

EIGENFUNCTIONS OF DIRAC OPERATORS AT THE THRESHOLD ENERGIES

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Abstract. We show that the eigenspaces of the Dirac operator $H = \alpha \cdot (D - A(x)) + m\beta$ at the threshold energies $\pm m$ coincide with the direct sum of the zero space and the kernel of the Weyl-Dirac operator $\sigma \cdot (D - A(x))$. Based on this result, we describe the asymptotic limits of the eigenfunctions of the Dirac operator corresponding to these threshold energies. Also, we discuss the set of vector potentials for which the kernels of $H \mp m$ are non-trivial, i.e. $\text{Ker}(H \mp m) \neq \{0\}$.

Key words: Dirac operators, Weyl-Dirac operators, zero modes, asymptotic limits

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1. INTRODUCTION

This note¹ is concerned with eigenfunctions at the threshold energies of Dirac operators with positive mass. More precisely, the Dirac operator which we shall deal with is of the form

$$H = H_0 + Q = \alpha \cdot D + m\beta + Q(x), \quad D = \frac{1}{i} \nabla_x, \quad x \in \mathbb{R}^3. \quad (1.1)$$

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \quad (j = 1, 2, 3)$$

with the 2×2 zero matrix $\mathbf{0}$ and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix}.$$

The constant m is assumed to be positive.

Throughout this note we assume that $Q(x)$ is a 4×4 Hermitian matrix-valued function. In addition to this, we shall later need several different assumptions on $Q(x)$ under which the operator $Q = Q(x) \times$ becomes relatively compact with respect to the operator H_0 . Under these assumptions, the essential spectrum of the Dirac operator H is given by the union of the intervals $(-\infty, -m]$ and $[m, +\infty)$:

$$\sigma_{\text{ess}}(H) = (-\infty, -m] \cup [m, +\infty).$$

This fact implies that the discrete spectrum of H is contained in the spectral gap $(-m, m)$:

$$\sigma_{\text{dis}}(H) \subset (-m, m).$$

In other words, discrete eigenvalues with finite multiplicity may exist in the spectral gap.

By the threshold energies of H , we mean the values $\pm m$, the edges of the essential spectrum $\sigma_{\text{ess}}(H)$. These values are normally excluded in scattering theory. However, they are of particular importance and of interest from the physics point of view. See Pickl and Dürr [17] and Pickl [16].

The aim of this note is to investigate asymptotic behaviors of (square-integrable) eigenfunctions corresponding to the threshold energies $\pm m$. From the mathematical point of view, the values $\pm m$ are critical in

¹This note is based on joint work with Professor Yoshimi Saitō, University of Alabama at Birmingham, USA.

the following sense: eigenfunctions corresponding to eigenvalues in the spectral gap decrease rapidly at infinity; on the contrary, generalized eigenfunctions corresponding to energies in the intervals $(-\infty, -m) \cup (m, +\infty)$ behave like plane waves at infinity, hence stay away from zero. For this criticality, it is interesting and important, from the mathematical point of view as well, to examine the asymptotic behaviors of eigenfunctions corresponding to the threshold energies $\pm m$.

A closely related question to the aim mentioned above is the one about the existence of Q 's which yield eigenfunctions of the operators H at the threshold energies $\pm m$. Our answers to this question are the same as those which were given to the question about the existence of magnetic fields giving rise to zero modes for Weyl-Dirac operators $\sigma \cdot (D - A(x))$: see Adam, Muratori and Nash [1], [2], [3], Balinsky and Evans [4], [5], [6], and Elton [8]. Namely, there exist infinitely many Q 's which yield eigenfunctions of the operators H at the threshold energies $\pm m$, but the set of such Q 's is still rather sparse in a certain sense.

We should note that one can regard the operator (1.1) as a generalization of the Dirac operator of the form

$$\alpha \cdot (D - A(x)) + m\beta + q(x)I_4, \quad (1.2)$$

where (q, A) is an electromagnetic potential, by taking $Q(x)$ to be $-\alpha \cdot A(x) + q(x)I_4$. To formulate the main results of the present note, we shall need to deal with the operator (1.2) in the case where $m = 0$ and $q(x) \equiv 0$. In this case, the operator (1.2) becomes of the form

$$\alpha \cdot (D - A(x)) = \begin{pmatrix} \mathbf{0} & \sigma \cdot (D - A(x)) \\ \sigma \cdot (D - A(x)) & \mathbf{0} \end{pmatrix}.$$

In this way, the Weyl-Dirac operator

$$T = \sigma \cdot (D - A(x)) \quad (1.3)$$

mentioned above naturally appears in our setting.

Also, we should like to note that the operator (1.1) generalizes the Dirac operator of the form

$$\alpha \cdot D + m(x)\beta + q(x)I_4, \quad (1.4)$$

where $m(x)$, considered to be a variable mass, converges to a positive constant m_∞ at infinity in an appropriate manner.

Spectral properties of the operator (1.4) have been extensively studied under various assumptions on $m(x)$ and $q(x)$ in recent years. See Kalf and Yamada [13], Kalf, Okaji and Yamada [14], Schmidt and Yamada [22], Pladdy [18] and Yamada [23].

Notation.

By $L^2 = L^2(\mathbb{R}^3)$, we mean the Hilbert space of square-integrable functions on \mathbb{R}^3 , and we introduce a Hilbert space \mathcal{L}^2 by $\mathcal{L}^2 = [L^2(\mathbb{R}^3)]^4$, where the inner product is given by

$$(f, g)_{\mathcal{L}^2} = \sum_{j=1}^4 (f_j, g_j)_{L^2}$$

for $f = {}^t(f_1, f_2, f_3, f_4)$ and $g = {}^t(g_1, g_2, g_3, g_4)$.

By $L^{2,s}(\mathbb{R}^3)$, we mean the weighted L^2 space defined by

$$L^{2,s}(\mathbb{R}^3) := \{ u \mid \langle x \rangle^s u \in L^2(\mathbb{R}^3) \}$$

with the inner product

$$(u, v)_{L^{2,s}} := \int_{\mathbb{R}^3} \langle x \rangle^{2s} u(x) \overline{v(x)} dx,$$

where

$$\langle x \rangle = \sqrt{1 + |x|^2}.$$

We introduce the Hilbert space $\mathcal{L}^{2,s} = [L^{2,s}(\mathbb{R}^3)]^4$ with the inner product

$$(f, g)_{\mathcal{L}^{2,s}} = \sum_{j=1}^4 (f_j, g_j)_{L^{2,s}}.$$

By $H^1(\mathbb{R}^3)$ we denote the Sobolev space of order 1, and by \mathcal{H}^1 we mean the Hilbert space $[H^1(\mathbb{R}^3)]^4$. By $S(\mathbb{R}^3)$, we mean the Schwartz class of rapidly decreasing functions on \mathbb{R}^3 , and we set $\mathcal{S} = [S(\mathbb{R}^3)]^4$.

When we mention the Weyl-Dirac operator $T = \sigma \cdot (D - A(x))$, we must handle two-vectors (two components spinors) which will be denoted by φ, ψ , etc. Note that the Hilbert space for the Weyl-Dirac operator is $[L^2(\mathbb{R}^3)]^2$.

2. MASSLESS DIRAC OPERATORS

In this section, we shall treat the Dirac operator (1.1) in the massless case $m = 0$ under Assumption (Q) below. Namely, we shall consider the operator

$$H = H_0 + Q = \alpha \cdot D + Q(x). \quad (2.1)$$

We need discussions about the operator (2.1) in order to formulate the main results of this note, which will be stated in section 3.

Assumption (Q).

Each element $q_{jk}(x)$ ($j, k = 1, \dots, 4$) of $Q(x)$ is a measurable function satisfying

$$|q_{jk}(x)| \leq C_q \langle x \rangle^{-\rho} \quad (\rho > 1) \quad (2.2)$$

where C_q is a positive constant.

One should note that, under Assumption(Q), the Dirac operator (2.1) is a self-adjoint operator in \mathcal{L}^2 with $\text{Dom}(H) = \mathcal{H}^1$. The self-adjoint realization will be denoted by H again. With an abuse of notation, we shall write Hf in the distributional sense for $f \in \mathcal{S}'$ whenever it makes sense.

DEFINITION. By a zero mode, we mean a function $f \in \text{Dom}(H)$ which satisfies

$$Hf = 0.$$

By a zero resonance, we mean a function $f \in \mathcal{L}^{2,-s} \setminus \mathcal{L}^2$, for some $s \in (0, 3/2]$, which satisfies $Hf = 0$ in the distributional sense.

It is evident that a zero mode of H is an eigenfunction of H corresponding to the eigenvalue 0, i.e., a zero mode is an element of $\text{Ker}(H)$, the kernel of the self-adjoint operator H .

We now state results which will be needed in section 3.

Theorem 2.1. *Suppose Assumption (Q) is satisfied. Let f be a zero mode of the operator (2.1). Then for any $\omega \in \mathbb{S}^2$*

$$\lim_{r \rightarrow +\infty} r^2 f(r\omega) = -\frac{i}{4\pi} (\alpha \cdot \omega) \int_{\mathbb{R}^3} Q(y) f(y) dy, \quad (2.3)$$

where the convergence is uniform with respect to $\omega \in \mathbb{S}^2$.

In connection with the expression $f(r\omega)$ in (2.3), it is worthy to note that every zero mode is a continuous function. This fact was shown in [20].

Theorem 2.2 below means that zero resonances do not exist under the restrictions on ρ and s imposed in the theorem.

Theorem 2.2. *Suppose Assumption (Q) is satisfied with $\rho > 3/2$. If f belongs to $\mathcal{L}^{2,-s}$ for some s with $0 < s \leq \min\{3/2, \rho - 1\}$ and satisfies $Hf = 0$ in the distributional sense, then $f \in \mathcal{H}^1$.*

For the proofs of Theorems 2.1 and 2.2, see [19] and [20].

As can be easily understood from the discussions in the introduction, the Dirac operator $H = \alpha \cdot D + Q(x)$ in (2.1) is a natural generalization of the Weyl-Dirac operator $T = \sigma \cdot (D - A(x))$. Accordingly, we obtain results on the Weyl-Dirac operator as corollaries to Theorems 2.1 and 2.2. To state these theorems, we have to make an assumption on the vector potential $A(x)$, in accordance with Assumption (Q).

Assumption (A1).

Each element $A_j(x)$ ($j = 1, 2, 3$) of $A(x)$ is a real-valued measurable function satisfying

$$|A_j(x)| \leq C_a \langle x \rangle^{-\rho} \quad (\rho > 1) \quad (2.4)$$

where C_a is a positive constant.

Assumption (A1) assures that $T = \sigma \cdot (D - A(x))$ is a self-adjoint operator in $[L^2(\mathbb{R}^3)]^2$ with domain $[H^1(\mathbb{R}^3)]^2$.

Theorem 2.3. *Suppose Assumption (A1) is satisfied. Let ψ be a zero mode of the Weyl-Dirac operator $T = \sigma \cdot (D - A(x))$. Then for any $\omega \in \mathbb{S}^2$*

$$\begin{aligned} & \lim_{r \rightarrow +\infty} r^2 \psi(r\omega) \\ &= \frac{i}{4\pi} \int_{\mathbb{R}^3} \{(\omega \cdot A(y))I_2 + i\sigma \cdot (\omega \times A(y))\} \psi(y) dy, \end{aligned} \quad (2.5)$$

where the convergence is uniform with respect to $\omega \in \mathbb{S}^2$.

Theorem 2.4. *Suppose Assumption (A1) is satisfied with $\rho > 3/2$. If ψ belongs to $[L^{2,-s}(\mathbb{R}^3)]^2$ for some s with $0 < s \leq \min\{3/2, \rho - 1\}$ and satisfies $T\psi = 0$ in the distributional sense, then $\psi \in [H^1(\mathbb{R}^3)]^2$.*

3. DIRAC OPERATORS WITH POSITIVE MASS

In this section, we shall restrict ourselves to the Dirac operators with a vector potential

$$H = \alpha \cdot (D - A(x)) + m\beta, \quad (3.1)$$

where $m > 0$.

One of our main results characterizes eigenfunctions of the Dirac operator H in (3.1) at the threshold eigenvalues $\pm m$ in terms of zero modes of the Weyl-Dirac operator T in (1.3).

Theorem 3.1. *Suppose Assumption (A1) is satisfied. Then*

- (i) $f \in \text{Ker}(H - m) \iff \exists \psi \in \text{Ker}(T)$ such that $f = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$;
- (ii) $f \in \text{Ker}(H + m) \iff \exists \psi \in \text{Ker}(T)$ such that $f = \begin{pmatrix} 0 \\ \psi \end{pmatrix}$.

It is of some interest to point out that Theorem 3.1 implies that eigenfunctions and eigenspaces corresponding to the threshold eigenvalues $\pm m$ are independent of m . Also we point out that Theorem 3.1 implies

$$\text{Ker}(H - m) = \text{Ker}(T) \oplus \{0\}, \quad (3.2)$$

$$\text{Ker}(H + m) = \{0\} \oplus \text{Ker}(T). \quad (3.3)$$

It is immediate that Theorem 3.1, together with Theorem 2.3, yields the following corollary.

Corollary 3.1. *Suppose Assumption (A1) is verified. Let $u_\psi(\omega)$ be the continuous function on \mathbb{S}^2 defined by (2.5).*

- (i) *If $f \in \text{Ker}(H - m)$, then f is continuous on \mathbb{R}^3 and satisfies that for any $\omega \in \mathbb{S}^2$*

$$\lim_{r \rightarrow +\infty} r^2 f(r\omega) = \begin{pmatrix} u_\psi(\omega) \\ 0 \end{pmatrix},$$

where the convergence is uniform with respect to $\omega \in \mathbb{S}^2$.

- (ii) *If $f \in \text{Ker}(H + m)$, then f is continuous on \mathbb{R}^3 and satisfies that for any $\omega \in \mathbb{S}^2$*

$$\lim_{r \rightarrow +\infty} r^2 f(r\omega) = \begin{pmatrix} 0 \\ u_\psi(\omega) \end{pmatrix},$$

where the convergence is uniform with respect to $\omega \in \mathbb{S}^2$.

The conclusions of Theorem 3.1 are valid under weaker assumptions than Assumption (A1). Indeed, we shall introduce two assumptions, which are quite different from each other. The one (Assumption (A2) below) needs continuity of the vector potentials, but it allows a slightly slower decay at infinity. The other one (Assumption (A3) below) allows the vector potentials with local singularities.

Assumption (A2).

Each element $A_j(x)$ is a real-valued continuous function satisfying

$$A_j(x) = o(|x|^{-1}) \quad (|x| \rightarrow +\infty). \quad (3.4)$$

Assumption (A3).

Each element $A_j(x)$ is a real-valued measurable function satisfying

$$A_j \in L^3(\mathbb{R}^3). \quad (3.5)$$

It is obvious that under Assumption (A2), $-\alpha \cdot A$ is a bounded self-adjoint operator in \mathcal{L}^2 , hence the Dirac operator (3.1) is a self-adjoint operator in \mathcal{L}^2 with $\text{Dom}(H) = \mathcal{H}^1$.

As for the Dirac operator (3.1) under Assumption (A3), one can show that $\alpha \cdot (D - A(x)) + m\beta$ is a relatively compact perturbation of the operator $\alpha \cdot D + m\beta$. Therefore we find that the formal expression (3.1) admits the self-adjoint realization in \mathcal{L}^2 with $\text{Dom}(H) = \mathcal{H}^1$.

Theorem 3.2. *Suppose either of Assumption (A2) or Assumption (A3) is satisfied. Then all the conclusions of Theorem 3.1 hold.*

We should like to mention relevant works on the Weyl-Dirac operator T by Balinsky and Evans, and Elton. In their work [5], Balinsky and Evans showed sparseness of the set of vector potentials and derived an estimate of the dimension of the subspace consisting of zero modes of T under Assumption (A3). In a similar spirit to [5], Elton [8] investigated, under Assumption (A2), the local structure of the set of vector potentials which produce zero modes with multiplicity $k \geq 0$.

Combining the results in [8] with Theorem 3.2 above, we get Theorem 3.3 below. To formulate the theorem, we need to introduce a Banach space of vector potentials

$$\mathcal{A} = \{ A \mid A_j(x) \in C^0(\mathbb{R}^3, \mathbb{R}), A_j(x) = o(|x|^{-1}) \text{ as } |x| \rightarrow +\infty \}$$

equipped with the norm

$$\|A\| = \sup_{x \in \mathbb{R}^3} \langle x \rangle |A(x)|.$$

Theorem 3.3. *Suppose Assumption (A2) is satisfied. Let*

$$\mathcal{Z}_k^\pm = \{ A \in \mathcal{A} \mid \dim (\text{Ker}(H \mp m)) = k \}$$

for $k = 0, 1, 2, \dots$. Then

- (i) $\mathcal{Z}_k^+ = \mathcal{Z}_k^-$ for any k ;
- (ii) \mathcal{Z}_0^\pm is an open dense subset of \mathcal{A} ;
- (iii) for any k and any open subset $\Omega \subset \mathbb{R}^3$, $\Omega \neq \emptyset$,

$$[C_0^\infty(\Omega)]^3 \cap \mathcal{Z}_k^\pm \neq \emptyset.$$

In a similar fashion, we combine the results in [5] with Theorem 3.2 above, and we get the following

Theorem 3.4. *Suppose Assumption (A3) is satisfied. Then*

- (i) the sets $\{ A \in [L^3(\mathbb{R}^3)]^3 \mid \text{Ker}(H \mp m) = \{0\} \}$ contain open dense subsets of $[L^3(\mathbb{R}^3)]^3$;
- (ii) $\dim(\text{Ker}(H - m)) = \dim(\text{Ker}(H + m)) \leq c_0 \int_{\mathbb{R}^3} |A(x)|^3 dx$ for some constant c_0 .

The proofs of the theorems in this section will be found in our coming paper [21].

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