Perturbation Theory in Celestial Mechanics

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Contents

1 Glossary

KAM theory: it provides the persistence of quasi–periodic motions under a small perturbation of an integrable system. KAM theory can be applied under quite general assumptions, i.e. a non–degeneracy of the integrable system and a diophantine condition of the frequency of motion. It yields a constructive algorithm to evaluate the strength of the perturbation ensuring the existence of invariant tori.

Perturbation theory: it provides an approximate solution of the equations of motion of a nearly–integrable system.

Spin–orbit problem: a model composed by a rigid satellite rotating about an internal axis and orbiting around a central point–mass planet; a spin–orbit resonance means that the ratio between the revolutional and rotational periods is rational.

Three–body problem: a system composed by three celestial bodies (e.g. Sun–planet–satellite) assumed as point–masses subject to the mutual gravitational attraction. The restricted three–body problem assumes that the mass of one of the bodies is so small that it can be neglected.

2 Definition

Perturbation theory aims to find an approximate solution of nearly–integrable systems, namely systems which are composed by an integrable part and by a small perturbation. The key point of perturbation theory is the construction of a suitable canonical transformation which removes the perturbation to higher orders. A typical example of a nearly–integrable system is provided by a two–body model perturbed by the gravitational influence of a third body whose mass is much smaller than the mass of the central body. Indeed, the solution of the three–body problem greatly stimulated the development of perturbation theories. The solar system dynamics has always been a testing ground for such theories, whose applications range from the computation of the ephemerides of natural bodies to the development of the trajectories of artificial satellites.

3 Introduction

The two–body problem can be solved by means of Kepler's laws, according to which for negative energies the point–mass planets move on ellipses with the Sun located in one of the two foci. The dynamics becomes extremely complicated when adding the gravitational influence of another body. Indeed Poincaré showed ([12]) that the three–body problem does not admit a sufficient number of prime integrals which allow to integrate the problem. Nevertheless a special attention deserves the so–called *restricted* three–body problem, namely when the mass of one of the three bodies is so small that its influence on the others can be neglected. In this case one can assume that the primaries move on Keplerian ellipses around their common barycenter; if the mass of one of the primaries is much larger than the other (as it is the case in any Sun–planet sample), the motion of the minor body is governed by nearly–integrable equations, where the integrable part represents the interaction with the major body, while the perturbation is due to the influence of the other primary. A typical example is provided by the motion of an asteroid under the gravitational attraction of the Sun and Jupiter. The small body may be taken not to influence the motion of the primaries, which are assumed to move on elliptic trajectories. The dynamics of the asteroid is essentially driven by the Sun and perturbed by Jupiter, since the Jupiter–Sun mass–ratio amounts to about 10^{-3}. The solution of this kind of problem stimulated the work of the scientists, especially in the XVIII and XIX centuries. Indeed, Lagrange, Laplace, Leverrier, Delaunay, Tisserand and Poincaré developed perturbation theories which are at the basis of the study of the dynamics of celestial bodies, from the computation of the ephemerides to the recent advances in flight dynamics. For example, on the basis of perturbation theory Delaunay ([8]) developed a theory of the Moon, providing very refined ephemerides. Celestial Mechanics greatly motivated the advances of perturbation theories as witnessed by the discovery of Neptune: its position was theoretically predicted by John Adams and by Jean Urbain Leverrier on the basis of perturbative computations; following the suggestion provided by the theoretical investigations, Neptune was finally discovered on 23 September 1846 by the astronomer Johann Gottfried Galle.

The aim of perturbation theory is to implement a canonical transformation which allows to find the solution of a nearly–integrable system within a better degree of approximation (see section 4 and references [2], [4], [9], [10], [11], [13], [14]). Let us denote the frequency vector of the system by $\underline{\omega}$ (see [Hamiltonian Normal Forms?], [KAM theory?]), which we assume to belong to $\mathbf{R}^n$, where n is the number of degrees of freedom of the system. Classical perturbation theory can be implemented provided that the frequency vector satisfies a non–resonant relation, which means that there do not exist a vector $\underline{m} \in \mathbf{Z}^n$ such that $\underline{\omega} \cdot \underline{m} \equiv \sum_{j=1}^{n} \omega_j m_j = 0$. In case there exists such commensurability condition, a resonant perturbation theory can be developed as outlined in section 5. In general, the three–body problem (and, more extensively, the N–body problem) is described by a *degenerate* Hamiltonian system, which means that the integrable part (i.e., the Keplerian approximation) depends on a subset of the action variables. In this case a degenerate perturbation theory must be implemented as explained in section 5.3. For all the above perturbation theories (classical, resonant and degenerate) an application to Celestial Mechanics is given: the precession of the perihelion of Mercury, orbital resonances within a three–body framework, the precession of the equinoxes.
Even if the non–resonance condition is satisfied, the quantity $\underline{\omega} \cdot \underline{m}$ can become arbitrarily small, giving rise to the so–called *small divisor* problem; indeed, these terms appear at the denominator of the series defining the canonical transformations necessary to implement perturbation theory and therefore they might prevent the convergence of the series. In order to overcome the small divisor problem, a breakthrough came with the work of Kolmogorov ([31]), later extended to different mathematical settings by Arnold ([16]) and Moser ([37]). The overall theory is known with the acronym of KAM theory. As far as concrete estimates on the allowed size of the perturbation are concerned, the original versions of the theory gave discouraging results, which were extremely far from the physical measurements of the parameters involved in the proof. Nevertheless the implementation of computer–assisted KAM proofs allowed to obtain results which are in good agreement with the reality. Concrete estimates with applications to Celestial

Mechanics are reported in section 6.
In the framework of nearly–integrable systems a very important role is provided by periodic orbits, which might be used to approximate the dynamics of quasi–periodic trajectories; for example, a truncation of the continued fraction expansion of an irrational frequency provides a sequence of rational numbers, which are associated to periodic orbits eventually approximating a quasi–periodic torus. A classical computation of periodic orbits using a perturbative approach is provided in section 7, where an application to the determination of the libration in longitude of the Moon is reported.

4 Classical perturbation theory

4.1 The classical theory

Consider a nearly–integrable Hamiltonian function of the form

$$H(\underline{I},\underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I},\underline{\varphi}) \ , \tag{1}$$

where h and f are analytic functions of $\underline{I} \in V$ (V open set of $\mathbf{R}^n$) and $\underline{\varphi} \in \mathbf{T}^n$ ($\mathbf{T}^n$ is the standard n–dimensional torus), while $\varepsilon > 0$ is a small parameter which measures the strength of the perturbation. The aim of perturbation theory is to construct a canonical transformation, which allows to remove the perturbation to higher orders in the perturbing parameter. To this end, let us look for a canonical change of variables (i.e., with symplectic Jacobian matrix) $\mathcal{C} : (\underline{I},\underline{\varphi}) \to (\underline{I}',\underline{\varphi}')$, such that the Hamiltonian (1) takes the form

$$H'(\underline{I}',\underline{\varphi}') \ = \ H \circ \mathcal{C}(\underline{I},\underline{\varphi}) \ \equiv \ h'(\underline{I}') + \varepsilon^2 f'(\underline{I}',\underline{\varphi}') \ , \tag{2}$$

where h' and f' denote the new unperturbed Hamiltonian and the new perturbing function. To achieve such result we need to proceed along the following steps: build a suitable canonical transformation close to the identity, perform a Taylor series expansion in the perturbing parameter, require that the unknown transformation removes the dependence on the angle variables up to second order terms, expand in Fourier series in order to get an explicit form of the canonical transformation.

The change of variables is defined by the equations

$$\begin{aligned} \underline{I} \ &= \ \underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}',\underline{\varphi})}{\partial \underline{\varphi}} \\ \underline{\varphi}' \ &= \ \underline{\varphi} + \varepsilon \frac{\partial \Phi(\underline{I}',\underline{\varphi})}{\partial \underline{I}'} \ , \end{aligned} \tag{3}$$

where $\Phi(\underline{I}',\underline{\varphi})$ is an unknown generating function, which is determined so that (1) takes the form (2). Decompose the perturbing function as

$$f(\underline{I},\underline{\varphi}) = f_0(\underline{I}) + \tilde{f}(\underline{I},\underline{\varphi}) \ ,$$

where f_0 is the average over the angle variables and $\tilde{f}$ is the remainder function defined through $\tilde{f}(\underline{I},\underline{\varphi}) \equiv f(\underline{I},\underline{\varphi}) - f_0(\underline{I})$. Define the *frequency vector* $\underline{\omega} = \underline{\omega}(\underline{I})$ as

$$\underline{\omega}(\underline{I}) \equiv \frac{\partial h(\underline{I})}{\partial \underline{I}} \ .$$

Inserting the transformation (3) in (1) and expanding in Taylor series around $\varepsilon = 0$ up to the second order, one gets

$$\begin{aligned} & h(\underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}) + \varepsilon f(\underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}, \varphi) \\ = \ & h(\underline{I}') + \underline{\omega}(\underline{I}') \cdot \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \varepsilon f_0(\underline{I}') + \varepsilon \tilde{f}(\underline{I}', \underline{\varphi}) + O(\varepsilon^2) \ . \end{aligned}$$

The new Hamiltonian is integrable up to $O(\varepsilon^2)$ provided that the function Φ satisfies:

$$\underline{\omega}(\underline{I}') \cdot \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \tilde{f}(\underline{I}', \underline{\varphi}) = \underline{0} \ . \tag{4}$$

In such case the new integrable part becomes

$$h'(\underline{I}') \ = \ h(\underline{I}') + \varepsilon f_0(\underline{I}') \ ,$$

which provides a better integrable approximation with respect to (1). The solution of (4) yields the explicit expression of the generating function. In fact, let us expand Φ and $\tilde{f}$ in Fourier series as

$$\begin{aligned} \Phi(\underline{I}', \underline{\varphi}) \ &= \sum_{\underline{m} \in \mathbf{Z}^n \setminus \{\underline{0}\}} \hat{\Phi}_{\underline{m}}(\underline{I}') \ e^{i \underline{m} \cdot \underline{\varphi}} \ , \\ \tilde{f}(\underline{I}', \underline{\varphi}) \ &= \sum_{\underline{m} \in \mathcal{I}} \hat{f}_{\underline{m}}(\underline{I}') \ e^{i \underline{m} \cdot \underline{\varphi}} \ , \end{aligned} \tag{5}$$

where $\mathcal{I}$ denotes the set of integer vectors corresponding to the non–vanishing Fourier coefficients of $\tilde{f}$. Inserting the above expansions in (4) one obtains

$$i \sum_{\underline{m} \in \mathbf{Z}^n \setminus \{\underline{0}\}} \underline{\omega}(\underline{I}') \cdot \underline{m} \ \hat{\Phi}_{\underline{m}}(\underline{I}') \ e^{i \underline{m} \cdot \underline{\varphi}} = - \sum_{\underline{m} \in \mathcal{I}} \hat{f}_{\underline{m}}(\underline{I}') \ e^{i \underline{m} \cdot \underline{\varphi}} \ ,$$

which provides

$$\hat{\Phi}_{\underline{m}}(\underline{I}') = - \frac{\hat{f}_{\underline{m}}(\underline{I}')}{i \ \underline{\omega}(\underline{I}') \cdot \underline{m}} \ .$$

Casting together the above formulae, the generating function is given by

$$\Phi(\underline{I}', \underline{\varphi}) = i \sum_{\underline{m} \in \mathcal{I}} \frac{\hat{f}_{\underline{m}}(\underline{I}')}{\underline{\omega}(\underline{I}') \cdot \underline{m}} \ e^{i \underline{m} \cdot \underline{\varphi}} \ . \tag{6}$$

We stress that this algorithm is constructive in the sense that it provides an explicit expression for the generating function and for the transformed Hamiltonian. We remark that (6) is well defined unless there exists an integer vector $\underline{m} \in \mathcal{I}$ such that

$$\underline{\omega}(\underline{I}') \cdot \underline{m} = 0 \ .$$

On the contrary, if $\underline{\omega}$ is rationally independent, there are no zero divisors in (6), though these terms can become arbitrarily small with a proper choice of the vector $\underline{m}$. This problem is known as the *small divisor problem*, which can prevent the implementation of perturbation theory (see [Hamiltonian Normal Forms?], [KAM theory]).

4.2 The precession of the perihelion of Mercury

As an example of the implementation of classical perturbation theory we consider the computation of the precession of the perihelion in a (restricted, planar, circular) three–body model, taking as sample the planet Mercury. The computation requires the introduction of Delaunay action–angle variables, the definition of the three–body Hamiltonian, the expansion of the perturbing function and the implementation of classical perturbation theory (see [7], [15]).

4.2.1 Delaunay action–angle variables

We consider two bodies, say P_0 and P_1 with masses, respectively, m_0, m_1; let $M \equiv m_0 + m_1$ and let $\mu > 0$ be a positive parameter. Let r be the orbital radius and φ be the longitude of P_1 with respect to P_0; let (I_r, I_φ) be the momenta conjugated to (r, φ). In these coordinates the two–body problem Hamiltonian takes the form

$$H_{2b}(I_r, I_\varphi, r, \varphi) = \frac{1}{2\mu}\ (I_r^2 + \frac{I_\varphi^2}{r^2}) - \frac{\mu M}{r}\ . \tag{7}$$

On the orbital plane we introduce the planar Delaunay action–angle variables $(\Lambda, \Gamma, \lambda, \gamma)$ as follows ([21]). Let E denote the total mechanical energy; then:

$$I_r = \sqrt{2\mu E + \frac{2\mu^2 M}{r} - \frac{I_\varphi^2}{r^2}}\ .$$

Since (7) does not depend on φ, setting $\Gamma = I_\varphi$ and $\Lambda = \sqrt{-\frac{\mu^3 M^2}{2E}}$, we introduce a generating function of the form

$$F(\Lambda, \Gamma, r, \varphi) = \int \sqrt{-\frac{\mu^4 M^2}{\Lambda^2} + \frac{2\mu^2 M}{r} - \frac{\Gamma^2}{r^2}}\ dr + \Gamma\varphi\ .$$

From the definition of Λ the new Hamiltonian H_{2D} becomes

$$H_{2D}(\Lambda, \Gamma, \lambda, \gamma) = -\frac{\mu^3 M^2}{2\Lambda^2}\ ,$$

where (Λ, Γ) are the Delaunay action variables; by Kepler's laws one finds that (Λ, Γ) are related to the semimajor axis a and to the eccentricity e of the Keplerian orbit of P_1 around P_0 by the formulae:

$$\Lambda = \mu\sqrt{Ma}\ , \qquad \Gamma = \Lambda\sqrt{1 - e^2}\ .$$

Concerning the conjugated angle variables, we start by introducing the eccentric anomaly u as follows: build the auxiliary circle of the ellipse, draw the line through P_1 perpendicular to the semimajor axis whose intersection with the auxiliary circle forms at the origin an angle u with the semimajor axis. By definition of the generating function, one finds

$$\lambda = \frac{\partial F}{\partial \Lambda} = \int \frac{\mu^4 M^2}{\Lambda^3 \sqrt{-\frac{\mu^4 M^2}{\Lambda^2} + \frac{2\mu^2 M}{r} - \frac{\Gamma^2}{r^2}}}\ dr = u - e \sin u\ ,$$

which defines the mean anomaly λ in terms of the eccentric anomaly u.

In a similar way, if f denotes the true anomaly related to the eccentric anomaly by $\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{u}{2}$, then one has:

$$\gamma = \frac{\partial F}{\partial \Gamma} = \varphi - \int \frac{\Gamma}{r^2\sqrt{-\frac{\mu^4 M^2}{\Lambda^2} + \frac{2\mu^2 M}{r} - \frac{\Gamma^2}{r^2}}}\, dr = \varphi - f \ ,$$

which represents the argument of perihelion of P_1, i.e. the angle between the perihelion line and a fixed reference line.

4.2.2 The restricted, planar, circular, three–body problem

Let P_0, P_1, P_2 be three bodies of masses, respectively, m_0, m_1, m_2. We assume that m_1 is much smaller than m_0 and m_2 (restricted problem) and that the motion of P_2 around P_0 is circular. We also assume that the three bodies always move on the same plane. We choose the free parameter μ as $\mu \equiv \frac{1}{m_0^{2/3}}$, so that the two–body Hamiltonian becomes $H_{2D} = -\frac{1}{2\Lambda^2}$, while we introduce the perturbing parameter as $\varepsilon \equiv \frac{m_2}{m_0^{2/3}}$ ([21]). Set the units of measure so that the distance between P_0 and P_2 is one and so that $m_0 + m_2 = 1$. Taking into account the interaction of P_2 on P_1, the Hamiltonian function governing the three–body problem becomes

$$H_{3b}(\Lambda, \Gamma, \lambda, \gamma, t) \;=\; -\frac{1}{2\Lambda^2} + \varepsilon\Big(r_1\cos(\varphi - t) - \frac{1}{\sqrt{1 + r_1^2 - 2r_1\cos(\varphi - t)}}\Big) \ ,$$

where r_1 is the distance between P_0 and P_1. The first term of the perturbation comes out from the choice of the reference frame, while the second term is due to the interaction with the external body. Since $\varphi - t = f + \gamma - t$, we perform the canonical change of variables

$$\begin{aligned} L &= \Lambda \qquad & \ell &= \lambda \\ G &= \Gamma \qquad & g &= \gamma - t \ , \end{aligned}$$

which provides the following two degrees–of–freedom Hamiltonian

$$H_{3D}(L, G, \ell, g) = -\frac{1}{2L^2} - G + \varepsilon R(L, G, \ell, g) \ , \tag{8}$$

where

$$R(L, G, \ell, g) \equiv r_1\cos(\varphi - t) - \frac{1}{\sqrt{1 + r_1^2 - 2r_1\cos(\varphi - t)}} \tag{9}$$

with r_1 and $\varphi - t$ must be expressed in terms of the Delaunay variables (L, G, ℓ, g). Notice that when $\varepsilon = 0$ one obtains the integrable Hamiltonian function $h(L, G) \equiv -\frac{1}{2L^2} - G$ with associated frequency vector $\underline{\omega} = (\frac{\partial h}{\partial L}, \frac{\partial h}{\partial G}) = (\frac{1}{L^3}, -1)$.

4.2.3 Expansion of the perturbing function

We expand the perturbing function (9) in terms of the Legendre polynomials P_j obtaining

$$R \;=\; -\frac{1}{r_1}\sum_{j=2}^{\infty} P_j(\cos(\varphi - t))\, \frac{1}{r_1^j} \ .$$

The explicit expressions of the first few Legendre polynomials are:

$$
\begin{aligned}
P_2(\cos(\varphi - t)) &= \frac{1}{4} + \frac{3}{4}\cos 2(\varphi - t) \\
P_3(\cos(\varphi - t)) &= \frac{3}{8}\cos(\varphi - t) + \frac{5}{8}\cos 3(\varphi - t) \\
P_4(\cos(\varphi - t)) &= \frac{9}{64} + \frac{5}{16}\cos 2(\varphi - t) + \frac{35}{64}\cos 4(\varphi - t) \\
P_5(\cos(\varphi - t)) &= \frac{15}{64}\cos(\varphi - t) + \frac{35}{128}\cos 3(\varphi - t) + \frac{63}{128}\cos 5(\varphi - t)\ .
\end{aligned}
$$

We invert Kepler's equation $\ell = u - e\sin u$ to the second order in the eccentricity as

$$
u = \ell + e\sin\ell + \frac{e^2}{2}\sin(2\ell) + O(e^3)\ ,
$$

from which one gets

$$
\begin{aligned}
\varphi - t &= g + \ell + 2e\sin\ell + \frac{5}{4}e^2\sin 2\ell + O(e^3) \\
r_1 &= a\left(1 + \frac{1}{2}e^2 - e\cos\ell - \frac{1}{2}e^2\cos 2\ell\right) + O(e^3)\ .
\end{aligned}
$$

Then, up to inessential constants the perturbing function can be expanded as

$$
\begin{aligned}
R &= R_{00}(L,G) + R_{10}(L,G)\cos\ell + R_{11}(L,G)\cos(\ell + g) \\
&+ R_{12}(L,G)\cos(\ell + 2g) + R_{22}(L,G)\cos(2\ell + 2g) \\
&+ R_{32}(L,G)\cos(3\ell + 2g) + R_{33}(L,G)\cos(3\ell + 3g) \\
&+ R_{44}(L,G)\cos(4\ell + 4g) + R_{55}(L,G)\cos(5\ell + 5g) + \ldots\ ,
\end{aligned}
\tag{10}
$$

where the coefficients R_{ij} are given by the following expressions (recall that $e = \sqrt{1 - \frac{G^2}{L^2}}$):

$$
\begin{aligned}
R_{00} &= -\frac{L^4}{4}(1 + \frac{9}{16}L^4 + \frac{3}{2}e^2) + \ldots\ , & R_{10} &= \frac{L^4 e}{2}(1 + \frac{9}{8}L^4) + \ldots \\
R_{11} &= -\frac{3}{8}L^6(1 + \frac{5}{8}L^4) + \ldots\ , & R_{12} &= \frac{L^4 e}{4}(9 + 5L^4) + \ldots \\
R_{22} &= -\frac{L^4}{4}(3 + \frac{5}{4}L^4) + \ldots\ , & R_{32} &= -\frac{3}{4}L^4 e + \ldots \\
R_{33} &= -\frac{5}{8}L^6(1 + \frac{7}{16}L^4) + \ldots\ , & R_{44} &= -\frac{35}{64}L^8 + \ldots \\
R_{55} &= -\frac{63}{128}L^{10} + \ldots
\end{aligned}
\tag{11}
$$

4.2.4 Computation of the precession of the perihelion

We identify the three bodies P_0, P_1, P_2 with the Sun, Mercury and Jupiter. Taking ε as perturbing parameter, we implement a first order perturbation theory, which provides a new integrable Hamiltonian function of the form

$$
h'(L', G') = -\frac{1}{2L'^2}\ -\ G'\ +\ \varepsilon\ R_{00}(L', G')\ .
$$

From Hamilton's equations one obtains

$$\dot{g} = \frac{\partial h'(L',G')}{\partial G'} = -1 + \varepsilon \frac{\partial R_{00}(L',G')}{\partial G'} \ ;$$

neglecting $O(e^3)$ in R_{00} and recalling that $g = \gamma - t$, one has

$$\dot{\gamma} = \varepsilon \ \frac{\partial R_{00}(L',G')}{\partial G'} = \frac{3}{4}\varepsilon L'^2 G' \ .$$

Notice that to the first order in ε one has $L' = L$, $G' = G$. The astronomical data are $m_0 = 2 \cdot 10^{30}$ kg, $m_2 = 1.9 \cdot 10^{27}$ kg, which give $\varepsilon = 9.49 \cdot 10^{-4}$; setting to one the Jupiter–Sun distance one has $a = 0.0744$, while $e = 0.2056$. Taking into account that the orbital period of Jupiter amounts to about 11.86 years, one obtains

$$\dot{\gamma} = 154.65 \ \frac{\text{arcsecond}}{\text{century}} \ ,$$

which represents the contribution due to Jupiter to the precession of perihelion of Mercury. The value found by Leverrier on the basis of the data available in the year 1856 was of 152.59 arcsecond/century ([6]).

5 Resonant perturbation theory

5.1 The resonant theory

Let us consider an Hamiltonian system with n degrees of freedom of the form

$$H(\underline{I}, \underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I}, \underline{\varphi})$$

and let $\omega_j(\underline{I}) = \frac{\partial h(\underline{I})}{\partial I_j}$ $(j = 1, ..., n)$ be the frequencies of the motion, which we assume to satisfy ℓ, $\ell < n$, resonance relations of the form

$$\underline{\omega} \cdot \underline{m}_k = 0 \qquad \text{for } k = 1, ..., \ell \ ,$$

for suitable rational independent integer vectors $\underline{m}_1$, ..., $\underline{m}_\ell$. A resonant perturbation theory can be implemented to eliminate the non–resonant terms. More precisely, the aim is to construct a canonical transformation $\mathcal{C} : (\underline{I}, \underline{\varphi}) \to (\underline{J}', \underline{\vartheta}')$ such that the transformed Hamiltonian takes the form

$$H'(\underline{J}', \underline{\vartheta}') = h'(\underline{J}', \vartheta'_1, ..., \vartheta'_\ell) + \varepsilon^2 f'(\underline{J}', \underline{\vartheta}') \ , \tag{12}$$

where h' depends only on the resonant angles $\vartheta'_1, ..., \vartheta'_\ell$. To this end, let us first introduce the angles $\underline{\vartheta} \in \mathbf{T}^n$ as

$$\begin{aligned} \vartheta_j &= \underline{m}_j \cdot \underline{\varphi} \qquad j = 1, ..., \ell \\ \vartheta_k &= \underline{m}_k \cdot \underline{\varphi} \qquad k = \ell + 1, ..., n \ , \end{aligned}$$

where the first ℓ angle variables are the resonant angles, while the latter $n - \ell$ angle variables are defined as suitable linear combinations so to make the transformation canonical together with the following change of coordinates on the actions $\underline{J} \in \mathbf{R}^n$:

$$\begin{aligned} I_j &= \underline{m}_j \cdot \underline{J} \qquad j = 1, ..., \ell \\ I_k &= \underline{m}_k \cdot \underline{J} \qquad k = \ell + 1, ..., n \ . \end{aligned}$$

The aim is to construct a canonical transformation which removes (to higher order) the dependence on the short–period angles $(\vartheta_{\ell+1}, ..., \vartheta_n)$, while the lowest order Hamiltonian will necessarily depend upon the resonant angles. Let us decompose the perturbation as

$$f(\underline{J}, \underline{\vartheta}) = \overline{f}(\underline{J}) + f_r(\underline{J}, \vartheta_1, ..., \vartheta_\ell) + f_n(\underline{J}, \underline{\vartheta}) \ , \tag{13}$$

where $\overline{f}$ is the average of the perturbation over the angles, f_r is the part depending on the resonant angles and f_n is the non–resonant part. In analogy to the classical perturbation theory, we implement a canonical transformation of the form

$$\begin{aligned} \underline{J} &= \underline{J}' + \varepsilon \frac{\partial \Phi}{\partial \underline{\vartheta}}(\underline{J}', \underline{\vartheta}) \\ \underline{\vartheta}' &= \underline{\vartheta} + \varepsilon \frac{\partial \Phi}{\partial \underline{J}'}(\underline{J}', \underline{\vartheta}) \ , \end{aligned}$$

such that the new Hamiltonian takes the form (12). Taking into account (13) and developing up to the second order in the perturbing parameter, one obtains:

$$\begin{aligned} & h(\underline{J}' + \varepsilon \frac{\partial \Phi}{\partial \underline{\vartheta}}) + \varepsilon f(\underline{J}', \underline{\vartheta}) + O(\varepsilon^2) \\ = \ & h(\underline{J}') + \varepsilon \sum_{k=1}^{n} \frac{\partial h}{\partial J_k} \frac{\partial \Phi}{\partial \vartheta_k} + \varepsilon \overline{f}(\underline{J}') + \varepsilon f_r(\underline{J}', \vartheta_1, ..., \vartheta_\ell) + \varepsilon f_n(\underline{J}', \underline{\vartheta}) + O(\varepsilon^2) \ . \end{aligned}$$

Equating same orders of ε one gets that

$$h'(\underline{J}', \vartheta_1', ..., \vartheta_\ell') = h(\underline{J}') + \varepsilon \overline{f}(\underline{J}') + \varepsilon f_r(\underline{J}', \vartheta_1', ..., \vartheta_\ell') \ , \tag{14}$$

provided that

$$\sum_{k=1}^{n} \omega_k' \frac{\partial \Phi}{\partial \vartheta_k} = -f_n(\underline{J}', \underline{\vartheta}) \ , \tag{15}$$

where $\omega_k' = \omega_k'(\underline{J}') \equiv \frac{\partial h(\underline{J}')}{\partial J_k'}$. The solution of (15) gives the generating function, which allows to reduce the Hamiltonian to the required form (12); as a consequence the conjugated action variables, say $J_{\ell+1}'$, ..., J_n', are constants of the motion up to the second order in ε. We conclude by mentioning that using the new frequencies ω_k', the resonant relations take the form $\omega_k' = 0$ for $k = 1, ..., \ell$.

5.2 Three–body resonance

We consider the three–body Hamiltonian (8) with perturbing function (10)–(11) and let $\underline{\omega} \equiv (\omega_\ell, \omega_g)$ be the frequency of motion. We assume that the frequency vector satisfies the resonance relation

$$\omega_\ell + 2\omega_g = 0 \ .$$

According to the theory described in the previous section we perform the canonical change of variables

$$\begin{aligned} \vartheta_1 &= \ell + 2g \qquad & J_1 &= \frac{1}{2} G \\ \vartheta_2 &= 2\ell \qquad & J_2 &= \frac{1}{2} L - \frac{1}{4} G \ . \end{aligned}$$

In the new coordinates the unperturbed Hamiltonian becomes

$$h'(\underline{J}) \equiv -\frac{1}{2(J_1+2J_2)^2} + 2J_1 \ ,$$

with frequency vector $\underline{\omega}' \equiv \frac{\partial h'(\underline{J})}{\partial \underline{J}}$, while the perturbation takes the form

$$\begin{aligned} R(J_1,J_2,\vartheta_1,\vartheta_2) &\equiv R_{00}(\underline{J}) + R_{10}(\underline{J})\cos(\frac{1}{2}\vartheta_2) + R_{11}(\underline{J})\cos(\frac{1}{2}\vartheta_1+\frac{1}{4}\vartheta_2) \\ &+ R_{12}(\underline{J})\cos(\vartheta_1) + R_{22}(\underline{J})\cos(\vartheta_1+\frac{1}{2}\vartheta_2) \\ &+ R_{32}(\underline{J})\cos(\vartheta_1+\vartheta_2) + R_{33}(\underline{J})\cos(\frac{3}{2}\vartheta_1+\frac{3}{4}\vartheta_2) \\ &+ R_{44}(\underline{J})\cos(2\vartheta_1+\vartheta_2) + R_{55}(\underline{J})\cos(\frac{5}{2}\vartheta_1+\frac{5}{4}\vartheta_2) + ... \end{aligned}$$

with the coefficients R_{ij} as in (11). Let us decompose the perturbation as $R = \overline{R}(\underline{J}) + R_r(\underline{J},\vartheta_1) + R_n(\underline{J},\underline{\vartheta})$, where $\overline{R}(\underline{J})$ is the average over the angles, $R_r(\underline{J},\vartheta_1) = R_{12}(\underline{J})\cos(\vartheta_1)$ is the resonant part, while R_n contains all the remaining non–resonant terms. We look for a canonical transformation close to the identity with generating function $\Phi = \Phi(\underline{J}',\underline{\vartheta})$ such that

$$\underline{\omega}'(\underline{J}') \cdot \frac{\partial \Phi(\underline{J}',\underline{\vartheta})}{\partial \underline{\vartheta}} = -R_n(\underline{J}',\underline{\vartheta}) \ ,$$

which is well defined since $\underline{\omega}'$ is non–resonant for the Fourier components appearing in R_n. Finally, according to (14) the new unperturbed Hamiltonian is given by

$$h'(\underline{J}',\vartheta_1') \equiv h(\underline{J}') + \varepsilon R_{00}(\underline{J}') + \varepsilon R_{12}(\underline{J}')\cos\vartheta_1' \ .$$

5.3 Degenerate perturbation theory

A special case of resonant perturbation theory is obtained when considering a degenerate Hamiltonian function with n degrees of freedom of the form

$$H(\underline{I},\underline{\varphi}) = h(I_1,...,I_d) + \varepsilon f(\underline{I},\underline{\varphi}) \ , \qquad d<n \ ; \tag{16}$$

notice that the integrable part depends on a subset of the action variables, being degenerate in I_{d+1}, ...,I_n. In this case we look for a canonical transformation $\mathcal{C} : (\underline{I},\underline{\varphi}) \to (\underline{I}',\underline{\varphi}')$ such that the transformed Hamiltonian becomes

$$H'(\underline{I}',\underline{\varphi}') = h'(\underline{I}') + \varepsilon h_1'(\underline{I},\varphi_{d+1}',...,\varphi_n') + \varepsilon^2 f'(\underline{I}',\underline{\varphi}') \ , \tag{17}$$

where the part $h'+\varepsilon h_1'$ admits d integrals of motion. Let us decompose the perturbing function in (16) as

$$f(\underline{I},\underline{\varphi}) = \overline{f}(\underline{I}) + f_d(\underline{I},\varphi_{d+1},..,\varphi_n) + \tilde{f}(\underline{I},\underline{\varphi}) \ , \tag{18}$$

where $\overline{f}$ is the average over the angle variables, f_d is independent on φ_1, ..., φ_d and $\tilde{f}$ is the remainder. As in the previous sections we want to determine a near–to–identity canonical transformation $\Phi = \Phi(\underline{I}', \underline{\varphi})$ of the form (3), such that in view of (18) the Hamiltonian (16) takes the form (17). One obtains

$$
\begin{aligned}
& h(I_1', ..., I_d') + \varepsilon \sum_{k=1}^{d} \frac{\partial h}{\partial I_k} \frac{\partial \Phi}{\partial \varphi_k} + \varepsilon \overline{f}(I') + \varepsilon f_d(\underline{I}', \varphi_{d+1}, ..., \varphi_n) + \varepsilon \tilde{f}(\underline{I}', \underline{\varphi}) + O(\varepsilon^2) \\
= \ & h'(\underline{I}') + \varepsilon h_1'(\underline{I}', \varphi_{d+1}, ..., \varphi_n) + O(\varepsilon^2) \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
h'(\underline{I}') &= h(I_1', ..., I_d') + \varepsilon \overline{f}(I') \\
h_1'(\underline{I}', \varphi_{d+1}, ..., \varphi_n) &= f_d(\underline{I}', \varphi_{d+1}, ..., \varphi_n) \ ,
\end{aligned}
$$

while Φ is determined solving the equation

$$
\sum_{k=1}^{d} \frac{\partial h}{\partial I_k} \frac{\partial \Phi}{\partial \varphi_k} + \tilde{f}(\underline{I}', \underline{\varphi}) = 0 \ .
$$

Expanding Φ and $\tilde{f}$ in Fourier series as in (5) one obtains that Φ is given by (6) where $\underline{\omega} \cdot \underline{m} = \sum_{k=1}^{d} m_k \omega_k$, being $\omega_k = 0$ for $k = d+1, ..., n$. The generating function is well defined provided that $\underline{\omega} \cdot \underline{m} \neq 0$ for any $\underline{m} \in \mathcal{I}$, which is equivalent to require that

$$
\sum_{k=1}^{d} m_k \omega_k \neq 0 \qquad \text{for } \ \underline{m} \in \mathcal{I} \ .
$$

5.4 The precession of the equinoxes

An example of the application of the degenerate perturbation theory in Celestial Mechanics is provided by the computation of the precession of the equinoxes.
We consider a triaxial rigid body moving in the gravitational field of a primary body. We introduce the following reference frames with common origin in the barycenter of the rigid body: $(O, \underline{i}_1^{(i)}, \underline{i}_2^{(i)}, \underline{i}_3^{(i)})$ is an inertial reference frame, $(O, \underline{i}_1^{(b)}, \underline{i}_2^{(b)}, \underline{i}_3^{(b)})$ is a body frame oriented along the direction of the principal axes of the ellipsoid, $(O, \underline{i}_1^{(s)}, \underline{i}_2^{(s)}, \underline{i}_3^{(s)})$ is the spin reference frame with the vertical axis along the direction of the angular momentum. Let (J, g, ℓ) be the Euler angles formed by the body and spin frames, and let $(K, h, 0)$ be the Euler angles formed by the spin and inertial frames. The angle K is the obliquity (representing the angle between the spin and inertial vertical axes), while J is the non–principal rotation angle (representing the angle between the spin and body vertical axes).
This problem is conveniently described in terms of the following set of action–angle variables introduced by Andoyer in [1] (see also [24]). Let $\underline{M}_0$ be the angular momentum and let $M_0 \equiv |\underline{M}_0|$; the action variables are defined as

$$
\begin{aligned}
G &\equiv \underline{M}_0 \cdot \underline{i}_3^{(s)} = M_0 \\
L &\equiv \underline{M}_0 \cdot \underline{i}_3^{(b)} = G \cos J \\
H &\equiv \underline{M}_0 \cdot \underline{i}_3^{(i)} = G \cos K \ ,
\end{aligned}
$$

while the corresponding angle variables are the quantities (g, ℓ, h) introduced before.

We limit ourselves to consider the gyroscopic case in which $I_1 = I_2 < I_3$ are the principal moments of inertia of the rigid body E around the primary S; let m_E and m_S be their masses and let $|E|$ be the volume of E. We assume that E orbits on a Keplerian ellipse around S with semimajor axis a and eccentricity e, while λ_E and $\underline{r}_E$ denote the longitude and instantaneous orbital radius (due to the assumption of Keplerian motion λ_E and $\underline{r}_E$ are known functions of the time). The Hamiltonian describing the motion of E around S is given by ([23])

$$\mathcal{H}(L,G,H,\ell,g,h,t) = \frac{G^2}{2I_1} + \frac{I_1 - I_3}{2I_1 I_3} L^2 + V(L,G,H,\ell,g,h,t) \ ,$$

where the perturbation is implicitly defined by

$$V \equiv - \int_E \frac{\tilde{G} m_S m_E}{|\underline{r}_E + \underline{x}|} \frac{d\underline{x}}{|E|} \ ,$$

$\tilde{G}$ being the gravitational constant. Setting $r_E = |\underline{r}_E|$ and $x = |\underline{x}|$, we can expand V using the Legendre polynomials as

$$V = - \frac{\tilde{G} m_S m_E}{r_E} \int_E \frac{d\underline{x}}{|E|} \Big[1 - \frac{\underline{x} \cdot \underline{r}_E}{r_E^2} + \frac{1}{2r_E^2} (3 \frac{(\underline{x} \cdot \underline{r}_E)^2}{r_E^2} - x^2) \Big] + O((\frac{x}{\underline{r}_E})^3) \ .$$

We further assume that $J = 0$ (i.e. $G = L$) so that E rotates around a principal axis. Let G_0 and H_0 be the initial values of G and H; if α denotes the angle bewteen $\underline{r}_E$ and $\underline{k}$, the perturbing function can be written as

$$V = \frac{3}{2} \eta \omega \frac{G_0^2}{H_0} \frac{(1 - e \cos \lambda_E)^3}{(1 - e^2)^3} \cos^2 \alpha$$

with $\eta = \frac{I_3 - I_1}{I_3}$ and $\omega = \frac{\tilde{G} m_S}{a^3} \ I_3 \ \frac{H_0}{G_0^2}$. Elementary computations show that

$$\cos \alpha = \sin(\lambda_E - h) \ \sqrt{1 - \frac{H^2}{G^2}} \ .$$

Neglecting first order terms in the eccentricity, we approximate $\frac{(1 - e \cos \lambda_E)^3}{(1-e^2)^3}$ with one. A first order degenerate perturbation theory provides that the new unperturbed Hamiltonian is given by

$$\mathcal{K}(G,H) = \frac{G^2}{2I_3} + \frac{3}{2} \eta \omega \ \frac{G_0^2}{H_0} \frac{G^2 - H^2}{2G^2} \ .$$

Therefore the average angular velocity of precession is given by

$$\dot{h} = \frac{\partial \mathcal{K}(G,H)}{\partial H} = - \frac{3}{2} \eta \omega \ \frac{G_0^2}{H_0} \ \frac{H}{G^2} \ .$$

At $t = 0$ it is

$$\dot{h} = - \frac{3}{2} \eta \omega = - \frac{3}{2} \eta \omega_y^2 \omega_d^{-1} \cos K \ , \qquad (19)$$

where we used $\omega = \omega_y^2 \omega_d^{-1} \cos K$ with ω_y being the frequency of revolution and ω_d the frequency of rotation.

In the case of the Earth, the astronomical measurements show that $\eta = \frac{1}{298.25}$, $K = 23.45^o$. The contribution due to the Sun is thus obtained inserting $\omega_y = 1$ year, $\omega_d = 1$ day in (19), which yields $\dot{h}^{(S)} = -2.51857 \cdot 10^{-12}$ rad/sec, corresponding to a retrograde precessional period of 79107.9 years. A similar computation shows that the contribution of the Moon amounts to $\dot{h}^{(L)} = -5.49028 \cdot 10^{-12}$ rad/sec, corresponding to a precessional period of 36289.3 years. The total amount is obtained as the sum of $\dot{h}^{(S)}$ and $\dot{h}^{(L)}$, providing an overall retrograde precessional period of 24877.3 years.

6 Invariant tori

6.1 Invariant KAM surfaces

We consider an n–dimensional nearly–integrable Hamiltonian function

$$H(\underline{I}, \underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I}, \underline{\varphi}) \ ,$$

defined in a $2n$–dimensional phase space $\mathcal{M} \equiv V \times \mathbf{T}^n$, where V is an open bounded region of $\mathbf{R}^n$. A *KAM torus* associated to H is an n–dimensional invariant surface on which the flow is described parametrically by a coordinate $\underline{\theta} \in \mathbf{T}^n$ such that the *conjugated flow* is linear, namely $\underline{\theta} \in \mathbf{T}^n \to \underline{\theta} + \underline{\omega} t$ where $\underline{\omega} \in \mathbf{R}^n$ is a Diophantine vector, i.e. there exist $\gamma > 0$ and $\tau > 0$ such that

$$|\underline{\omega} \cdot \underline{m}| \geq \frac{\gamma}{|\underline{m}|^\tau} \ , \qquad \forall \ \underline{m} \in \mathbf{Z}^n \backslash \{0\} \ .$$

Kolmogorov's theorem ([31], see also [KAM theory]) ensures the persistence of invariant tori with diophantine frequency, provided ε is sufficiently small and provided the unperturbed Hamiltonian is non–degenerate, i.e. for a given torus $\{\underline{I}_0\} \times \mathbf{T}^n \subset \mathcal{M}$

$$\det h''(\underline{I}_0) \equiv \det \left(\frac{\partial^2 h}{\partial I_i \partial I_j}(\underline{I}_0) \right)_{i,j=1,\dots,n} \neq 0 \ . \tag{20}$$

The condition (20) can be replaced by the isoenergetic non–degeneracy condition introduced by Arnold ([16])

$$\det \begin{pmatrix} h''(\underline{I}_0) & h'(\underline{I}_0) \\ h'(\underline{I}_0) & \underline{0} \end{pmatrix} \neq 0 \ , \tag{21}$$

which ensures the existence of KAM tori on the energy level corresponding to the unperturbed energy $h(\underline{I}_0)$, say $\mathcal{M}_0 \equiv \{(\underline{I}, \underline{\varphi}) \in \mathcal{M} : \ H(\underline{I}, \underline{\varphi}) = h(\underline{I}_0)\}$. In the context of the n–body problem Arnold ([16]) addressed the question of the existence of a set of initial conditions with positive measure such that, if the initial position and velocities of the bodies belong to this set, then the mutual distances remain perpetually bounded. A positive answer is provided by Kolmogorov's theorem in the framework of the planar, circular, restricted three–body problem, since the integrable part of the Hamiltonian (8) satisfies the isoenergetic non–degeneracy condition (21); denoting by (L_0, G_0) the initial values of the Delaunay's action variables, if ε is sufficiently small, there exist KAM tori for (8) on the energy level $\mathcal{M}_0 \equiv \{H_{3D} = -\frac{1}{2L_0^2} - G_0\}$. In particular, the motion of the perturbed body remains forever bounded from the orbits of the primaries. Indeed, a stronger statement is also valid: due to the fact that the two–dimensional KAM surfaces separate the three dimensional energy levels, any trajectory starting between two KAM tori remains forever trapped in the region between such tori.

In the framework of the three–body problem, Arnold ([16]) stated the following result: "*If the masses, eccentricities and inclinations of the planets are sufficiently small, then for the majority of initial conditions the true motion is conditionally periodic and differs little from Lagrangian motion with suitable initial conditions throughout an infinite interval of time* $-\infty < t < \infty$". Arnold provided a complete proof for the case of three coplanar bodies, while the spatial three–body problem was investigated by Laskar and Robutel in [32], [38] using Poincaré variables, the Jacobi's "reduction of the nodes" (see, e.g., [5]) and Birkhoff's normal form ([14], [2], [3]). The full proof of Arnold's theorem was provided in [26], based on Herman's results on the planetary problem; it makes use of Poincaré variables restricted to the symplectic manifold of vertical total angular momentum.

Explicit estimates on the perturbing parameter ensuring the existence of KAM tori were given by M. Hénon ([30]); he showed that direct applications of KAM theory to the three–body problem lead to analytical results which are much smaller than the astronomical observations. For example, the application of Arnold's theorem to the restricted three–body problem is valid provided the mass–ratio of the primaries is less than 10^{-333}. This result can be improved up to 10^{-48} by applying Moser's theorem, but it is still very far from the actual Jupiter–Sun mass–ratio which amounts to about 10^{-3}. In the context of concrete estimates, a big improvement comes from the synergy between KAM theory and computer–assisted proofs, based on the application of *interval arithmetic* which allows to keep rigorously track of the rounding–off and propagation errors introduced by the machine. Computer–assisted KAM estimates were implemented in a number of cases in Celestial Mechanics, like the three–body problem and the spin–orbit model as briefly recalled in the following subsections.

Another interesting example of the interaction between the analytical theory and the computer implementation is provided by the analysis of the stability of the triangular Lagrangian points; in particular, the stability for exponentially long times is obtained using Nekhoroshev theory combined with computer–assisted implementations of Birkhoff normal form (see, e.g., [17], [22], [25], [27], [28], [29], [36], [39]).

6.2 Rotational tori for the spin–orbit problem

We study the motion of a rigid triaxial satellite around a central planet under the following assumptions ([18]):

i) the orbit of the satellite is Keplerian,

ii) the spin–axis is perpendicular to the orbital plane,

iii) the spin–axis coincides with the smallest physical axis,

iv) external perturbations as well as dissipative forces are neglected.

Let $I_1 < I_2 < I_3$ be the principal moments of inertia; let a, e be the semimajor axis and eccentricity of the Keplerian ellipse; let r and f be the instantaneous orbital radius and the true anomaly of the satellite; let x be the angle between the longest axis of the triaxial satellite and the periapsis line. The equation of motion governing the spin–orbit model is given by:

$$\ddot{x} + \frac{3}{2}\frac{I_2 - I_1}{I_3}(\frac{a}{r})^3 \sin(2x - 2f) = 0 \ . \tag{22}$$

Due to assumption *i*), the quantities r and f are known functions of the time. Expanding the second term of (22) in Fourier–Taylor series and neglecting terms of order 6 in the eccentricity, setting $y \equiv \dot{x}$ one obtains that the equation of motion corresponds to Hamilton's equations associated to the Hamiltonian

$$\begin{aligned} H(y,x,t) &\equiv \frac{y^2}{2} - \varepsilon \, [(-\frac{e}{4} + \frac{e^3}{32} - \frac{5}{768}e^5) \; \cos(2x-t) \; + \\ &+ \; (\frac{1}{2} - \frac{5}{4}e^2 + \frac{13}{32}e^4) \; \cos(2x-2t) \; + \; (\frac{7}{4}e - \frac{123}{32}e^3 + \frac{489}{256}e^5) \; \cos(2x-3t) \; + \\ &+ \; (\frac{17}{4}e^2 - \frac{115}{12}e^4) \; \cos(2x-4t) \; + \; (\frac{845}{96}e^3 - \frac{32525}{1536}e^5) \; \cos(2x-5t) \; + \\ &+ \; \frac{533}{32}e^4 \cos(2x-6t) \; + \; \frac{228347}{7680}e^5 \; \cos(2x-7t) \;] \; , \end{aligned} \tag{23}$$

where $\varepsilon \equiv \frac{3}{2}\frac{I_2-I_1}{I_3}$ and we have chosen the units so that $a = 1$, $\frac{2\pi}{T_{rev}} = 1$, where T_{rev} is the period of revolution. Let p, q be integers with $q \neq 0$; a $p:q$ resonance occurs whenever $\langle \dot{x} \rangle = \frac{p}{q}$, meaning that during q orbital revolutions, the satellite makes on average p rotations. Being the phase–space three–dimensional, the two–dimensional KAM tori separate the phase–space into invariant regions, thus providing the stability of the trapped orbits. In particular, let $\mathcal{P}(\frac{p}{q})$ be the periodic orbit associated to the $p:q$ resonance; its stability is guaranteed by the existence of trapping rotational tori with frequencies $\mathcal{T}(\omega_1)$ and $\mathcal{T}(\omega_2)$ with $\omega_1 < \frac{p}{q} < \omega_2$. For example, one can consider the sequences of irrational rotation numbers

$$\Gamma_k^{(p/q)} \equiv \frac{p}{q} - \frac{1}{k+\alpha}, \qquad \Delta_k^{(p/q)} \equiv \frac{p}{q} + \frac{1}{k+\alpha}, \quad k \in \mathbf{Z}, \; k \geq 2$$

with $\alpha \equiv \frac{\sqrt{5}-1}{2}$. In fact, the continued fraction expansion of $\frac{1}{k+\alpha}$ is given by $\frac{1}{k+\alpha} = [0, k, 1^\infty]$. Therefore, both $\Gamma_k^{(p/q)}$ and $\Delta_k^{(p/q)}$ are *noble* numbers (i.e. with continued fraction expansion definitely equal to one); by number theory they satisfy the diophantine condition and bound $\frac{p}{q}$ from below and above.

As a concrete sample we consider the synchronous spin–orbit resonance ($p = q = 1$) of the Moon, whose physical values of the parameters are $\varepsilon \equiv 3.45 \cdot 10^{-4}$ and $e = 0.0549$. The stability of the motion is guaranteed by the existence of the surfaces $\mathcal{T}(\Gamma_{40}^{(1)})$ and $\mathcal{T}(\Delta_{40}^{(1)})$, which is obtained implementing a computer–assisted KAM theory for the realistic values of the parameters. The result provides the confinement of the synchronous periodic orbit in a limited region of the phase space.

6.3 Librational tori for the spin–orbit problem

The existence of invariant librational tori around a spin–orbit resonance can be obtained as follows ([19]). Let us consider the 1:1 resonance corresponding to Hamilton's equations associated to (23). First one implements a canonical transformation to center around the synchronous periodic orbit; after expanding in Taylor series, one diagonalizes the quadratic terms, thus obtaining a harmonic oscillator plus higher degree (time–dependent) terms. Finally, it is convenient to transform the Hamiltonian using the action–angle variables (I, φ) of the harmonic oscillator. After these symplectic changes of variables one is led to a Hamiltonian of the form

$$H(I,\varphi,t) \; \equiv \; \omega I \; + \; \varepsilon \overline{h}(I) \; + \; \varepsilon R(I,\varphi,t) \; , \qquad I \in \mathbf{R}, \; (\varphi,t) \in T^2 \; ,$$

where $\omega \equiv \omega(\varepsilon)$ is the frequency of the harmonic oscillator, while $\overline{h}(I)$ and $R(I,\varphi,t)$ are suitable functions, precisely polynomials in the action (or the in the square of the action). Then apply a Birkhoff normal form (see [Hamiltonian Normal Forms?], [Normal Forms?]) up to the order k ($k = 5$ in [19]) to obtain the following Hamiltonian:

$$H_k(I', \varphi', t) \;=\; \omega I' \;+\; \varepsilon\, h_k(I'; \varepsilon) \;+\; \varepsilon^{k+1}\, R_k(I', \varphi', t)\ .$$

Finally, implementing a computer–assisted KAM theorem one gets the following result: consider the Moon–Earth case with $\varepsilon_{obs} = 3.45 \cdot 10^{-4}$ and $e = 0.0549$; there exists an invariant torus around the synchronous resonance corresponding to a libration of $8^o.79$ for any $\varepsilon \leq \varepsilon_{obs}/5.26$. The same strategy applied to different samples, e.g. the Rhea–Saturn pair, allows to prove the existence of librational invariant tori around the synchronous resonance for values of the parameters in full agreement with the observational measurements ([19]).

6.4 Rotational tori for the restricted three–body problem

The planar, circular, restricted three–body problem has been considered in [21], where the stability of the asteroid 12 Victoria has been investigated under the gravitational influence of the Sun and Jupiter. On a fixed energy level invariant KAM tori trapping the motion of Victoria have been established for the astronomical value of the Jupiter–Sun mass–ratio (about 10^{-3}). After an expansion of the perturbing function and a truncation to a suitable order (see [21]), the Hamiltonian function describing the motion of the asteroid is given in Delaunay's variables by

$$H(L, G, \ell, g) \equiv -\frac{1}{2L^2} - G - \varepsilon f(L, G, \ell, g)\ ,$$

where setting $a \equiv L^2$, $e = \sqrt{1 - \frac{G^2}{L^2}}$, the perturbation is given by

$$\begin{aligned} f(L,G,\ell,g) = 1 \;\; &+\;\; \frac{a^2}{4} + \frac{9}{64}\,a^4 + \frac{3}{8}\,a^2e^2 - \Big(\frac{1}{2} + \frac{9}{16}a^2\Big)\,a^2 e\ \cos\ell \\ &+\;\; \Big(\frac{3}{8}\,a^3 + \frac{15}{64}a^5\Big)\ \cos(\ell+g) - \Big(\frac{9}{4} + \frac{5}{4}a^2\Big)\,a^2 e\ \cos(\ell + 2g) \\ &+\;\; \Big(\frac{3}{4}\,a^2 + \frac{5}{16}\,a^4\Big)\ \cos(2\,\ell + 2\,g) + \frac{3}{4}\,a^2 e\ \cos(3\,\ell + 2\,g) \\ &+\;\; \Big(\frac{5}{8}\,a^3 + \frac{35}{128}\,a^5\Big)\ \cos(3\,\ell + 3\,g) + \frac{35}{64}\,a^4\ \cos(4\,\ell + 4\,g) \\ &+\;\; \frac{63}{128}a^5 \cos(5\ell + 5g)\ . \end{aligned}$$

For the asteroid Victoria the orbital elements are $a_V \simeq 2.334$ AU, $e_V \simeq 0.220$, which give the observed values of the Delaunay's action variables as $L_V = 0.670$, $G_V = 0.654$. The energy level is taken as

$$E_{\rm V}^{(0)} \equiv -\frac{1}{2L_{\rm V}^2} - G_{\rm V} \simeq -1.768\ , \quad E_{\rm V}^{(1)} \equiv -\langle f(L_V, G_V, \ell, g)\rangle \simeq -1.060\ , \quad E_{\rm V}(\varepsilon) \equiv E_{\rm V}^{(0)} + \varepsilon E_{\rm V}^{(1)}.$$

The osculating energy level of the Sun–Jupiter–Victoria model is defined as

$$E_{\rm V}^* \equiv E_{\rm V}(\varepsilon_J) = E_{\rm V}^{(0)} + \varepsilon_J E_{\rm V}^{(1)} \simeq -1.769\ .$$

We now look for two invariant tori bounding the observed values of L_V and G_V. To this end, let $\tilde{L}_\pm = L_V \pm 0.001$ and let

$$\underline{\tilde{\omega}}_\pm = \Big(\frac{1}{\tilde{L}_\pm^3}, -1\Big) \equiv (\tilde{\alpha}_\pm, -1)\ .$$

To obtain diophantine frequencies, the continued fraction expansion of $\tilde{\alpha}_\pm$ is modified adding a tail of one's after the order 5; this procedure gives the diophantine numbers $\alpha_\pm$ which define the bounding frequencies as $\underline{\omega}_\pm = (\alpha_\pm, -1)$. By a computer–assisted KAM theorem, the stability of the asteroid Victoria is a consequence of the following result ([21]): for $|\varepsilon| \leq 10^{-3}$ the unperturbed tori can be analytically continued into invariant KAM tori for the perturbed system on the energy level $H^{-1}(E_{\mathrm{V}}(\varepsilon))$, keeping fixed the ratio of the frequencies. Therefore the orbital elements corresponding to the semimajor axis and to the eccentricity of the asteroid Victoria stay forever ε–close to their unperturbed values.

6.5 Planetary problem

The dynamics of the planetary problem composed by the Sun, Jupiter and Saturn is investigated in [33], [34] and [35]. In [33] the secular dynamics of the following model is studied: after the Jacobi's reduction of the nodes, the 4–dimensional Hamiltonian is averaged over the fast angles and its series expansion is considered up to the second order in the masses. This procedure provides a Hamiltonian function with two degrees of freedom, describing the slow motion of the parameters characterizing the Keplerian approximation (i.e., the eccentricities and the arguments of perihelion). Afterwards, action–angle coordinates are introduced and a partial Birkhoff normalization is performed. Finally, a computer–assisted implementation of a KAM theorem yields the existence of two invariant tori bounding the secular motions of Jupiter and Saturn for the observed values of the parameters.
The approach sketched above is extended in [35] so to include the description of the fast variables, like the semi–major axes and the mean longitudes of the planets. Indeed, the preliminary average on the fast angles is now performed without eliminating the terms with degree greater or equal than two with respect to the fast actions. The canonical transformations involving the secular coordinates can be adapted to produce a good initial approximation of an invariant torus for the reduced Hamiltonian of the three–body planetary problem. This is the starting point of the procedure for constructing the Kolmogorov's normal form which is numerically shown to be convergent. In [34] the same result of [35] has been obtained for a fictitious planetary solar system composed by two planets with masses equal to 1/10 of those of Jupiter and Saturn.

7 Periodic orbits

7.1 Construction of periodic orbits

One of the most intriguing conjectures of Poincaré concerns the pivotal role of the periodic orbits in the study of the dynamics; more precisely, he states that given a particular solution of Hamilton's equations one can always find a periodic solution (possibly with very long period) such that the difference between the two solutions is small for an arbitrary long time. The literature on periodic orbits is extremely wide (see, e.g., [3], [7], [10], [14], [15] and references therein); here we present the construction of periodic orbits implementing a perturbative approach (see [20]) as shown by Poincaré in [12]. We describe such method taking as example the spin–orbit Hamiltonian (23) that we write in a compact form as $H(y, x, t) \equiv \frac{y^2}{2} - \varepsilon f(x, t)$ for a suitable function $f = f(x, t)$; the corresponding Hamilton's equations are

$$\dot{x} = y$$

$$\dot{y} = \varepsilon f_x(x,t) \ . \tag{24}$$

A spin–orbit resonance of order $p:q$ is a periodic solution of period $T = 2\pi q$ ($q \in \mathbf{Z}\backslash\{0\}$), such that

$$\begin{aligned} x(t+2\pi q) &= x(t) + 2\pi p \\ y(t+2\pi q) &= y(t) \ . \end{aligned} \tag{25}$$

From (24) the solution can be written in integral form as

$$\begin{aligned} y(t) &= y(0) + \varepsilon \int_0^t f_x(x(s),s)\ ds \\ x(t) &= x(0) + y(0)t + \varepsilon \int_0^t \int_0^\tau f_x(x(s),s)\ ds\ d\tau = x(0) + \int_0^t y(s)\ ds \ ; \end{aligned}$$

combining the above equations with (25) one obtains

$$\begin{aligned} \int_0^{2\pi q} f_x(x(s),s) ds &= 0 \\ \int_0^{2\pi q} y(s) ds - 2\pi p &= 0 \ . \end{aligned} \tag{26}$$

Let us write the solution as the series

$$\begin{aligned} x(t) &\equiv \overline{x} + \overline{y}t + \varepsilon x_1(t) + ... \\ y(t) &\equiv \overline{y} + \varepsilon y_1(t) + ... \ , \end{aligned} \tag{27}$$

where $x(0) = \overline{x}$ and $y(0) = \overline{y}$ are the initial conditions, while $x_1(t)$, $y_1(t)$ are the first order terms in ε. Expanding the initial conditions in power series of ε, one gets:

$$\begin{aligned} \overline{x} &= \overline{x}_0 + \varepsilon \overline{x}_1 + \varepsilon^2 \overline{x}_2 + ... \\ \overline{y} &= \overline{y}_0 + \varepsilon \overline{y}_1 + \varepsilon^2 \overline{y}_2 + ... \end{aligned} \tag{28}$$

Inserting (27) and (28) in (24), equating same orders in ε and taking into account the periodicity condition (26), one can find the following explicit expressions for $x_1(t)$, $y_1(t)$, $\overline{y}_0$, $\overline{y}_1$:

$$\begin{aligned} y_1(t) &= y_1(t;\overline{y},\overline{x}) = \int_0^t f_x(\overline{x}_0 + \overline{y}_0 s, s)\ ds \\ x_1(t) &= x_1(t;\overline{y},\overline{x}) = \int_0^t y_1(s)\ ds \\ \overline{y}_0 &= \frac{p}{q} \\ \overline{y}_1 &= -\frac{1}{2\pi q} \int_0^{2\pi q} \int_0^t f_x(\overline{x}_0 + \overline{y}_0 s, s)\ ds\ dt \ . \end{aligned}$$

Furthermore, $\overline{x}_0$ is determined as a solution of

$$\int_0^{2\pi q} f_x(\overline{x}_0 + \overline{y}_0 s, s)\ ds = 0 \ ,$$

while $\overline{x}_1$ is given by

$$\overline{x}_1 = -\frac{1}{\int_0^{2\pi q} f_{xx}^0 dt} \left[\overline{y}_1 \int_0^{2\pi q} t f_{xx}^0\ dt + \int_0^{2\pi q} f_{xx}^0\ x_1(t)\ dt \right] \ ,$$

where, for shortness, we have written $f_{xx}^0 = f_{xx}(\overline{x}_0 + \overline{y}_0 t, t)$.

7.2 The libration in longitude of the Moon

The previous computation of the $p : q$ periodic solution can be used to evaluate the libration in longitude of the Moon. More precisely, setting $p = q = 1$ one obtains

$$\begin{aligned}
\overline{x}_0 &= 0 \\
\overline{y}_0 &= 1 \\
x_1(t) &= 0.232086t - 0.218318\sin(t) - 6.36124 \cdot 10^{-3}\sin(2t) \\
&- 3.21314 \cdot 10^{-4}\sin(3t) - 1.89137 \cdot 10^{-5}\sin(4t) \\
&- 1.18628 \cdot 10^{-6}\sin(5t) \\
y_1(t) &= 0.232086 - 0.218318\cos(t) - 0.0127225\cos(2t) \\
&- 9.63942 \cdot 10^{-4}\cos(3t) - 7.56548 \cdot 10^{-5}\cos(4t) \\
&- 5.93138 \cdot 10^{-6}\cos(5t) \\
\overline{x}_1 &= 0 \\
\overline{y}_1 &= -0.232086 \ ,
\end{aligned}$$

where we used $e = 0.05494$, $\varepsilon = 3.45 \cdot 10^{-4}$. Therefore the synchronous periodic solution, computed up to the first order in ε, is given by

$$\begin{aligned}
x(t) &= \overline{x}_0 + \overline{y}_0 t + \varepsilon x_1(t) = t - 7.53196 \cdot 10^{-5}\sin(t) - 2.19463 \cdot 10^{-6}\sin(2t) \\
&- 1.10853 \cdot 10^{-7}\sin(3t) - 6.52523 \cdot 10^{-9}\sin(4t) \\
&- 4.09265 \cdot 10^{-10}\sin(5t) \\
y(t) &= \overline{y}_0 t + \varepsilon y_1(t) = 1 - 7.53196 \cdot 10^{-5}\cos(t) - 4.38926 \cdot 10^{-6}\cos(2t) \\
&- 3.3256 \cdot 10^{-7}\cos(3t) - 2.61009 \cdot 10^{-8}\cos(4t) \\
&- 2.04633 \cdot 10^{-9}\cos(5t) \ .
\end{aligned}$$

It turns out that the libration in longitude of the Moon, provided by the quantity $x(t) - t$, is of the order of $7 \cdot 10^{-5}$ in agreement with the observational data.

8 Future directions

The end of the XX century has been greatly marked by astronomical discoveries, which changed the shape of the solar system as well as of the entourage of other stars. In particular, the detection of many small bodies beyond the orbit of Neptune has moved forward the edge of the solar system and it has increased the number of its population. Hundreds objects have been observed to move in a ring beyond Neptune, thus forming the so–called Kuiper's belt. Its components show a great variety of behaviors, like resonance clusterings, regular orbits, scattered trajectories. Furthermore, far outside the solar system, the astronomical observations of extrasolar planetary systems have opened new scenarios with a great variety of dynamical behaviors. In these contexts classical and resonant perturbation theories will deeply contribute to provide a fundamental insight of the dynamics and will play a prominent role in explaining the different configurations observed within the Kuiper's belt as well as within extrasolar planetary systems.

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