Strong time operators associated with generalized Hamiltonians

Fumio Hiroshima*†, Sotaro Kuribayashi‡
and Yasumichi Matsuzawa§

Abstract

Let the pair of operators, (H, T), satisfy the weak Weyl relation:

$$Te^{-itH} = e^{-itH}(T + t),$$

where H is self-adjoint and T is closed symmetric. Suppose that $g \in C^2(\mathbb{R} \setminus K)$ for some $K \subset \mathbb{R}$ with Lebesgue measure zero and that $\lim_{|\lambda|\to\infty} g(\lambda)e^{-\beta\lambda^2} = 0$ for all $\beta > 0$. Then we can construct a closed symmetric operator D such that $(g(H), D)$ also obeys the weak Weyl relation.

1 Weak Weyl relation and strong time operators

1.1 Introduction

The energy of a quantum system can be realized as a self-adjoint operator on some Hilbert space, whereas time t is treated as a parameter, and not intuitively as an operator. So, since the foundation of quantum mechanics, the energy-time uncertainty relation has had a different basis than that underlying the position-momentum uncertainty relation.

Let Q be the multiplication operator defined by $(Qf)(x) = xf(x)$ with maximal domain $\mathrm{D}(Q) = \{f \in L^2(\mathbb{R}) | \int |x|^2 f(x)^2 dx < \infty\}$ and let $P = -id/dx$ be the weak derivative with domain $H^1(\mathbb{R})$. In quantum mechanics, the position operator Q and the

*F.H. thanks for Grant-in-Aid for Science Research (B) 20340032 from JSPS for financial support.
†Graduate School of Mathematics, Kyushu University 812-8581, Fukuoka, Japan
‡Graduate School of Mathematics, Kyushu University 812-8581, Fukuoka, Japan
§Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan

momentum operator P in $L^2(\mathbb{R})$ obey the Weyl relation: $e^{-isP}e^{-itQ} = e^{-ist}e^{-itQ}e^{-isP}$ for $s, t \in \mathbb{R}$. From this we can derive the so-called weak Weyl relation:

$$Qe^{-itP} = e^{-itP}(Q + t), \quad t \in \mathbb{R}, \tag{1.1}$$

and moreover the canonical commutation relation $[P, Q] = -iI$ also holds. The strong time operator T is defined as an operator satisfying (1.1) with Q and P replaced by T and the Hamiltonian H of the quantum system under consideration, respectively.

More precisely, we explain the weak Weyl relation (1.1) as follows. Let $\mathscr{H}$ be a Hilbert space over the complex field $\mathbb{C}$. We denote by $\mathrm{D}(L)$ the domain of an operator L. We say that the pair (H, T) consisting of a self-adjoint operator H and a symmetric operator T on $\mathscr{H}$ obeys the weak Weyl relation if and only if, for all $t \in \mathbb{R}$,

(1) $e^{-itH}\mathrm{D}(T) \subset \mathrm{D}(T)$;

(2) $Te^{-itH}\Phi = e^{-itH}(T + t)\Phi$ for $\Phi \in \mathrm{D}(T)$.

Here T is referred to as a strong time operator associated with H and we denote it by T_H for T. Note that a strong time operator is not unique. Although from the weak Weyl relation it follows that $[H, T_H] = -iI$, the converse is not true; a pair (A, B) satisfying $[A, B] = -iI$ does not necessarily obey the Weyl relation or the weak Weyl relation. If T_H is self-adjoint, then it is known that

$$e^{-isT_H}e^{-itH} = e^{-ist}e^{-itH}e^{-isT_H} \tag{1.2}$$

holds. In particular when Hilbert space $\mathscr{H}$ is separable, by the von Neumann uniqueness theorem the Weyl relation (1.2) implies that H and T_H are unitarily equivalent to $\oplus^n P$ and $\oplus^n Q$ with some n, respectively. This asserts that any strong time operators associated with a semibounded H on a separable Hilbert space are symmetric non-self-adjoint. These facts may implicitly suggest that strong time operators are not "observable".

A time operator but not necessarily strong associated with a self-adjoint operator H is defined as an operator T for which $[H, T] = -iI$. As was mentioned above, although a strong time operator is automatically a time operator, the converse is not true. For example there is no strong time operator associated with the harmonic oscillator $\frac{1}{2}(P^2 + \omega^2 Q^2)$, whereas its time operator is formally given by

$$\frac{1}{2\omega}(\arctan(\omega P^{-1}Q) + \arctan(\omega Q P^{-1})).$$

See e.g. [AM08-b, Gal02, Gal04, LLH96, Dor84, Ros69]. The concept of time operators was derived in the framework for the energy-time uncertainty relation in [KA94]. See also e.g. [Fuj80, FWY80, GYS81-1, GYS81-2]. A strong connection with the decay of survival probability was pointed out by [Miy01], where the weak Weyl relation was introduced and then strong time operators were discussed. Moreover it was drastically generalized in [Ara05] and some uniqueness theorems are established in [Ara08].

This paper is inspired by [Miy01, Section VII] and [AM08-a]. In particular Arai and Matsuzawa [AM08-a] developed machinery for reconstructing a pair of operators obeying the weak Weyl relation from a given pair (H, T_H); in particular, they constructed a strong time operator associated with $\log|H|$. The main result of the paper is an extension of this work and we derive a time operator associated with general Hamiltonian $g(H)$.

1.2 Description of the main results

By (1.1) the strong time operator T_P associated with P is unique and is given by

$$T_P = Q. \tag{1.3}$$

For the self-adjoint operator $(1/2)P^2$ in $L^2(\mathbb{R})$, it is established that

$$T_{(1/2)P^2} = \frac{1}{2}(P^{-1}Q + QP^{-1}) \tag{1.4}$$

is an associated strong time operator referred to as the Aharonov-Bohm operator. Comparing (1.3) with (1.4) we arrive at

$$T_{(1/2)P^2} = \frac{1}{2}\left(f'(P)^{-1}T_P + T_P f'(P)^{-1}\right), \tag{1.5}$$

where $f(\lambda) = (1/2)\lambda^2$. We wish to extend formula (1.5) for more general f's and for any (H, T_H).

More precisely let g be some Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$. We want to construct a map $\mathscr{T}(g)$ such that $\mathscr{T}(g)T_H = T_{g(H)}$ and to show that

$$T_{g(H)} = \frac{1}{2}(g'(H)^{-1}T_H + T_H g'(H)^{-1}).$$

We denote the set of n times continuously differentiable functions on $\Omega \subset \mathbb{R}$ with compact support by $C_0^n(\Omega)$. Throughout, we suppose that the following assumptions hold.

Assumption 1.1 (H,T) *obeys the weak Weyl relation and* T *is a closed symmetric operator.*

Note that if (H,T) satisfies the weak Weyl relation, then so does $(H,\overline{T})$.

Assumption 1.2 *(1)* $g \in C^2(\mathbb{R}\setminus K)$ *for some* $K \subset \mathbb{R}$ *with Lebesgue measure zero; (2) The Lebesgue measure of the set of zero points* $\{\lambda \in \mathbb{R}\setminus K | g'(\lambda)=0\}$ *is zero; (3)* $\lim_{|\lambda|\to\infty} g(\lambda)e^{-\beta\lambda^2}=0$ *for all* $\beta>0$.

We fix (H,T), $K \subset \mathbb{R}$ and $g \in C^2(\mathbb{R}\setminus K)$ in what follows. For a measurable function ρ, $\rho(H)$ is defined by $\rho(H)=\int \rho(\lambda)dE_\lambda$ for the spectral resolution E_λ of H. Let Z be the set of singular points of $1/g'$:

$$Z=\{\lambda \in \mathbb{R}\setminus K | g'(\lambda)=0\} \cup K,$$

which has Lebesgue measure zero. Define the dense subspace $X_n^{\mathscr{D}}$, $0 \le n \le \infty$, $\mathscr{D} \subset \mathscr{H}$, in $\mathscr{H}$ by

$$X_n^{\mathscr{D}} = \text{L.H.}\{\rho(H)\phi | \rho \in C_0^n(\mathbb{R}\setminus Z), \phi \in \mathscr{D}\}, \tag{1.6}$$

where L.H.$\{\cdots\}$ denotes the linear hull of $\{\cdots\}$ and $C_0^0=C_0$. The next proposition is fundamental.

Proposition 1.3 [Ara05] *Let* $f \in C^1(\mathbb{R})$ *and let both* f *and* f' *be bounded. Then* $f(H)\mathrm{D}(T) \subset \mathrm{D}(T)$ *and*

$$Tf(H)\phi = f(H)T\phi + if'(H)\phi, \quad \phi \in \mathrm{D}(T). \tag{1.7}$$

PROOF: First suppose that $f \in C_0^\infty(\mathbb{R})$. Let $\check{f}$ denote the inverse Fourier transform of f. Then for $\psi \in \mathrm{D}(T)$,

$$\begin{aligned}(T\psi, f(H)\phi) &= (2\pi)^{-1/2}\int_{\mathbb{R}} (T\psi, e^{-i\lambda H}\phi)\check{f}(\lambda)d\lambda \\ &= (2\pi)^{-1/2}\int_{\mathbb{R}} \check{f}(\lambda)(\psi, e^{-i\lambda H}(T+\lambda)\phi)d\lambda = (\psi, (f(H)T+if'(H))\phi).\end{aligned}$$

So (1.7) follows for $f \in C_0^\infty(\mathbb{R})$. By a limiting argument on f and the fact that T is closed, (1.7) follows for $f \in C^1(\mathbb{R})$ such that f and f' are bounded. **qed**

This proposition suggests that *informally*

$$Te^{-itg(H)}\phi = e^{-itg(H)}T\phi + tg'(H)e^{-itg(H)}\phi$$

and then $Tg'(H)^{-1}e^{-itg(H)}\phi = e^{-itg(H)}(Tg'(H)^{-1} + t)\phi$. Symmetrizing $Tg'(H)^{-1}$, we expect that a strong time operator associated with $g(H)$ will be given by

$$T_{g(H)} = \frac{1}{2}(g'(H)^{-1}T + Tg'(H)^{-1}). \tag{1.8}$$

In order to establish (1.8), the remaining problem is to check the domain argument and to extend Proposition 1.3 for unbounded f and f'.

Lemma 1.4 *It follows that*

(1) $T : X_n^{\mathrm{D}(T)} \to X_{n-1}^{\mathscr{H}}$ *for* $1 \le n \le \infty$.

(2) $g'(H)^{-1} : \begin{cases} X_n^{\mathscr{D}} \to X_1^{\mathscr{D}}, & 1 \le n \le \infty, \\ X_0^{\mathscr{D}} \to X_0^{\mathscr{D}}, & n = 0, \end{cases}$ *for any* $\mathscr{D} \subset \mathscr{H}$.

Proof: Let $\Phi = \rho(H)\phi \in X_n^{\mathrm{D}(T)}$. By Proposition 1.3, $\Phi \in \mathrm{D}(T)$ and we have $T\Phi = i\rho'(H)\phi + \rho(H)T\phi$. Then (1) follows. Note that $\rho/g' \in C_0^1(\mathbb{R} \setminus K)$ for $\rho \in C_0^n(\mathbb{R} \setminus K)$ with $n \ge 1$, and $\rho/g' \in C_0(\mathbb{R} \setminus Z)$ for $\rho \in C_0(\mathbb{R} \setminus K)$. Then (2) follows. **qed**

Define the symmetric operator $\widetilde{D}$ by

$$\widetilde{D} = \frac{1}{2}(g'(H)^{-1}T + Tg'(H)^{-1})\Big\lceil_{X_1^{\mathrm{D}(T)}}. \tag{1.9}$$

$\widetilde{D}$ is well defined by Lemma 1.4. Since the domain of the adjoint of $\widetilde{D}$ includes the dense subspace $X_1^{\mathrm{D}(T)}$, then $\widetilde{D}$ is closable. We define

$$D = \frac{1}{2}\overline{(g'(H)^{-1}T + Tg'(H)^{-1})\lceil_{X_1^{\mathrm{D}(T)}}}. \tag{1.10}$$

The main theorem is as follows.

Theorem 1.5 *Suppose Assumptions 1.1 and 1.2. Then* $(g(H), D)$ *obeys the weak Weyl relation.*

Example 1.6 *Examples of strong time operators are as follows:*

(1) g *is a polynomial.*

(2) *Let* $g(\lambda) = \log|\lambda|$. *Then a strong time operator associated with* $\log|H|$ *is*

$$\frac{1}{2}\overline{(HT + TH)\lceil_{X_1^{\mathrm{D}(T)}}}.$$

This time operator is derived in [AM08-a].

(3) *Let* $(H,T)=(P,Q)$ *and* $g(\lambda)=\sqrt{\lambda^2+m^2}$, $m\geq 0$. *Then a strong time operator associated with* $H(P)=\sqrt{P^2+m^2}$ *is*

$$\frac{1}{2}\overline{(H(P)P^{-1}Q+QP^{-1}H(P))\lceil_{\mathrm{D}(X_1^{\mathrm{D}(Q)})}}.$$

$H(P)$ is a semi-relativistic Schrödinger operator.

(4) *Strong time operators associated with (3) and* P^2 *can be generalized. Let* $H_\alpha(P)=(P^2+m^2)^{\alpha/2}$, $\alpha\in\mathbb{R}\setminus\{0\}$. *Then a strong time operator associated with* $H_\alpha(P)$ *is given by*

$$\frac{1}{2\alpha}\overline{((P^2+m^2)P^{-1}H_\alpha(P)^{-1}Q+QH_\alpha(P)^{-1}P^{-1}(P^2+m^2))\lceil_{\mathrm{D}(X_1^{\mathrm{D}(Q)})}}.$$

2 Proof of Theorem 1.5

In order to prove Theorem 1.5 we approximate g with some bounded functions. Define

$$g_\beta(\lambda)=g(\lambda)e^{-\beta\lambda^2},\quad \beta\geq 0. \tag{2.1}$$

Lemma 2.1 *Let* $\Phi\in X_1^{\mathrm{D}(T)}$. *Then for sufficiently small* $\beta\geq 0$ *(β possibly depending on Φ),*

(1) $\Phi\in\mathrm{D}(g_\beta'(H)^{-1})$ *and* $g_\beta'(H)^{-1}\Phi\in\mathrm{D}(T)$;

(2) $e^{-itg_\beta(H)}g_\beta'(H)^{-1}\Phi\in\mathrm{D}(T)$;

(3) $T\Phi\in\mathrm{D}(g_\beta'(H)^{-1})$;

(4) $e^{-itg_\beta(H)}\Phi\in\mathrm{D}(T)$ *and* $Te^{-itg_\beta(H)}\Phi\in\mathrm{D}(g_\beta'(H)^{-1})$.

Proof: Let $\Phi=\rho(H)\phi\in X_1^{\mathrm{D}(T)}$ with $\rho\in C_0^1(\mathbb{R}\setminus Z)$ and $\phi\in\mathrm{D}(T)$. Put $\mathscr{K}=\mathrm{supp}\rho$. Note that $Z\not\subset\mathscr{K}$. Then in the case of $\beta=0$, g_β' has no zero point on $\mathscr{K}$. We have

$$m<\inf_{\lambda\in\mathscr{K}}|g'(\lambda)|\leq\sup_{\lambda\in\mathscr{K}}|g'(\lambda)|<M$$

for some $m>0$ and $M>0$. Let $Z_\beta=\{\lambda\in\mathbb{R}\setminus K|g_\beta'(\lambda)=0\}$. Let $a\in Z_\beta$. Then $g'(a)/a=2\beta$ from the definition of g_β. However $\inf_{\lambda\in\mathscr{K}}|g'(\lambda)/\lambda|>c$ for some $c>0$. Thus for β such that

$$0<\beta<c/2, \tag{2.2}$$

g'_β has no zero points in $\mathscr{K}$. Hence $\rho/g'_\beta \in C_0^1(\mathbb{R}\setminus Z)$ and then $\Phi \in \mathrm{D}(g'_\beta(H)^{-1})$. By Lemma 1.3, $g'_\beta(H)^{-1}\Phi = g'_\beta(H)^{-1}\rho(H)\phi \in \mathrm{D}(T)$ if (2.2) holds, and (1) follows.

We can also see that $e^{-itg_\beta}\rho/g'_\beta \in C_0^1(\mathbb{R}\setminus Z)$ and that its derivative is bounded if (2.2) holds. Then $e^{-itg_\beta(H)}g'_\beta(H)^{-1}\Phi \in \mathrm{D}(T)$ follows by Lemma 1.3 and (2) follows.

Since $T\rho(H)\phi = i\rho'(H)\phi + \rho(H)T\phi$, $\rho, \rho' \in C_0^1(\mathbb{R}\setminus Z)$ and $\rho/g_\beta, \rho'/g_\beta \in C_0^1(\mathbb{R}\setminus Z)$, we have $T\Phi \in \mathrm{D}(g'_\beta(H)^{-1})$ if (2.2) holds, and (3) follows.

Finally we show (4). Since $h = e^{-itg_\beta}\rho \in C_0^1(\mathbb{R}\setminus Z)$ and its derivative is bounded, $e^{-itg_\beta(H)}\Phi \in \mathrm{D}(T)$ and $Th(H)\phi = ih'(H)\phi + h(H)T\phi$ follows. Here $h' \in C_0(\mathbb{R}\setminus Z)$. From this we have $Th(H)\phi \in \mathrm{D}(g'_\beta(H)^{-1})$. **qed**

Define

$$D_\beta = \frac{1}{2}(g'_\beta(H)^{-1}T + Tg'_\beta(H)^{-1}).$$

Note that for each $\Phi \in X_1^{\mathrm{D}(T)}$, by taking sufficiently small β, we can see that $\Phi \in \mathrm{D}(D_\beta)$.

Lemma 2.2 *Let $\Phi \in X$. Then for sufficiently small β (possibly depending on Φ),*

$$D_\beta e^{-itg_\beta(H)}\Phi = e^{-itg_\beta(H)}(D_\beta + t)\Phi.$$

PROOF: We divide the proof into three steps.

(Step 1)

$$Te^{-itg_\beta(H)}g'_\beta(H)^{-1}\Phi = e^{-itg_\beta(H)}(Tg'_\beta(H)^{-1} + t)\Phi. \tag{2.3}$$

Proof: From Lemma 1.3 it follows that $e^{-itg_\beta(H)}\mathrm{D}(T) \subset \mathrm{D}(T)$ and

$$Te^{-itg_\beta(H)}\Phi = e^{-itg_\beta(H)}(T + tg'_\beta(H))\Phi. \tag{2.4}$$

Since we have already shown in the previous lemmas that $\Phi \in \mathrm{D}(g'_\beta(H)^{-1})$ and $g'_\beta(H)^{-1}\Phi \in \mathrm{D}(e^{-itg_\beta(H)}T) \cap \mathrm{D}(Te^{-itg_\beta(H)})$, we can substitute $g'_\beta(H)^{-1}\Phi$ for Φ in (2.4). Then (2.3) follows.

(Step2)

$$g'_\beta(H)^{-1}Te^{-itg_\beta(H)}\Phi = e^{-itg_\beta(H)}g'_\beta(H)^{-1}T\Phi + te^{-itg_\beta(H)}\Phi. \tag{2.5}$$

Proof: Let $\Phi \in X_1^{\mathrm{D}(T)}$ and $\Psi \in X_1^{\mathrm{D}(T)}$. (2.3) implies that

$$(\Phi, Te^{-itg_\beta(H)}g'_\beta(H)^{-1}\Psi - e^{-itg_\beta(H)}Tg'_\beta(H)^{-1}\Psi) = t(\Phi, e^{-itg_\beta(H)}\Psi).$$

By Lemma 1.4, we can take the adjoint of both sides above. Then (2.5) follows if we transform t to $-t$.

(Step3) Combining (2.3) and (2.5), we have the lemma. **qed**

Lemma 2.3 *Let* $\Phi \in X_1^{\mathrm{D}(T)}$. *Then* $e^{itg(H)}\Phi \in \mathrm{D}(T)$ *and*

$$De^{-itg(H)}\Phi = e^{-itg(H)}(D+t)\Phi. \tag{2.6}$$

Proof: It is enough to show that

$$g'_\beta(H)^{-1}Te^{-itg_\beta(H)}\Phi \to g'(H)^{-1}Te^{-itg(H)}\Phi, \tag{2.7}$$

$$Tg'_\beta(H)^{-1}e^{-itg_\beta(H)}\Phi \to Tg'(H)^{-1}e^{-itg(H)}\Phi, \tag{2.8}$$

$$e^{-itg_\beta(H)}g'_\beta(H)^{-1}T\Phi \to e^{-itg(H)}g'(H)^{-1}T\Phi, \tag{2.9}$$

$$e^{-itg_\beta(H)}Tg'_\beta(H)^{-1}\Phi \to e^{-itg(H)}Tg'(H)^{-1}\Phi \tag{2.10}$$

strongly as $\beta \to 0$. Let $h_\beta = e^{-itg_\beta}\rho \in C_0^1(\mathbb{R}\setminus Z)$. Then

$$g'_\beta(H)^{-1}Th_\beta(H)\phi = g'_\beta(H)^{-1}(ih'_\beta(H) + h_\beta(H)T)\Phi.$$

We have

$$\|g'_\beta(H)^{-1}h'_\beta(H)\phi - g'(H)^{-1}h'_0(H)\phi\|^2 = \int_{\mathbb{R}}\left|\frac{h'_\beta(\lambda)}{g'_\beta(\lambda)} - \frac{h'_0(\lambda)}{g'(\lambda)}\right|^2 d\|E_\lambda\phi\|^2 \to 0,$$

$$\|g'_\beta(H)^{-1}h_\beta(H)T\phi - g'(H)^{-1}h_0(H)T\phi\|^2 = \int_{\mathbb{R}}\left|\frac{h_\beta(\lambda)}{g'_\beta(\lambda)} - \frac{h_0(\lambda)}{g'(\lambda)}\right|^2 d\|E_\lambda T\phi\|^2 \to 0$$

as $\beta \to 0$ by dominated convergence. Thus (2.7) follows.

Let $k_\beta = e^{-itg_\beta}\rho/g'_\beta \in C_0^1(\mathbb{R}\setminus Z)$. Then

$$Tg'_\beta(H)^{-1}e^{-itg_\beta(H)}\rho(H)\phi = ik'_\beta(H)\phi + k_\beta(H)T\phi.$$

We have

$$\|k'_\beta(H)\phi - k'_0(H)\phi\|^2 = \int_{\mathbb{R}}|k'_\beta(\lambda) - k'_0(\lambda)|^2 d\|E_\lambda\phi\|^2 \to 0,$$

$$\|k_\beta(H)T\phi - k_0(H)T\phi\|^2 = \int_{\mathbb{R}}|k_\beta(\lambda) - k_0(\lambda)|^2 d\|E_\lambda T\phi\|^2 \to 0$$

as $\beta \to 0$. Thus (2.8) follows. (2.9) is trivial to see.

Finally we show (2.10). Let $l_\beta = \rho/g'_\beta \in C_0^1(\mathbb{R}\setminus Z)$. Then

$$e^{-itg_\beta(H)}Tg'_\beta(H)^{-1}\Phi = e^{-itg_\beta(H)}(il'_\beta(H) + l_\beta T)\phi.$$

Then

$$\|e^{-itg_\beta}l'_\beta(H)\phi - e^{-itg(H)}l'_0(H)\phi\|^2 = \int_{\mathbb{R}} |e^{-itg_\beta(\lambda)}l'_\beta(\lambda) - e^{-itg(\lambda)}l'_0(\lambda)|^2 d\|E_\lambda\phi\|^2 \to 0,$$

$$\|e^{-itg_\beta}l_\beta(H)T\phi - e^{-itg(H)}l_0(H)T\phi\|^2 = \int_{\mathbb{R}} |e^{-itg_\beta(\lambda)}l_\beta(\lambda) - e^{-itg(\lambda)}l_0(\lambda)|^2 d\|E_\lambda T\phi\|^2 \to 0$$

as $\beta \to 0$. Thus the proof is complete. **qed**

Proof of Theorem 1.5:

Let $\Phi \in \mathrm{D}(D)$. There exists $\Phi_n \in X_1^{\mathrm{D}(T)}$ such that $\Phi_n \to \Phi$ and $D\Phi_n \to D\Phi$ as $n \to \infty$ strongly. By Lemma 2.3, for each Φ_n, $De^{-itg(H)}\Phi_n = e^{-itg(H)}(D+t)\Phi_n$ holds. Since D is closed, the theorem follows by a limiting argument. **qed**

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