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EQUIVALENCE OF FAMILIES OF DIFFEOMORPHISMS ON BANACH SPACES

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ABSTRACT. We consider two families of C^∞-diffeomorphisms (with hyperbolic linear part at 0) on a Banach space. Suppose that these two families formally conjugate at 0. We prove that they admit local conjugation, which is infinitely smooth in both, the space variable and the family parameter. In particular, subject to non-resonance condition, there exists a family of local C^∞ linearizations of the family of diffeomorphisms. The linearizing family has C^∞ smoothness in parameter. The results are proved under the assumption that the Banach space allows a special extension of the maps. We discuss corresponding properties of Banach spaces.

1. INTRODUCTION

Approximations by normal forms and especially linear approximation are convenient simplifications of a physical system.

Most of the physical systems are assumed to be analytic. The study of analytic conjugation was initiated by Poincaré and extended in numerous works. But it was shown in the work of Bruno ([Br]) that the class of systems, which admits analytic reduction to the Poincaré-Dulac normal form is small even in $\mathbb{R}^n$. For that reason we study this problem in the class of C^∞ transformations.

This question was studied in numerous works (for example, in [S], [Ch], [B]). The first step in the proofs of these results in $\mathbb{R}^n$ is the condition of formal solvability in C^∞. The latter generates an algebraic expression, called resonance condition. For a generic diffeomorphism, the non-resonance assumption is a necessary condition for removing resonant non-linear terms.

Although, resonance is an obstacle for a smooth conjugation in dimensions higher than two, for a contracting (or expanding) diffeomorphisms in $\mathbb{R}^n$ this assumption can be removed (see [H] or [B4] for a more general result). The situation is qualitatively different in Banach spaces. It was shown in the work of [RS-M] that the non-resonance assumption in Banach space is essential even for contractions. They construct an example of a contraction diffeomorphism in infinite dimensions that is not C^1 linearizable.

In this paper we study the question of smooth conjugation of two families of diffeomorphisms with hyperbolic linear part Λ. The diffeomorphisms are defined on a Banach space B_1, and the family parameter is also an element of another Banach space, B_2.

If we restrict our attention to the question of smooth dependence of linearizing homeomorphisms on the parameter, we do not need to make a non-resonance assumption for some class of diffeomorphisms. It was shown in [R] that for a family of diffeomorphisms with non-linearities of the second order, there exists a family of linearizing homeomorphisms which have C^1 dependence on the parameter. Unfortunately, the methods, used in the paper can only be used for diffeomorphisms with resonances. Also, we only proved C^1 differentiability. The proof for higher regularity would be very routine.

In this paper we show that, subject to formal conjugation, a family of germs with hyperbolic linear part Λ admits a C^∞ local conjugation between the members of the family. This conjugation has C^∞ dependence on the parameter of the family.

As a consequence, we deduce that subject to the non-resonance condition, there exists a C^∞ local linearization, which has smooth dependence on the parameter of the family of diffeomorphisms.

One of the technical difficulties is the question of C^∞ extension of a local map to the whole space B_1. It is connected to the question of existence of "hat" functions, i.e., $\tau(x) \in C^\infty$ with bounded support. (One can also see the assumption about "hat" function in the proof of C^1 conjugation for contracting diffeomorphisms on Banach spaces in the paper of RS-M1.) If $\tau(x)$ is such a function, then the extension of f, call it $\hat{f}$ can be defined as $\hat{f}(x) = f(\tau(x) \cdot x)$. In this expression $\tau(x) \cdot x$ defines a map $K(x) : B_1 \to B_1$ $(K(x) = \tau(x) \cdot x)$ with the following properties:

1. $K(x) = x$ for $||x|| < 1$
2. $K \in C^\infty$ and $||K^{(j)}(x)|| \leq c_j, \ j = 0, 1, ...$

Therefore, we can see that for existence of extension $\hat{f}$ it is enough to have map K, defined on the Banach space.

It turns out that there exist Banach spaces (for example, $C([0, 1])$) without "hat" functions, but with K maps. Moreover, the question of extension of a map remains open. It is not know whether there are Banach spaces without K maps.

Another technical difficulty, associated with infinite dimensional space, is the fact that we cannot construct a compact set of maps on this space, to which we would apply Banach Contraction Principle. A special method for this situation is introduced in Section 3.

The methods, used in this proof, were developed in the two works of G. Belitskii: [B1], [B2]. The technique of the first article allows us to consider the problem in Banach spaces, where we use a special metric, induced by a countable collection of norms. The special metric (with "inner" contraction) permits us to use the Banach Contraction Principle on the bounded set of smooth maps. The idea to use the Banach Contraction Principle for the study of smooth dependence on parameters comes from the second paper, [B2].

2. The main theorem

Lets consider a linear hyperbolic operator Λ on a Banach space B_1. It divides B_1 into contracting and expanding subspaces L_+ and L_- (see, for example [N]). Lets assume that there exist K_+ and K_-, defined on L_+ and L_- correspondingly, with the properties 1-2, discussed above. Then, it is clear that the map $K : B_1 \to B_1$ can be defined as $K = (K_+, K_-)$. But it is unknown, whether the existence of K on B_1 implies the existence of $K_\pm$ on the subspaces. Thus, assuming that there exists a pair of maps $K_\pm$, we can prove the following theorem.

Theorem 1. *Assume that there exist $K_\pm$ on $L_\pm \subset B_1$. Let*

$$F(x,t) = \begin{cases} \Lambda x + f(x,t) \\ t \end{cases}$$

and

$$G(x,t) = \begin{cases} \Lambda x + g(x,t) \\ t \end{cases}$$

be two $C^\infty(B_1 \times B_2)$ diffeomorphisms on the product of Banach spaces B_1 and B_2. Assume that Λ is hyperbolic, $f(x,t) = o_t(x)$, $g(x,t) = o_t(x)$ and $f(x,t) - g(x,t) = o_t(x^n)$ for all $n \in \mathbb{N}$, as $||x|| \to 0$, for small values of $||t||$.

Then, there exists a $C^\infty(B_1 \times B_2)$ diffeomorphism

$$H(x,t) = \begin{cases} x + h(x,t) \\ t \end{cases},$$

with $h(x,t) = o_t(x^n)$ for all $n \in \mathbb{N}$, as $||x|| \to 0$, such that for small values of $||x||$ and $||t||$

$$\text{(1)} \qquad\qquad\qquad H \circ F = G \circ H.$$

Proof. It is known that the map F has local stable and unstable manifolds W_U and W_S, which are C^∞ graphs over L_+ and L_- correspondingly. We can smoothly straighten W_U and W_S. I. e., we shell assume that $W_U = L_+$ and $W_S = L_-$.

We are interested in the local properties of the maps F and G. Thus, we can modify the maps F and G with the help of K, so that the new maps coincide with the original maps in a small δ neighborhood of the origin. From now on we shell assume that

$$\text{(2)} \qquad\qquad F(x,t) = \begin{cases} \Lambda x + f(\delta \cdot K(x/\delta),t), \\ t \end{cases}$$

and

$$\text{(3)} \qquad\qquad G(x,t) = \begin{cases} \Lambda x + g(\delta \cdot K(x/\delta),t), \\ t. \end{cases}$$

The first derivatives of the non-linear parts of these maps are close to 0 and all other derivatives are bounded. Thus, we obtained the desired extensions of these maps to $B_1 \times B_2$.

It is enough to prove the existence of a C^∞ solution h for the equation

$$\text{(4)} \qquad\qquad (\text{Id} + h) \circ (\Lambda + f) = (\Lambda + g) \circ (\text{Id} + h).$$

Consider closed subspaces of C^∞ maps:

$$\mathcal{E}_{L_+} = \{\phi \in C^\infty(B_1 \times B_2) : \ \phi^{(k)}_{x \in L_+} = 0, \ k \in \mathbb{N}\}$$

and

$$\mathcal{E}_{L_-} = \{\phi \in C^\infty(B_1 \times B_2) : \ \phi^{(k)}_{x \in L_-} = 0, \ k \in \mathbb{N}\}$$

We shell say that ϕ is flat on L_- (or L_+), if $\phi \in \mathcal{E}_{L_-}$ (or $\phi \in \mathcal{E}_{L_+}$).
The equation (4) can be reduced to

$$\text{(5)} \quad h\left(\Lambda x + f(x,t),t\right) - \Lambda h(x,t) - \left(g\left(x + h(x,t),t\right) - g(x)\right) = (g(x) - f(x))$$

The right side of this equation (call it γ) is flat at the origin. By Lemma 9, γ can be written as a sum of two maps, γ_+ and γ_-, flat on L_- and L_+ correspondingly. We shell search for a solution h of equation 5 as a sum of two solutions, $h_+ \in \mathcal{E}_{L_+}$ and $h_- \in \mathcal{E}_{L_-}$, where h_+ satisfies

$$h_+(x,t) = \Lambda h_+\big((\Lambda x + f(x,t))^{-1}, t\big)$$

(6)
$$+ g\Big(\big(\Lambda x + f(x,t)\big)^{-1} + h_+\big((\Lambda x + f(x,t))^{-1}, t\big), t\Big)$$

$$- g((\Lambda x + f(x,t))^{-1}) + \gamma_+\big((\Lambda x + f(x,t))^{-1}, t\big).$$

The second addend, h_- shell satisfy the equation

$$h_-(x,t) = \Lambda^{-1} h_-\big(\Lambda x + f(x,t), t\big) + \Lambda^{-1} g(x + h_-(x,t) + h_+(x,t), t)$$
$$- \Lambda^{-1} g(x + h_+(x,t), t) - \Lambda^{-1} \gamma_-(x,t),$$

or, if we put $\hat{g}(x,t) = \Lambda^{-1} g\big(x + h_+(x,t), t\big)$,

(7)
$$h_-(x,t) = \Lambda^{-1} h_-\big(\Lambda x + f(x,t), t\big) + \hat{g}(x + h_-(x,t), t)$$
$$- \hat{g}(x,t) - \Lambda^{-1} \gamma_-(x,t).$$

The existence of solutions to these equations follows from Lemma 5 $\qquad\square$

It is known that one can define a "hat" function on a Hilbert space. It can be done in the following way: $\tau(x) := \hat{\tau}(\|x\|^2)$, where $\hat{\tau}(t)$ is a "hat" function in $\mathbb{R}$. Moreover, there exists a pair of maps $K_\pm$, which can be defined as $K_\pm := (x_\pm \cdot \tau(x_\pm))$, and there exists a map $K = (K_+, K_-)$. Then, the following corollary follows immediately from Theorem 1.

Corollary 2. *Assume that B_1 is a Hilbert space. Let*

$$F(x,t) = \left\{ \begin{array}{l} \Lambda x + f(x,t) \\ t \end{array} \right.$$

and

$$G(x,t) = \left\{ \begin{array}{l} \Lambda x + g(x,t) \\ t \end{array} \right.$$

be two $C^\infty(B_1 \times B_2)$ diffeomorphisms on the product of Banach spaces B_1 and B_2. Assume that Λ is hyperbolic, $f(x,t) = o_t(x)$, $g(x,t) = o_t(x)$ and $f(x,t) - g(x,t) = o_t(x^n)$ for all $n \in \mathbb{N}$, as $\|x\| \to 0$, for small values of $\|t\|$.

Then, there exists a $C^\infty(B_1 \times B_2)$ diffeomorphism

$$H(x,t) = \left\{ \begin{array}{l} x + h(x,t) \\ t \end{array} \right. ,$$

with $h(x,t) = o_t(x^n)$ for all $n \in \mathbb{N}$, as $\|x\| \to 0$, such that for small values of $\|x\|$ and $\|t\|$

(8)
$$H \circ F = G \circ H.$$

Remark 3. Note, that there exists K on $C([0,1])$, but it is unclear whether there exist $K_\pm$ on the subspaces.

Remark 4. Let $B_1 = C([0,1])$. Consider two families of purely contracting (or expanding) diffeomorphisms on $C([0,1]) \times B_2$. Then, if the two families formally conjugate, they smoothly conjugate in the class $C([0,1]) \times B_2$.
The same conclusion is true for finite dimensional Banach spaces.

3. Solvability in flat maps

The following lemma is presented for the "negative" subspaces L_-, $\mathcal{E}_{L_-}$, etc. Proof for the "positive" subspaces is similar.

Lemma 5. *Consider the equation*

$$
\begin{aligned}
\phi(x,t) = {} & A\phi\big(\Lambda x + f(x,t),t\big) \\
& + g\Big(\Lambda x + f(x,t) + \phi((\Lambda x + f(x,t),t)),t\Big) \\
& - g\big(\Lambda x + f(x,t),t\big) + \rho(x,t)
\end{aligned}
\tag{9}
$$

with a linear operator A, hyperbolic linear operator Λ and a map ρ, which is flat on the linear stable invariant manifold $L_- \subset B_1$ of Λ. Then, there exists flat on L_- solution $\phi \in C^\infty(B_1 \times B_2)$.

Proof. We shell search for a fixed point of the operator T, defined by the right hand side of the equation 9:

$$
\begin{aligned}
T_\phi = {} & A\phi\big(\Lambda x + f(x,t),t\big) \\
& + g\Big(\Lambda x + f(x,t) + \phi((\Lambda x + f(x,t),t)),t\Big) - g\big(\Lambda x + f(x,t),t\big) + \rho(x,t)
\end{aligned}
\tag{10}
$$

Consider space $\mathcal{E}_{L_-}$ of smooth maps, flat on the manifold L_-. To simplify our notations, we shell call it $\mathcal{E}$

Define a norm

$$
N_{k,p}(\phi) := \max_{0 \leq j \leq k} \sup \left\{ \frac{||\phi^{(j)}(x,t)||}{||\xi||^p}, \ \phi \in \mathcal{E}, \ \xi \in L_-, \ x \in B_1, \ t \in B_2 \right\}.
$$

Proposition 6. *There exist a real number $q < 1$ and sequences of $b_{k,p} \in \mathbb{R}_+$ and $p_k \in \mathbb{N}$, such that for all $p,k \in \mathbb{N}$, $p > p_k$ and $\phi, \psi \in \mathcal{E}$*

1. $N_{0,p}(T_\phi) \leq q N_{0,p}(\phi) + N_{0,p}(\rho)$,
   $N_{k,p}(T_\phi) \leq q N_{k,p}(\phi) + b_{k,p} N_{k-1,p}(\phi) + N_{k,p}(\rho)$ *for $k = 1,2,\dots$ and*

2. $N_{0,p}(T_\phi - T_\psi) \leq q N_{0,p}(\phi - \psi)$,
   $N_{k,p}(T_\phi - T_\psi) \leq q N_{k,p}(\phi - \psi) + b_{k,p} N_{k-1,p}(\phi - \psi)$ *for $k = 1,2,\dots$*

Proof. First, we shell do the estimates for the part 2. The proof will require inductive arguments on the index k. It is easy to see that for $k = 0$

$$
\begin{aligned}
N_{0,p}(T_\phi - T_\psi) \leq {} & ||A|| N_{0,p}\Big(\phi\big(\Lambda x + f(x,t),t\big) - \psi\big(\Lambda x + f(x,t),t\big)\Big) \\
& + \int_0^1 g'_{\sigma(s)} ds \cdot N_{0,p}\Big(\phi\big(\Lambda x + f(x,t),t\big) - \psi\big(\Lambda x + f(x,t),t\big)\Big).
\end{aligned}
$$

Here $\sigma : [0,1] \to B_1 \times B_2$ denotes a path, connecting the points

$$
\sigma(0) = \big(\Lambda x + f(x,t) + \phi((\Lambda x + f(x,t),t)\big)
$$

and

$$
\sigma(1) = \Big(\Lambda x + f(x,t) + \psi((\Lambda x + f(x,t),t)),t\Big).
$$

Then,

$$
N_{0,p}(T_\phi) \leq \big(||A|| + \sup_{x,\ t} ||g'(x,t)||\big) \cdot N_{0,p}\big(\phi(\Lambda x + f(x,t),t) - \psi(\Lambda x + f(x,t),t)\big).
$$

Let us recall that choosing sufficiently small δ in the K function, we can make $\sup_{x,\,t} ||g'(x,t)||$ less than an arbitrary Δ. (See formula 3.) Then,

$$N_{0,p}(T_\phi - T_\psi) \le \big(||A|| + \Delta\big) \cdot N_{0,p}\big(\phi(x,t) - \psi(x,t)\big)$$

$$\cdot \Big(\sup_{x,\,\xi,\,t} \frac{||\big(\Lambda x + f(x,t),t\big)_{|L_-}||}{||\xi||} \Big)^p.$$

Since the spectrum of $\Lambda_{|L_-}$ is less than 1, and with sufficiently small δ in the K function we can make $||f(x,t)||$ as small as we want (see formula 2), there exists $\alpha < 1$ such that

$$\sup_{x,\,\xi,\,t} \frac{||\big(\Lambda x + f(x,t),t\big)_{|L_-}||}{||\xi||} < \alpha < 1.$$

Then, there exists p_0, such that for all $p > p_0$

$$\big(||A|| + \Delta\big) \cdot \Big(\sup_{x,\,\xi,\,t} \frac{||\big(\Lambda x + f(x,t),t\big)_{|L_-}||}{||\xi||} \Big)^p < q << 1.$$

Thus,

$$N_{0,p}(T_\phi - T_\psi) \le q N_{0,p}(\phi - \psi).$$

The estimates for $k = 1$ are more complex.

$$N_{1,p}(T_\phi - T_\psi) \le \Big(||A|| \cdot N_{1,p}\big(\phi(x,t) - \psi(x,t)\big) \cdot ||\big(\Lambda x + f(x,t)\big)'||$$

$$+ \sup_{x,\,t} ||g'|| \, N_{1,p}\big(\phi(x,t) - \psi(x,t)\big)$$

$$+ \sup_{x,\,t} ||g''|| \, N_{0,p}\big(\phi(x,t) - \psi(x,t)\big)\Big)$$

$$\cdot \Big(\sup_{x,\,\xi,\,t} \frac{||\big(\Lambda x + f(x,t)\big)_{|L_-}||}{||\xi||} \Big)^p$$

$$\le \big(||A|| \cdot ||\big(\Lambda x + f(x,t)\big)'|| + \epsilon\big) N_{1,p}\big(\phi(x,t) - \psi(x,t)\big)$$

$$\cdot \Big(\sup_{x,\,\xi,\,t} \frac{||\big(\Lambda x + f(x,t)\big)_{|L_-}||}{||\xi||} \Big)^p + b_{1,p} \, N_{0,p}\big(\phi(x,t) - \psi(x,t)\big).$$

Here the term $b_{1,p} N_{0,p}\big(\phi(x) - \psi(x)\big)$ (with some positive constant $b_{1,p}$) appears in the estimate, as a result of differentiation of the product of $\int_0^1 g' ds \cdot F(x)$. There exists p_1, such that for all $p > p_1$

$$\Big(||A|| \cdot \big|\big(\Lambda x + f(x,t)\big)'\big| + \epsilon\Big) \cdot \Big(\sup_{x,\,\xi,\,t} \frac{||\big(\Lambda x + f(x,t)\big)_{|L_-}||}{||\xi||} \Big)^p < q < 1.$$

Then, it follows that

$$N_{1,p}(T_\phi - T_\psi) < q N_{1,p}(\phi - \psi) + b_{1,p} N_{0,p}(\phi - \psi).$$

Suppose that the estimates are correct for $i = 1, ..., k-1$. Then, for $i = k$

$$N_{k,p}(T_\phi - T_\psi) \le \Big(\big(||A|| + ||g'||\big) \cdot N_{k,p}\big(\phi(x,t) - \psi(x,t)\big) \cdot ||\big(\Lambda x + f(x,t)\big)'||^k$$

$$+ b_{k,p} \, N_{k-1,p}\big(\phi(x,t) - \psi(x,t)\big)\Big)$$

$$\cdot \Big(\sup_{x,\,\xi,\,t} \frac{||\big(\Lambda x + f(x,t)\big)_{|L_-}||}{||\xi||} \Big)^p.$$

Here the term $b_{k,p}N_{k-1,p}\big(\phi(x)-\psi(x)\big)$ (with some positive constant $b_{k,p}$) is related to taking the kth derivative of the compusition $(\phi - \psi) \circ F$, and estimating all addends with $(\phi - \psi)',...,(\phi - \psi)^{(k-1)}$, plus the sum of the derivatives:

$$\sum_{j=0}^{k-1} \frac{k!}{j!(k-j)!} ||g^{(j+1)}|| \cdot \sup_{x,\ \xi,\ t} \frac{||(\phi - \psi)^{(k-j)}(x,t)||}{||\xi||^p}.$$

There exists p_k, such that for all $p > p_k$

$$\big(||A|| + ||g'||\big) \cdot \big|\big(\Lambda x + f(x,t)\big)'\big|^k \cdot \Big(\sup_{x,\ \xi,\ t} \frac{||\big(\Lambda x + f(x,t)\big)_{|L_-}||}{||\xi||}\Big)^p < q < 1.$$

Thus,

$$N_{k,p}(T_\phi - T_\psi) < qN_{k,p}(\phi - \psi) + b_{k,p}N_{k-1,p}(\phi - \psi).$$

Thus, we proved part 2.

Part 1 follows easily from part 2. Indeed,

$$N_{0,p}(T_\phi) = N_{0,p}(T_\phi - T_0 + \rho) \le N_{0,p}(T_\phi - T_0) + N_{0,p}(\rho)$$
$$< qN_{0,p}(\phi) + N_{0,p}(\rho).$$

and for $k = 1, 2, ...$

$$N_{k,p}(T_\phi) = N_{k,p}(T_\phi - T_0 + \rho) \le N_{k,p}(T_\phi - T_0) + N_{k,p}(\rho)$$
$$< qN_{k,p}(\phi) + b_{k,p}N_{k-1,p}(\phi) + N_{k,p}(\rho).$$

$\square$

Using the above estimates, we introduce a new set of norms $||.||_{k,p}$, defined with the help of $N_{k,p}$, such that the operator T is a contraction in $||.||_{k,p}$.
Let $||.||_{k,p}$ be defined on $\mathcal{E}$ according to the following rule:

$$||\phi||_{0,p} := N_{0,p}(\phi)$$
$$||\phi||_{1,p} := N_{1,p}(\phi) + A^1_{1,p}N_{0,p}(\phi)$$
$$\vdots$$
$$||\phi||_{k,p} := N_{k,p}(\phi) + A^1_{k,p}N_{k-1,p}(\phi) + A^2_{k,p}N_{k-2,p}(\phi) + ... + A^k_{k,p}N_{0,p}(\phi).$$

Choosing appropriate constants $A^j_{k,p}$, one can show that the operator T is a contraction in each norm $||.||_{k,p}$.

Proposition 7. *Let q be as in Proposition 6. Fix $\epsilon > 0$ such that $q + \epsilon < 1$. Let the constants $A^j_{k,p}$ $(j = 1, ..., k)$ in the definition of the norms $||.||_{k,p}$ satisfy the following inequalities:*

(11)
$$A^1_{k,p} > \frac{b_{k,p}}{\epsilon}$$
$$A^2_{k,p} > \frac{A^1_{k,p}b_{k-1,p}}{\epsilon}$$
$$\vdots$$
$$A^k_{k,p} > \frac{A^{k-1}_{k,p}b_{1,p}}{\epsilon}$$

Then, for each pair of $\phi, \psi \in \mathcal{E}$ and for each norm $||.||_{k,p}$ $(p > p_k)$, the operator T admits the following estimates:

$1'$. $||T_\phi||_{k,p} < (q+\epsilon)||\phi||_{k,p} + ||\rho||_{k,p}$ *and*

$2'$. $||T_\phi - T_\psi||_{k,p} < (q+\epsilon)||\phi - \psi||_{k,p}.$

Proof. We begin with the second part of the proposition. In this proof we shell also use induction on the k-index.

$$||T_\phi - T_\psi||_{0,p} = N_{0,p}(T_\phi - T_\psi) < qN_{0,p}(\phi - \psi) = q||\phi - \psi||_{0,p}.$$

Thus, it is a contraction in $||.||_{0,p}$.

$$\begin{aligned}||T_\phi - T_\psi||_{1,p} &= N_{1,p}(T_\phi - T_\psi) + A_{1,p}^1 N_{0,p}(T_\phi - T_\psi)\\ &< qN_{1,p}(\phi - \psi) + b_{1,p}N_{0,p}(\phi - \psi) + A_{1,p}^1 qN_{0,p}(\phi - \psi)\\ &= qN_{1,p}(\phi - \psi) + (A_{1,p}^1 q + b_{1,p})N_{0,p}(\phi - \psi)\\ &< (q+\epsilon)N_{1,p}(\phi - \psi) + A_{1,p}^1(q+\epsilon)N_{0,p}(\phi - \psi),\end{aligned}$$

if $A_{1,p}^1 > \frac{b_{1,p}}{\epsilon}$. Finally, note that

$$(q+\epsilon)N_{1,p}(\phi - \psi) + A_{1,p}^1(q+\epsilon)N_{0,p}(\phi - \psi) = (q+\epsilon)||\phi - \psi||_{1,p}.$$

Thus, in this norm, the operator T is a contraction as well.

Assume that the assertion of the Proposition is correct for all $i = 1, ..., k-1$. We shell prove it for $i = k$, i.e., we shell estimate $||T_\phi - T_\psi||_{k,p}$.

$$\begin{aligned}||T_\phi - T_\psi||_{k,p} &= N_{k,p}(T_\phi - T_\psi) + A_{k,p}^1 N_{k-1,p}(T_\phi - T_\psi)\\ &\quad + A_{k,p}^2 N_{k-2,p}(T\phi - T\psi) + ... + A_{k,p}^k N_{0,p}(T\phi - T\psi)\\ &< qN_{k,p}(\phi - \psi) + b_{k,p}N_{k-1,p}(\phi - \psi)\\ &\quad + A_{k,p}^1[qN_{k-1,p}(\phi - \psi) + b_{k-1,p}N_{k-2,p}(\phi - \psi)]\\ &\quad + A_{k,p}^2[qN_{k-2,p}(\phi - \psi) + b_{k-2,p}N_{k-3,p}(\phi - \psi)]\\ &\vdots\\ &\quad + A_{k,p}^{k-1}[qN_{1,p}(\phi - \psi) + b_{1,p}N_{0,p}(\phi - \psi)]\\ &\quad + A_{k,p}^k qN_{0,p}(\phi - \psi)\\ &= qN_{k,p}(\phi - \psi)\\ &\quad + (A_{k,p}^1 q + b_{k,p})N_{k-1,p}(\phi - \psi)\\ &\quad + (A_{k,p}^2 q + A_{k,p}^1 b_{k-1,p})N_{k-2,p}(\phi - \psi)\\ &\vdots\\ &\quad + (A_{k,p}^{k-1} q + A_{k,p}^{k-2} b_{2,p})N_{1,p}(\phi - \psi)\\ &\quad + (A_{k,p}^k q + A_{k,p}^{k-1} b_{1,p})N_{0,p}(\phi - \psi)\end{aligned}$$

From inequalities 11 it immediately follows that

$$(A_{k,p}^1 q + b_{k,p}) < (q+\epsilon)A_{k,p}^1$$

and

$$A_{k,p}^{k-j} q + A_{k,p}^{k-j-1} b_{j+1,p} < (q+\epsilon)A_{k,p}^{k-j}$$

for $j = 0, ..., k - 2$. Then,

$$
\begin{aligned}
||T_\phi - T_\psi||_{k,p} &< q N_{k,p}(\phi - \psi) \\
&\quad + A_{k,p}^1 (q + \epsilon) N_{k-1,p}(\phi - \psi) \\
&\quad + A_{k,p}^2 (q + \epsilon) N_{k-2,p}(\phi - \psi) \\
&\quad \vdots \\
&\quad + A_{k,p}^{k-1}(q + \epsilon) N_{1,p}(\phi - \psi) \\
&\quad + A_{k,p}^k (q + \epsilon) N_{0,p}(\phi - \psi), \\
&< (q + \epsilon)[N_{k,p} + A_{k,p}^1 N_{k-1,p} + ... + A_{k,p}^k N_{0,p}] \\
&= (q + \epsilon)||\phi - \psi||_{k,p}
\end{aligned}
$$

Thus the first part of the theorem is proved. The second part follows immediately from the first part. Indeed,

$$
||T_\phi||_{k,p} = ||T_\phi - (T_0 - \rho)||_{k,p} \le ||T_\phi - T_0||_{k,p} + ||\rho||_{k,p} \le (q + \epsilon)||\phi||_{k,p} + ||\rho||_{k,p}.
$$

$\square$

Now, we shell define a closed disk, such that it is invariant under the action of T. Choose a sequence of real positive numbers $c_{k,p}$, and define the closed disc D in the space $\mathcal{E}$ as

$$
D = \{\phi \in \mathcal{E} : ||\phi||_{k,p} \le c_{k,p}\}.
$$

Assume that for some small $\epsilon > 0$, the constants $c_{k,p}$ are such that

$$
c_{k,p} \ge \frac{||\rho||_{k,p}}{1 - q - \epsilon}.
$$

Corollary 8.

$$
T(D) \subset D.
$$

Proof. The proof is an immediate consequence of the estimate $1^{'}$ in Proposition 7.

$\square$

Thus, we have proved that in the countable collection of norms $||.||_{k,p}$, the operator T is a contraction that maps the closed disc D into itself. Using these inequalities, we can now construct a metric on $\mathcal{E}$ with the properties, similar to those, discussed for the norms $||.||_{k,p}$. Then, we can use Banach Contraction Principle to show the existence of a fixed point of the operator T in the disc D.

Let us define the metric on D. Fix a positive $R \in \mathbb{R}$. Let $\epsilon_{k,p} < 1$ be a sequence of positive real numbers, such that $c_{k,p}\epsilon_{k,p} < R$ (here $c_{k,p}$ are the constants that define the disc D). Then,

$$
\rho(\phi, \psi) := \sum_{k,p > p_k} \frac{1}{2^{kp}} \cdot \frac{\epsilon_{k,p}||\phi - \psi||_{k,p}}{1 + \epsilon_{k,p}||\phi - \psi||_{k,p}}, \quad \phi, \psi \in D.
$$

Obviously, D is a complete metric space.

Consider a function $\kappa(t) = \frac{t}{1+t}$, defined for positive values of t. This is an increasing function, it is less than 1 for bounded values of t. Put

$$
t := \epsilon_{k,p}||\phi - \psi||_{k,p} < R.
$$

Since T is a contraction in $||.||_{k,p}$, and $\kappa(t)$ is an increasing function,

$$\rho(T_\phi, T_\psi) < \sum_{k,p > p_k} \frac{1}{2^{kp}} \cdot \frac{(q+\epsilon)\epsilon_{k,p}||\phi - \psi||_{k,p}}{1 + (q+\epsilon)\epsilon_{k,p}||\phi - \psi||_{k,p}}.$$

Also for bounded values of t there exists $0 < r < 1$ such that

$$\kappa((q+\epsilon)t) < r\kappa(t).$$

In other words,

$$\frac{(q+\epsilon)\epsilon_{k,p}||\phi - \psi||_{k,p}}{1 + (q+\epsilon)\epsilon_{k,p}||\phi - \psi||_{k,p}} < r\frac{\epsilon_{k,p}||\phi - \psi||_{k,p}}{1 + \epsilon_{k,p}||\phi - \psi||_{k,p}}.$$

We proved that $T : D \to D$ acts as a contraction operator on D:

$$\rho(T_\phi, T_\psi) < r\rho(\phi, \psi)$$

The Banach Contraction Principle implies the existence of a fixed point in D. This fixed point is the solution of the equation 10. $\square$

In the proof of Theorem 1 we used the fact that any map, which is flat at the origin can be represented as a sum of two maps, flat on the linear subspaces. In the next lemma we discuss how to split γ into the sum of γ_+ and γ_-.

Lemma 9. *Let $\gamma(x, t) \in C^\infty(B_1 \times B_2)$ be flat at 0, i.e., $\gamma^{(k)}(0, t) = 0$ for all $k \in \mathbb{N}$. Let L_+ and L_- be subspaces of B_1, such that $L_+ \times L_- = B_1$. Suppose that there exists a C^∞ K map on one of the subspaces. Then, there exist maps γ_+ and γ_-, such that $\gamma = \gamma_- + \gamma_+$, and γ_+ is flat on $L_- \times B_2$ and γ_- is flat on $L_+ \times B_2$.*

Proof. Lets write $x = (\xi, \eta) \in B_1$ with $\xi \in L_+$, $\eta \in L_-$. For definiteness, lets assume that there exists C^∞ K map on L_-. Then, we can explicitly construct γ_+ map.

Let $\delta_j \neq 0$ be so small that the following series converges in C^∞ topology.

$$\gamma_+ := \sum_{j=0}^\infty \frac{\partial^j \gamma}{\partial \eta^j}(\xi, 0, t)\big(K(\delta_j^{-1}\eta), ..., K(\delta_j^{-1}\eta)\big) \cdot \frac{\delta_j^j}{j!}$$

Define γ_- as

$$\gamma_- = \gamma - \gamma_+$$

Then, it is easy to check that γ_+ is flat on $L_- \times B_2$ and γ_- is flat on $L_+ \times B_2$. Indeed,

$$\frac{\partial^p \gamma_+}{\partial \xi^p}(0, \eta, t) = \sum_{j=0}^\infty \frac{\partial^j \gamma}{\partial \eta^j}(0, 0, t)\big(K(\delta_j^{-1}\eta), ..., K(\delta_j^{-1}\eta)\big) \cdot \frac{\delta_j^j}{j!}$$

$$= 0\big(K(\delta_j^{-1}\eta), ..., K(\delta_j^{-1}\eta)\big) \cdot \frac{\delta_j^j}{j!} = 0.$$

Also,

$$\frac{\partial^q \gamma_-}{\partial \eta^q}(\xi, 0, t) = \frac{\partial^q \gamma}{\partial \eta^q}(\xi, 0, t) - \sum_{j=0}^\infty \frac{\partial^q}{\partial \eta^q}\Big(\frac{\partial^j \gamma}{\partial \eta^j}(\xi, 0, t)\big(K(\delta_j^{-1}\eta), ..., K(\delta_j^{-1}\eta)\big)\Big) \cdot \frac{\delta_j^j}{j!}$$

Recall that $K^{(j)}(0) = \mathbf{0}$ for $j \neq 1$ and $K'(0) = \mathbf{1}$. Then,

$$\frac{\partial^q \gamma_-}{\partial \eta^q}(\xi, 0, t) = \frac{\partial^q \gamma}{\partial \eta^q}(\xi, 0, t) - \frac{\partial^q \gamma}{\partial \eta^q}(\xi, 0, t) \cdot \mathbf{1} \cdot \delta_k^{-k} \cdot \delta_k^k = 0$$

$\square$

4. Linearization

In this section we shell study smooth dependence of linearization on parameters of the family of germs. We shell also prove that the linearization is a C^∞ map on Banach space, if the non-resonance condition is satisfied.

Now we shell formally define the resonance for linear operators on Banach spaces. Consider the following linear operator L_k, defined on the space of homogeneous polynomial maps of degree k.

$$L_k : P^{(k)}[B_1] \rightarrow P^{(k)}[B_1],$$

$$L_k\phi(x) := \Lambda\phi(x) - \phi(\Lambda x).$$

Definition 10. Suppose that for all $k \in \mathbb{N}$, $k > 1$, L_k is invertible on $P^{(k)}[B_1]$. Then, we say that the operator Λ has no resonances.

In particular, when B_1 is a n-dimensional space and Λ is diagonal, the non-resonance condition is equivalent to the following statement:
For any $\lambda_j \in \sigma(\Lambda)$ and for any combination of $p_1, ..., p_n \in \mathbb{N}$, $p_1 + ... + p_n > 1$

$$\lambda_j - \lambda_1^{p_1}...\lambda_n^{p_n} \neq 0.$$

Now we present C^∞ linearization theorem for a family of germs on Banach spaces. The proof of this theorem easily follows from Theorem 1.

Theorem 11. *Let*

$$F(x,t) = \left\{ \begin{array}{l} \Lambda x + f(x,t) \\ t \end{array} \right.$$

be a $C^\infty(B_1 \times B_2)$ diffeomorphism on the product of Banach spaces B_1 and B_2. Assume that Λ is hyperbolic, $f(x,t) = o_t(x)$, as $||x|| \rightarrow 0$, for small values of $||t||$. Assume that there exist $K_\pm$ maps on the corresponding subspaces $L_\pm \subset B_1$, and Λ has no resonances.

Then, there exists a $C^\infty(B_1 \times B_2)$ diffeomorphisms

$$H(x,t) = \left\{ \begin{array}{l} x + h(x,t) \\ t \end{array} \right. ,$$

with $h(x,t) = o_t(x)$, as $||x|| \rightarrow 0$, such that for small values of $||x||$ and $||t||$

$$(12) \qquad\qquad\qquad H \circ F = LH.$$

Proof. Let $H(x,t) = x + h(x,t)$. Then, the equation 12 can be simplified:

$$(13) \qquad\qquad f(x,t) + h(\Lambda x + f(x,t)) = \Lambda h(x,t)$$

In order to get a flat residue, let us formally differentiate both sides of this equation

$$(14) \qquad\qquad \left[f(x,t) + h(\Lambda x + f(x,t))\right]_{x=0}^{(k)} = \Lambda h^{(k)}(x,t)_{x=0}$$

and solve it for $h^{(k)}(0,t)$. The solution $h^{(k)}(0,t)$ exists, if Λ has no resonances.

Now, with the help of the Borel lemma, we can construct a map $\hat{h}(x,t)$ on the whole space with the same derivatives at 0. Since there exist K map on B_1, the Borel lemma is valid on the space B_1. The proof of the Borel lemma for Banach spaces can be found in [B3] (these ideas were also used in [B1]). Lets write $h(x,t)$ as the sum of $\hat{h}(x,t)$ and a flat map $\phi(x,t)$. Then the equation 13 can be written as

$$\Lambda\phi(x,t) - \phi(\Lambda x + f(x,t)) = f(x,t) + \hat{h}(\Lambda x + f(x,t)) - \Lambda\hat{h}(x,t).$$

The right hand side of this equation is flat at 0. Then, by Lemma 5 we can solve it for ϕ.   □

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