Spectral analysis for convolution operators on locally compact groups

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Abstract

We consider operators H_μ of convolution with measures μ on locally compact groups. We characterize the spectrum of H_μ by constructing auxiliary operators whose kernel contain the pure point and singular subspaces of H_μ, respectively. The proofs rely on commutator methods.

1 Introduction

Any selfadjoint operator H in a Hilbert space $\mathcal{H}$, with spectral measure E_H and spectrum $\sigma(H)$, is reduced by an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}}(H) \oplus \mathcal{H}_{\mathrm{p}}(H) \oplus \mathcal{H}_{\mathrm{sc}}(H),$$

that we briefly recall (*cf.* [30, Sec. 7.4]). Denote by $\mathrm{Bor}(\mathbb{R})$ the family of Borel subsets of $\mathbb{R}$. Then, for any $f \in \mathcal{H}$, one has the positive Borel measure

$$\nu_H^f : \mathrm{Bor}(\mathbb{R}) \to [0, \infty), \quad A \mapsto \nu_H^f(A) := \|E_H(A)f\|^2 = \langle f, E_H(A)f \rangle.$$

We say that *f belongs to the spectral subspace* $\mathcal{H}_{\mathrm{p}}(H)$ if ν_H^f is pure point, *f belongs to the spectral subspace* $\mathcal{H}_{\mathrm{ac}}(H)$ if ν_H^f is absolutely continuous with respect to the Lebesgue measure, and *f belongs to the spectral subspace* $\mathcal{H}_{\mathrm{sc}}(H)$ if ν_H^f is singularly continuous with respect to the Lebesgue measure. One also uses the notations $\mathcal{H}_{\mathrm{c}}(H) := \mathcal{H}_{\mathrm{ac}}(H) \oplus \mathcal{H}_{\mathrm{sc}}(H)$ for the *continuous subspace of H* and $\mathcal{H}_{\mathrm{s}}(H) := \mathcal{H}_{\mathrm{p}}(H) \oplus \mathcal{H}_{\mathrm{sc}}(H)$ for the *singular subspace of H*. The sets $\sigma_{\mathrm{p}}(H) := \sigma(H|_{\mathcal{H}_{\mathrm{p}}(H)})$, $\sigma_{\mathrm{ac}}(H) := \sigma(H|_{\mathcal{H}_{\mathrm{ac}}(H)})$, $\sigma_{\mathrm{sc}}(H) := \sigma(H|_{\mathcal{H}_{\mathrm{sc}}(H)})$, $\sigma_{\mathrm{c}}(H) := \sigma(H|_{\mathcal{H}_{\mathrm{c}}(H)})$ and $\sigma_{\mathrm{s}}(H) := \sigma(H|_{\mathcal{H}_{\mathrm{s}}(H)})$ are called *pure point spectrum*, *absolutely continuous spectrum*, *continuous spectrum*, and *singular spectrum of H*, respectively.

An important issue in spectral theory consists in determining the above spectral subspaces or subsets for concrete selfadjoint operators. Under various assumptions this has been performed for important classes of operators: Schrödinger and more general partial differential operators, Toeplitz operators, Wiener-Hopf operators, and many others. Since the mathematical literature on this subject is considerable, it seems pointless to try to indicate references.

In the present article we consider locally compact groups X, abelian or not, and convolution operators H_μ, acting on $\mathsf{L}^2(X)$, defined by suitable measures μ belonging to $\mathsf{M}(X)$, the Banach *-algebra of complex Radon measures on X. The case $\mu = \chi_S$, the characteristic function of a compact generating subset, leads to Hecke operators associated to the left regular representation (notice that our groups need not to be discrete). The precise definitions and statements are gathered in the next section. Essentially, our result consists in determining

subspaces $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^2$ of $\mathsf{L}^2(X)$, explicitly defined in terms of μ and the family $\mathrm{Hom}(X, \mathbb{R})$ of continuous group morphisms $\Phi : X \to \mathbb{R}$, such that $\mathcal{H}_\mathrm{p}(H_\mu) \subset \mathcal{K}_\mu^1$ and $\mathcal{H}_\mathrm{s}(H_\mu) \subset \mathcal{K}_\mu^2$. The cases $\mathcal{K}_\mu^1 = \{0\}$ or $\mathcal{K}_\mu^2 = \{0\}$ are interesting; in the first case H_μ has no eigenvalues, and in the second case H_μ is purely absolutely continuous. The subspaces $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^2$ can be calculated explicitly only in very convenient situations. Rather often we are only able to show that they differ from $\mathcal{H}$.

In Section 3 we prove the results stated and discussed in Section 2. The proofs rely on a modification of a positive commutator technique called *the method of the weakly conjugate operator*. This method, an unbounded version of the Kato-Putnam theorem [26, Thm. XIII.28], developed and used in various situations [5, 6, 19, 23, 24, 25, 27], is recalled in Section 3.1. The last section is devoted to examples.

We refer to [2, 3, 4, 7, 8, 9, 11, 15, 17, 18, 20, 21, 22, 29] for some related works on the spectral theory of operators on groups and graphs. Some of these articles put into evidence (Hecke-type) operators with large singular or singular continuous components. In [25], where analogous technics are used, one gets restrictions on the singular spectrum for adjacency operators on certain classes of graphs (which could be of non-Cayley type).

2 The main result

We give in this section the statement of our main result for convolution operators on arbitrary locally compact groups (LCG). The reader is referred to [10, 16] for general information on the theory of LCG.

2.1 Statement of the main result

Let X be a LCG with identity e, center $Z(X)$ and modular function Δ. Let us fix a left Haar measure λ on X, using the notation $\mathrm{d}x := \mathrm{d}\lambda(x)$. The associated right Haar measure ρ is defined by $\rho(E) := \lambda(E^{-1})$ for each Borel subset E of X. Whenever X is compact, λ is normalized, *i.e.* $\lambda(X) = 1$. On discrete groups the counting measure (assigning mass 1 to every point) is considered. The notation *a.e.* stands for "almost everywhere" and refers to the Haar measure λ. The Lebesgue space $\mathsf{L}^p(X) \equiv \mathsf{L}^p(X, \mathrm{d}\lambda)$, $1 \le p \le \infty$, of X with respect to λ is endowed with the usual norm

$$\|f\|_p := \left(\int_X \mathrm{d}x \, |f(x)|^p \right)^{1/p}.$$

We are interested in convolution of functions by measures. Namely, we consider for every measure $\mu \in \mathsf{M}(X)$ and every function $f \in \mathsf{L}^p(X)$, $1 \le p < \infty$, the convolution of μ and f given (essentially) by

$$(\mu * f)(x) := \int_X \mathrm{d}\mu(y) \, f(y^{-1}x) \quad \text{for } a.e. \ x \in X.$$

It is known [16, Thm. 20.12] that $\mu * f \in \mathsf{L}^p(X)$ and that $\|\mu * f\|_p \le \|\mu\| \, \|f\|_p$, where $\|\mu\| := |\mu|(X)$ is the norm of the measure μ. Since we are mainly concerned with the hilbertian theory, we consider in the sequel the convolution operator H_μ, $\mu \in \mathsf{M}(X)$, acting in the Hilbert space $\mathcal{H} := \mathsf{L}^2(X)$:

$$H_\mu f := \mu * f, \quad f \in \mathcal{H}.$$

The operator H_μ is bounded with norm $\|H_\mu\| \le \|\mu\|$, and it admits an adjoint operator H_μ^* equal to H_{μ^*}, the convolution operator by $\mu^* \in \mathsf{M}(X)$ defined by $\mu^*(E) = \overline{\mu(E^{-1})}$. If the measure μ is absolutely continuous w.r.t. the left Haar measure λ, so that $\mathrm{d}\mu = a \, \mathrm{d}\lambda$ with $a \in \mathsf{L}^1(X)$, then μ^* is also absolutely continuous w.r.t. λ and $\mathrm{d}\mu^* = a^* \mathrm{d}\lambda$, where $a^*(x) := \overline{a(x^{-1})}\Delta(x^{-1})$ for *a.e.* $x \in X$. In such a case we simply write H_a for $H_{a\,\mathrm{d}\lambda}$. We shall always assume that H_μ is selfadjoint, *i.e.* that $\mu = \mu^*$.

Let $\mathcal{U}(\mathcal{H})$ stands for the group of unitary operators in $\mathcal{H}$ and let $L : X \to \mathcal{U}(\mathcal{H})$ be the left regular representation of X. Then H_μ is equal to the strong operator integral

$$H_\mu = \int_X \mathrm{d}\mu(y) \, L(y),$$

and $\mu \mapsto H_\mu$ is the integrated form of L.

We recall that given two measures $\mu, \nu \in \mathsf{M}(X)$, their convolution $\mu * \nu \in \mathsf{M}(X)$ is defined by the relation [10, Eq. 2.34] generalizing the usual convolution of L^1-functions:

$$\int_X \mathrm{d}(\mu * \nu)(x)\, g(x) := \int_X \int_X \mathrm{d}\mu(x)\mathrm{d}\nu(y)\, g(xy) \qquad \forall g \in C_0(X),$$

where $C_0(X)$ denotes the C^*-algebra of continuous complex functions on X decaying at infinity. The inequality $\|\mu * \nu\| \le \|\mu\|\,\|\nu\|$ holds.

Given $\mu \in \mathsf{M}(X)$, let $\varphi : X \to \mathbb{R}$ be such that the linear functional

$$F : C_0(X) \to \mathbb{C}, \quad g \mapsto \int_X \mathrm{d}\mu(x)\, \varphi(x)g(x)$$

is bounded. Then there exists a unique measure in $M(X)$ associated to F, due to the Riesz-Markov representation theorem. We write $\varphi\mu$ for this measure, and we simply say that φ is such that $\varphi\mu \in \mathsf{M}(X)$.

Let us call *real character* any continuous group morphism $\Phi : X \to \mathbb{R}$. Their set forms a real vector space $\mathrm{Hom}(X, \mathbb{R})$, which can be infinite dimensional.

Definition 2.1. *Let $\mu = \mu^* \in \mathsf{M}(X)$.*

(a) *A real character Φ is* semi-adapted *to μ if $\Phi\mu, \Phi^2\mu \in \mathsf{M}(X)$, and $(\Phi\mu) * \mu = \mu * (\Phi\mu)$. The set of real characters that are semi-adapted to μ is denoted by $\mathrm{Hom}_\mu^1(X, \mathbb{R})$.*

(b) *A real character Φ is* adapted *to μ if Φ is semi-adapted to $\mu, \Phi^3\mu \in \mathsf{M}(X)$, and $(\Phi\mu) * (\Phi^2\mu) = (\Phi^2\mu) * (\Phi\mu)$. The set of real characters that are adapted to μ is denoted by $\mathrm{Hom}_\mu^2(X, \mathbb{R})$.*

Let $\mathcal{K}_\mu^j := \bigcap_{\Phi \in \mathrm{Hom}_\mu^j(X,\mathbb{R})} \ker(H_{\Phi\mu})$, for $j = 1, 2$; then our main result is the following.

Theorem 2.2. *Let X be a LCG and let $\mu = \mu^* \in \mathsf{M}(X)$. Then*

$$\mathcal{H}_{\mathrm{p}}(H_\mu) \subset \mathcal{K}_\mu^1 \qquad \text{and} \qquad \mathcal{H}_{\mathrm{s}}(H_\mu) \subset \mathcal{K}_\mu^2.$$

A more precise result is obtained in a particular situation.

Corollary 2.3. *Let X be a LCG and let $\mu = \mu^* \in \mathsf{M}(X)$. Assume that there exists a real character Φ adapted to μ such that Φ^2 is equal to an nonzero constant on $\mathrm{supp}(\mu)$. Then H_μ has a purely absolutely continuous spectrum, with the possible exception of an eigenvalue located at the origin, with eigenspace $\ker(H_\mu) = \ker(H_{\Phi\mu})$.*

Corollary 2.3 specially applies to adjacency operators on certain classes of Cayley graphs, which are Hecke-type operators in the regular representation, thus convolution operators on discrete groups.

Remark 2.4. *Using the method of the weakly conjugate operator, some extra results (as a Limiting Absorption Principle, global smooth operators, perturbations of H_μ) can also be obtained. For simplicity we do not include them here, even if they can be inferred quite straightforwardly from [5] and [6]. Improvements in the assumptions are also possible, but with the cost of more complicated statements and proofs. Proposition 2.1 in [6] shows the generality of the method.*

2.2 Comments and remarks

(A) One obstacle in applying Theorem 2.2 is the fact that certain locally compact groups admit few nonzero real characters, maybe none.

We say that $x \in X$ is *compact*, and we write $x \in B(X)$, if the closed subgroup generated by x is compact. If the order of $x \in B(X)$ is finite, then x is clearly a compact element (but in non-discrete groups there could be others). Although $B(X)$ is the union of all the compact subgroups of X, it is in general neither a subgroup, nor a closed set in X. We write $\mathscr{B}(X)$ for the closed subgroup generated by $B(X)$.

A continuous group morphism sends compact subgroups to compact subgroups. But the unique compact subgroup of $\mathbb{R}$ is $\{0\}$. Thus a real character on X annihilates $\mathscr{B}(X)$. It is not clear in general that the "smallness"

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of the vector space $\mathrm{Hom}(X, \mathbb{R})$ is related to a tendancy for convolution operators on X to have a substantial singular subspace, but for certain classes of groups this is indeed the case. For example, if X is compact, then $X = \mathscr{B}(X)$, $\mathrm{Hom}(X, \mathbb{R}) = \{0\}$ and all operators H_μ, $\mu \in \mathsf{M}(X)$, are pure point (see (B) below).

(B) It is not at all exceptional for a convolution operator to have eigenvalues. For example, if a type I representation U is contained in the left regular representation L, then any function $a \in \mathsf{L}^1(X)$ which is transformed by (the integrated form of) U into a compact operator will lead to a convolution operator H_a having eigenvalues.

To consider just an exteme case, let us assume that X is a CCR group and that L is completely reducible. Then H_a can be written as a direct sum of compact operators, thus it has pure point spectrum. These conditions are fulfilled in the very particular case of compact groups. Actually, in this case, the irreducible representations are all finite-dimensional, so even convolution operators by elements of $\mathsf{M}(X)$ are pure point.

(C) The occurrence in Theorem 2.2 of the subspaces $\mathcal{K}_\mu^j$ is not as mysterious as it could seem at first sight. For example, if $\mu = \delta_e$, then $\Phi\mu = 0$ for any $\Phi \in \mathrm{Hom}(X, \mathbb{R})$, so that $\mathcal{K}_\mu^j = \mathcal{H}$. Accordingly $H_\mu = 1$, with spectrum composed of the single eigenvalue 1 with corresponding eigenspace $\mathcal{H}$.

Another simple example is obtained by considering X compact. On one hand the single real character is $\Phi = 0$, with associated subspaces $\mathcal{K}_\mu^j = \mathcal{H}$ for any $\mu = \mu^* \in \mathsf{M}(X)$. On the other hand we know from (B) that $\mathcal{H}_\mathrm{p}(H_\mu)$ is also equal to $\mathcal{H}$.

If the support of μ is contained in a subgroup Y of X with $0 < \lambda(Y) < \infty$, then a direct calculation shows that the associated characteristic function χ_Y is an eigenvector of H_μ with eigenvalue $\mu(Y)$. Actually, since $(\Phi\mu)(Y) = 0$ for any $\Phi \in \mathrm{Hom}(X, \mathbb{R})$ with $\Phi\mu \in \mathsf{M}(X)$, $\mathbb{C}\chi_Y$ is contained in $\ker(H_{\Phi\mu})$.

3 Proof of the main result

The proof of Theorem 2.2 relies on an abstract method, that we briefly recall in a simple form.

3.1 The method of the weakly conjugate operator

The method of the weakly conjugate operator works for unbounded operators, but for our purposes it will be enough to assume H bounded. It also produces estimations on the boundary values of the resolvent and information on wave operators, but we shall only concentrate on spectral results.

We start by introducing some notations. The symbol $\mathcal{H}$ stands for a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and put $\mathscr{B}(\mathcal{H}) := \mathscr{B}(\mathcal{H}, \mathcal{H})$. We assume that $\mathcal{H}$ is endowed with a strongly continuous unitary group $\{W_t\}_{t \in \mathbb{R}}$. Its selfadjoint generator is denoted by A and has domain $\mathcal{D}(A)$. In most of the applications A is unbounded.

Definition 3.1. *A bounded selfadjoint operator H in $\mathcal{H}$ belongs to $C^1(A; \mathcal{H})$ if one of the following equivalent condition is satisfied:*

(i) the map $\mathbb{R} \ni t \mapsto W_{-t} H W_t \in \mathscr{B}(\mathcal{H})$ is strongly differentiable,

(ii) the sesquilinear form

$$\mathcal{D}(A) \times \mathcal{D}(A) \ni (f, g) \mapsto i \langle Hf, Ag \rangle - i \langle Af, Hg \rangle \in \mathbb{C}$$

is continuous when $\mathcal{D}(A)$ is endowed with the topology of $\mathcal{H}$.

We denote by B the strong derivative in (i) calculated at $t = 0$, or equivalently the bounded selfadjoint operator associated with the extension of the form in (ii). The operator B provides a rigorous meaning to the commutator $i[H, A]$. We shall write $B > 0$ if B is positive and injective, namely if $\langle f, Bf \rangle > 0$ for all $f \in \mathcal{H} \setminus \{0\}$.

Definition 3.2. *The operator A is* weakly conjugate to *the bounded selfadjoint operator H if $H \in C^1(A; \mathcal{H})$ and $B \equiv i[H, A] > 0$.*

For $B > 0$ let us consider the completion $\mathcal{B}$ of $\mathcal{H}$ with respect to the norm $\|f\|_{\mathcal{B}} := \langle f, Bf \rangle^{1/2}$. The adjoint space $\mathcal{B}^*$ of $\mathcal{B}$ can be identified with the completion of $B\mathcal{H}$ with respect to the norm $\|g\|_{\mathcal{B}^*} := \langle g, B^{-1}g \rangle^{1/2}$. One has then the continuous dense embeddings $\mathcal{B}^* \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{B}$, and B extends to an isometric operator from $\mathcal{B}$ to $\mathcal{B}^*$. Due to these embeddings it makes sense to assume that $\{W_t\}_{t\in\mathbb{R}}$ restricts to a C_0-group in $\mathcal{B}^*$, or equivalently that it extends to a C_0-group in $\mathcal{B}$. Under this assumption (tacitly assumed in the sequel) we keep the same notation for these C_0-groups. The domain of the generator of the C_0-group in $\mathcal{B}$ (resp. $\mathcal{B}^*$) endowed with the graph norm is denoted by $\mathcal{D}(A, \mathcal{B})$ (resp. $\mathcal{D}(A, \mathcal{B}^*)$). In analogy with Definition 3.1 the requirement $B \in C^1(A; \mathcal{B}, \mathcal{B}^*)$ means that the map $\mathbb{R} \ni t \mapsto W_{-t}BW_t \in \mathscr{B}(\mathcal{B}, \mathcal{B}^*)$ is strongly differentiable, or equivalently that the sesquilinear form

$$\mathcal{D}(A, \mathcal{B}) \times \mathcal{D}(A, \mathcal{B}) \ni (f, g) \mapsto i \langle f, BAg \rangle - i \langle Af, Bg \rangle \in \mathbb{C}$$

is continuous when $\mathcal{D}(A, \mathcal{B})$ is endowed with the topology of $\mathcal{B}$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{B}$ and $\mathcal{B}^*$.

Theorem 3.3. *Assume that A is weakly conjugate to H and that $B \equiv i[H, A]$ belongs to $C^1(A; \mathcal{B}, \mathcal{B}^*)$. Then the spectrum of H is purely absolutely continuous.*

Note that the method should be conveniently adapted when absolute continuity is expected only in a subspace of the Hilbert space. This is the case considered in the sequel.

3.2 Proof of Theorem 2.2

In this section we construct suitable weakly conjugate operators in the framework of section 2, and we prove our main result. For that purpose, let us fix a real character $\Phi \in \mathrm{Hom}(X, \mathbb{R})$ and a measure $\mu = \mu^* \in \mathsf{M}(X)$. We shall keep writing Φ for the associated operator of multiplication in $\mathcal{H}$. In most of the applications this operator is unbounded; its domain is equal to $\mathcal{D}(\Phi) \equiv \{f \in \mathcal{H} \mid \Phi f \in \mathcal{H}\}$.

One ingredient of our approach is the fact that multiplication by morphisms behaves like a derivation with respect to the convolution product: for suitable functions or measures $f, g : X \to \mathbb{C}$, one has $\Phi(f * g) = (\Phi f) * g + f * (\Phi g)$. Using this observation we show in the next lemma that the commutator $i[H_\mu, \Phi]$ (constructed as in Definition 3.1) is related to the operator $H_{\Phi\mu}$. This provides a partial explanation of our choice of the "semi-adapted" and "adapted" conditions.

Lemma 3.4.   *(a) If Φ is semi-adapted to μ, then $H_\mu \in C^1(\Phi; \mathcal{H})$, and $i[H_\mu, \Phi] = -iH_{\Phi\mu} \in \mathscr{B}(\mathcal{H})$. Similarly, $-iH_{\Phi\mu} \in C^1(\Phi, \mathcal{H})$, and $i[-iH_{\Phi\mu}, \Phi] = -H_{\Phi^2\mu} \in \mathscr{B}(\mathcal{H})$. Moreover, the equality $[H_\mu, H_{\Phi\mu}] = 0$ holds.*

(b) If Φ is adapted to μ, then $-H_{\Phi^2\mu} \in C^1(\Phi, \mathcal{H})$, and the equality $[H_{\Phi\mu}, H_{\Phi^2\mu}] = 0$ holds.

Proof. (a) Let Φ be semi-adapted to μ and let $f \in \mathcal{D}(\Phi)$. Then one has $\mu, \Phi\mu \in \mathsf{M}(X)$ and $f, \Phi f \in \mathcal{H}$. Thus $\mu * f \in \mathcal{D}(\Phi)$, and the equality $\Phi(H_\mu f) = (\Phi\mu) * f + \mu * (\Phi f)$ holds in $\mathcal{H}$. It follows that $i(H_\mu\Phi - \Phi H_\mu)$ is well-defined on $\mathcal{D}(\Phi)$ and is equal to $-iH_{\Phi\mu}$. Hence Condition (ii) of Definition 3.1 is fulfilled.

The proof that $-iH_{\Phi\mu}$ belongs to $C^1(\Phi, \mathcal{H})$ and that $i[-iH_{\Phi\mu}, \Phi] = -H_{\Phi^2\mu}$ is similar. Finally the equality $[H_\mu, H_{\Phi\mu}] = 0$ is clearly equivalent to the requirement $(\Phi\mu) * \mu = \mu * (\Phi\mu)$.

(b) The proof is completely analogous to that of point (a). $\qquad\square$

If Φ is semi-adapted to μ, we set $K := i[H_\mu, \Phi] = -iH_{\Phi\mu}$ and $L := i[K, \Phi] = i[-iH_{\Phi\mu}, \Phi] = -H_{\Phi^2\mu}$ (for the sake of simplicity, we omit to write the dependence of these operators in Φ and μ). The first part of the previous lemma states that H_μ and K belongs to $C^1(\Phi; \mathcal{H})$. In particular, it follows that K leaves invariant the domain $\mathcal{D}(\Phi)$, and the operator

$$\mathcal{A} := \tfrac{1}{2}(\Phi K + K\Phi)$$

is well-defined and symmetric on $\mathcal{D}(\Phi)$. Similarly, if Φ is adapted to μ, the second part of the lemma states that L belongs to $C^1(\Phi; \mathcal{H})$. Therefore the operator L leaves $\mathcal{D}(\Phi)$ invariant, and the operator

$$\mathcal{A}' := \tfrac{1}{2}(\Phi L + L\Phi)$$

is well-defined and symmetric on $\mathcal{D}(\Phi)$.

Lemma 3.5. *(a) If Φ is semi-adapted to μ, then the operator $\mathcal{A}$ is essentially selfadjoint on $\mathcal{D}(\Phi)$. The domain of its closure A is $\mathcal{D}(A) = \mathcal{D}(\Phi K) = \{f \in \mathcal{H} \mid \Phi K f \in \mathcal{H}\}$ and A acts on $\mathcal{D}(A)$ as the operator $\Phi K - \frac{i}{2}L$.*

(b) If Φ is adapted to μ, then the operator $\mathcal{A}'$ is essentially selfadjoint on $\mathcal{D}(\Phi)$. The domain of its closure A' is $\mathcal{D}(A') = \mathcal{D}(\Phi L) = \{f \in \mathcal{H} \mid \Phi L f \in \mathcal{H}\}$.

Proof. One just has to reproduce the proof of [11, Lemma 3.1], replacing their couple (N, S) by (Φ, K) for the point (a) and by (Φ, L) for the point (b). $\qquad\square$

In the next lemma we collect some results on commutators with A or A'. The commutation relations exhibited in Lemma 3.4, *i.e.* $[H_\mu, K] = 0$ if Φ is semi-adapted to μ and $[K, L] = 0$ if Φ is adapted to μ, are essential.

Lemma 3.6. *If Φ is semi-adapted to μ, then*

(a) The quadratic form $\mathcal{D}(A) \ni f \mapsto i\langle H_\mu f, Af \rangle - i\langle Af, H_\mu f \rangle$ extends uniquely to the bounded form defined by the operator K^2,

(b) The quadratic form $\mathcal{D}(A) \ni f \mapsto i\langle K^2 f, Af \rangle - i\langle Af, K^2 f \rangle$ extends uniquely to the bounded form defined by the operator $KLK + \frac{1}{2}\left(K^2 L + LK^2\right)$ (which reduces to $2KLK$ if Φ is adapted to μ),

(c) If Φ is adapted to μ, then the quadratic form $\mathcal{D}(A') \ni f \mapsto i\langle Kf, A'f \rangle - i\langle A'f, Kf \rangle$ extends uniquely to the bounded form defined by the operator L^2.

The proof is straightforward. Computations may be performed on the core $\mathcal{D}(\Phi)$. These results imply that $H_\mu \in C^1(A; \mathcal{H})$, $K^2 \in C^1(A; \mathcal{H})$ and (when Φ is adapted) $K \in C^1(A'; \mathcal{H})$. Using these results we now establish a relation between the kernels of the operators H_μ, K and L.

Lemma 3.7. *If Φ is semi-adapted to μ, then one has*

$$\ker(H_\mu) \subset \mathcal{H}_{\mathrm{p}}(H_\mu) \subset \ker(K) \subset \mathcal{H}_{\mathrm{p}}(K).$$

If Φ is adapted to μ, one also has

$$\mathcal{H}_{\mathrm{p}}(K) \subset \ker(L) \subset \mathcal{H}_{\mathrm{p}}(L).$$

Proof. Let f be an eigenvector of H_μ. Due to the Virial Theorem [1, Proposition 7.2.10] and the fact that H_μ belongs to $C^1(A; \mathcal{H})$, one has $\langle f, i[H_\mu, A]f \rangle = 0$. It follows by Lemma 3.6.(a) that $0 = \langle f, K^2 f \rangle = \|Kf\|^2$, *i.e.* $f \in \ker(K)$. The inclusion $\mathcal{H}_{\mathrm{p}}(H_\mu) \subset \ker(K)$ follows. Similarly, by using A' instead of A and Lemma 3.6.(c) one gets (when Φ is adapted) the inclusion $\mathcal{H}_{\mathrm{p}}(K) \subset \ker(L)$, and the lemma is proved. $\qquad\square$

Assume now that Φ is semi-adapted to μ. Then we can decompose the Hilbert space $\mathcal{H}$ into the direct sum $\mathcal{H} = \mathcal{K} \oplus \mathcal{G}$, where $\mathcal{K} := \ker(K)$ and $\mathcal{G}$ is the closure of the range $K\mathcal{H}$. It is easy to see that H_μ and K are reduced by this decomposition and that their restrictions to the Hilbert space $\mathcal{G}$ are bounded selfadjoint operators. In the next lemma we prove that this decomposition of $\mathcal{H}$ also reduces the operator A if Φ is adapted to μ.

Lemma 3.8. *If Φ is adapted to μ, then the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{G}$ reduces the operator A. The restriction of A to $\mathcal{G}$ defines a selfadjoint operator denoted by A_0.*

Proof. We already know that on $\mathcal{D}(A) = \mathcal{D}(\Phi K)$ one has $A = \Phi K - \frac{i}{2}L$. By using Lemma 3.7 it follows that $\mathcal{K} \subset \ker(A) \subset \mathcal{D}(A)$. Then one trivially checks that (i) $A\left[\mathcal{K} \cap \mathcal{D}(A)\right] \subset \mathcal{K}$, (ii) $A\left[\mathcal{G} \cap \mathcal{D}(A)\right] \subset \mathcal{G}$ and (iii) $\mathcal{D}(A) = [\mathcal{K} \cap \mathcal{D}(A)] + [\mathcal{G} \cap \mathcal{D}(A)]$, which means that A is reduced by the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{G}$. Thus by [30, Theorem 7.28] the restriction of A to $\mathcal{D}(A_0) \equiv \mathcal{D}(A) \cap \mathcal{G}$ is selfadjoint in $\mathcal{G}$. $\qquad\square$

Proof of Theorem 2.2. We know from Lemma 3.7 that $\mathcal{H}_\mathrm{p}(H_\mu) \subset \ker(-iH_{\Phi\mu})$ for each $\Phi \in \mathrm{Hom}^1_\mu(X, \mathbb{R})$. This obviously implies the first inclusion of the theorem.

Let us denote by H_0 and K_0 the restrictions to $\mathcal{G}$ of the operators H_μ and K. We shall prove in points (i)-(iii) below that if Φ is adapted to μ, then the method of the weakly conjugate operator, presented in Section 3.1, applies to the operators H_0 and A_0 in the Hilbert space $\mathcal{G}$. It follows then that $\mathcal{G} \subset \mathcal{H}_\mathrm{ac}(H_\mu)$, and *a fortiori* that $\mathcal{H}_\mathrm{sc}(H_\mu) \subset \mathcal{K} = \ker(-iH_{\Phi\mu})$. Since this result holds for each $\Phi \in \mathrm{Hom}^2_\mu(X, \mathbb{R})$, the second inclusion of the theorem follows straightforwardly.

(i) Lemma 3.6.(a) implies that $i(H_0 A_0 - A_0 H_0)$ is equal in the form sense to K_0^2 on $\mathcal{D}(A_0) \equiv \mathcal{D}(A) \cap \mathcal{G}$. Therefore the corresponding quadratic form extends uniquely to the bounded form defined by the operator K_0^2. This implies that H_0 belongs to $C^1(A_0; \mathcal{G})$.

(ii) Since $B_0 := i[H_0, A_0] \equiv K_0^2 > 0$ in $\mathcal{G}$, the operator A_0 is weakly conjugate to H_0. So we define the space $\mathcal{B}$ as the completion of $\mathcal{G}$ with respect to the norm $\|f\|_\mathcal{B} := \langle f, B_0 f \rangle^{1/2}$. The adjoint space of $\mathcal{B}$ is denoted by $\mathcal{B}^*$ and can be identified with the completion of $B_0\mathcal{G}$ with respect to the norm $\|f\|_{\mathcal{B}^*} := \langle f, B_0^{-1} f \rangle^{1/2}$. It can also be expressed as the closure of the subspace $K\mathcal{H} = K_0\mathcal{G}$ with respect to the same norm $\|f\|_{\mathcal{B}^*} = \||K_0|^{-1} f\|$. Due to Lemma 3.6.(b) the quadratic form $\mathcal{D}(A_0) \ni f \mapsto i \langle B_0 f A_0 f \rangle - i \langle A_0 f, B_0 f \rangle$ extends uniquely to the bounded form defined by the operator $2K_0 L_0 K_0$, where L_0 is the restriction of L to $\mathcal{G}$. We write $i[B_0, A_0]$ for this extension, which clearly defines an element of $\mathscr{B}(\mathcal{B}, \mathcal{B}^*)$.

(iii) Let $\{W_t\}_{t \in \mathbb{R}}$ be the unitary group in $\mathcal{G}$ generated by A_0. We check now that this group extends to a C_0-group in $\mathcal{B}$. This easily reduces to proving that for any $t \in \mathbb{R}$ there exists a constant $\mathrm{c}(t) \geq 0$ such that $\|W_t f\|_\mathcal{B} \leq \mathrm{c}(t)\|f\|_\mathcal{B}$ for all $f \in \mathcal{D}(A_0)$. Due to point (ii) one has for each $f \in \mathcal{D}(A_0)$

$$\|W_t f\|_\mathcal{B}^2 = \langle f, B_0 f \rangle + \int_0^t \mathrm{d}\tau \, \langle W_\tau f, i[B_0, A_0] W_\tau f \rangle \leq \|f\|_\mathcal{B}^2 + 2\|L_0\| \int_0^{|t|} \mathrm{d}\tau \, \|W_\tau f\|_\mathcal{B}^2 \,.$$

Since $\mathcal{G} \hookrightarrow \mathcal{B}$, the function $(0, |t|) \ni \tau \mapsto \|W_\tau f\|_\mathcal{B}^2 \in \mathbb{R}$ is bounded. Thus we get the inequality $\|W_t f\|_\mathcal{B} \leq \mathrm{e}^{|t|\|L_0\|} \|f\|_\mathcal{B}$ by using a simple form of the Gronwall Lemma. Therefore $\{W_t\}_{t \in \mathbb{R}}$ extends to a C_0-group in $\mathcal{B}$, and by duality $\{W_t\}_{t \in \mathbb{R}}$ also defines a C_0-group in $\mathcal{B}^*$. This concludes the proof of the fact that B_0 extends to an element of $C^1(A_0; \mathcal{B}, \mathcal{B}^*)$. Thus all hypotheses of Theorem 3.3 are satisfied and this gives the result. $\square$

Proof of Corollary 2.3. Since $L = -H_{\Phi^2\mu}$ is proportional with H_μ, one has

$$\ker(H_\mu) = \mathcal{H}_\mathrm{p}(H_\mu) \subset \mathcal{H}_\mathrm{s}(H_\mu) \subset \ker(H_{\Phi\mu})$$

due to Lemma 3.7 and Theorem 2.2. Using this with μ replaced by $i\Phi\mu$, one easily gets the identity $\ker(H_{\Phi\mu}) = \ker(H_\mu)$. Therefore

$$\ker(H_\mu) = \mathcal{H}_\mathrm{p}(H_\mu) = \mathcal{H}_\mathrm{s}(H_\mu) = \ker(H_{\Phi\mu}),$$

and the claim is proved. $\square$

4 Examples

4.1 Perturbations of central measures

In this section, we exploit commutativity in a non-commutative setting by using central measures. The group X is assumed to be unimodular.

By definition, the *central measures* are the elements of the center $Z[\mathsf{M}(X)]$ of the convolution Banach *-algebra $\mathsf{M}(X)$. They can be characterized by the condition $\mu(yEy^{-1}) = \mu(E)$ for any $y \in X$ and any Borel set $E \subset X$. The *central* (or *class*) *functions* are the elements of $Z[\mathsf{M}(X)] \cap \mathsf{L}^1(X) = Z[\mathsf{L}^1(X)]$. Thus a characteristic function χ_E is central iff $\lambda(E) < \infty$ and E is *invariant* under all inner automorphisms.

The relevant simple facts are the following: if μ is central, $\Phi \in \mathrm{Hom}(X, \mathbb{R})$ and $\Phi\mu \in \mathsf{M}(X)$, then $\Phi\mu$ is also central (this follows from the identity $\Phi(yxy^{-1}) = \Phi(y) + \Phi(x) - \Phi(y) = \Phi(x), \forall x, y \in X$). On the other hand, if μ is arbitrary but supported on $\mathscr{B}(X)$, then $\Phi\mu = 0$ for any real character Φ. Thus all the commutation relations in Definition 2.1 are satisfied, and one gets from Theorem 2.2 the following result:

Corollary 4.1. *Let X be a unimodular LCG, let $\mu_0 = \mu_0^* \in \mathsf{M}(X)$ be a central measure, and let $\mu_1 = \mu_1^* \in \mathsf{M}(X)$ with $\mathrm{supp}(\mu_1) \subset \mathscr{B}(X)$. Then*

$$\mathcal{H}_{\mathrm{p}}(H_{\mu_0+\mu_1}) \subset \bigcap_{\substack{\Phi \in \mathrm{Hom}(X,\mathbb{R}) \\ \Phi\mu_0, \Phi^2\mu_0 \in \mathsf{M}(X)}} \ker(H_{\Phi\mu_0})$$

and

$$\mathcal{H}_{\mathrm{s}}(H_{\mu_0+\mu_1}) \subset \bigcap_{\substack{\Phi \in \mathrm{Hom}(X,\mathbb{R}) \\ \Phi\mu_0, \Phi^2\mu_0, \Phi^3\mu_0 \in \mathsf{M}(X)}} \ker(H_{\Phi\mu_0}).$$

In order to get more explicit results, we restrict ourselves in the next section to a convenient class of LCG, generalizing both abelian and compact groups.

4.2 Convolution operators on central groups

Following [12], we say that X is *central* (or *of class* $[Z]$) if the quotient $X/Z(X)$ is compact. Central groups possess a specific structure [12, Thm. 4.4]: If X is central, then X isomorphic to a direct product $\mathbb{R}^d \times H$, where H contains a compact open normal subgroup.

Proposition 4.2. *Let X be a central group and $\mu_0 = \mu_0^* \in \mathsf{M}(X)$ a central measure such that $\mathrm{supp}(\mu_0)$ is compact and not included in $\mathscr{B}(X)$. Let $\mu_1 = \mu_1^* \in \mathsf{M}(X)$ with $\mathrm{supp}(\mu_1) \subset \mathscr{B}(X)$ and set $\mu := \mu_0 + \mu_1$. Then $\mathcal{H}_{\mathrm{ac}}(H_\mu) \neq \{0\}$.*

Proof. Central groups are unimodular [13, Prop. p. 366], and $\Phi\mu, \Phi^2\mu, \Phi^3\mu \in \mathsf{M}(X)$ for any $\Phi \in \mathrm{Hom}(X,\mathbb{R})$, due to the hypotheses. Furthermore we know by [12, Thm. 5.7] that $B(X) = \mathscr{B}(X)$ is a closed normal subgroup of X and that $X/\mathscr{B}(X)$ is isomorphic to the direct product $\mathbb{R}^d \times D$, where D is a discrete torsion-free abelian group. But the groups $\mathbb{R}^d \times D$ are exactly those for which the real characters separate points [12, Cor. p. 335]. Therefore for any $x \in \mathrm{supp}(\mu_0) \setminus \mathscr{B}(X)$ there exists $\Phi \in \mathrm{Hom}(X,\mathbb{R})$ such that $\Phi(x) \neq 0$. Thus $H_{\Phi\mu}$ is a nonzero convolution operator, and the claim follows by Corollary 4.1. $\qquad\square$

In a central group X there exists plenty of central compactly supported measures. For instance there always exists in X [12, Thm. 4.2] a neighbourhood base of e composed of compact sets $S = S^{-1}$ which are invariant (under the inner automorphisms), *i.e.* central groups belong to the class [SIN]. Therefore the measures $\mu_0 = \chi_S \, \mathrm{d}\lambda$ satisfy $\mu_0 = \mu_0^*$ and are subject to Proposition 4.2. Actually this also applies to central characteristic functions χ_S with "large" S, since in X any compact set is contained in a compact invariant neighbourhood of the identity [13, Lemma p. 365]. One can also exihibit central measures satisfying Proposition 4.2 defined by continuous functions. Indeed we know by [13, Thm. 1.3] that for any neighbourhood U of the identity e of a central group X there exists a non-negative continuous central function a_U, with $\mathrm{supp}(a_U) \subset U$ and $a_U(e) > 0$.

A simple way to construct examples is as follows. Let $X := K \times Y$, where K is a compact group with Haar measure λ_K and Y is an abelian LCG with Haar measure λ_Y. Clearly X is central and $\mathscr{B}(X) = K \times \mathscr{B}(Y)$. Let $\mathcal{E}$ be a finite family of invariant subsets of K such that each $E \in \mathcal{E}$ satisfies $\lambda_K(E) > 0$ and $E^{-1} \in \mathcal{E}$. For each $E \in \mathcal{E}$, let I_E be a compact subset of Y such that $\lambda_Y(I_E) > 0$ and $(I_E)^{-1} = I_{E^{-1}}$. Suppose also that I_{E_0} is not a subset of $\mathscr{B}(Y)$ for some $E_0 \in \mathcal{E}$. Then one easily shows that the set $S := \bigcup_{E \in \mathcal{E}} E \times I_E$ satisfies the following properties: S is compact, $S = S^{-1}$, S is invariant, and S not included in $\mathscr{B}(X)$. Thus $\mathcal{H}_{\mathrm{ac}}(H_{\chi_S+\mu_1}) \neq \{0\}$ for any $\mu_1 = \mu_1^* \in \mathsf{M}(X)$ with $\mathrm{supp}(\mu_1) \subset K \times \mathscr{B}(Y)$, due to Proposition 4.2.

The following two examples are applications of the preceding construction.

Example 4.3. *Let $X := S_3 \times \mathbb{Z}$, where S_3 is the symmetric group of degree 3. The group S_3 has a presentation $\langle a, b \mid a^2, b^2, (ab)^3 \rangle$, and its conjugacy classes are $E_1 = E_1^{-1} = \{e\}$, $E_2 = E_2^{-1} = \{a, b, aba\}$ and $E_3 = E_3^{-1} = \{ab, ba\}$. Set $\mathcal{E} := \{E_2, E_3\}$ and choose I_{E_1}, I_{E_2} two finite symetric subsets of $\mathbb{Z}$, each of them containing at least two elements. Clearly these sets satisfy all the requirements of the above construction. Thus $\mathcal{H}_{\mathrm{ac}}(H_{\chi_S}) \neq \{0\}$ if $S := \bigcup_{E \in \mathcal{E}} E \times I_E$.*

Example 4.4. *Let $X = \mathrm{SU}(2) \times \mathbb{R}$, where $\mathrm{SU}(2)$ is the group (with Haar measure λ_2) of 2×2 unitary matrices of determinant $+1$. For each $\vartheta \in [0, \pi]$ let $C(\vartheta)$ be the conjugacy class of the matrix $\mathrm{diag}(\mathrm{e}^{i\vartheta}, \mathrm{e}^{-i\vartheta})$ in $\mathrm{SU}(2)$. A direct calculation (using for instance Euler angles) shows that $\lambda_2 \left(\bigcup_{\vartheta \in J} C(\vartheta) \right) > 0$ for each $J \subset [0, \pi]$ with nonzero Lebesgue measure. Set $E_1 := \bigcup_{\vartheta \in (0,1)} C(\vartheta)$, $E_2 := \bigcup_{\vartheta \in (2,\pi)} C(\vartheta)$, $\mathcal{E} := \{E_1, E_2\}$, $I_{E_1} := (-1, 1)$, and $I_{E_2} := (-3, -2) \cup (2, 3)$. Clearly these sets and many others satisfy all the requirements of the above construction. Thus $\mathcal{H}_{\mathrm{ac}}(H_{\chi_S}) \neq \{0\}$ if $S := \bigcup_{E \in \mathcal{E}} E \times I_E$.*

A nice example of a central group which is not the product of a compact and an abelian group can be found in [14, Ex. 4.7].

In a simple situation one even gets purely absolutely continuous operators; this should be compared with the discussion in Section 2.2:

Example 4.5. *Let X be a central group, let $z \in Z(X) \setminus \mathscr{B}(X)$, and set $\mu := \delta_z + \delta_{z^{-1}} + \mu_1$ for some $\mu_1 = \mu_1^* \in \mathsf{M}(X)$ with $\mathrm{supp}(\mu_1) \subset \mathscr{B}(X)$. Then μ satisfies the hypotheses of Proposition 4.2, and we can choose $\Phi \in \mathrm{Hom}(X, \mathbb{R})$ such that $\Phi(z) = \frac{1}{2}\Phi(z^2) \neq 0$ (note in particular that $z \notin \mathscr{B}(X)$ iff $z^2 \notin \mathscr{B}(X)$ and that $\Phi\mu_1 = 0$). Thus $\mathcal{H}_{\mathrm{s}}(H_\mu) \subset \ker(H_{\Phi\mu})$. But $f \in \mathcal{H}$ belongs to $\ker(H_{\Phi\mu}) = \ker\left(H_{\Phi(\delta_z + \delta_{z^{-1}})}\right)$ iff $f(z^{-1}x) = f(zx)$ for a.e. $x \in X$. This periodicity w.r.t. the non-compact element z^2 easily implies that the L^2-function f should vanish a.e. and thus that $\mathcal{H}_{\mathrm{ac}}(H_\mu) = \mathcal{H}$.*

4.3 Abelian groups

We consider in this section the case of *locally compact abelian groups* (LCAG), whose theory can be found in the monograph [16]. LCAG are particular cases of central groups. Their convolution algebra $\mathsf{M}(X)$ is abelian, so spectral results on convolution operators can be deduced from the preceding section. We shall not repete them here, but rather invoke duality to obtain properties of a class of multiplication operators on the dual group $\widehat{X}$.

Let X stands for a LCAG with elements $x, y, z, \ldots$, and let $\widehat{X}$ be the dual group of X, *i.e.* the set of characters of X endowed with the topology of compact convergence on X. The elements of $\widehat{X}$ are denoted by $\xi, \eta, \zeta, \ldots$ and we shall use the notation $\langle x, \xi \rangle$ for the expression $\xi(x)$. The Fourier transform m of a measure $\mu \in \mathsf{M}(X)$ is given by

$$m(\xi) \equiv [\mathscr{F}(\mu)](\xi) := \int_X \mathrm{d}\mu(x) \, \overline{\langle x, \xi \rangle}, \quad \xi \in \widehat{X}.$$

We recall from [16, Thm. 23.10] that m belongs to the C^*-algebra $BC(\widehat{X})$ of bounded continuous complex functions on $\widehat{X}$, and that $\|m\|_\infty \leq \|\mu\|$ (showing that the bound $\|H_\mu\| \leq \|\mu\|$ is not optimal in general). Actually the subspace $\mathscr{F}(\mathsf{M}(X))$ is dense in $BC(\widehat{X})$, and the subspace $\mathscr{F}(\mathsf{L}^1(X))$ is densely contained in $C_0(\widehat{X})$, the ideal of $BC(\widehat{X})$ composed of continuous complex functions on $\widehat{X}$ vanishing at infinity. For a suitably normalized Haar measure on $\widehat{X}$, the Fourier transform also defines a unitary isomorphism from $\mathcal{H}$ onto $\mathsf{L}^2(\widehat{X})$, which we denote by the same symbol. It maps unitarily H_μ on the operator M_m of multiplication with $m = \mathscr{F}(\mu)$. Moreover $\mu = \mu^*$, iff m is real, and

$$\sigma(H_\mu) = \sigma(M_m) = \overline{m(\widehat{X})}, \quad \sigma_{\mathrm{p}}(H_\mu) = \sigma_{\mathrm{p}}(M_m) = \overline{\{s \in \mathbb{R} \mid \lambda_\wedge (m^{-1}(s)) > 0\}},$$

where $\lambda_\wedge$ is any Haar measure on $\widehat{X}$.

This does not solve the problem of determining the nature of the spectrum, at least for three reasons. First, simple or natural conditions on μ could be obscured when using the Fourier transform; the function $m = \mathscr{F}(\mu)$ could be difficult to compute or to evaluate. Second, the dual group $\widehat{X}$ can be complicated. We are not aware of general results on the nature of the spectrum of multiplication operators on LCAG. Third, even for $\widehat{X} = \mathbb{R}^d$, the spectral theory of multiplication operators is quite subtle. For the particular case $\widehat{X} = \mathbb{R}^d$, one finds in [1, Sec. 7.1.4 & 7.6.2] refined results both on the absolute continuity and on the occurence of singular continuous spectrum for multiplication operators.

Let us recall that there is an almost canonical identification of $\mathrm{Hom}(X,\mathbb{R})$ with the vector space $\mathrm{Hom}(\mathbb{R},\widehat{X})$ of all continuous one-parameter subgroups of $\widehat{X}$. For a given real character Φ, we denote by $\varphi \in \mathrm{Hom}(\mathbb{R},\widehat{X})$ the unique element satisfying

$$\langle x, \varphi(t) \rangle = \mathrm{e}^{it\Phi(x)}, \qquad \forall t \in \mathbb{R}, \; x \in X.$$

Definition 4.6. *The function $m : \widehat{X} \to \mathbb{C}$ is* differentiable at $\xi \in \widehat{X}$ along the one-parameter subgroup $\varphi \in \mathrm{Hom}(\mathbb{R},\widehat{X})$ *if the function $\mathbb{R} \ni t \mapsto m(\xi + \varphi(t)) \in \mathbb{C}$ is differentiable at $t = 0$. In such a case we write $(d_\varphi m)(\xi)$ for $\frac{\mathrm{d}}{\mathrm{d}t} m(\xi + \varphi(t))\big|_{t=0}$. Higher order derivatives, when existing, are denoted by $d_\varphi^k m$, $k \in \mathbb{N}$.*

This definition triggers a formalism which has some of the properties of the differential calculus on $\mathbb{R}^d$. However a differentiable function might not be continuous. Moreover, if $\widehat{X}$ is totally disconnected, then the theory is trivial: Every complex function defined on $\widehat{X}$ is differentiable with respect to the single trivial element of $\mathrm{Hom}(\mathbb{R},\widehat{X})$, and the derivative is always zero. If $\mu \in \mathsf{M}(X)$ is such that $\Phi\mu \in \mathsf{M}(X)$, then [28, p. 68] $m = \mathscr{F}(\mu)$ is differentiable at any point ξ along the one-parameter subgroup φ and $-i\mathscr{F}(\Phi\mu) = d_\varphi m$.

Let us fix a bounded continuous function $m : \widehat{X} \to \mathbb{R}$ such that $\mathscr{F}^{-1}(m) \in \mathsf{M}(X)$. We say that the one-parameter subgroup $\varphi : \mathbb{R} \to \widehat{X}$ is in $\mathrm{Hom}_m^1(\mathbb{R},\widehat{X})$ if m is twice differentiable w.r.t. φ and $d_\varphi m, d_\varphi^2 m \in \mathscr{F}(\mathsf{M}(X))$. If, in addition, m is thrice differentiable w.r.t. φ and $d_\varphi^3 m \in \mathscr{F}(\mathsf{M}(X))$ too, we say that φ belongs to $\mathrm{Hom}_m^2(\mathbb{R},\widehat{X})$. Then next result follows directly from Corollary 4.1.

Corollary 4.7. *Let X be a LCAG and let m_0, m_1 be real functions with $\mathscr{F}^{-1}(m_0), \mathscr{F}^{-1}(m_1) \in \mathsf{M}(X)$ and $\mathrm{supp}(\mathscr{F}^{-1}(m_1)) \subset \mathscr{B}(X)$. Then*

$$\mathcal{H}_\mathrm{p}(M_{m_0+m_1}) \subset \bigcap_{\varphi \in \mathrm{Hom}_{m_0}^1(\mathbb{R},\widehat{X})} \ker(M_{d_\varphi m_0})$$

and

$$\mathcal{H}_\mathrm{s}(M_{m_0+m_1}) \subset \bigcap_{\varphi \in \mathrm{Hom}_{m_0}^2(\mathbb{R},\widehat{X})} \ker(M_{d_\varphi m_0}).$$

It is worth noting that for X abelian, the following assertions are equivalent [16, Thm. 24.34 & Cor. 24.35]: (i) $\mathscr{B}(X) = \{e\}$, (ii) the real characters separate points, (iii) the dual group $\widehat{X}$ is connected, (iv) X is isomorphic to $\mathbb{R}^d \times D$, where D is a discrete torsion-free abelian group.

Up to our knowledge, Corollary 4.7 is not known in the present generality. It is a by-product of a theory working in a non-commutative framework and it is obviously far from being optimal. We hope to treat the spectral analysis of (unbounded) multiplication operators on LCAG in greater detail in a forthcoming publication.

One may interpret our use of $\mathrm{Hom}(X,\mathbb{R})$ in Theorem 2.2 as an attempt to involve "smoothness" and "derivatives" in spectral theory for groups which might not be abelian or might not have a given Lie structure.

4.4 Semidirect products

Let N, G be two discrete groups with G abelian (for which we use additive notations), and let $\tau : G \to \mathrm{Aut}(N)$ be a group morphism. Let $X := N \times_\tau G$ be the τ-semidirect produt of N by G. The multiplication in X is defined by

$$(n, g)(m, h) := (n\tau_g(m), g + h),$$

so that

$$(n, g)^{-1} = (\tau_{-g}(n^{-1}), -g).$$

In the sequel we only consider real characters $\Phi \in \mathrm{Hom}(X,\mathbb{R})$ of the form $\Phi = \phi \circ \pi$, where $\phi \in \mathrm{Hom}(G,\mathbb{R})$ and $\pi : X \to G$ is the canonical morphism given by $\pi(n, g) := g$.

Proposition 4.8. *Let $a_0 = a_0^* : X \to \mathbb{C}$ have a finite support and satisfy*

$$\sum_{\substack{n_1, n_2 \in N \\ m = n_1 \tau_{g_1}(n_2)}} a_0(n_1, g_1)a_0(n_2, g_2) = \sum_{\substack{n_1, n_2 \in N \\ m = n_1 \tau_{g_2}(n_2)}} a_0(n_1, g_2)a_0(n_2, g_1), \quad \forall g_1, g_2 \in G, \; \forall m \in N. \qquad (4.1)$$

Let $a_1 = a_1^* : X \to \mathbb{C}$ have a finite support contained in $\mathscr{B}(X)$ and be such that $a_1 * a_0 = a_0 * a_1$. Then

$$\mathcal{H}_{\mathrm{s}}(H_{a_0+a_1}) \subset \bigcap_{\phi \in \mathrm{Hom}(G,\mathbb{R})} \ker\left(H_{(\phi \circ \pi)a_0}\right).$$

Proof. Since $a := a_0 + a_1$ has a finite support, we only have to check that

$$\Phi a * a - a * \Phi a = \Phi a * \Phi^2 a - \Phi^2 a * \Phi a = 0 \quad \text{for any } \Phi = \phi \circ \pi, \ \phi \in \mathrm{Hom}(G,\mathbb{R}). \tag{4.2}$$

Since $a_1 * a_0 = a_0 * a_1$ and $\Phi a_1 = 0$ for each $\Phi \in \mathrm{Hom}(X,\mathbb{R})$, we are easily reduced to check (4.2) only for a_0. Let $(m,h) \in X$; then a direct calculation gives

$$(\Phi a_0 * a_0 - a_0 * \Phi a_0)(m,h) = \sum_{g \in G} \phi(2g - h) \sum_{\substack{n_1,n_2 \in N \\ m = n_1 \tau_g(n_2)}} a_0(n_1,g)a_0(n_2,h - g).$$

This leads to the identity

$$(\Phi a_0 * a_0 - a_0 * \Phi a_0)(m,h) = -(\Phi a_0 * a_0 - a_0 * \Phi a_0)(m,h)$$

by using Condition (4.1) and the change of variable $g' := h - g$. Thus $\Phi a_0 * a_0 - a_0 * \Phi a_0 = 0$.

By a similar argument one obtains that $\Phi a_0 * \Phi^2 a_0 - \Phi^2 a_0 * \Phi a_0 = 0$ (the extra factor involved in this computation is symmetric with respect to the change of variables). $\square$

The proposition tells us that H_a has a non-trivial absolutely continuous component if there exists a real character $\phi \in \mathrm{Hom}(G,\mathbb{R})$ such that $(\phi \circ \pi)a \neq 0$. Consequently, as soon as $\mathrm{supp}(a)$ is not included in $N \times \mathscr{B}(G)$, we are done. For instance if $G = \mathbb{Z}^d$, we simply have to ask for the existence of an element $(n,g) \in \mathrm{supp}(a)$ with $g \neq 0$. In the remaining part we indicate several situations to which Proposition 4.8 applies; the perturbation a_1 is left apart for simplicity.

Procedure 4.9. *Let G_0 be a finite subset of G such that $G_0 = -G_0$. For each $g \in G_0$ let N_g be a finite subset of N such that $\tau_g(N_{-g}) = N_g^{-1}$ for each $g \in G_0$. Set*

$$S := \bigsqcup_{g \in G_0} N_g \times \{g\}.$$

This is a convenient description of the most general finite subset of X satisfying $S^{-1} = S$ (which is equivalent to $\chi_S = \chi_S^*$). Condition (4.1) amounts to

$$\#\{(n_1,n_2) \in N_{g_1} \times N_{g_2} \mid m = n_1\tau_{g_1}(n_2)\} = \#\{(n_1,n_2) \in N_{g_2} \times N_{g_1} \mid m = n_1\tau_{g_2}(n_2)\} \tag{4.3}$$

for each $g_1, g_2 \in G_0$ and $m \in N$. Under these assumptions Proposition 4.8 applies and H_{χ_S} has a non-trivial absolutely continuous component if $G_0 \cap [G \setminus \mathscr{B}(G)] \neq \varnothing$.

One can ensure in various situations that S is a system of generators (which is needed to assign a *connected* Cayley graph to (X, S)). This happens, for instance, if G_0 generates G, $\cup_{g \in G_0} N_g$ generates N and the unit e of N belongs to N_g for each $g \in G_0$.

Example 4.10. *Let $N := S_3 = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$, $G := \mathbb{Z}$, $G_0 := \{-1, 1\}$, $N_{-1} := \{a, aba\}$, $N_1 := \{a, b\}$, and $\tau_g(n) := a^g n a^{-g}$ for each $g \in \mathbb{Z}$ and $n \in S_3$. Then direct calculations show that all the assumptions in Procedure 4.9 are verified. Hence $\mathcal{H}_{\mathrm{ac}}(H_{\chi_S}) \neq \{0\}$ for $X := S_3 \times_\tau \mathbb{Z}$ if $S = \{(a, -1), (aba, -1), (a, 1), (b, 1)\}$. Actually, by applying Corollary 2.3 with $\Phi(n, g) := g$, one finds that the single possible component of the singular spectrum of H_{χ_S} is an eigenvalue located at the origin. However a careful inspection shows us that H_{χ_S} is injective, and thus that $\mathcal{H}_{\mathrm{ac}}(H_{\chi_S}) = \mathcal{H}$.*

A rather simple (but not trivial) possibility consists in taking $N_g \equiv N_0$ independent on $g \in G_0$ in Procedure 4.9. In such a case $S = N_0 \times G_0$ and $\chi_S = \chi_{N_0} \otimes \chi_{G_0}$ (which does not implies in general that H_{χ_S} is a tensor product of operators). If we also assume that N_0 is invariant under the set $\{\tau_g \mid g \in G_0\}$, then all the necessary assumptions are satisfied. For instance, the invariance of N_0 permits to define a bijection between the two sets in Formula (4.3). A particular case would be to choose $G := \mathbb{Z}^d$, with $G_0 := \{(\pm 1, \dots, 0), \dots, (0, \dots, \pm 1)\}$. By using the real morphism $\phi : \mathbb{Z}^d \to \mathbb{R}$ defined by $\phi(\pm 1, \dots, 0) =: \pm 1, \dots, \phi(0, \dots, \pm 1) =: \pm 1$, one can apply Corollary 2.3 to conclude that, except the possible eigenvalue 0, H_{χ_S} is purely absolutely continuous.

The following two examples are applications of the preceding construction.

Example 4.11. *We consider a simple type of wreath product. Take G a discrete abelian group and put $N := R^J$, where R is an arbitrary discrete group and J is a finite set on which G acts by $(g, j) \mapsto g(j)$. Then $\tau_g(\{r_j\}_{j \in J}) := \{r_{g(j)}\}_{j \in J}$ defines an action of G on R^J, thus we can construct the semidirect product $R^J \times_\tau G$. If $G_0 = -G_0 \subset G$ and $R_0 = R_0^{-1} \subset R$ are finite subsets with $G_0 \cap [G \setminus \mathscr{B}(G)] \neq \varnothing$, then $N_0 := R_0^J$ satisfies all the required conditions. Thus $\mathcal{H}_{\mathrm{ac}}(H_{\chi_S}) \neq \{0\}$ if $S := N_0 \times G_0$.*

Example 4.12. *Let G be a discrete abelian group, let N be the free group generated by the family $\{a_1, \dots, a_n\}$, and set $N_0 := \{a_1^{\pm 1}, \dots, a_k^{\pm 1}\}$. Choose a finite set $G_0 = -G_0 \subset G$ with $G_0 \cap [G \setminus \mathscr{B}(G)] \neq \varnothing$ and an action τ on N such that the conditions on $S := N_0 \times G_0$ are satisfied (for instance τ_g, $g \in G_0$, may act by permutation on the generators). Then H_{χ_S} has a non-trivial absolutely continuous part.*

Virtually the methods of this article could also be applied to non-split group extensions.

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References

[1] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. C_0-*groups, commutator methods and spectral theory of N-body Hamiltonians*, volume 135 of *Progress in Math.* Birkhäuser, Basel, 1996.

[2] L. Bartholdi and R. I. Grigorchuk. On the spectrum of Hecke type operators related to some fractal groups. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):5–45, 2000.

[3] L. Bartholdi and W. Woess. Spectral computations on lamplighter groups and Diestel-Leader graphs. *J. Fourier Anal. Appl.*, 11(2):175–202, 2005.

[4] C. Béguin, A. Valette, and A. Zuk. On the spectrum of a random walk on the discrete Heisenberg group and the norm of Harper's operator. *J. Geom. Phys.*, 21(4):337–356, 1997.

[5] A. Boutet de Monvel, G. Kazantseva, and M. Măntoiu. Some anisotropic Schrödinger operators without singular spectrum. *Helv. Phys. Acta*, 69(1):13–25, 1996.

[6] A. Boutet de Monvel and M. Măntoiu. The method of the weakly conjugate operator. In *Inverse and algebraic quantum scattering theory (Lake Balaton, 1996)*, volume 488 of *Lecture Notes in Phys.*, pages 204–226, Berlin, 1997. Springer.

[7] J. Breuer. Singular continuous spectrum for the Laplacian on a certain sparse tree. preprint on `math.SP/0604552`.

[8] J. Breuer. Singular continuous and dense point spectrum for sparse tree with finite dimensions. preprint on `math.SP/0608159`.

[9] W. Dicks and T. Schick. The spectral measure of certain elements of the complex group ring of a wreath product. *Geom. Dedicata*, 93:121–137, 2002.

[10] G. B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1995.

[11] G. Georgescu and S. Golénia. Isometries, Fock spaces, and spectral analysis of Schrödinger operators on trees. *J. Funct. Anal.*, 227(2):389–429, 2005.

[12] S. Grosser and M. Moskowitz. On central topological groups. *Trans. Amer. Math. Soc.*, 127:317–340, 1967.

[13] S. Grosser and M. Moskowitz. Representation theory of central topological groups. *Trans. Amer. Math. Soc.*, 129:361–390, 1967.

[14] S. Grosser and M. Moskowitz. Compactness conditions in topological groups. *J. Reine Angew. Math.*, 246:1–40, 1971.

[15] R. I. Grigorchuk and A. Żuk. The lamplighter group as a group generated by a 2-state automaton, and its spectrum. *Geom. Dedicata*, 87(1-3):209–244, 2001.

[16] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*, volume 115 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[17] P. de la Harpe, A. Guyan Robertson, and A. Valette. On the spectrum of the sum of generators for a finitely generated group. *Israel J. Math.*, 81(1-2):65–96, 1993.

[18] P. de la Harpe, A. Guyan Robertson, and A. Valette. On the spectrum of the sum of generators of a finitely generated group. II. *Colloq. Math.*, 65(1):87–102, 1993.

[19] A. Iftimovici and M. Măntoiu. Limiting absorption principle at critical values for the Dirac operator. *Lett. Math. Phys.*, 49(3):235–243, 1999.

[20] M. Kambites, P. V. Silva, and B. Steinberg. The spectra of lamplighter groups and Cayley machines. *Geom. Dedicata*, 120:193–227, 2006.

[21] H. Kesten. Symmetric random walks on groups. *Trans. Amer. Math. Soc.*, 92:336–354, 1959.

[22] L. Malozemov and A. Teplyaev. Pure point spectrum of the Laplacians on fractal graphs. Pure point spectrum of the Laplacians on fractal graphs. *J. Funct. Anal.*, 129:390–405, 1995.

[23] M. Măntoiu and M. Pascu. Global resolvent estimates for multiplication operators. *J. Operator Th.*, 36(2):283–294, 1996.

[24] M. Măntoiu and S. Richard. Absence of singular spectrum for Schrödinger operators with anisotropic potentials and magnetic fields. *J. Math. Phys.*, 41(5):2732–2740, 2000.

[25] M. Măntoiu, S. Richard, and R. Tiedra de Aldecoa. Spectral analysis for adjacency operators on graphs. preprint on `math-ph/0603020` and to appear in Ann. H. Poincaré.

[26] M. Reed and B. Simon. *Methods of modern mathematical physics, IV: Analysis of Operators*. Academic Press, New York, 1978.

[27] S. Richard. Some improvements in the method of the weakly conjugate operator. *Lett. Math. Phys.*, 76(1):27–36, 2006.

[28] J. Riss. Eléments de calcul différentiel et théorie des distributions sur les groupes abéliens localement compacts. *Acta Math.*, 89:45–105, 1953.

[29] B. Simon. Operators with singular continuous spectrum VI: Graph Laplaciens and Laplace-Beltrami operators. *Proc. Amer. Math. Soc.*, 124:1177–1182, 1996.

[30] J. Weidmann. *Linear operators in Hilbert spaces*. Springer-Verlag, New York, 1980.