}^{(k)}(\cdot)) = \sup_{t\in[0,T]}\eta_k(\gamma^{(k)}(t),\widetilde{\gamma}^{(k)}(t))$ is defined in [6]. Then, the topology τ_{prod} is introduced on $\oplus_{k\in\mathbb{N}}C([0,T],\mathcal{L}^1_k)$, given by the product of topologies generated by the metrics $\widehat{\eta}_k$ on $C([0,T],\mathcal{L}^1_k)$.

Proposition 3.1. *The sequence of marginal densities $\widetilde{\Gamma}_{N,t} = \{\widetilde{\gamma}^{(k)}_{N,t}\}_{k=1}^N$ is compact with respect to the product topology τ_{prod} generated by the metrics η_k from [6]. For any limit point $\Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k\geq 1}$, each $\gamma^{(k)}_{\infty,t}$ is symmetric under permutations, is positive, and $\mathrm{Tr}\gamma^{(k)}_{\infty,t} \leq 1$ for every $k \geq 1$.*

Proof. The proof is completely analogous to the one given for a related result in [6], and for Theorem 4.1 in [13]. We summarize the main steps.

Using a Cantor diagonal argument, it is sufficient to prove the compactness of $\widetilde{\gamma}^{(k)}_{N,t}$ for a fixed k. This is achieved by proving equicontinuity of $\Gamma_{N,t} = \{\widetilde{\gamma}^{(k)}_{N,t}\}_{k=1}^N$ with respect to the metric η_k. It is sufficient to prove that for every observable $J^{(k)}$ from a dense subset of $\mathcal{K}_k$ and for every $\epsilon > 0$, there exists $\delta = \delta(J^{(k)},\epsilon)$ such that

$$\sup_{N\geq 1}\Big| \mathrm{Tr}J^{(k)}\big(\widetilde{\gamma}^{(k)}_{N,t}-\widetilde{\gamma}^{(k)}_{N,s}\big)\Big| < \epsilon$$

for all $t, s \in [0,T]$ with $|t-s| < \delta$. To this end, the norm

$$\|\!| J^{(k)} \|\!| \;=\; \sup_{\underline{p}'_k} \int d\underline{p}_k \prod_{j=1}^{k} \langle p_j \rangle \langle p'_j \rangle \Big(\, |\widehat{J}^{(k)}(\underline{p}_k;\underline{p}'_k)| + |\widehat{J}^{(k)}(\underline{p}'_k;\underline{p}_k)| \, \Big) \tag{3.2}$$

is considered in [6, 13], and it is observed that the set of all $J^{(k)} \in \mathcal{K}_k$ for which this norm is finite, is dense in $\mathcal{K}_k$.

The claim of the proposition then follows from

$$\sup_{N\geq 1} \Big| \, \mathrm{Tr} J^{(k)} \, \big(\, \widetilde{\gamma}^{(k)}_{N,t} - \widetilde{\gamma}^{(k)}_{N,s} \, \big) \, \Big| \; < C \, \|\!| J^{(k)} \|\!| \, |t-s| \tag{3.3}$$

which is proved in the same manner as in [6, 13]. □

Theorem 3.2. *Assume that* $\Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k=1}^{\infty} \in \oplus_{k\geq} C([0,T], \mathcal{L}^1_k)$ *is a limit point of* $\widetilde{\Gamma}_{N,t} = \{\widetilde{\gamma}^{(k)}_{N,t}\}_{k=1}^{N}$ *with respect to the product topology* τ_{prod}*. Then,* $\Gamma_{\infty,t}$ *is a solution of the infinite hierarchy*

$$\gamma^{(k)}_{\infty,t} \;=\; \mathcal{U}^{(k)}(t) \, \gamma^{(k)}_{\infty,0} - \, i \, b_0 \sum_{j=1}^{k} \int_0^t ds \, \mathcal{U}^{(k)}(t-s) \, B_{j;k+1,k+2} \, \gamma^{(k+2)}_{\infty,s} \tag{3.4}$$

with initial data $\gamma^{(k)}_{\infty,t} = \big|\phi\big\rangle\big\langle\phi\big|^{\otimes k}$.

Proof. Here again, the proof can be adopted straightforwardly from [13]. We outline the main steps.

Let us fix $k \geq 1$. As in [13], by passing to a subsequence we can assume that for every $J^{(k)} \in \mathcal{K}_k$ we have:

$$\sup_{t\in[0,T]} \mathrm{Tr} \; J^{(k)} \left(\widetilde{\gamma}^{(k)}_{N,t} - \gamma^{(k)}_{\infty,t} \right) \to 0, \qquad \text{as } N \to \infty. \tag{3.5}$$

We shall prove (3.4) by testing the limit point $\gamma^{(k)}_{\infty,t}$ against an observable belonging to a dense set in $\mathcal{K}_k$. In particular, choose an arbitrary $J^{(k)} \in \mathcal{K}_k$ such that $\|\!| J^{(k)} \|\!| < \infty$ (where the definition of the norm $\|\!| \cdot \|\!|$ is given by (3.2)). It suffices to prove that

$$\mathrm{Tr} J^{(k)} \gamma^{(k)}_{\infty,0} = \mathrm{Tr} J^{(k)} \big|\phi\big\rangle\big\langle\phi\big|^{\otimes k} \tag{3.6}$$

and

$$\begin{aligned} \mathrm{Tr} \; J^{(k)} \gamma^{(k)}_{\infty,t} &= \mathrm{Tr} \; J^{(k)} \mathcal{U}^{(k)}(t) \, \gamma^{(k)}_{\infty,0} \\ &\quad - \, i \, b_0 \sum_{j=1}^{k} \int_0^t ds \, \mathrm{Tr} \; J^{(k)} \mathcal{U}^{(k)}(t-s) \, B_{j;k+1,k+2} \, \gamma^{(k+2)}_{\infty,s}. \end{aligned} \tag{3.7}$$

First, we note that (3.6) follows from (3.5). On the other hand in order to prove (3.7) we rewrite the BBGKY hierarchy (1.8) in the integral form as follows:

$$\mathrm{Tr}\; J^{(k)}\widetilde{\gamma}^{(k)}_{N,t} \tag{3.8}$$

$$= \mathrm{Tr}\; J^{(k)}\mathcal{U}^{(k)}(t)\,\widetilde{\gamma}^{(k)}_{N,0} \tag{3.9}$$

$$- \;\frac{i}{N^2}\sum_{1\le i<j<\ell\le k}\int_0^t ds\, \mathrm{Tr}\; J^{(k)}\mathcal{U}^{(k)}(t-s)[V_N(x_i-x_j,x_i-x_\ell),\widetilde{\gamma}^{(k)}_{N,s}] \tag{3.10}$$

$$- \;\frac{i(N-k)}{N^2}\sum_{1\le i<j\le k}\int_0^t ds\, \mathrm{Tr}\; J^{(k)}\mathcal{U}^{(k)}(t-s)[V_N(x_i-x_j,x_i-x_{k+1}),\widetilde{\gamma}^{(k+1)}_{N,s}] \tag{3.11}$$

$$- \;\frac{i(N-k)(N-k-1)}{N^2}\sum_{j=1}^{k}\int_0^t ds\, \mathrm{Tr}\; J^{(k)}\mathcal{U}^{(k)}(t-s) \qquad [V_N(x_j-x_{k+1},x_j-x_{k+2}),\widetilde{\gamma}^{(k+2)}_{N,s}]. \tag{3.12}$$

Now we observe the following:

- As $N\to\infty$, the term (3.8) converges to the term on the lhs of (3.7), thanks to (3.5).
- Also thanks to (3.5), the term (3.9) converges to the first term on the rhs of (3.7).
- The terms (3.10) and (3.11) vanish as $N\to\infty$.

Hence it suffices to prove that (3.12) converges to the last term on the rhs of (3.7), as $N\to\infty$. Also since the contributions in (3.12) proportional to $\frac{k(k-1)}{N^2}$ as well as those proportional to $\frac{k}{N}$ and to $\frac{k-1}{N}$ vanish as $N\to\infty$, we only need to prove that, for fixed T, k and $J^{(k)}$ we have:

$$\sup_{s\le t\le T}\Big|\mathrm{Tr} J^{(k)}\mathcal{U}^{(k)}(t-s) \Big(V_N(x_j-x_{k+1},x_j-x_{k+2})\widetilde{\gamma}^{(k+2)}_{N,s} - b_0\delta(x_j-x_{k+1})\delta(x_j-x_{k+2})\gamma^{(k+2)}_{\infty,s}\Big)\Big| \to 0, \tag{3.13}$$

as $N\to\infty$, which can be proved in a similar way as the expression (6.6) in [13]. □

The cutoff parametrized by $\kappa>0$ that is introduced in (2.19) can be removed by the same limiting procedure as in [6], see also [13]. We quote the main steps for the convenience of the reader, from [6, 13].

For the limiting hierarchy $\widetilde{\Gamma}_{N,t}\to\Gamma_{\infty,t}$ as $N\to\infty$, it is proven below that for every $\kappa>0$, $\widehat{\eta}(\widehat{\gamma}^{(k)}_{N,t},|\phi_t\rangle\langle\phi_t|^{\otimes k})\to 0$ as $N\to\infty$, for every fixed k. This also implies the convergence

$$\widetilde{\gamma}^{(k)}_{N,t} \;\to\; |\phi_t\rangle\langle\phi_t|^{\otimes k} \tag{3.14}$$

in the weak* topology of $\mathcal{L}^1_k$.

It remains to be proven that also $\gamma^{(k)}_{N,t} \to |\phi_t\rangle\langle\phi_t|^{\otimes k}$. To this end, one may assume $\kappa > 0$ to be sufficiently small such that

$$\Big| \mathrm{Tr} J^{(k)} \left(\gamma^{(k)}_{N,t} - \widetilde{\gamma}^{(k)}_{N,t} \right) \Big| \le \| J^{(k)} \| \, \| \Psi_N - \widetilde{\Psi}_N \| < C\kappa \le \frac{\epsilon}{2} , \tag{3.15}$$

uniformly in N. This follows from $\|\Psi_N - \widetilde{\Psi}_N\| < C\kappa$, uniformly in N, which can be easily verified. On the other hand, for all $N > N_0$ with N_0 sufficiently large, we have

$$\Big| \mathrm{Tr} J^{(k)} \left(\widetilde{\gamma}^{(k)}_{N,t} - |\phi_t\rangle\langle\phi_t|^{\otimes k} \right) \Big| \le \frac{\epsilon}{2} , \tag{3.16}$$

due to the convergence of $\widetilde{\gamma}^{(k)}_{N,t}$ described above. This implies that for arbitrary $\epsilon > 0$,

$$\Big| \mathrm{Tr} J^{(k)} \left(\gamma^{(k)}_{N,t} - |\phi_t\rangle\langle\phi_t|^{\otimes k} \right) \Big| \le \epsilon , \tag{3.17}$$

for all $N > N_0$. Thus, for every $t \in [0, T]$ and every fixed k, $\gamma^{(k)}_{N,t} \to |\phi_t\rangle\langle\phi_t|^{\otimes k}$ in the weak* topology of $\mathcal{L}^1_k$. Because the limiting density is an orthogonal projection, this is equivalent to the convergence in trace norm topology. For details, we refer to [6, 13], from which we have quoted the above results. Combined with the proof of (3.14) given below, this establishes Theorem 1.1.

4. A priori energy bounds on the limiting hierarchy

In this section we prove some spatial bounds for the limit points $\{\gamma^{(k)}_{\infty,t}\}_{k\ge1}$ that shall be used in order to prove uniqueness of the hierarchy.

More precisely, first we state the a-priori bound which follows from the estimates (2.20) for $\tilde{\gamma}^{(k)}_{N,t}$.

Proposition 4.1. *If $\Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k\ge1}$ is a limit point of the sequence $\tilde{\Gamma}_{N,t} = \{\tilde{\gamma}^{(k)}_{N,t}\}_{k=1}^N$ with respect to the product topology τ_{prod}, then there exists $C > 0$ such that*

$$\mathrm{Tr}(1 - \Delta_1) \cdots (1 - \Delta_k) \gamma^{(k)}_{\infty,t} \le C^k, \tag{4.1}$$

for all $k \ge 1$.

Proof. The proof follows from the fact that the a-priori estimates (2.20) for $\tilde{\gamma}^{(k)}_{N,t}$ hold uniformly in N. □

As in [13], we prove uniqueness of the infinite hierarchy following the approach introduced by Klainerman and Machedon [12]. In order to apply the approach of [12] we establish another a-priori bound on the limiting density. Such a bound is formulated in Theorem 4.2 below. In what follows $S^{(k,\alpha)}$ denotes

$$S^{(k,\alpha)} = \prod_{j=1}^{k} (1 - \Delta_{x_j})^{\frac{\alpha}{2}} (1 - \Delta_{x'_j})^{\frac{\alpha}{2}} .$$

Theorem 4.2. *Suppose that $d \in \{1,2\}$. If $\Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k\geq 1}$ is a limit point of the sequence $\tilde{\Gamma}_{N,t} = \{\tilde{\gamma}^{(k)}_{N,t}\}_{k=1}^N$ with respect to the product topology τ_{prod}, then, for every $\alpha < 1$ if $d = 2$, and every $\alpha \leq 1$ if $d = 1$, there exists $C > 0$ such that*

$$\|S^{(k,\alpha)} B_{j;k+1,k+2} \gamma^{(k+2)}_{\infty,t}\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \leq C^{k+2}, \tag{4.2}$$

for all $k \geq 1$ and all $t \in [0,T]$.

Proof. We modify the proof of an analogous result presented in Theorem 5.2 of [13]. We note that for the argument employed here, the fact is used that $\gamma^{(\ell)}_{\infty,t}$ is positive, and thus, especially, hermitean. We note that Theorem 5.1 below states a similar result, but for a different quantity than $\gamma^{(\ell)}_{\infty,t}$ which may be neither positive nor hermitean. Thus, the proof of Theorem 5.1 is based on a different approach that necessitates a *lower* bound on α, instead of an upper bound as required here.

By (4.1) it suffices to prove

$$\|S^{(k,\alpha)} B_{j;k+1,k+2} \gamma^{(k+2)}_{\infty,t}\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \leq \mathrm{Tr}(1-\Delta_1)...(1-\Delta_{k+2})\gamma^{(k+2)}_{\infty,t}. \tag{4.3}$$

We will consider the case $k = 1, j = 1$ (the argument for $k \geq 2$ can be carried out in a similar way). We start by calculating the Fourier transform of $B_{1;2,3}\gamma^{(3)}_{\infty,t}$:

$$\begin{aligned}
&\widehat{B_{1;2,3}\gamma^{(3)}_{\infty,t}}(p;p') \\
&= \int dx_1\, dx_1' e^{-ix_1\cdot p} e^{ix_1'\cdot p'} \int dx_2\, dx_2'\, dx_3\, dx_3' \\
&\quad \delta(x_1-x_2)\delta(x_1-x_2')\delta(x_1-x_3)\delta(x_1-x_3')\gamma^{(3)}(x_1,x_2,x_3;x_1',x_2',x_3') \\
&= \int dq\, d\kappa\, dr\, ds \int dx_1\, dx_1'\, dx_2\, dx_2'\, dx_3\, dx_3' \\
&\quad e^{-ix_1\cdot p} e^{ix_1'\cdot p'} e^{iq(x_1-x_2)} e^{-i\kappa(x_1-x_2')} e^{ir(x_1-x_3)} e^{-is(x_1-x_3')} \gamma^{(3)}_{\infty,t}(x_1,x_2,x_3;x_1',x_2',x_3') \\
&= \int dq\, d\kappa\, dr\, ds \int dx_1\, dx_1'\, dx_2\, dx_2'\, dx_3\, dx_3' \\
&\quad e^{-ix_1\cdot(p-q+\kappa-r+s)} e^{-ix_2\cdot q} e^{-ix_3\cdot r} e^{ix_1'\cdot p'} e^{ix_2'\cdot\kappa} e^{ix_3'\cdot s} \gamma^{(3)}_{\infty,t}(x_1,x_2,x_3;x_1',x_2',x_3') \\
&= \int dq\, d\kappa\, dr\, ds\, \widehat{\gamma^{(3)}_{\infty,t}}(p-q+\kappa-r+s,q,r;p',\kappa,s).
\end{aligned} \tag{4.4}$$

Hence

$$\begin{aligned}
&S^{(1,\alpha)}\widehat{B_{1;2,3}\gamma^{(3)}_{\infty,t}}(p;p') \\
&\quad = \langle p\rangle^\alpha \langle p'\rangle^\alpha \int dq\, d\kappa\, dr\, ds\, \widehat{\gamma^{(3)}_{\infty,t}}(p-q+\kappa-r+s,q,r;p',\kappa,s)\,,
\end{aligned} \tag{4.5}$$

which in turn implies

$$\begin{aligned}
&\|S^{(1,\alpha)} B_{1;2,3}\gamma^{(3)}_{\infty,t}(p;p')\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \\
&\quad = \int dp\, dp'\, dq_1\, dq_2\, d\kappa_1\, d\kappa_2\, dr_1\, dr_2\, ds_1\, ds_2 \langle p\rangle^{2\alpha}\langle p'\rangle^{2\alpha} \\
&\qquad\qquad \widehat{\gamma^{(3)}_{\infty,t}}(p-q_1+\kappa_1-r_1+s_1,q_1,r_1;p_1',\kappa_1,s_1) \\
&\qquad\qquad\quad \widehat{\gamma^{(3)}_{\infty,t}}(p-q_2+\kappa_2-r_2+s_2,q_2,r_2;p_2',\kappa_2,s_2).
\end{aligned} \tag{4.6}$$

Substituting

$$\widehat{\gamma_{\infty,t}^{(3)}}(p_1,p_2,p_3;p_1',p_2',p_3') = \sum_j \lambda_j \psi_j(p_1,p_2,p_3)\bar{\psi}_j(p_1',p_2',p_3') \tag{4.7}$$

into (4.6) and keeping in mind that $\lambda_j \geq 0$ for all j and $\sum_j \lambda_j \leq 1$ thanks to $\gamma^{(k+2)}$ being a non-negative trace-class operator with trace at most one, we obtain:

$$\begin{aligned}
&\|S^{(1,\alpha)}B_{1;2,3}\gamma_{\infty,t}^{(3)}(p;p')\|_{L^2(\mathbb{R}^d\times\mathbb{R}^d)}^2 \\
&= \sum_{i,j}\lambda_i\lambda_j s \int dp\; dp'\; dq_1\; dq_2\; d\kappa_1\; d\kappa_2\; dr_1\; dr_2\; ds_1\; ds_2 \big\langle p\big\rangle^{2\alpha}\big\langle p'\big\rangle^{2\alpha} \\
&\qquad \psi_j(p-q_1+\kappa_1-r_1+s_1,q_1,r_1)\bar{\psi}_j(p',\kappa_1,s_1) \\
&\qquad\quad \psi_j(p-q_2+\kappa_2-r_2+s_2,q_2,r_2)\bar{\psi}_j(p',\kappa_2,s_2).
\end{aligned} \tag{4.8}$$

We observe that for $l=1,2$ we have

$$\big\langle p\big\rangle^{\alpha} \leq C\left[\big\langle p-q_l+\kappa_l-r_l+s_l\big\rangle^{\alpha}+\big\langle q_l\big\rangle^{\alpha}+\big\langle r_l\big\rangle^{\alpha}+\big\langle \kappa_l\big\rangle^{\alpha}+\big\langle s_l\big\rangle^{\alpha}\right]$$

which implies that

$$\begin{aligned}
\big\langle p\big\rangle^{2\alpha} \;\leq\;& C\left[\big\langle p-q_1+\kappa_1-r_1+s_1\big\rangle^{\alpha}+\big\langle q_1\big\rangle^{\alpha}+\big\langle r_1\big\rangle^{\alpha}+\big\langle \kappa_1\big\rangle^{\alpha}+\big\langle s_1\big\rangle^{\alpha}\right] \\
&\times\left[\big\langle p-q_2+\kappa_2-r_2+s_2\big\rangle^{\alpha}+\big\langle q_2\big\rangle^{\alpha}+\big\langle r_2\big\rangle^{\alpha}+\big\langle \kappa_2\big\rangle^{\alpha}+\big\langle s_2\big\rangle^{\alpha}\right].
\end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8), we obtain 16 terms. We will illustrate how to control one of them, the remaining cases are similar. Using a weighted Schwarz inequality, we find

$$\begin{aligned}
&\int dp\; dp'\; dq_1\; dq_2\; d\kappa_1\; d\kappa_2\; dr_1\; dr_2\; ds_1\; ds_2 \\
&\qquad \big\langle p'\big\rangle^{2\alpha}\big\langle p-q_1+\kappa_1-r_1+s_1\big\rangle^{\alpha}\big\langle p-q_2+\kappa_2-r_2+s_2\big\rangle^{\alpha} \\
&\qquad \psi_j(p-q_1+\kappa_1-r_1+s_1,q_1,r_1)\bar{\psi}_j(p',\kappa_1,s_1) \\
&\qquad\quad \psi_j(p-q_2+\kappa_2-r_2+s_2,q_2,r_2)\bar{\psi}_j(p',\kappa_2,s_2) \\
&\leq\; I+II,
\end{aligned}$$

where

$$\begin{aligned}
I = &\int dp\; dp'\; dq_1\; dq_2\; d\kappa_1\; d\kappa_2\; dr_1\; dr_2\; ds_1\; ds_2 \\
&\quad \frac{\big\langle p'\big\rangle^{2\alpha}\big\langle p-q_1+\kappa_1-r_1+s_1\big\rangle^{2}\big\langle q_1\big\rangle^{2}\big\langle r_1\big\rangle^{2}\big\langle k_2\big\rangle^{2}\big\langle s_2\big\rangle^{2}}{\big\langle p-q_2+\kappa_2-r_2+s_2)^2\big\rangle^{2-2\alpha}\big\langle q_2\big\rangle^{2}\big\langle r_2\big\rangle^{2}\big\langle k_1\big\rangle^{2}\big\langle s_1\big\rangle^{2}} \\
&\quad |\psi_j(p-q_1+\kappa_1-r_1+s_1,q_1,r_1)|^2\,|\psi_j(p',\kappa_2,s_2)|^2,
\end{aligned}$$

and

$$\begin{aligned}
II = &\int dp\; dp'\; dq_1\; dq_2\; d\kappa_1\; d\kappa_2\; dr_1\; dr_2\; ds_1\; ds_2 \\
&\quad \frac{\big\langle p'\big\rangle^{2\alpha}\big\langle p-q_2+\kappa_2-r_2+s_2\big\rangle^{2}\big\langle q_2\big\rangle^{2}\big\langle r_2\big\rangle^{2}\big\langle k_1\big\rangle^{2}\big\langle s_1\big\rangle^{2}}{\big\langle p-q_1+\kappa_1-r_1+s_1\big\rangle^{2-2\alpha}\big\langle q_1\big\rangle^{2}\big\langle r_1\big\rangle^{2}\big\langle k_2\big\rangle^{2}\big\langle s_2\big\rangle^{2}} \\
&\quad |\psi_j(p-q_2+\kappa_2-r_2+s_2,q_2,r_2)|^2\,|\psi_j(p',\kappa_1,s_1)|^2.
\end{aligned}$$

Below we illustrate how to estimate I. The expression II can be estimated in a similar manner. We will use the bound

$$\int_{\mathbb{R}^d} \frac{dy}{\langle P-y\rangle^{2-2\alpha}\langle y\rangle^2} \leq \frac{C}{\langle P\rangle^{2-2\alpha}}, \tag{4.10}$$

which is valid for $d=1$ if $\alpha \leq 1$, and for $d=2$ if $\alpha<1$; it is easily obtained by rescaling $y \to \langle P\rangle y$. Ti estimate I, we integrate over q_2, using (4.10), followed by integrating over r_2, using (4.10) again, to obtain:

$$I \leq \int dp\, dp'\, dq_1\, d\kappa_1\, d\kappa_2\, dr_1\, ds_1\, ds_2 \frac{\langle p'\rangle^{2\alpha}\langle p-q_1+\kappa_1-r_1+s_1\rangle^2\langle q_1\rangle^2\langle r_1\rangle^2\langle k_2\rangle^2\langle s_2\rangle^2}{\langle p+\kappa_2+s_2\rangle^{2-2\alpha}\langle k_1\rangle^2\langle s_1\rangle^2} |\psi_j(p-q_1+\kappa_1-r_1+s_1,q_1,r_1)|^2\,|\psi_j(p',\kappa_2,s_2)|^2 .$$

The change of variable $\tilde{p}=p-q_1+\kappa_1-r_1+s_1$ gives

$$\begin{aligned} I \;&\leq\; \int d\tilde{p}\, dp'\, dq_1\, d\kappa_1\, d\kappa_2\, dr_1\, ds_1\, ds_2 \frac{\langle \tilde{p}\rangle^2\langle q_1\rangle^2\langle r_1\rangle^2\langle p'\rangle^2\langle k_2\rangle^2\langle s_2\rangle^2}{\langle \tilde{p}+q_1-\kappa_1+r_1-s_1+\kappa_2+s_2\rangle^{2-2\alpha}\langle k_1\rangle^2\langle s_1\rangle^2} |\psi_j(\tilde{p},q_1,r_1)|^2\,|\psi_j(p',\kappa_2,s_2)|^2 \\ &\leq\; C_\alpha \int d\tilde{p}\, dq_1\, dr_1\, \langle \tilde{p}\rangle^2\langle q_1\rangle^2\langle r_1\rangle^2\, |\psi_j(\tilde{p},q_1,r_1)|^2 \int dp'\, d\kappa_2\, ds_2\, \langle p'\rangle^2\langle \kappa_2\rangle^2\langle s_2\rangle^2\, |\psi_j(p',\kappa_2,s_2)|^2 . \end{aligned} \tag{4.11}$$

To obtain (4.11) we have used that, as a consequence of (4.10),

$$C_\alpha \;=\; \sup_{P\in\mathbb{R}^d} \int \frac{dy\,dz}{\langle P-y-z\rangle^{2-2\alpha}\langle y\rangle^2\langle z\rangle^2} \;<\; \infty\,, \tag{4.12}$$

for all $\alpha \leq 1$ if $d=1$, and all $\alpha<1$ if $d=2$.

The other 15 contributions to (4.8) can be obtained in a similar way. Therefore, using the above analysis and (4.8), we conclude that

$$\begin{aligned} &\|S^{(1,\alpha)}B_{1;2,3}\gamma^{(3)}_{\infty,t}(p;p')\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \\ &\quad\leq\; C\sum_{i,j}\lambda_i\lambda_j \int d\tilde{p}\, dq_1\, dr_1 \langle \tilde{p}\rangle^2\langle q_1\rangle^2\langle r_1\rangle^2\, |\psi_j(\tilde{p},q_1,r_1)|^2 \int dp'\, d\kappa_2\, ds_2 \langle p'\rangle^2\langle \kappa_2\rangle^2\langle s_2\rangle^2\, |\psi_j(p',\kappa_2,s_2)|^2 \\ &\quad\leq\; \left[\int d\tilde{p}\, dq_1\, dr_1 \langle \tilde{p}\rangle^2\langle q_1\rangle^2\langle r_1\rangle^2 \left|\widehat{\gamma^{(3)}_{\infty,t}}(\tilde{p},q_1,r_1)\right|^2\right]^2 \\ &\quad=\; C\left[\,\mathrm{Tr}(1-\Delta_1)(1-\Delta_2)(1-\Delta_3)\,\gamma^{(3)}_{\infty,t}\right]^2, \end{aligned} \tag{4.13}$$

which gives (4.3) in the case $k=1$, $j=1$. □

In addition to the above results, we derive a third type of spatial bounds, which is more restrictive in terms of the condition on α (it requires $\alpha > \frac{d}{2}$). Note that for $d = 1$ we can afford this range of α. In particular, we shall use this new bound iteratively in the proof of uniqueness of the limiting hierarchy when $d = 1$. The proof of the bound is inspired by the proof of a space-time bound for the freely evolving limiting hierarchy given in Theorem 1.3 of [12]. However, the bound that we derive here is obtained for any $\gamma^{(k)}_{\infty,t}$.

Theorem 4.3. *Suppose that $d \geq 1$. If $\Gamma_{\infty,t} = \{\gamma^{(k)}_{\infty,t}\}_{k\geq 1}$ is a limit point of the sequence $\tilde{\Gamma}_{N,t} = \{\tilde{\gamma}^{(k)}_{N,t}\}_{k=1}^{N}$ with respect to the product topology τ_{prod}, then, for every $\alpha > \frac{d}{2}$ there exists a constant $C = C(\alpha)$ such that the estimate*

$$\Big\| S^{(k,\alpha)} B_{j;k+1,k+2} \gamma^{(k+2)}_{\infty,t} \Big\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \leq C \Big\| S^{(k+2,\alpha)} \gamma^{(k+2)}_{\infty,t} \Big\|_{L^2(\mathbb{R}^{d(k+2)}\times\mathbb{R}^{d(k+2)})} \tag{4.14}$$

holds.

Proof. Let $(\underline{u}_k, \underline{u}'_k)$, $\underline{q} := (q_1, q_2)$, and $\underline{q}' := (q'_1, q'_2)$ denote the Fourier conjugate variables corresponding to $(\underline{x}_k, \underline{x}'_k)$, (x_{k+1}, x_{k+2}), and (x'_{k+1}, x'_{k+2}), respectively.

Without any loss of generality, we may assume that $j = 1$ in $B_{j;k+1,k+2}$. Then, we have

$$\begin{aligned} &\Big\| S^{(k,\alpha)} B_{1;k+1,k+2} \gamma^{(k+2)}_{\infty,t} \Big\|^2_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} \\ &= \int d\underline{u}_k \, d\underline{u}'_k \prod_{j=1}^{k} \langle u_j \rangle^{2\alpha} \langle u'_j \rangle^{2\alpha} \\ &\qquad \Big(\int d\underline{q} \, d\underline{q}' \, \widehat{\gamma}^{(k+2)}_{\infty,t}(t, u_1 + q_1 + q_2 - q'_1 - q'_2, u_2, \dots, u_k, \underline{q}; \underline{u}'_k, \underline{q}') \Big)^2 . \end{aligned} \tag{4.15}$$

where now, the Fourier transform in only performed in the spatial coordinates. Applying the Schwarz inequality, we find the upper bound

$$\begin{aligned} &\leq \int d\underline{u}_k \, d\underline{u}'_k \, I'_\alpha(\tau, \underline{u}_k, \underline{u}'_k) \int d\underline{q} \, d\underline{q}' \\ &\qquad \langle u_1 + q_1 + q_2 - q'_1 - q'_2 \rangle^{2\alpha} \langle q_1 \rangle^{\alpha} \langle q_2 \rangle^{2\alpha} \langle q'_1 \rangle^{\alpha} \langle q'_2 \rangle^{2\alpha} \prod_{j=2}^{k} \langle u_j \rangle^{2\alpha} \prod_{j'=1}^{k} \langle u'_{j'} \rangle^{2\alpha} \\ &\qquad \Big| \widehat{\gamma}^{(k+2)}_{\infty,t}(t, u_1 + q_1 + q_2 - q'_1 - q'_2, u_2, \dots, u_k, \underline{q}; \underline{x}'_k, \underline{q}') \Big|^2 \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} &I'_\alpha(\underline{u}_k, \underline{u}'_k) \\ &:= \int d\underline{q} d\underline{q}' \, \frac{\langle u_1 \rangle^{2\alpha}}{\langle u_1 + q_1 + q_2 - q'_1 - q'_2 \rangle^{2\alpha} \langle q_1 \rangle^{2\alpha} \langle q_2 \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \langle q'_2 \rangle^{2\alpha}} . \end{aligned} \tag{4.17}$$

Using

$$\langle u_1 \rangle^{2\alpha} \leq C \Big[\langle u_1 + q_1 + q_2 - q'_1 - q'_2 \rangle^{2\alpha} + \langle q_1 \rangle^{2\alpha} + \langle q_2 \rangle^{2\alpha} + \langle q'_1 \rangle^{2\alpha} + \langle q'_2 \rangle^{2\alpha} \Big] , \tag{4.18}$$

and shifting some of the momentum variables, one immediately obtains that

$$I'_\alpha(\underline{u}_k, \underline{u}'_k) \, < \, C \int d\underline{q}d\underline{q}' \, \frac{1}{\langle q_1 \rangle^{2\alpha} \langle q_2 \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \langle q'_2 \rangle^{2\alpha}} \,, \tag{4.19}$$

which is finite for all

$$\alpha \, > \, \frac{d}{2} \, . \tag{4.20}$$

This proves the claim. □

5. Bounds on the freely evolving infinite hierarchy

In this section, we prove bounds on the infinite hierarchy for $b_0 = 0$, i.e., in the absence of particle interactions; see (1.9) for the definition of b_0. These will be used for the recursive estimation of terms appearing in the Duhamel expansions studied in Section 7. Our approach is similar to the one of Klainerman and Machedon in [12]. In dimension $d = 2$, we prove spacetime bounds in complete analogy to [12, 13] which are global in time [1].

From here on and for the rest of this paper, we will write

$$\gamma^{(r)}(t, \underline{x}_k; \underline{x}'_k) \, \equiv \, \gamma^{(r)}_{\infty,t}(t, \underline{x}_k; \underline{x}'_k) \tag{5.1}$$

which is notationally more convenient for the discussion of spacetime norms.

Theorem 5.1. *Assume that $d = 2$ and $\frac{5}{6} < \alpha < 1$. Let $\gamma^{(k+2)}$ denote the solution of*

$$i\partial_t \gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}) \, + \, (\Delta_{\underline{x}_{k+2}} - \Delta_{\underline{x}'_{k+2}})\gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}) \, = \, 0 \tag{5.2}$$

with initial condition

$$\gamma^{(k+2)}(0, \cdot\,) \, = \, \gamma_0^{(k+2)} \in \mathcal{H}^\alpha \, . \tag{5.3}$$

Then, there exists a constant $C = C(\alpha)$ such that

$$\begin{aligned} &\Big\| \, S^{(k,\alpha)} B_{j;k+1,k+2} \gamma^{(k+2)} \, \Big\|_{L^2_{t,\underline{x}_k,\underline{x}'_k}(\mathbb{R}\times\mathbb{R}^{2(k+2)}\times\mathbb{R}^{2(k+2)})} \\ &\qquad \leq \, C \Big\| \, S^{(k+2,\alpha)} \gamma_0^{(k+2)} \, \Big\|_{L^2_{\underline{x}_{k+2},\underline{x}'_{k+2}}(\mathbb{R}^{2(k+2)}\times\mathbb{R}^{2(k+2)})} \end{aligned} \tag{5.4}$$

holds.

Proof. We give a proof using the arguments of [12, 13]. We note that the arguments presented in the proof of Theorem 4.2 cannot be straightforwardly employed here because here, $B_{j;k+1,k+2}\gamma^{(k+2)}$ are not hermitean so that (4.7) is not available.

Let $(\tau, \underline{u}_k, \underline{u}'_k)$, $\underline{q} := (q_1, q_2)$, and $\underline{q}' := (q'_1, q'_2)$ denote the Fourier conjugate variables corresponding to $(t, \underline{x}_k, \underline{x}'_k)$, (x_{k+1}, x_{k+2}), and (x'_{k+1}, x'_{k+2}), respectively.

[1] In dimension $d = 1$, the argument used for $d = 2$ would produce a divergent bound; accordingly, when $d = 1$ we shall use the a priori bounds obtained in Theorem 4.3.

Without any loss of generality, we may assume that $j=1$ in $B_{j;k+1,k+2}$. Then, abbreviating

$$\delta(\cdots) \;:=\; \delta(\,\tau+(u_1+q_1+q_2-q_1'-q_2')^2+\sum_{j=2}^{k}u_j^2+|\underline{q}|^2-|\underline{u}_k'|^2-|\underline{q}'|^2\,) \tag{5.5}$$

we find

$$\begin{aligned}
&\Big\|\,S^{(k,\alpha)}B_{1;k+1,k+2}\gamma^{(k+2)}\,\Big\|^2_{L^2_{t,\underline{x}_k,\underline{x}_k'}(\mathbb{R}\times\mathbb{R}^{2(k+2)}\times\mathbb{R}^{2(k+2)})}\\
&=\;\int_{\mathbb{R}}d\tau\int d\underline{u}_k d\underline{u}_k'\prod_{j=1}^{k}\big\langle u_j\big\rangle^{2\alpha}\big\langle u_j'\big\rangle^{2\alpha} \qquad (5.6)\\
&\qquad\Big(\int d\underline{q}d\underline{q}'\,\delta(\cdots)\widehat{\gamma}^{(k+2)}(\tau,u_1+q_1+q_2-q_1'-q_2',u_2,\dots,u_k,\underline{q};\underline{u}_k',\underline{q}')\Big)^2.
\end{aligned}$$

Using the Schwarz estimate, this is bounded by

$$\begin{aligned}
&\leq\;\int_{\mathbb{R}}d\tau\int d\underline{u}_k d\underline{u}_k'\,I_\alpha(\tau,\underline{u}_k,\underline{u}_k')\int d\underline{q}d\underline{q}'\,\delta(\cdots)\\
&\quad\big\langle u_1+q_1+q_2-q_1'-q_2'\big\rangle^{2\alpha}\big\langle q_1\big\rangle^{2\alpha}\big\langle q_2\big\rangle^{2\alpha}\big\langle q_1'\big\rangle^{2\alpha}\big\langle q_2'\big\rangle^{2\alpha}\prod_{j=2}^{k}\big\langle u_j\big\rangle^{2\alpha}\prod_{j'=1}^{k}\big\langle u_{j'}'\big\rangle^{2\alpha}\\
&\qquad\Big|\,\widehat{\gamma}^{(k+2)}(\tau,u_1+q_1+q_2-q_1'-q_2',u_2,\dots,u_k,\underline{q};\underline{u}_k',\underline{q}')\,\Big|^2 \qquad (5.7)
\end{aligned}$$

where

$$\begin{aligned}
&I_\alpha(\tau,\underline{u}_k,\underline{u}_k') \qquad (5.8)\\
&\quad:=\int d\underline{q}\,d\underline{q}'\,\frac{\delta(\cdots)\,\big\langle u_1\big\rangle^{2\alpha}}{\big\langle u_1+q_1+q_2-q_1'-q_2'\big\rangle^{2\alpha}\big\langle q_1\big\rangle^{2\alpha}\big\langle q_2\big\rangle^{2\alpha}\big\langle q_1'\big\rangle^{2\alpha}\big\langle q_2'\big\rangle^{2\alpha}}\,.
\end{aligned}$$

Similarly as in [12, 13], we observe that

$$\big\langle u_1\big\rangle^{2\alpha}\;\leq\;C\Big[\,\big\langle u_1+q_1+q_2-q_1'-q_2'\big\rangle^{2\alpha}+\big\langle q_1\big\rangle^{2\alpha}+\big\langle q_2\big\rangle^{2\alpha}+\big\langle q_1'\big\rangle^{2\alpha}+\big\langle q_2'\big\rangle^{2\alpha}\,\Big], \tag{5.9}$$

so that

$$I_\alpha(\tau,\underline{u}_k,\underline{u}_k')\;\leq\;\sum_{\ell=1}^{5}J_\ell \tag{5.10}$$

where J_ℓ is obtained from bounding the numerator of (5.8) using (5.9), and from canceling the ℓ-th term on the rhs of (5.9) with the corresponding term in the denominator of (5.8). Thus, for instance,

$$J_1\;<\;\int d\underline{q}\,d\underline{q}'\,\frac{\delta(\cdots)}{\big\langle q_1\big\rangle^{2\alpha}\big\langle q_2\big\rangle^{2\alpha}\big\langle q_1'\big\rangle^{2\alpha}\big\langle q_2'\big\rangle^{2\alpha}}\,, \tag{5.11}$$

and each of the terms J_ℓ with $\ell=2,\dots,5$ can be brought into a similar form by appropriately translating one of the momenta q_i, q_j'.

Further following [12, 13], we observe that the argument of the delta distribution equals

$$\tau + (u_1 + q_1 + q_2 - q_1')^2 + \sum_{j=2}^{k} u_j^2 + |\underline{q}|^2 - |\underline{u}_k'|^2 - (q_1')^2 - 2(u_1 + q_1 + q_2 - q_1') \cdot q_2' \,,$$

and we integrate out the delta distribution using the component of q_2' parallel to $(u_1 + q_1 + q_2 - q_1')$. This leads to the bound

$$J_1 \;<\; C_\alpha C \int d\underline{q} dq_1' \, \frac{1}{|u_1 + q_1 + q_2 - q_1'| \langle q_1 \rangle^{2\alpha} \langle q_2 \rangle^{2\alpha} \langle q_1' \rangle^{2\alpha}} \tag{5.12}$$

where

$$C_\alpha \;:=\; \int_{\mathbb{R}} \frac{d\zeta}{\langle \zeta \rangle^{2\alpha}} \,. \tag{5.13}$$

Clearly, C_α is finite for any $\alpha > \frac{1}{2}$.

To bound J_1, we pick a spherically symmetric function $h \geq 0$ with rapid decay away from the unit ball in $\mathbb{R}^2$, such that $h^\vee(x) \geq 0$ decays rapidly outside of the unit ball in $\mathbb{R}^2$, and

$$\frac{1}{\langle q \rangle^{2\alpha}} \;<\; \Big(h * \frac{1}{|\cdot|^{2\alpha}} \Big)(q) \,. \tag{5.14}$$

(for example, $h(u) = c_1 e^{-c_2 u^2}$, for suitable constants c_1, c_2); since $\alpha < 1$, the right hand side is in $L^\infty(\mathbb{R}^2)$. Then,

$$\begin{aligned} J_1 \;&<\; C_\alpha C \Big\langle \Big(\frac{1}{|\cdot|} * (h * \frac{1}{|\cdot|^{2\alpha}}) \Big) * (h * \frac{1}{|\cdot|^{2\alpha}}) \,,\, (h * \frac{1}{|\cdot|^{2\alpha}}) \Big\rangle_{L^2(\mathbb{R}^2)} \\ &=\; C_\alpha C \int dx \Big(\frac{1}{|\cdot|} \Big)^\vee (x) \Big((h * \frac{1}{|\cdot|^{2\alpha}})^\vee (x) \Big)^3 \\ &=\; C_\alpha C' \int dx \, \frac{1}{|x|} (h^\vee(x))^3 \Big(\frac{1}{|x|^{2-2\alpha}} \Big)^3 . \end{aligned} \tag{5.15}$$

The integral on the last line is finite if the singularity at $x = 0$ is integrable. In dimension $d = 2$, this is the case if

$$\alpha \;>\; \frac{5}{6} \,. \tag{5.16}$$

Finiteness of the integral for the region $|x| \gg 1$ is obtained from the decay of $h^\vee$. We remark that if $0 < 1 - \alpha \ll 1$, the upper bound (5.14) may overestimate the left hand side by as much as a factor $\frac{1}{1-\alpha} \gg 1$ pointwise in q, for small $|q|$, due to the singularity of $\frac{1}{|\cdot|^{2(1-\alpha)}}$ at zero. But the integral in (5.15) is uniformly bounded in the limit $\alpha \nearrow 1$, implying that the argument is robust. The terms $J_2, \dots, J_5$ can be bounded in a similar manner. For more details, we refer to [12, 13]. This proves the statement of the theorem. □

6. Uniqueness of solutions of the infinite hierarchy

Collecting our results derived in the previous sections, we now prove the uniqueness of solutions of the infinite hierarchy.

We recall the notation $\Delta_{\pm}^{(k)} = \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k}$ and $\Delta_{\pm,x_j} = \Delta_{x_j} - \Delta_{x'_j}$.

Let us fix a positive integer r. Using Duhamel's formula we can express $\gamma^{(r)}$ in terms of the iterates $\gamma^{(r+2)}, \gamma^{(r+4)}, ..., \gamma^{(r+2n)}$ as follows:

$$\begin{aligned}\gamma^{(r)}(t_r, \cdot) &= \int_0^{t_r} e^{i(t_r - t_{r+2})\Delta_{\pm}^{(r)}} B_{r+2}(\gamma^{(r+2)}(t_{r+2}))\, dt_{r+2} \\ &= \int_0^{t_r} \int_0^{t_{r+2}} e^{i(t_r - t_{r+2})\Delta_{\pm}^{(r)}} B_{r+2} e^{i(t_{r+2} - t_{r+4})\Delta_{\pm}^{(r+2)}} B_{r+4}(\gamma^{(r+4)}(t_{r+4}))\, dt_{r+2}\, dt_{r+4} \\ &= ... \\ &= \int_0^{t_r} ... \int_0^{t_{r+2n}} J^r(\underline{t}_{r+2n})\, dt_{r+2}...dt_{r+2n}, \qquad (6.1)\end{aligned}$$

where

$$\begin{aligned}&\underline{t}_{r+2n} = (t_r, t_{r+2}, ..., t_{r+2n}), \\ &J^r(\underline{t}_{r+2n}) = e^{i(t_r - t_{r+2})\Delta_{\pm}^{(r)}} B_{r+2} ... e^{i(t_{r+2(n-1)} - t_{r+2n})\Delta_{\pm}^{(r+2(n-1))}} B_{r+2n}(\gamma^{(r+2n)}(t_{r+2n})).\end{aligned}$$

Our main result is the following theorem.

Theorem 6.1. *Assume that $d \in \{1, 2\}$ and $t_r \in [0, T]$. The estimate*

$$\Big\| \int_0^{t_r} ... \int_0^{t_{r+2n}} J^r(\underline{t}_{r+2n})\, dt_{r+2}...dt_{r+2n} \Big\|_{L^2(\mathbb{R}^{dr} \times \mathbb{R}^{dr})} < C^r\, (C_0 T)^n \qquad (6.2)$$

holds for constants C, C_0 independent of r and T.

Theorem 6.1 implies that for sufficiently small T,

$$\Big\| \int_D J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2}...dt_{r+2n} \Big\|_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} \to 0 \qquad (6.3)$$

as $n \to \infty$. Since n is arbitrary, we conclude that $\gamma^{(r)}(t_r, \cdot) = 0$, given the initial condition $\gamma^{(r)}(0, \cdot) = 0$. This establishes the uniqueness of $\gamma^{(r)}(t_r, \cdot)$, and since r is arbitrary, we conclude that the solution of the infinite hierarchy is unique.

The proof of Theorem 6.1 will occupy sections 7 and 8.

7. Combinatorics of contractions

In this section, we organize the Duhamel expansion with respect to the individual terms in the operators $B_{r+2\ell}$. This is obtained from an extension of the method of Klainerman-Machedon introduced in [12].

Recalling that $B_{k+2} = \sum_{j=1}^{k} B_{j;k+1,k+2}$ we can rewrite $J^r(\underline{t}_{r+2n})$ as

$$J^r(\underline{t}_{r+2n}) = \sum_{\mu \in M} J^r(\underline{t}_{r+2n}; \mu), \tag{7.1}$$

where

$$\begin{aligned} J^r(\underline{t}_{r+2n}; \mu) =& e^{i(t_r - t_{r+2})\Delta_{\pm}^{(r)}} B_{\mu(r+1);r+1,r+2} e^{i(t_{r+2}-t_{r+4})\Delta_{\pm}^{(r+2)}} \ldots \\ & e^{i(t_{r+2(n-1)} - t_{r+2n})\Delta_{\pm}^{(r+2(n-1))}} B_{\mu(r+2n-1);r+2n-1,r+2n}(\gamma^{(r+2n)}(t_{r+2n})), \end{aligned}$$

and μ is a map from $\{r+1, r+2, ..., r+2n-1\}$ to $\{r, r+1, ..., r+2n-2\}$ such that $\mu(2) = 1$ and $\mu(j) < j$ for all j. Here M denotes the set of all such mappings μ.

We observe that such a mapping μ can be represented by highlighting one nonzero entry in each column of the $(r+2n-2) \times n$ matrix:

$$\begin{bmatrix} \mathbf{B_{1;r+1,r+2}} & B_{1;r+3,r+4} & \ldots & \mathbf{B_{1;r+2n-1,r+2n}} \\ \ldots & \mathbf{B_{2;r+3,r+4}} & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ B_{r;r+1,r+2} & B_{r;r+3,r+4} & \ldots & \ldots \\ 0 & B_{r+1;r+3,r+4} & \ldots & \ldots \\ 0 & B_{r+2;r+3,r+4} & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & B_{r+2n-2;r+2n-1,r+2n} \end{bmatrix}. \tag{7.2}$$

Since we can rewrite (6.1) as

$$\gamma^{(r)}(t_r, \cdot) = \int_0^{t_r} \ldots \int_0^{t_{r+2n}} \sum_{\mu \in M} J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} \ldots dt_{r+2n}, \tag{7.3}$$

the integrals of the following type are of interest to us:

$$I(\mu, \sigma) = \int_{t_r \geq t_{\sigma(r+2)} \geq \ldots \geq t_{\sigma(r+2n)}} J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} \ldots dt_{r+2n}, \tag{7.4}$$

where σ is a permutation of $\{r+2, r+4, ..., r+2n\}$. We would like to associate to such an integral a matrix, which will help us visualize $B_{\mu(r+2j-1);r+2j-1,r+2j}$'s as well as σ at the same time. More precisely, to $I(\mu,\sigma)$ we associate the matrix

$$\begin{bmatrix} t_{\sigma^{-1}(r+2)} & t_{\sigma^{-1}(r+4)} & ... & t_{\sigma^{-1}(r+2n)} \\ \mathbf{B_{1;r+1,r+2}} & B_{1;r+3,r+4} & ... & \mathbf{B_{1;r+2n-1,r+2n}} \\ ... & \mathbf{B_{2;r+3,r+4}} & ... & ... \\ ... & ... & ... & ... \\ B_{r;r+1,r+2} & B_{r;r+3,r+4} & ... & ... \\ 0 & B_{r+1;r+3,r+4} & ... & ... \\ 0 & B_{r+2;r+3,r+4} & ... & ... \\ ... & 0 & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & ... & B_{r+2n-2;r+2n-1,r+2n} \end{bmatrix}$$

whose columns are labeled 1 through n and whose rows are labeled $0, 1, ...r+2n-2$.

As in [12] we introduce a board game on the set of such matrices. In particular, the following move shall be called an "acceptable move": If $\mu(r+2j+1) < \mu(r+2j-1)$, the player as allowed to do the following four changes at the same time:

- exchange the highlightened entries in columns j and $j+1$,
- exchange the highlightened entries in rows $r+2j-1$ and $r+2j+1$,
- exchange the highlightened entries in rows $r+2j$ and $r+2j+2$,
- exchange $t_{\sigma^{-1}(r+2j)}$ and $t_{\sigma^{-1}(r+2j+2)}$.

As in [12], the importance of this game is visible from the following lemma:

Lemma 7.1. *If (μ,σ) is transformed into (μ',σ') by an acceptable move, then $I(\mu,\sigma) = I(\mu',\sigma')$.*

Proof. We modify the proof of Lemma 3.1 in [12] accordingly. Let us start by fixing an integer $j \geq 3$. Then select two integers i and l such that $i < l < j < j+1$.

Suppose $I(\mu,\sigma)$ and $I(\mu',\sigma')$ are as follows

$$\begin{aligned} I(\mu,\sigma) &= \int_{t_r \geq ... \geq t_{\sigma(r+2j)} \geq t_{\sigma(r+2j+2)} \geq ... \geq t_{\sigma(r+2n)} \geq 0} J^r(\underline{t}_{r+2n},\mu)\, dt_{r+2}...dt_{r+2n} \\ &= \int_{t_r \geq ... \geq t_{\sigma(r+2j)} \geq t_{\sigma(r+2j+2)} \geq ... \geq t_{\sigma(r+2n)} \geq 0} ... e^{i(t_{r+2j-2}-t_{r+2j})\Delta_{\pm}^{(r+2j-2)}} \\ &\qquad B_{l;r+2j-1,r+2j} e^{i(t_{r+2j}-t_{r+2j+2})\Delta_{\pm}^{(r+2j)}} B_{i;r+2j+1,r+2j+2} \\ &\qquad e^{i(t_{r+2j+2}-t_{r+2j+4})\Delta_{\pm}^{(r+2j+2)}} (...)\, dt_{r+2}...dt_{r+2n}, \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} I(\mu',\sigma') &= \int_{t_r \geq \ldots \geq t_{\sigma'(r+2j)} \geq t_{\sigma'(r+2j+2)} \geq \ldots \geq t_{\sigma'(r+2n)} \geq 0} J^r(\underline{t}_{r+2n},\mu')\, dt_{r+2} \ldots dt_{r+2n} \\ &= \int_{t_r \geq \ldots \geq t_{\sigma'(r+2j)} \geq t_{\sigma'(r+2j+2)} \geq \ldots \geq t_{\sigma'(r+2n)} \geq 0} \ldots e^{i(t_{r+2j-2}-t_{r+2j})\Delta_\pm^{(r+2j-2)}} \\ &\quad B_{i;r+2j-1,r+2j} e^{i(t_{r+2j}-t_{r+2j+2})\Delta_\pm^{(r+2j)}} B_{l;r+2j+1,r+2j+2} \\ &\quad e^{i(t_{r+2j+2}-t_{r+2j+4})\Delta_\pm^{(r+2j+2)}} (\ldots)'\, dt_{r+2} \ldots dt_{r+2n}. \end{aligned} \tag{7.6}$$

Here ... in (7.5) and (7.6) coincide. On the other hand any $B_{r+2j-1;s,s+1}$ in (...) of (7.5) becomes $B_{r+2j+1;s,s+1}$ in $(\ldots)'$ of (7.6) and any $B_{r+2j+1;s,s+1}$ in (...) of (7.5) becomes $B_{r+2j-1;s,s+1}$ in $(\ldots)'$ of (7.6). Also any $B_{r+2j;s,s+1}$ in (...) of (7.5) becomes $B_{r+2j+2;s,s+1}$ in $(\ldots)'$ of (7.6) and any $B_{r+2j+2;s,s+1}$ in (...) of (7.5) becomes $B_{r+2j;s,s+1}$ in $(\ldots)'$ of (7.6).

We shall prove that

$$I(\mu,\sigma) = I(\mu',\sigma'). \tag{7.7}$$

As in [12] we introduce the operators P and $\tilde{P}$. In our context they are introduced as follows:

$$\begin{aligned} P &= B_{l;r+2j-1,r+2j} e^{i(t_{r+2j}-t_{r+2j+2})\Delta_\pm^{(r+2j)}} B_{i;r+2j+1,r+2j+2}, \\ \tilde{P} &= B_{i;r+2j+1,r+2j+2} e^{-i(t_{r+2j}-t_{r+2j+2})\tilde{\Delta}_\pm^{(r+2j)}} B_{l;r+2j-1,r+2j}, \end{aligned}$$

where

$$\tilde{\Delta}_\pm^{(r+2j)} = \Delta_\pm^{(r+2j)} - \Delta_{\pm,x_{r+2j}} - \Delta_{\pm,x_{r+2j-1}} + \Delta_{\pm,x_{r+2j+1}} + \Delta_{\pm,x_{r+2j+2}}.$$

First, let us prove that

$$\begin{aligned} & e^{i(t_{r+2j-2}-t_{r+2j})\Delta_\pm^{(r+2j-2)}} P e^{i(t_{r+2j+2}-t_{r+2j+4})\Delta_\pm^{(r+2j+2)}} \\ &\quad = e^{i(t_{r+2j-2}-t_{r+2j+2})\Delta_\pm^{(r+2j-2)}} \tilde{P} e^{i(t_{r+2j}-t_{r+2j+4})\Delta_\pm^{(r+2j+2)}}. \end{aligned} \tag{7.8}$$

In order to do that we observe that

$$\Delta_\pm^{(r+2j)} = \Delta_{\pm,x_i} + (\Delta_\pm^{(r+2j)} - \Delta_{\pm,x_i}).$$

Hence the factor $e^{i(t_{r+2j}-t_{r+2j+2})\Delta_\pm^{(r+2j)}}$ appearing in the definition of P can be rewritten as

$$\begin{aligned} & e^{i(t_{r+2j}-t_{r+2j+2})\Delta_\pm^{(r+2j)}} \\ &\quad = e^{i(t_{r+2j}-t_{r+2j+2})\Delta_{\pm,x_i}} e^{i(t_{r+2j}-t_{r+2j+2})(\Delta_\pm^{(r+2j)}-\Delta_{\pm,x_i})}, \end{aligned} \tag{7.9}$$

which in turn allows us to see (after two basic commutations) that P equals to:

$$\begin{aligned} P = & e^{i(t_{r+2j}-t_{r+2j+2})\Delta_{\pm,x_i}} B_{l;r+2j-1,r+2j} B_{i;r+2j+1,r+2j+2} \\ & e^{i(t_{r+2j}-t_{r+2j+2})(\Delta_\pm^{(r+2j)}-\Delta_{\pm,x_i})}. \end{aligned} \tag{7.10}$$

Therefore using (7.10), the LHS of (7.8) can be rewritten as

$$\begin{aligned}
& e^{i(t_{r+2j-2}-t_{r+2j})\Delta_\pm^{(r+2j-2)}} P e^{i(t_{r+2j+2}-t_{r+2j+4})\Delta_\pm^{(r+2j+2)}} \\
= \; & e^{i(t_{r+2j-2}-t_{r+2j})\Delta_\pm^{(r+2j-2)}} e^{i(t_{r+2j}-t_{r+2j+2})\Delta_{\pm,x_i}} \\
& B_{l;r+2j-1,r+2j} B_{i;r+2j+1,r+2j+2} \\
& e^{i(t_{r+2j}-t_{r+2j+2})(\Delta_\pm^{(r+2j)}-\Delta_{\pm,x_i})} e^{i(t_{r+2j+2}-t_{r+2j+4})\Delta_\pm^{(r+2j+2)}} \\
= \; & e^{i(t_{r+2j-2}-t_{r+2j})\Delta_\pm^{(r+2j-2)}} e^{i(t_{r+2j}-t_{r+2j+2})\Delta_{\pm,x_i}} \\
& B_{l;r+2j-1,r+2j} B_{i;r+2j+1,r+2j+2} \\
& e^{i(t_{r+2j+2}-t_{r+2j+4})(\Delta_{\pm,x_i}+\Delta_{\pm,r+2j+1}+\Delta_{\pm,r+2j+2})} \\
& \qquad e^{i(t_{r+2j}-t_{r+2j+4})(\Delta_{\pm,x_1}+\ldots+\hat{\Delta}_{\pm,x_i}+\ldots+\Delta_{\pm,r+2j})}, \qquad (7.11)
\end{aligned}$$

where $\hat{\Delta}_{\pm,x_i}$ denotes that the term $\Delta_{\pm,x_i}$ is missing.
On the other hand, we can rewrite $\tilde{\Delta}_\pm^{(r+2j)}$ as

$$\begin{aligned}
\tilde{\Delta}_\pm^{(r+2j)} &= \Delta_\pm^{(r+2j)} - \Delta_{\pm,x_{r+2j}} - \Delta_{\pm,x_{r+2j-1}} + \Delta_{\pm,x_{r+2j+1}} + \Delta_{\pm,x_{r+2j+2}} \\
&= \Delta_\pm^{(r+2j-2)} + \Delta_{\pm,x_{r+2j+1}} + \Delta_{\pm,x_{r+2j+2}} \\
&= (\Delta_\pm^{(r+2j-2)} - \Delta_{\pm,x_i}) + (\Delta_{\pm,x_i} + \Delta_{\pm,x_{r+2j+1}} + \Delta_{\pm,x_{r+2j+2}}).
\end{aligned}$$

Hence the factor $e^{-i(t_{r+2j}-t_{r+2j+2})\tilde{\Delta}_\pm^{(r+2j)}}$ appearing in the definition of $\tilde{P}$ can be rewritten as:

$$\begin{aligned}
& e^{-i(t_{r+2j}-t_{r+2j+2})\tilde{\Delta}_\pm^{(r+2j)}} \\
= \; & e^{-i(t_{r+2j}-t_{r+2j+2})(\Delta_\pm^{(r+2j-2)}-\Delta_{\pm,x_i})} e^{-i(t_{r+2j}-t_{r+2j+2})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{r+2j+1}}+\Delta_{\pm,x_{r+2j+2}})},
\end{aligned}$$

which in turn implies that (after two basic commutations) $\tilde{P}$ equals

$$\begin{aligned}
\tilde{P} = & e^{-i(t_{r+2j}-t_{r+2j+2})(\Delta_\pm^{(r+2j-2)}-\Delta_{\pm,x_i})} B_{i;r+2j+1,r+2j+2} B_{l;r+2j-1,r+2j} \\
& e^{-i(t_{r+2j}-t_{r+2j+2})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{r+2j+1}}+\Delta_{\pm,x_{r+2j+2}})}. \qquad (7.12)
\end{aligned}$$

Thus using (7.12), the RHS of (7.8) can be written as

$$\begin{aligned}
& e^{i(t_{r+2j-2}-t_{r+2j+2})\Delta_\pm^{(r+2j-2)}} \tilde{P} e^{i(t_{r+2j}-t_{r+2j+4})\Delta_\pm^{(r+2j+2)}} \\
= \; & e^{i(t_{r+2j-2}-t_{r+2j+2})\Delta_\pm^{(r+2j-2)}} e^{-i(t_{r+2j}-t_{r+2j+2})(\Delta_\pm^{(r+2j-2)}-\Delta_{\pm,x_i})} \\
& B_{i;r+2j+1,r+2j+2} B_{l;r+2j-1,r+2j} \\
& e^{-i(t_{r+2j}-t_{r+2j+2})(\Delta_{\pm,x_i}+\Delta_{\pm,x_{r+2j+1}}+\Delta_{\pm,x_{r+2j+2}})} e^{i(t_{r+2j}-t_{r+2j+4})\Delta_\pm^{(r+2j+2)}} \\
= \; & e^{i(t_{r+2j-2}-t_{r+2j})\Delta_\pm^{(r+2j-2)}} e^{i(t_{r+2j}-t_{r+2j+2})\Delta_{\pm,x_i}} \\
& B_{i;r+2j+1,r+2j+2} B_{l;r+2j-1,r+2j} \\
& e^{i(t_{r+2j+2}-t_{r+2j+4})(\Delta_{\pm,x_i}+\Delta_{\pm,r+2j+1}+\Delta_{\pm,r+2j+2})} \\
& \qquad e^{i(t_{r+2j}-t_{r+2j+4})(\Delta_{\pm,x_1}+\ldots+\hat{\Delta}_{\pm,x_i}+\ldots+\Delta_{\pm,r+2j})}. \qquad (7.13)
\end{aligned}$$

We combine (7.11) and (7.13) to obtain (7.8).

Now we are ready to prove (7.7). We observe that thanks to the symmetry the value of $I(\mu,\sigma)$ does not change if in (7.5) we perform the following two exchanges in the arguments of $\gamma^{(r+2n)}$ only:

- exchange $(x_{r+2j-1}, x'_{r+2j-1})$ with $(x_{r+2j+1}, x'_{r+2j+1})$
- exchange (x_{r+2j}, x'_{r+2j}) with $(x_{r+2j+2}, x'_{r+2j+2})$.

After these two exchanges we use (7.8) and the definition of $\tilde{P}$ to rewrite (7.5) as:

$$\begin{aligned}
&I(\mu,\sigma)\\
&=\int_{t_r\geq\ldots\geq t_{\sigma(r+2j)}\geq t_{\sigma(r+2j+2)}\geq\ldots\geq t_{\sigma(r+2n)}\geq 0}\ldots\\
&\qquad e^{i(t_{r+2j-2}-t_{r+2j})\Delta_{\pm}^{(r+2j-2)}}Pe^{i(t_{r+2j+2}-t_{r+2j+4})\Delta_{\pm}^{(r+2j+2)}}(\ldots)'\,dt_{r+2}\ldots dt_{r+2n}\\
&=\int_{t_r\geq\ldots\geq t_{\sigma(r+2j)}\geq t_{\sigma(r+2j+2)}\geq\ldots\geq t_{\sigma(r+2n)}\geq 0}\ldots\\
&\qquad e^{i(t_{r+2j-2}-t_{r+2j+2})\Delta_{\pm}^{(r+2j-2)}}\tilde{P}e^{i(t_{r+2j}-t_{r+2j+4})\Delta_{\pm}^{(r+2j+2)}}(\ldots)'\,dt_{r+2}\ldots dt_{r+2n}\\
&=\int_{t_r\geq\ldots\geq t_{\sigma(r+2j)}\geq t_{\sigma(r+2j+2)}\geq\ldots\geq t_{\sigma(r+2n)}\geq 0}\int_{\mathbb{R}^{d(r+2n+2)}}\ldots e^{i(t_{r+2j-2}-t_{r+2j+2})\Delta_{\pm}^{(r+2j-2)}}\\
&\qquad \delta_{i;r+2j+1,r+2j+2}e^{-i(t_{r+2j}-t_{r+2j+2})\tilde{\Delta}_{\pm}^{(r+2j)}}\delta_{l;r+2j-1,r+2j}\\
&\qquad e^{i(t_{r+2j}-t_{r+2j+4})\Delta_{\pm}^{(r+2j+2)}}(\ldots)'\,dt_{r+2}\ldots dt_{r+2n},
\end{aligned}\tag{7.14}$$

where $\delta_{j;k+1,k+2}$ denotes the kernel of the operator $B_{i;k+1,k+2}$ i.e.

$$\begin{aligned}
\delta_{j;k+1,k+2} &= \delta(x_j-x_{k+1})\delta(x_j-x'_{k+1})\delta(x_j-x_{k+2})\delta(x_j-x'_{k+2})\\
&\quad -\delta(x'_j-x_{k+1})\delta(x'_j-x'_{k+1})\delta(x'_j-x_{k+2})\delta(x'_j-x'_{k+2})\,.
\end{aligned}\tag{7.15}$$

Now in (7.14) we perform the change of variables that exchanges

$$(t_{r+2j-1}, x_{r+2j-1}, x'_{r+2j-1})\quad\text{and}\quad(t_{r+2j+1}, x_{r+2j+1}, x'_{r+2j+1})$$

as well as

$$(t_{r+2j}, x_{r+2j}, x'_{r+2j})\quad\text{and}\quad(t_{r+2j+2}, x_{r+2j+2}, x'_{r+2j+2})\,.$$

Under the same change of variables $\tilde{\Delta}^{(r+2j)}$ which is equal to

$$\begin{aligned}
\tilde{\Delta}^{(r+2j)} &= \Delta_{\pm}^{(r+2j)}-\Delta_{\pm,x_{r+2j}}-\Delta_{\pm,x_{r+2j-1}}+\Delta_{\pm,x_{r+2j+1}}+\Delta_{\pm,x_{r+2j+2}}\\
&= \Delta_{\pm}^{(r+2j-2)}+\Delta_{\pm,x_{r+2j-1}}+\Delta_{\pm,x_{r+2j}}\\
&\qquad -\Delta_{\pm,x_{r+2j}}-\Delta_{\pm,x_{r+2j-1}}+\Delta_{\pm,x_{r+2j+1}}+\Delta_{\pm,x_{r+2j+2}}\\
&= \Delta_{\pm}^{(r+2j-2)}+\Delta_{\pm,x_{r+2j+1}}+\Delta_{\pm,x_{r+2j+2}}
\end{aligned}$$

becomes $\Delta_{\pm}^{(r+2j-2)} + \Delta_{\pm,x_{r+2j-1}} + \Delta_{\pm,x_{r+2j}}$ that equals $\Delta_{\pm}^{(r+2j)}$. Therefore, after we perform this change of variables in (7.14), we obtain

$$\begin{aligned} I(\mu,\sigma) &= \int_{t_r \geq \cdots \geq t_{\sigma'(r+2j+2)} \geq t_{\sigma'(r+2j)} \geq \cdots \geq t_{\sigma(r+2n)}} \cdots e^{i(t_{r+2j-2}-t_{r+2j})\Delta_{\pm}^{(r+2j-2)}} \\ &\qquad B_{i;r+2j-1,r+2j} e^{-i(t_{r+2j+2}-t_{r+2j})\Delta_{\pm}^{(r+2j)}} B_{l;r+2j+1,r+2j+2} \\ &\qquad e^{i(t_{r+2j+2}-t_{r+2j+4})\Delta_{\pm}^{(r+2j+2)}} (\cdots)' \, dt_{r+2} \cdots dt_{r+2n} \\ &= I(\mu',\sigma'), \end{aligned} \tag{7.16}$$

where $\sigma' = (r+2j, r+2j+2) \circ \sigma$. Here (a,b) denotes the permutation which reverses a and b. Hence (7.7) is proved. □

Let us consider the set N of those matrices in M which are in so-called "upper echelon" form. Here, as in [12], we say that a matrix of the type (7.2) is in upper echelon form if each highlighted entry in a row is to the left of each highlighted entry in a lower row. For example, the following matrix is in upper echelon form (with $r = 1$ and $n = 3$):

$$\begin{bmatrix} \mathbf{B_{1;2,3}} & B_{1;4,5} & B_{1;6,7} \\ 0 & \mathbf{B_{2;4,5}} & B_{2;6,7} \\ 0 & B_{3;4,5} & B_{3;6,7} \\ 0 & 0 & \mathbf{B_{4;6,7}} \\ 0 & 0 & B_{5;6,7} \end{bmatrix}.$$

In the same way as in Lemma 3.2 in [12] one can prove that in our context:

Lemma 7.2. *For each matrix in M there is a finite number of acceptable moves that transforms the matrix into upper echelon form.*

Let $C_{r,n}$ denote the number of upper echelon matrices of the size $(r+2n-2) \times n$. The following lemma gives an upper bound on $C_{r,n}$.

Lemma 7.3. *The following holds:*

$$C_{r,n} \leq 2^{r+3n-2}.$$

Proof. As in [12] the proof proceeds in two steps.

Step 1 First, we bring all highlighted entries to the first row. In such a way the first row is partitioned into subsets that consist of elements that were originally in the same row. Let us denote by P_n the number of possible partitions of the first row into these subsets. Then

$$P_n \leq 2^n,$$

ax explained in [12]. One can see this by first observing that

$$P_n = 1 + P_1 + \cdots + P_{n-1}, \tag{7.17}$$

which in turn can be verified by counting the number of the elements in the last subset. More precisely, if the last subset has 0 elements that gives exactly one contribution toward P_n. In general, if the last subset has k elements, then the rest of $n-k$ elements of the first row can be partitioned into P_{n-k} ways. Hence (7.17) follows.

Step 2 Now we reassemble the matrix obtained in the previous step by lowering the first subset into the first used row, the second subset into the second used row etc. If a given partition of the first row has exactly i subsets, then these subsets can be lowered in an order preserving way to the available $r+2n-2$ rows in $\binom{r+2n-2}{i}$ ways.

Now we combine Steps 1 and 2 to conclude

$$C_{r,n} \leq P_n \sum_{i=1}^{n} \binom{r+2n-2}{i} \leq 2^{r+3n-2}.$$

□

Let μ_{es} be a matrix in N. We write $\mu \sim \mu_{es}$ if μ can be transformed into μ_{es} in finitely many acceptable moves. It can be seen that:

Theorem 7.4. *Suppose $\mu_{es} \in N$. Then there exists a subset of $[0, t_r]^n$, denoted by D, such that*

$$\sum_{\mu \sim \mu_{es}} \int_0^{t_r} ... \int_0^{t_{r+2n}} J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} ... dt_{r+2n} = \int_D J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} ... dt_{r+2n}.$$

Proof. Here we give an outline of the proof, which is analogous to the proof of a similar result stated in Theorem 3.4 in [12].

We consider the integral

$$I(\mu, id) = \int_0^{t_r} ... \int_0^{t_{r+2n}} J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} ... dt_{r+2n}.$$

and perform finitely many acceptable moves on the corresponding matrix determined by (μ, id) until we transform it to the special upper echelon matrix associated with (μ_{es}, σ). Then Lemma 7.1 guarantees that

$$I(\mu, id) = I(\mu_{es}, \sigma).$$

As in [12], if (μ_1, id) and (μ_2, id) with $\mu_1 \neq \mu_2$ produce the same echelon form μ_{es}, then the corresponding permutations σ_1 and σ_2 must be distinct. Hence, to determine D, we need to identify all permutations σ that occur in a connection with a given class of equivalence μ_{es}. Then D can be chosen to be the union of all $\{t_r \geq t_{\sigma(r+2)} \geq t_{\sigma(r+4)} \geq ... \geq t_{\sigma(r+2n)}\}$. □

8. Proof of Theorem 6.1

In combination with Theorem 7.4, the following result immediately implies Theorem 6.1.

Theorem 8.1. *Assume that $d \in \{1, 2\}$ and $t_r \in [0, T]$. The estimate*

$$\Big\| \int_D J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} ... dt_{r+2n} \Big\|_{L^2(\mathbb{R}^{dr} \times \mathbb{R}^{dr})} < C^r\, (C_0 T)^n \tag{8.1}$$

holds for a constant C_0 independent of r and T.

Proof. We first address the case of dimension $d = 1$. We infer from Theorem 4.3 that in this case,

$$\Big\| S^{(k,\alpha)} B_{j;k+1,k+2} \mathcal{U}^{(k+2)}(t_{k+1} - t_{k+2}) \widetilde{\gamma}^{(k+2)} \Big\|_{L^2(\mathbb{R}^k \times \mathbb{R}^k)} \\ \leq C \Big\| S^{(k+2,\alpha)} \mathcal{U}^{(k+2)}(t_{k+1} - t_{k+2}) \widetilde{\gamma}^{(k+2)} \Big\|_{L^2(\mathbb{R}^{k+2} \times \mathbb{R}^{k+2})} \tag{8.2}$$

for any $\alpha > \frac{1}{2}$.

We find that, for $t_r \in [0, T]$,

$$\begin{aligned}
&\Big\| \int_D J^r(\underline{t}_{r+2n}, \mu)\, dt_{r+2} ... dt_{r+2n} \Big\|_{L^2(\mathbb{R}^r \times \mathbb{R}^r)} \\
&\leq \; T^n \sup_{t_{r+2},\dots,t_{r+2n} \in [0,T]} \Big\| S^{(r,\alpha)} B_{\mu(r+1);r+1,r+2} \qquad (8.3)\\
&\qquad\qquad \mathcal{U}^{(r+2)}(t_{r+2} - t_{r+4}) \widetilde{\gamma}^{(r+2)} \Big\|_{L^2(\mathbb{R}^r \times \mathbb{R}^r)} \\
&\leq \; C T^n \sup_{t_{r+2},\dots,t_{r+2n} \in [0,T]} \Big\| S^{(r+2,\alpha)} \mathcal{U}^{(r+2)}(t_{r+2} - t_{r+4}) \widetilde{\gamma}^{(r+2)} \Big\|_{L^2(\mathbb{R}^{r+2} \times \mathbb{R}^{r+2})} \qquad (8.4)\\
&\leq \; C T^n \sup_{t_{r+2},\dots,t_{r+2n} \in [0,T]} \Big\| \mathcal{U}^{(r+2)}(t_{r+2} - t_{r+4}) S^{(r+2,\alpha)} \widetilde{\gamma}^{(r+2)} \Big\|_{L^2(\mathbb{R}^{r+2} \times \mathbb{R}^{r+2})} \qquad (8.5)\\
&= \; C T^n \sup_{t_{r+2},\dots,t_{r+2n} \in [0,T]} \Big\| S^{(r+2,\alpha)} B_{\mu(r+3);r+3,r+4} \qquad (8.6)\\
&\qquad\qquad \mathcal{U}^{(r+2)}(t_{r+2} - t_{r+4}) \widetilde{\gamma}^{(r+4)} \Big\|_{L^2(\mathbb{R}^{r+2} \times \mathbb{R}^{r+2})} \\
&= \; C T^n \sup_{t_{r+4},\dots,t_{r+2n} \in [0,T]} \Big\| S^{(r+2,\alpha)} B_{\mu(r+3);r+3,r+4} \qquad (8.7)\\
&\qquad\qquad \mathcal{U}^{(r+4)}(t_{r+4} - t_{r+6}) \widetilde{\gamma}^{(r+4)} \Big\|_{L^2(\mathbb{R}^{r+2} \times \mathbb{R}^{r+2})} \\
&\leq \; \cdots \\
&\leq \; C^{n-1} T^n \sup_{t_{r+2n} \in [0,T]} \Big\| S^{(r+2n,\alpha)} B_{\mu(r+2n);r+2n,r+2n+1} \gamma^{(r+2n)} \Big\|_{L^2(\mathbb{R}^{r+2n} \times \mathbb{R}^{r+2n})} \qquad (8.8)\\
&\leq \; C^r \, (C\,T)^n \,, \qquad (8.9)
\end{aligned}$$

where in order to bound (8.3) we have employed the estimate (8.2), and subsequently used that any free evolution operator $\mathcal{U}^{(\ell)}$ commutes with any $S^{(j,\alpha)}$, since both are Fourier multiplication operators. Then, to obtain (8.6) we use unitarity of $\mathcal{U}^{(r+2)}$, and to get (8.7), we observe that the norm in (8.6) is independent of t_{r+2}. We then repeat these steps until all free evolution operators are eliminated, and we arrive at (8.8). In the last step, we use the a priori energy estimate provided by Theorem 4.2 for $d = 1$. Clearly, for T sufficiently small, we have that $(CT)^n \to 0$ as $n \to \infty$. This implies that $\gamma^{(1)} = 0$ in $[0, T]$ for zero initial condition, provided that $\frac{1}{2} < \alpha \leq 1$.

For $d = 2$, the proof proceeds precisely in the same way as in [12, 13], under the condition that $\frac{5}{6} < \alpha < 1$, by using nested Duhamel's formulas of section 7, recursive applications of the space-time bounds given in Theorem 5.1 and at the end by using the a priori spatial bound provided by Theorem 4.2. $\square$

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