A FLOQUET OPERATOR WITH PURE POINT SPECTRUM AND ENERGY INSTABILITY

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ABSTRACT. An example of Floquet operator with purely point spectrum and energy instability is presented. In the unperturbed energy eigenbasis its eigenfunctions are exponentially localized.

1. INTRODUCTION

It is not immediate whether a self-adjoint operator H with purely point spectrum implies absence of transport under the time evolution $U(t) = e^{-iHt}$; in fact, it is currently known examples of Schrödinger operators with such kind of spectrum and transport. In case of tight-binding models on $l^2(\mathbb{N})$ the transport is usually probed by the moments of order $m > 0$ of the position operator $Xe_k = ke_k$, that is,

$$X^m = \sum_{k\in\mathbb{N}} k^m \langle e_k, \cdot\rangle e_k, \tag{1}$$

where $e_k(j) = \delta_{kj}$ (Kronecker delta) is the canonical basis of $l^2(\mathbb{N})$. Analogous definition applies for $l^2(\mathbb{Z})$ and even higher dimensional spaces. Then, by definition, *transport* at ψ, also called *dynamical instability* or *dynamical delocalization*, occurs if for some m the function

$$t \mapsto \langle\psi(t), X^m\psi(t)\rangle, \quad \psi(t) := U(t)\psi, \tag{2}$$

is unbounded. If for all $m > 0$ the corresponding functions are bounded, one has *dynamical stability*, also called *dynamical localization*.

The first rigorous example of a Schrödinger operator with purely point spectrum and dynamical instability has appeared in [7], Appendix 2, what the authors have called "A Pathological Example;" in this case the tight binding Schrödinger operator h on $l^2(\mathbb{N})$ with a Dirichlet condition at $n = -1$ was

$$(hu)(n) = u(n+1) + u(n-1) + v(n)u(n)$$

with potential

$$v(n) = 3\cos(\pi\alpha n + \theta) + \lambda\delta_{n0}, \tag{3}$$

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that is, rank one perturbations of an instance of the almost Mathieu operator. An irrational number α was constructed so that for a.e. $\theta \in [0, 2\pi)$ and a.e. $\lambda \in [0, 1]$ the corresponding self-adjoint operator h has purely point spectrum with dynamical instability at e_0 (throughout, the term "a.e." without specification means with respect to the Lebesgue measure under consideration). More precisely, it was shown that for all $\epsilon > 0$

$$\limsup_{t\to\infty} \frac{1}{t^{2-\epsilon}} \langle \psi(t), X^2\psi(t)\rangle = \infty, \quad \psi(0) = e_0.$$

Compare with the absence of ballistic motion for point spectrum Hamiltonians [16]

$$\lim_{t\to\infty} \frac{1}{t^2} \langle \psi(t), X^2\psi(t)\rangle = 0.$$

Additional examples of this behavior are known, even for random potentials, but with a strong local correlations [10], as for the random dimer model in the Schrödinger case; there is also an adaptation [6] for the random Bernoulli Dirac operator with no correlation in the potential, although for the massless case.

The time evolution of a quantum system with time-dependent Hamiltonian is given by a strongly continuous family of unitary operators $U(t, r)$ (the propagator). For an initial condition ψ_0 at $t = 0$, its time evolution is given by $U(t, 0)\psi_0$. If the Hamiltonian is time-periodic with period T, then

$$U(t + T, r + T) = U(t, r), \quad \forall t, r,$$

and we have the Floquet operator $U_F := U(T, 0)$ defined as the evolution generated by the Hamiltonian over a period.

Quantum systems governed by a time periodic Hamiltonian have their dynamical stability often characterized in terms of the spectral properties of the corresponding Floquet operator. As in the autonomous case, the presence of continuous spectrum is a signature of unstable quantum systems; this is a consequence of the famous RAGE theorem, firstly proved for the autonomous case [15] and then for time-periodic Hamiltonians [8]. In principle, a Floquet operator with purely point spectrum would imply "stability," but one should be alerted by the above mentioned "pathological" examples in the autonomous case.

In this work we give an example of a Floquet operator with purely point spectrum and "energy instability," which can be considered the partner concept of dynamical instability in case of autonomous systems. We shall consider a particular choice in the family of Floquet operators studied in [3]; such operators describe the quantum dynamics of certain interesting physical models (see [1, 3] and references therein), and display a band structure with respect to an orthogonal basis $\{\varphi_k\}$ of $l^2(\mathbb{N})$ or $l^2(\mathbb{Z})$, constructed as eigenfunctions of an unperturbed energy operator. There are some conceptual differences with respect to the autonomous case mentioned before, since now the momentum X^m is defined in the energy space

$$X^m = \sum_{k\geq 1} k^m \langle \varphi_k, \cdot \rangle \varphi_k, \tag{4}$$

instead of the "physical space" $\mathbb{N}$. Thus, if for all $m > 0$ the functions

$$n \mapsto \langle \psi(n), X^m \psi(n) \rangle, \quad \psi(n) := U_F^n \psi, \quad n \geq 0, \tag{5}$$

are bounded we say there is *energy stability* or *energy localization*, while if at least one of them is unbounded we say the system presents *energy instability* or *energy delocalization*; the latter reflects a kind of "resonance."

Our construction is a fusion of the Floquet operator studied in [3], now under suitable additional rank one perturbations, and the arguments presented in [7] for model (3). For suitable values of parameters we shall get the following properties:

1. The resulting unitary operator $U_\lambda(\beta, \theta)^+$ (after the rank one perturbation; see Eq. (10)) still belongs to the family of Floquet operators considered in [3].
2. $U_\lambda(\beta, \theta)^+$ has purely point spectrum with exponentially localized eigenfunctions.
3. The time evolution along the Floquet operator $U_\lambda(\beta, \theta)^+$ of the initial condition φ_1 presents energy instability.

$U_\lambda(\beta, \theta)^+$ will be obtained as a rank one perturbation of the almost periodic class of operators studied in the Section 7 of [3] (we describe them ahead). In order to prove purely point spectrum, we borrow an argument from [9] that was used to prove localization for random unitary operators, and it combines spectral averaging and positivity of the Lyapunov exponent with polynomial boundedness of generalized eigenfunctions. In order to get dynamical instability, although we adapt ideas of [7], we underline that some results needed completely different proofs and they are not entirely trivial.

It is worth mentioning that in [19] a form of dynamical stability was obtained for discrete evolution along some Floquet operators (CMV matrices) related to random Verblunsky coefficients.

This paper is organized as follows. In Section 2 we present the model of Floquet operator we shall consider, some preliminary results and the main result is stated in Theorem 2. In Section 3 we shall prove that our Floquet operator is pure point. Section 4 is devoted to the proof of dynamical instability.

2. The Floquet Operator

We briefly recall the construction of the Floquet operator introduced in [3] based on the physical model discussed in [1]. The separable Hilbert space is $l^2(\mathbb{Z})$ and $\{\varphi_k\}_{k\in\mathbb{Z}}$ denote its canonical basis. Consider the set of 2×2

matrices defined for any $k \in \mathbb{Z}$ by

$$S_k = e^{-i\theta_k} \begin{pmatrix} re^{-i\alpha_k} & ite^{i\gamma_k} \\ ite^{-i\gamma_k} & re^{i\alpha_k} \end{pmatrix}$$

parameterized by the phases $\alpha_k,\ \gamma_k,\ \theta_k$ in the torus $\mathbb{T}$ and the real parameters t, r, the reflection and transition coefficients, respectively, linked by $r^2 + t^2 = 1$. Then, let P_j be the orthogonal projection onto the span of $\varphi_j,\ \varphi_{j+1}$ in $l^2(\mathbb{Z})$, and let $U_e,\ U_o$ be two 2×2 block diagonal unitary operators on $l^2(\mathbb{Z})$ defined by

$$U_e = \sum_{k \in \mathbb{Z}} P_{2k} S_{2k} P_{2k} \quad \text{and} \quad U_o = \sum_{k \in \mathbb{Z}} P_{2k+1} S_{2k+1} P_{2k+1}.$$

The matrix representation of U_e in the canonical basis is

$$U_e = \begin{pmatrix} \ddots & & & & \\ & S_{-2} & & & \\ & & S_0 & & \\ & & & S_2 & \\ & & & & \ddots \end{pmatrix},$$

and similarly for U_o, with S_{2k+1} in place of S_{2k}. The Floquet operator U is defined by

$$U = U_o U_e,$$

such that, for any $k \in \mathbb{Z}$,

$$\begin{aligned} U\varphi_{2k} &= irte^{-i(\theta_{2k}+\theta_{2k-1})}e^{-i(\alpha_{2k}-\gamma_{2k-1})}\varphi_{2k-1} \\ &\quad + r^2 e^{-i(\theta_{2k}+\theta_{2k-1})}e^{-i(\alpha_{2k}-\alpha_{2k-1})}\varphi_{2k} \\ &\quad + irte^{-i(\theta_{2k}+\theta_{2k+1})}e^{-i(\gamma_{2k}+\alpha_{2k+1})}\varphi_{2k+1} \\ &\quad - t^2 e^{-i(\theta_{2k}+\theta_{2k+1})}e^{-i(\gamma_{2k}+\gamma_{2k+1})}\varphi_{2k+2} \\ U\varphi_{2k+1} &= -t^2 e^{-i(\theta_{2k}+\theta_{2k-1})}e^{i(\gamma_{2k}+\gamma_{2k-1})}\varphi_{2k-1} \\ &\quad + irte^{-i(\theta_{2k}+\theta_{2k-1})}e^{i(\gamma_{2k}+\alpha_{2k-1})}\varphi_{2k} \\ &\quad + r^2 e^{-i(\theta_{2k}+\theta_{2k+1})}e^{i(\alpha_{2k}-\alpha_{2k+1})}\varphi_{2k+1} \\ &\quad + irte^{-i(\theta_{2k}+\theta_{2k+1})}e^{i(\alpha_{2k}-\gamma_{2k+1})}\varphi_{2k+2} \end{aligned} \tag{6}$$

The extreme cases where $rt = 0$ are spectrally trivial; in case $t = 0$, $r = 1$, U is pure point and if $t = 1$, $r = 0$, U is purely absolutely continuous (Proposition 3.1 in [3]). From now on we suppose $0 < r, t < 1$.

For the eigenvalue equation

$$U\psi = e^{iE}\psi$$

$$\psi = \sum_{k \in \mathbb{Z}} c_k \varphi_k, \qquad c_k, E \in \mathbb{C},$$

one gets the following relation between coefficients

$$\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} = T_k(E) \begin{pmatrix} c_{2k-2} \\ c_{2k-1} \end{pmatrix},$$

where the matrix $T_k(E)$ has elements

$$\begin{aligned}
T_k(E)_{11} &= -e^{-i(E+\gamma_{2k-1}+\gamma_{2k-2}+\theta_{2k-1}+\theta_{2k-2})}, \\
T_k(E)_{12} &= i\frac{r}{t}\left(e^{-i(E+\gamma_{2k-1}-\alpha_{2k-2}+\theta_{2k-1}+\theta_{2k-2})} - e^{-i(\gamma_{2k-1}-\alpha_{2k-1})}\right), \\
T_k(E)_{21} &= i\frac{r}{t}\Big(e^{-i(\theta_{2k-2}-\theta_{2k}+\gamma_{2k}+\gamma_{2k-1}+\gamma_{2k-2}+\alpha_{2k-1})} \\
&\quad -e^{-i(E+\theta_{2k-2}+\theta_{2k-1}+\gamma_{2k}+\gamma_{2k-1}+\gamma_{2k-2}+\alpha_{2k})}\Big), \qquad (7) \\
T_k(E)_{22} &= -\frac{1}{t^2}e^{i(E+\theta_{2k}+\theta_{2k-1}-\gamma_{2k}-\gamma_{2k-1})} \\
&\quad +\frac{r^2}{t^2}e^{-i(\gamma_{2k}+\gamma_{2k-1})}\left(e^{i(\theta_{2k}-\theta_{2k-2}+\alpha_{2k-2}-\alpha_{2k-1})} + e^{-i(\alpha_{2k}-\alpha_{2k-1})}\right) \\
&\quad -\frac{r^2}{t^2}e^{-i(E+\theta_{2k-2}+\theta_{2k-1}+\gamma_{2k}+\gamma_{2k-1}+\alpha_{2k}-\alpha_{2k-2})}
\end{aligned}$$

and

$$\det T_k(E) = e^{-i(\theta_{2k-2}-\theta_{2k}+\gamma_{2k}+2\gamma_{2k-1}+\gamma_{2k-2})}.$$

Given coefficients (c_0, c_1), for any $k \in \mathbb{N}^*$ one has

$$\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} = T_k(E)\ldots T_2(E)T_1(E) \begin{pmatrix} c_0 \\ c_1 \end{pmatrix},$$

$$\begin{pmatrix} c_{-2k} \\ c_{-2k+1} \end{pmatrix} = T_{-k+1}(E)^{-1}\ldots T_{-1}(E)^{-1}T_0(E)^{-1} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}.$$

In the physical setting [1], the natural Hilbert space is $l^2(\mathbb{N}^*)$, with $\mathbb{N}^*$ the set of positive integers, and the definition according with [3] of the Floquet operator, denoted by U^+, is

$$\begin{aligned}
U^+\varphi_1 &= re^{-i(\theta_0+\theta_1)}e^{-i\alpha_1}\varphi_1 + ite^{-i(\theta_0+\theta_1)}e^{-i\gamma_1}\varphi_2, \\
U^+\varphi_k &= U\varphi_k, \ k > 1 \qquad (8)
\end{aligned}$$

with $U\varphi_k$ as in (6). In this case the eigenvalue equation is

$$U^+\psi = e^{iE}\psi$$

with $\psi = \sum_{k=1}^{\infty} c_k\varphi_k$. Then starting from c_2, c_3, we have

$$\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} = T_k(E)\ldots T_2(E) \begin{pmatrix} c_2 \\ c_3 \end{pmatrix}, \qquad k = 2, 3, \ldots$$

where the transfer matrices $T_k(E)$ are given by (7), along with the additional one

$$\begin{pmatrix} c_2 \\ c_3 \end{pmatrix} = c_1 \begin{pmatrix} a_1(E) \\ a_2(E) \end{pmatrix},$$

where

$$\begin{aligned} a_1(E) &= \frac{i}{t}\left(e^{-i(E+\gamma_1+\theta_1+\theta_0)} - re^{-i(\gamma_1-\alpha_1)}\right) \\ a_2(E) &= -\frac{1}{t^2}e^{i(E+\theta_2+\theta_1-\gamma_2-\gamma_1)} \\ &\quad +\frac{r}{t^2}e^{-i(\gamma_2+\gamma_1)}\left(e^{i(\theta_2-\theta_0-\alpha_1)} + re^{-i(\alpha_2-\alpha_1)}\right) \\ &\quad -\frac{r}{t^2}e^{-i(E+\theta_0+\theta_1+\gamma_2+\gamma_1+\alpha_2)} \end{aligned}$$

For further details and generalizations of this class of unitary operators, we refer the reader to [3, 11, 12, 9]. In particular, when the phases are i.i.d. random variables, it was proved to hold in the unitary case typical results obtained for discrete one-dimensional random Schrödinger operators. For example, the availability of a transfer matrix formalism to express generalized eigenvectors allows to introduce a Lyapunov exponent, to prove a unitary version of Oseledec's Theorem and of Ishii-Pastur Theorem (and get absence of absolutely continuous spectrum in some cases).

Our main interest is on the almost periodic example

$$U \equiv U(\{\theta_k\}, \{\alpha_k\}, \{\gamma_k\}),$$

where the phases α_k are taken as constants, $\alpha_k = \alpha \ \ \forall k \in \mathbb{Z}$, while the γ_k's are arbitrary and can be replaced by $(-1)^{k+1}\alpha$ (see Lemma 3.2 in [3]). The almost periodicity due to the phases θ_k defined by $\theta_k = 2\pi\beta k + \theta$, where $\beta \in \mathbb{R}$, and $\theta \in [0, 2\pi)$. We denote U above by $U = U(\beta, \theta)$ and then for any $k \in \mathbb{Z}$ (see (6))

$$\begin{aligned} U(\beta,\theta)\varphi_{2k} &= irte^{-i(2\pi\beta(4k-1)+2\theta)}\varphi_{2k-1} \\ &\quad +r^2e^{-i(2\pi\beta(4k-1)+2\theta)}\varphi_{2k} \\ &\quad +irte^{-i(2\pi\beta(4k+1)+2\theta)}\varphi_{2k+1} \\ &\quad -t^2e^{-i(2\pi\beta(4k+1)+2\theta)}\varphi_{2k+2} \\ U(\beta,\theta)\varphi_{2k+1} &= -t^2e^{-i(2\pi\beta(4k-1)+2\theta)}\varphi_{2k-1} \\ &\quad +itre^{-i(2\pi\beta(4k-1)+2\theta)}\varphi_{2k} \\ &\quad +r^2e^{-i(2\pi\beta(4k+1)+2\theta)}\varphi_{2k+1} \\ &\quad +itre^{-i(2\pi\beta(4k+1)+2\theta)}\varphi_{2k+2} \end{aligned} \tag{9}$$

Let $U(\beta, \theta)^+$ be the corresponding operator on $l^2(\mathbb{N}^*)$ defined by (8). The following result was proved in [3].

Theorem 1. *(i) For β rational and each $\theta \in [0, 2\pi)$, $U(\beta, \theta)$ is purely absolutely continuous, $\sigma_{\rm sc}(U(\beta, \theta)^+) = \emptyset$, $\sigma_{\rm ac}(U(\beta, \theta)^+) = \sigma_{\rm ac}(U(\beta, \theta))$ and the point spectrum of $U(\beta, \theta)^+$ consists of finitely many simple eigenvalues in the resolvent set of $U(\beta, \theta)$.*
(ii) Let $T_k^\theta(E)$ be the transfer matrices at $E \in \mathbb{T}$ corresponding to $U(\beta, \theta)$.

For β irrational, the Lyapunov exponent $\gamma(E)$ satisfies, for almost all θ,

$$\gamma_\theta(E) = \lim_{N\to\infty} \frac{\ln \| \prod_{k=1}^N T_k^\theta(E)\|}{N} \geq \ln \frac{1}{t^2} > 0,$$

and so $\sigma_{ac}(U(\beta,\theta)) = \emptyset$. The same is true for $U(\beta,\theta)^+$.

Finally, we introduce our study model. We consider a rank one perturbation of $U(\beta,\theta)^+$, $\lambda \in [0, 2\pi)$ (see also [4])

$$U_\lambda(\beta,\theta)^+ := U(\beta,\theta)^+ e^{i\lambda P_{\varphi_1}} = U(\beta,\theta)^+ \left(\mathrm{I_d} + (e^{i\lambda} - 1)P_{\varphi_1}\right), \tag{10}$$

where $P_{\varphi_1}(\cdot) = \langle \varphi_1, \cdot\rangle\varphi_1$. We observe that

$$U(\beta,\theta)^+ \equiv U^+ \left(\{\theta_k\}_{k=0}^\infty, \{\alpha_k\}_{k=1}^\infty, \{\gamma_k\}_{k=1}^\infty\right)$$

and $U_\lambda(\beta,\theta)^+ \equiv U^+ \left(\{\tilde{\theta_k}\}_{k=0}^\infty, \{\tilde{\alpha_k}\}_{k=1}^\infty, \{\tilde{\gamma_k}\}_{k=1}^\infty\right)$ where $\tilde{\theta_0} = \theta_0 - \lambda$ and $\tilde{\theta_k} = \theta_k$, $\tilde{\alpha_k} = \alpha_k$, $\tilde{\gamma_k} = \gamma_k$ for $k \geq 1$. Hence, the perturbed operator $U_\lambda(\beta,\theta)^+$ also belongs to the family of Floquet operators studied in [3]. Note also that the Lyapunov exponent is independent on the parameter λ.

We can now state our main result:

Theorem 2. *(i) For β irrational, $U_\lambda(\beta,\theta)^+$ has only point spectrum for a.e. θ, $\lambda \in [0, 2\pi)$, and in the basis $\{\varphi_k\}$ its eigenfunctions decay exponentially.*
(ii) β can be chosen irrational so that

$$\limsup_{n\to\infty} \frac{\|X\left(U_\lambda(\beta,\theta)^+\right)^n \varphi_1\|^2}{F(n)} = \infty,$$

for all $\theta \in [0, 2\pi)$ and any $\lambda \in [\frac{\pi}{6}, \frac{\pi}{2}]$, where $F(n) = \frac{n^2}{\ln(2+n)}$ and X is the moment of order $m = 1$ given by (4).

Remarks. 1. Joining up (i) and (ii) of the theorem above we proved that for some β irrational, for a.e. $\theta \in [0, 2\pi)$ and $\lambda \in [\frac{\pi}{6}, \frac{\pi}{2}]$, $U_\lambda(\beta,\theta)^+$ has pure point spectrum and the function

$$n \mapsto \left\langle \left(U_\lambda(\beta,\theta)^+\right)^n \varphi_1, X^2 \left(U_\lambda(\beta,\theta)^+\right)^n \varphi_1\right\rangle$$

is unbounded. That is, we have pure point spectrum and dynamical instability.
2. One can modify the proof to replace the logarithm function $f(n) = \ln(2+n)$ for any monotone sequence f with $\lim_{n\to\infty} f(n) = \infty$.

3. Pure Point Spectrum

In this section we prove part (i) of Theorem 2. We need a preliminary lemma.

Lemma 1. *For any β and θ, the vector φ_1 is cyclic for $U(\beta,\theta)^+$.*

Proof. Fix β and θ. We indicate that any vector φ_k, $k \in \mathbb{N}^*$ can be written as a linear combination of the vectors $\left(U(\beta,\theta)^+\right)^n \varphi_1$, $n \in \mathbb{Z}$. Since $U(\beta,\theta)^+\varphi_1 = re^{-i(2\pi\beta+2\theta)}e^{-i\alpha}\varphi_1 + ite^{-i(2\pi\beta+2\theta)}e^{-i\alpha}\varphi_2$ then

$$\varphi_2 = -\frac{i}{t}e^{i(2\pi\beta+2\theta)}e^{i\alpha}U(\beta,\theta)^+\varphi_1 + \frac{ir}{t}\varphi_1. \tag{11}$$

Now

$$\left(U(\beta,\theta)^+\right)^{-1}\varphi_1 = a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3, \tag{12}$$

where a_1, a_2 and a_3 are nonzero complex numbers. Thus, using (11) and (12), suitable linear combination of $\left(U(\beta,\theta)^+\right)^{-1}\varphi_1$, φ_1 and $U(\beta,\theta)^+\varphi_1$ yields φ_3. Since $U(\beta,\theta)^+\varphi_2 = b_1\varphi_1 + b_2\varphi_2 + b_3\varphi_3 + b_4\varphi_4$ we obtain that φ_4 can be written as a linear combination desired. Due to the structure of $U(\beta,\theta)^+$, the process can be iterated to obtain any φ_k. □

We are in conditions to prove pure point spectrum for our model.

Proof. **(Theorem 2(i))** Fix β irrational and let $|\cdot|$ denote the Lebesgue measure on $[0,2\pi)$. By Theorem 1*(ii)*, for any $E \in [0,2\pi)$ there exists $\Omega(E) \subset [0,2\pi)$ with $|\Omega(E)| = 1$ such that

$$\gamma_\theta(E) > 0, \qquad \forall\, \theta \in \Omega(E).$$

Thus, by Fubini's Theorem,

$$\begin{aligned} 1 = \int_0^{2\pi} |\Omega(E)|\frac{dE}{2\pi} &= \int_0^{2\pi}\left(\int_0^{2\pi}\chi_{\Omega(E)}(\theta)\frac{d\theta}{2\pi}\right)\frac{dE}{2\pi} \\ &= \int_0^{2\pi}\left(\int_0^{2\pi}\chi_{\Omega(E)}(\theta)\frac{dE}{2\pi}\right)\frac{d\theta}{2\pi} \end{aligned}$$

and for θ in a set of measure one

$$\int_0^{2\pi}\chi_{\Omega(E)}(\theta)\frac{dE}{2\pi} = 1,$$

that is, $\theta \in \Omega(E)$ for almost all $E \in [0,2\pi)$. Then we get the existence of $\Omega_0 \subset [0,2\pi)$ with $|\Omega_0| = 1$ such that for any $\theta \in \Omega_0$ there exists $A_\theta \subset [0,2\pi)$ with $|A_\theta| = 0$ and

$$\gamma_\theta(E) > 0, \qquad \forall\, E \in A_\theta^c := [0,2\pi) \setminus A_\theta.$$

Let $\mu^k_{\theta,\lambda}$ be the spectral measures associated with

$$U_\lambda(\beta,\theta)^+ = \int_0^{2\pi} e^{iE}dF_{\theta,\lambda}(E)$$

and respectively vectors φ_k, so that for $k \in \mathbb{N}^*$ and all Borel sets $\Lambda \subset [0,2\pi)$

$$\mu^k_{\theta,\lambda}(\Lambda) = \langle \varphi_k, F_{\theta,\lambda}(\Lambda)\varphi_k\rangle.$$

Now, for rank one perturbations of unitary operators there is a spectral averaging formula as for rank one perturbations of self-adjoint operators (see [17, 20] for the self-adjoint case and [2, 4] for the unitary case). Thus, for any $f \in L^1([0, 2\pi))$ one has

$$\int_0^{2\pi} d\lambda \int_0^{2\pi} f(E) d\mu^1_{\theta,\lambda}(E) = \int_0^{2\pi} f(E) \frac{dE}{2\pi}. \tag{13}$$

Then, applying (13) with f the characteristic function of A_θ we obtain

$$\begin{aligned} 0 &= |A_\theta| = \int_0^{2\pi} \chi_{A_\theta}(E) \frac{dE}{2\pi} \\ &= \int_0^{2\pi} d\lambda \int_0^{2\pi} \chi_{A_\theta}(E) d\mu^1_{\theta,\lambda}(E) = \int_0^{2\pi} \mu^1_{\theta,\lambda}(A_\theta) d\lambda, \end{aligned}$$

and so $\mu^1_{\theta,\lambda}(A_\theta) = 0$ for almost all λ. Therefore, for each $\theta \in \Omega_0$, there is $J_\theta \subset [0, 2\pi)$ with $|J^c_\theta| = 0$ such that

$$\mu^1_{\theta,\lambda}(A_\theta) = 0, \qquad \forall\, \lambda \in J_\theta. \tag{14}$$

By Lemma 1 and (14), it follows that $F_{\theta,\lambda}(A_\theta) = 0$ for all $\theta \in \Omega_0$ and $\lambda \in J_\theta$. Moreover, let $S_{\theta,\lambda}$ denote the set of $E \in [0, 2\pi)$ so that the equation

$$U_\lambda(\beta, \theta)^+ \psi = e^{iE} \psi$$

has a nontrivial polynomially bounded solution. It is known that

$$F_{\theta,\lambda}\left([0, 2\pi) \setminus S_{\theta,\lambda}\right) = 0$$

(see [3, 9]). Thus we conclude that $S_{\theta,\lambda} \cap A^c_\theta$ is a support for $F_{\theta,\lambda}(\cdot)$ (see remark bellow) for all $\theta \in \Omega_0$ and $\lambda \in J_\theta$.

Now, if $E \in S_{\theta,\lambda} \cap A^c_\theta$ then $U_\lambda(\beta, \theta)^+ \psi = e^{iE} \psi$ has a nontrivial polynomially bounded solution ψ and $\gamma_\theta(E) > 0$. By construction $\gamma_{\theta,\lambda}(E) = \gamma_\theta(E)$ where $\gamma_{\theta,\lambda}(E)$ is the Lyapunov exponent associated with $U_\lambda(\beta, \theta)^+$. Thus, by Oseledec's Theorem, every solution which is polynomially bounded necessarily has to decay exponentially, so ψ is in $l^2(\mathbb{N}^*)$ and is an eigenfunction of $U_\lambda(\beta, \theta)^+$. Hence, we conclude that each $E \in S_{\theta,\lambda} \cap A^c_\theta$ is an eigenvalue of $U_\lambda(\beta, \theta)^+$ with corresponding eigenfunction decaying exponentially. As $l^2(\mathbb{N}^*)$ is separable, it follows that $S_{\theta,\lambda} \cap A^c_\theta$ is countable and then $F_{\theta,\lambda}(\cdot)$ has countable support for all $\theta \in \Omega_0$ and $\lambda \in J_\theta$. Thus $U_\lambda(\beta, \theta)^+$ has purely point spectrum for a.e. $\theta,\ \lambda \in [0, 2\pi)$. □

Remark. We say that a Borel set S supports the spectral projection $F(\cdot)$ if $F([0, 2\pi) \setminus S) = 0$.

4. Energy Instability

In this section we present the proof of Theorem 2*(ii)*. The initial strategy is that of Appendix 2 of [7], and Lemmas 2 and 3 ahead are similar to Lemmas B.1 and B.2 in [7]. However, some important technical issues needed quite different arguments. To begin with we shall discuss a series of preliminary lemmas, adapted to the unitary case from the self-adjoint setting.

4.1. **Preliminary Lemmas.** Let $P_{n\geq a}$ denote the projection onto those vectors supported by $\{n : n \geq a\}$, that is, for $\psi \in l^2(\mathbb{N}^*)$

$$(P_{n\geq a}\psi)(n) = \begin{cases} 0, & \text{if } \ n < a \\ \psi(n), & \text{if } \ n \geq a \end{cases},$$

and similarly for $P_{n<a}$. Let $f(n)$ be a monotone increasing sequence with $f(n) \to \infty$ as $n \to \infty$.

Lemma 2. *If there exists $T_m \to \infty$, $T_m \in \mathbb{N}$ for all m, such that*

$$\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta,\theta)^+\right)^j \varphi_1\|^2 \geq \frac{1}{f(T_m)^2}\ , \tag{15}$$

then

$$\limsup_{j\to\infty} \|X\left(U_\lambda(\beta,\theta)^+\right)^j \varphi_1\|^2 \frac{f(j)^5}{j^2} = \infty.$$

Proof. By hypothesis, for each $m \in \mathbb{N}$, there must be some $j_m \in [T_m, 2T_m]$ such that

$$\|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta,\theta)^+\right)^{j_m} \varphi_1\|^2 \geq \frac{1}{f(T_m)^2}$$

and then

$$\begin{aligned} \|X\left(U_\lambda(\beta,\theta)^+\right)^{j_m} \varphi_1\|^2 &= \sum_{n\in\mathbb{N}^*} n^2 |\left(U_\lambda(\beta,\theta)^+\right)^{j_m} \varphi_1(n)|^2 \\ &\geq \sum_{n\in\mathbb{N}^*} \left|\frac{T_m}{f(T_m)}\left(P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta,\theta)^+\right)^{j_m} \varphi_1\right)(n)\right|^2 \\ &\geq \frac{T_m^2}{f(T_m)^4}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{f(j_m)^5}{j_m^2}\|X\left(U_\lambda(\beta,\theta)^+\right)^{j_m} \varphi_1\|^2 &\geq \left(\frac{T_m}{j_m}\right)^2 \left(\frac{f(j_m)}{f(T_m)}\right)^4 f(j_m) \\ &\geq \frac{1}{4} f(j_m) \to \infty \end{aligned}$$

and the lemma is proved. ∎

In order to prove Theorem 2*(ii)* we want to apply the above lemma with $f(n) = (\ln(n+2))^{1/5}$. By keeping this goal in mind, the estimate in relation (15) is crucial as well as the following lemmas.

Lemma 3. *Let ξ be a unit vector, P a projection, and U a unitary operator. If $\xi = \eta + \psi$ with $\langle \eta, \psi \rangle = 0$, then*

$$\frac{1}{T+1}\sum_{j=T}^{2T} \|(\mathrm{I_d} - P)U^j\xi\|^2 \geq \|\psi\|^2 - 3\left(\frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\psi\|^2\right)^{1/2}. \tag{16}$$

Proof. Denote $D := \frac{1}{T+1}\sum_{j=T}^{2T}\|(\mathrm{I_d} - P)U^j\xi\|^2$. Then

$$\begin{aligned}
D &= \frac{1}{T+1}\sum_{j=T}^{2T}(1 - \|PU^j\xi\|^2) \\
&= \frac{1}{T+1}\sum_{j=T}^{2T}(\|\psi\|^2 + \|\eta\|^2 - \|PU^j(\eta+\psi)\|^2) \\
&= \|\eta\|^2 - \frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\eta\|^2 \\
&\quad + \|\psi\|^2 - \frac{1}{T+1}\sum_{j=T}^{2T}(\|PU^j\psi\|^2 + 2\mathrm{Re}\,(\langle PU^j\eta, PU^j\psi\rangle)) \\
&= A + B,
\end{aligned}$$

with $A = \|\eta\|^2 - \frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\eta\|^2$ and $B = \|\psi\|^2 - \frac{1}{T+1}\sum_{j=T}^{2T}(\|PU^j\psi\|^2 + 2\mathrm{Re}\,(\langle PU^j\eta, PU^j\psi\rangle))$.

Clearly, $\frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\eta\|^2 \le \|\eta\|^2 \le 1$, and the same is true with η replaced by ψ. Hence $A \ge 0$ and

$$\begin{aligned} B &= \|\psi\|^2 - \frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\psi\|^2 - \frac{2}{T+1}\sum_{j=T}^{2T}\mathrm{Re}\,(\langle PU^j\eta, PU^j\psi\rangle) \\ &\ge \|\psi\|^2 - \frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\psi\|^2 - \frac{2}{T+1}\sum_{j=T}^{2T}\|PU^j\eta\|\|PU^j\psi\| \\ &\ge \|\psi\|^2 - \Big(\frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\psi\|^2\Big)^{\frac12} \\ &\quad -2\Big(\frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\eta\|^2\Big)^{\frac12}\Big(\frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\psi\|^2\Big)^{\frac12} \\ &\ge \|\psi\|^2 - 3\Big(\frac{1}{T+1}\sum_{j=T}^{2T}\|PU^j\psi\|^2\Big)^{\frac12}. \end{aligned}$$

The result follows immediately. □

The following lemma is an adaptation to the discrete setup of a classical estimate found in Lemma 4.5, page 543 of [13].

Lemma 4. *Let $U = \int_0^{2\pi} e^{it}dE_U(t)$ be the spectral decomposition of a unitary operator U on the Hilbert space $\mathcal{H}$. Let $\xi \in \mathcal{H}$ be an absolutely continuous vector for U, i.e., the spectral measure μ_ξ, associated to U and ξ, is absolutely continuous with respect to Lebesgue measure, and denote by $g = \frac{d\mu_\xi}{dx} \in L^1([0,2\pi))$ the corresponding Radon-Nikodym derivative. Define*

$$|||\xi|||_U = \|g\|_\infty^{1/2}.$$

Then, for any $\eta \in \mathcal{H}$, one has

$$\sum_{j\in\mathbb{Z}} |\langle U^j\xi, \eta\rangle|^2 \le 2\pi\, |||\xi|||_U^2\, \|\eta\|^2.$$

If it is clear the unitary operator in question, then $|||\cdot|||$ will be used to indicate $|||\cdot|||_U$.

Proof. If $|||\xi||| = \infty$ then the result is clear. Suppose $|||\xi||| < \infty$ and take $\eta \in \mathcal{H}$. Denote by P_{ac} the spectral projection onto the absolutely continuous subspace $\mathcal{H}_{ac}$ of U, $\eta_0 = P_{ac}\eta$ and $\tilde g = \frac{d\mu_{\eta_0}}{d\lambda}$; then $\mu_{\xi,\eta}$ is absolutely continuous and its Radon-Nikodym derivative h is estimate by

$$|h(x)| \le (g\tilde g)^{\frac12}(x) = g^{\frac12}(x)\,\tilde g^{\frac12}(x) \le |||\xi|||\,\tilde g^{\frac12}(x).$$

Hence $h \in L^2([0,2\pi))$ with L^2 norm estimated by

$$\|h\|_2 \le |||\xi|||\Big(\int_0^{2\pi}\tilde g(x)dx\Big)^{\frac12} = |||\xi|||\Big(\int_0^{2\pi} d\mu_{\eta_0}\Big)^{\frac12}$$

$$= |||\xi||| \cdot \|\eta_0\| \leq |||\xi||| \cdot \|\eta\|.$$

Since

$$\langle U^j \xi, \eta \rangle = \int_0^{2\pi} e^{ijt} d\mu_{\xi,\eta}(t) = \int_0^{2\pi} e^{ijt} h(t) dt = \sqrt{2\pi} (\mathcal{F}h)(j),$$

it follows that

$$\sum_{j \in \mathbb{Z}} |\langle U^j \xi, \eta \rangle|^2 = \sum_{j \in \mathbb{Z}} 2\pi |(\mathcal{F}h)(j)|^2 = 2\pi \|h\|_2^2 \leq 2\pi |||\xi|||^2 \|\eta\|^2,$$

which is precisely the stated result. □

4.2. **Cauchy and Borel Transforms.** Given a probability measure μ on $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$, its Cauchy $F_\mu(z)$ and Borel $R_\mu(z)$ transforms are, respectively, for $z \in \mathbb{C}$ with $|z| \neq 1$,

$$F_\mu(z) = \int_{\partial\mathbb{D}} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$$

and

$$R_\mu(z) = \int_{\partial\mathbb{D}} \frac{1}{e^{it} - z} d\mu(t).$$

R_μ is related to F_μ by

$$F_\mu(z) = 2zR_\mu(z) + 1. \tag{17}$$

Moreover, F_μ has the following properties [18]:

- $\lim_{r\uparrow 1} F_\mu(re^{i\theta})$ exists for a.e. θ, and if one decomposes the measure in its absolutely continuous and singular parts

$$d\mu(\theta) = \omega(\theta) \frac{d\theta}{2\pi} + d\mu_s(\theta),$$

  then

$$\omega(\theta) = \lim_{r\uparrow 1} \text{Re } F_\mu(re^{i\theta}). \tag{18}$$

- θ_0 is a pure point of μ if and only if $\lim_{r\uparrow 1}(1 - r)\text{Re } F_\mu(re^{i\theta_0}) \neq 0$.
- $d\mu_s$ is supported on $\{\theta : \lim_{r\uparrow 1} F_\mu(re^{i\theta}) = \infty\}$.

Now, let U be a unitary operator on a separable Hilbert space $\mathcal{H}$ and ϕ a cyclic vector for U. Consider the rank one perturbation of U

$$U_\lambda = Ue^{i\lambda P_\phi} = U(\text{I}_\text{d} + (e^{i\lambda} - 1)P_\phi),$$

where $P_\phi(\cdot) = \langle \phi, \cdot \rangle \phi$ and $\lambda \in [0, 2\pi)$. Denote by $d\mu_\lambda$ the spectral measure associated with U_λ and ϕ, $F_\lambda = F_{\mu_\lambda}$ and $R_\lambda = R_{\mu_\lambda}$. We have the following relations between R_λ and R_0, F_λ and F_0:

Lemma 5. *For* $|z| \neq 1$

$$R_\lambda(z) = \frac{R_0(z)}{e^{i\lambda} + z(e^{i\lambda} - 1)R_0(z)} \tag{19}$$

and

$$F_\lambda(z) = \frac{(e^{i\lambda} - 1) + (e^{i\lambda} + 1)F_0(z)}{(e^{i\lambda} + 1) + (e^{i\lambda} - 1)F_0(z)} \tag{20}$$

In particular, for $\lambda \neq \pi$*,*

$$\mathrm{Re}\ F_\lambda(z) = \frac{(1 + y^2)\mathrm{Re}\ F_0(z)}{|1 + iyF_0(z)|^2}, \tag{21}$$

where $y = \frac{\sin\lambda}{1+\cos\lambda}$*, and for* $\lambda = \pi$

$$\mathrm{Re}\ F_\lambda(z) = \frac{\mathrm{Re}\ F_0(z)}{|F_0(z)|^2}. \tag{22}$$

Proof. Relation (19) was got in [4]. For checking (20) we use relations (17) and (19). In fact,

$$\begin{aligned}
F_\lambda(z) &= 2zR_\lambda(z) + 1 \\
&= 2z\frac{R_0(z)}{e^{i\lambda} + z(e^{i\lambda} - 1)R_0(z)} + 1 \\
&= \frac{e^{i\lambda} + z(e^{i\lambda} - 1)R_0(z) + 2zR_0(z)}{e^{i\lambda} + z(e^{i\lambda} - 1)R_0(z)} \\
&= \frac{e^{i\lambda} + z(e^{i\lambda} + 1)R_0(z)}{e^{i\lambda} + z(e^{i\lambda} - 1)R_0(z)} \\
&= \frac{2e^{i\lambda} + 2ze^{i\lambda}R_0(z) + 2zR_0(z)}{2e^{i\lambda} + 2ze^{i\lambda}R_0(z) - 2zR_0(z)} \\
&= \frac{e^{i\lambda} - 1 + e^{i\lambda} + 2e^{i\lambda}zR_0(z) + 1 + 2zR_0(z)}{e^{i\lambda} + 1 + e^{i\lambda} + 2e^{i\lambda}zR_0(z) - 1 - 2zR_0(z)} \\
&= \frac{(e^{i\lambda} - 1) + (e^{i\lambda} + 1)(1 + 2zR_0(z))}{(e^{i\lambda} + 1) + (e^{i\lambda} - 1)(1 + 2zR_0(z))} \\
&= \frac{(e^{i\lambda} - 1) + (e^{i\lambda} + 1)F_0(z)}{(e^{i\lambda} + 1) + (e^{i\lambda} - 1)F_0(z)}.
\end{aligned}$$

Now, for $\lambda \neq \pi$ we have $e^{i\lambda} + 1 \neq 0$ and then

$$\begin{aligned}
F_\lambda(z) &= \frac{\frac{e^{i\lambda}-1}{e^{i\lambda}+1} + F_0(z)}{1 + \left(\frac{e^{i\lambda}-1}{e^{i\lambda}+1}\right)F_0(z)} \\
&= \frac{iy + F_0(z)}{1 + iyF_0(z)} \times \frac{1 - iy\overline{F_0(z)}}{1 - iy\overline{F_0(z)}} \\
&= \frac{iy + F_0(z) - iy|F_0(z)|^2 + y^2\overline{F_0(z)}}{|1 + iyF_0(z)|^2},
\end{aligned}$$

where $\frac{e^{i\lambda}-1}{e^{i\lambda}+1} = iy$ and $y = \frac{\sin\lambda}{1+\cos\lambda}$. So, for $\lambda \neq \pi$,

$$\text{Re } F_\lambda(z) = \frac{(1+y^2)\text{Re } F_0(z)}{|1+iyF_0(z)|^2}$$

and (21) is obtained. For $\lambda = \pi$ we have $F_\lambda(z) = \frac{1}{F_0(z)}$ and (22) follows. □

Lemma 6. *Fix a rational number β. Then there exist $C_1 > 0$ and $C_2 < \infty$, and for each $\theta \in [0, 2\pi)$ and $\lambda \in [\frac{\pi}{6}, \frac{\pi}{2}]$ a decomposition*

$$\varphi_1 = \eta_{\theta,\lambda} + \psi_{\theta,\lambda}$$

so that

$$\langle \eta_{\theta,\lambda}, \psi_{\theta,\lambda} \rangle = 0, \tag{23}$$

$$\|\psi_{\theta,\lambda}\| \geq C_1, \tag{24}$$

$$|||\psi_{\theta,\lambda}|||_{U_\lambda(\beta,\theta)^+} \leq C_2 \tag{25}$$

(the notation $|||\cdot|||_U$ was introduced in Lemma 4).

Proof. We break the proof in some steps.

Step 1. By Theorem 1, since β is rational,

$$\sigma_{\text{sc}}(U(\beta,\theta)^+) = \emptyset, \quad \sigma_{\text{ac}}(U(\beta,\theta)^+) = \sigma_{\text{ac}}(U(\beta,\theta))$$

and the point spectrum of $U(\beta,\theta)^+$ consists of finitely many simple eigenvalues in the resolvent set of $U(\beta,\theta)$. Denote by $\mu_{\theta,\lambda}$ the spectral measure associated to $U_\lambda(\beta,\theta)^+$ and (the cyclic vector) φ_1, and by μ_θ the spectral measure associated to $U(\beta,\theta)^+$ and φ_1 (i.e., the case $\lambda = 0$). Write

$$d\mu_{\theta,\lambda}(E) = f_{\theta,\lambda}(E)\frac{dE}{2\pi} + d\mu_s^{\theta,\lambda}(E),$$

$$d\mu_\theta(E) = f_\theta(E)\frac{dE}{2\pi} + d\mu_p^\theta(E).$$

Step 2. Relation between $f_{\theta,\lambda}$ and f_θ: By Lemma 5, for $\lambda \neq \pi$ one has

$$\text{Re } F_{\mu_{\theta,\lambda}}(z) = \frac{(1+y^2)\text{Re } F_{\mu_\theta}(z)}{|1+iyF_{\mu_\theta}(z)|^2},$$

where $y = \frac{\sin\lambda}{1+\cos\lambda}$ and then

$$f_{\theta,\lambda}(E) = \frac{(1+y^2)f_\theta(E)}{|1+iy\lim_{r\uparrow 1} F_{\mu_\theta}(re^{iE})|^2},$$

for almost all E.

Step 3. Relation between f_θ and f_0: By (9) and (8) one gets

$$U(\beta,\theta)^+ = e^{-i2\theta}U(\beta,0)^+ \tag{26}$$

and using this relation it found that

$$\left(U(\beta,\theta)^+\right)^j = e^{-ij2\theta}\left(U(\beta,0)^+\right)^j$$

for all $j \in \mathbb{Z}$. Thus, by the spectral theorem, for any $j \in \mathbb{Z}$,

$$\int_0^{2\pi} e^{-ijE} f_\theta(E)\frac{dE}{2\pi} = \int_0^{2\pi} e^{-ijE} f_0(E-2\theta)\frac{dE}{2\pi}.$$

Hence

$$f_\theta(E) = f_0(E-2\theta) \tag{27}$$

for almost all E.

Step 4. Lower and upper bounds for $f_{\theta,\lambda}$: We have

$$\lim_{r\uparrow 1} F_{\mu_\theta}(re^{iE}) = f_\theta(E) + i\lim_{r\uparrow 1}\mathrm{Im}\ F_{\mu_\theta}(re^{iE})$$

and

$$\begin{aligned}\lim_{r\uparrow 1}\mathrm{Im}\ F_{\mu_\theta}(re^{iE}) \quad &= \quad \lim_{r\uparrow 1}\int_0^{2\pi}\mathrm{Im}\ \left(\frac{e^{it}+re^{iE}}{e^{it}-re^{iE}}\right) f_\theta(t)\frac{dt}{2\pi} \\ &\quad + \lim_{r\uparrow 1}\int_0^{2\pi}\mathrm{Im}\ \left(\frac{e^{it}+re^{iE}}{e^{it}-re^{iE}}\right) d\mu_p^\theta(t).\end{aligned}$$

If we denote

$$g_\theta(E) = \lim_{r\uparrow 1}\int_0^{2\pi}\mathrm{Im}\ \left(\frac{e^{it}+re^{iE}}{e^{it}-re^{iE}}\right) f_\theta(t)\frac{dt}{2\pi},$$

then by (27) we obtain $g_\theta(E) = g_0(E-2\theta)$ for almost all E. On the other hand, by (26) we have that E is an eigenvalue of $U(\beta,\theta)^+$ if and only if $E-2\theta$ is an eigenvalue of $U(\beta,0)^+$. Let $\{E_j^\theta\}_{j=1}^n$ be the set of eigenvalues of $U(\beta,\theta)^+$ (recall that $n<\infty$) and $d\mu_p^\theta = \sum_{j=1}^n \kappa_j^\theta \delta_{E_j^\theta}$ (δ_E is the Dirac measure at E). Then

$$\begin{aligned}\lim_{r\uparrow 1}\int_0^{2\pi}\mathrm{Im}\ \left(\frac{e^{it}+re^{iE}}{e^{it}-re^{iE}}\right) d\mu_p^\theta(t) \quad &= \quad \lim_{r\uparrow 1}\int_0^{2\pi}\frac{2r\sin(E-t)}{1+r^2-2r\cos(E-t)} d\mu_p^\theta(t) \\ &= \quad \lim_{r\uparrow 1}\sum_{j=1}^n \frac{2r\sin(E-E_j^\theta)\kappa_j^\theta}{1+r^2-2r\cos(E-E_j^\theta)} \\ &= \quad \sum_{j=1}^n \frac{2\sin(E-2\theta-E_j^0)\kappa_j^\theta}{\left|e^{iE_j^0}-e^{i(E-2\theta)}\right|^2}.\end{aligned}$$

Since $f_0 \in L^1([0,2\pi))$, by a result of [14] (Theorem 1.6 in Chapter III), the function g_0 is of weak L^1 type, i.e., g_0 is measurable and there exits a constant $C>0$ such that for all $T>0$ the Lebesgue measure

$$|\{E : |g_0(E)| \le T\}| \ge 1 - \frac{C}{T}. \tag{28}$$

Pick $S > 0$ such that $\Omega_S := \left\{E : \frac{1}{S} \leq f_0(E) \leq S\right\}$ satisfies $|\Omega_S| > 0$ and dist $(\Omega_S, \{E_j^0\}_{j=1}^n) = L > 0$. Then choose T sufficiently large such that

$$A := \Omega_S \cap \{E : |g_0(E)| \leq T\}$$

satisfies $|A| > 0$; by (28) this is possible. For $\theta \in [0, 2\pi)$ put

$$I_\theta := \{E \in [0, 2\pi) : E - 2\theta \in A\};$$

thus $|I_\theta| = |A| > 0$. Then, for all $\theta \in [0, 2\pi)$, $\lambda \in [0, \frac{\pi}{2}]$ (equivalently $y \in [0, 1]$) and almost all $E \in I_\theta$ one has

$$\begin{aligned}
\left|1 + iy \lim_{r\uparrow 1} F_{\mu_\theta}(re^{iE})\right| &\leq 1 + |y| \Bigg(f_\theta(E) + |g_\theta(E)| \\
&\quad + \left| \sum_{j=1}^n \frac{2\sin(E - 2\theta - E_j^0)\kappa_j^\theta}{|e^{iE_j^0} - e^{i(E-2\theta)}|^2} \right| \Bigg) \\
&\leq 1 + f_0(E - 2\theta) + |g_0(E - 2\theta)| + \sum_{j=1}^n \frac{2|\kappa_j^\theta|}{L^2} \\
&\leq 1 + S + T + \frac{2}{L^2}.
\end{aligned}$$

So, for all $\theta \in [0, 2\pi)$, $\lambda \in [0, \frac{\pi}{2}]$ and almost all $E \in I_\theta$

$$\begin{aligned}
f_{\theta,\lambda}(E) &= \frac{(1 + y^2) f_\theta(E)}{|1 + iy \lim_{r\uparrow 1} F_{\mu_\theta}(re^{iE}))|^2} \\
&\geq \frac{f_0(E - 2\theta)}{(1 + S + T + 2/L^2)^2} \\
&\geq \frac{1}{S(1 + S + T + 2/L^2)^2}.
\end{aligned}$$

In order to get un upper bound, note that

$$\left|1 + iy \lim_{r\uparrow 1} F_{\mu_\theta}(re^{iE})\right| \geq y f_\theta(E) \geq 0,$$

and so, for all $\theta \in [0, 2\pi)$, $\lambda \in [\frac{\pi}{6}, \frac{\pi}{2}]$ (equivalently $y \in \left[\frac{1}{2+\sqrt{3}}, 1\right]$) and almost all $E \in I_\theta$

$$\begin{aligned}
f_{\theta,\lambda}(E) &= \frac{(1 + y^2) f_\theta(E)}{|1 + iy \lim_{r\uparrow 1} F_{\mu_\theta}(re^{iE})|^2} \\
&\leq \frac{(1 + y^2) f_\theta(E)}{y^2 f_\theta(E)^2} = \frac{(1 + y^2)}{y^2 f_0(E - 2\theta)} \\
&\leq 2(2 + \sqrt{3})^2 S.
\end{aligned}$$

Summing up, for all $\theta \in [0, 2\pi)$, $\lambda \in [\frac{\pi}{6}, \frac{\pi}{2}]$ and almost all $E \in I_\theta$, we have proved that

$$\frac{1}{S(1+S+T+2/L^2)^2} \leq f_{\theta,\lambda}(E) \leq 2(2+\sqrt{3})^2 S. \tag{29}$$

Step 5. Conclusion: For $\lambda \in [\frac{\pi}{6}, \frac{\pi}{2}]$ and $\theta \in [0, 2\pi)$ let

$$\psi_{\theta,\lambda} = P_{I_\theta}^{\theta,\lambda} \varphi_1, \quad \eta_{\theta,\lambda} = (\mathrm{I_d} - P_{I_\theta}^{\theta,\lambda})\varphi_1,$$

where $P_{I_\theta}^{\theta,\lambda}$ is the spectral projection (of $U_\lambda(\beta,\theta)^+$) onto I_θ. Then for each $\theta \in [0, 2\pi)$ and $\lambda \in [\frac{\pi}{6}, \frac{\pi}{2}]$ we have the decomposition $\varphi_1 = \psi_{\theta,\lambda} + \eta_{\theta,\lambda}$ that satisfies (23).

By the construction in Step 4, we have that $A = I_0$ is in the absolutely continuous spectrum of $U(\beta, 0)^+$, so by (26) and the definition of I_θ it follows that I_θ is in the absolutely continuous spectrum of $U(\beta,\theta)^+$; thus using Birman-Krein's theorem on invariance of absolutely continuous spectrum under trace class perturbations, we conclude that I_θ belongs to the absolutely continuous spectrum of $U_\lambda(\beta,\theta)^+$ for all λ. Therefore by (29)

$$\begin{aligned} \|\psi_{\theta,\lambda}\|^2 &= \langle \psi_{\theta,\lambda}, \psi_{\theta,\lambda}\rangle = \langle P_{I_\theta}^{\theta,\lambda}\varphi_1, P_{I_\theta}^{\theta,\lambda}\varphi_1\rangle \\ &= \langle \varphi_1, P_{I_\theta}^{\theta,\lambda}\varphi_1\rangle = \int_0^{2\pi} \chi_{I_\theta}(E)\, d\mu_{\theta,\lambda} \\ &= \int_{I_\theta} f_{\theta,\lambda}(E)\frac{dE}{2\pi} \geq \frac{|A|}{2\pi S(1+S+T+2/L^2)^2} \end{aligned}$$

and (24) holds with

$$C_1 = \Big(\frac{|A|}{2\pi S(1+S+T+2/L^2)^2}\Big)^{1/2} > 0;$$

also

$$|||\psi_{\theta,\lambda}|||^2_{U_\lambda(\beta,\theta)^+} = |||P_{I_\theta}^{\theta,\lambda}\varphi_1|||^2_{U_\lambda(\beta,\theta)^+} = \|\chi_{I_\theta} f_{\theta,\lambda}\|_\infty \leq 2(2+\sqrt{3})^2 S$$

and (25) holds with $C_2 = (2(2+\sqrt{3})^2 S)^{1/2} < \infty$. The lemma is proved. □

4.3. **Variation of β.** The next lemma gives an estimate of the dependence of the dynamics on β. Its proof strongly uses the structure of $U_\lambda(\beta,\theta)^+$.

Lemma 7. *Let $\beta, \beta' \in \mathbb{R}$. Then, for $n \geq 1$,*

$$\left\|\left(U_\lambda(\beta,\theta)^+\right)^n \varphi_1 - \left(U_\lambda(\beta',\theta)^+\right)^n \varphi_1\right\| \leq 2 \times 4^n (2n^2 - n) 2\pi \left|\beta - \beta'\right|.$$

Proof. It is an induction. We have

$$\begin{aligned} U_\lambda(\beta,\theta)^+\varphi_j &= U(\beta,\theta)^+(\mathrm{I_d}+(e^{i\lambda}-1)P_{\varphi_1})\varphi_j \\ &= \begin{cases} U(\beta,\theta)^+\varphi_j & \text{if} \quad j>1 \\ U(\beta,\theta)^+\varphi_1+(e^{i\lambda}-1)U(\beta,\theta)^+\varphi_1 & \text{if} \quad j=1 \end{cases} \\ &= \begin{cases} U(\beta,\theta)^+\varphi_j & \text{if} \quad j>1 \\ e^{i\lambda}U(\beta,\theta)^+\varphi_1 & \text{if} \quad j=1 \end{cases} \end{aligned}$$

Thus

$$U_\lambda(\beta,\theta)^+\varphi_1 = e^{i\lambda}U(\beta,\theta)^+\varphi_1 = a_1e^{-i(2\pi\beta)}\varphi_1+a_2e^{-i(2\pi\beta)}\varphi_2$$

where $a_1 = re^{i\lambda}e^{-i(\alpha+2\theta)}$ and $a_2 = ite^{i\lambda}e^{-i(\alpha+2\theta)}$. Since

$$|e^{-ix}-e^{-ix'}| \le 2|x-x'| \tag{30}$$

and $|a_j| \le 1$, $j=1,2$, then

$$\begin{aligned} \left\|U_\lambda(\beta,\theta)^+\varphi_1-U_\lambda(\beta',\theta)^+\varphi_1\right\| &\le 2\left|e^{-i(2\pi\beta)}-e^{-i(2\pi\beta')}\right| \\ &\le 4\times 2|2\pi\beta-2\pi\beta'| = 2\times 4\times 2\pi\left|\beta-\beta'\right| \end{aligned}$$

and the lemma is proved for $n=1$.

Now

$$\begin{aligned} \left(U_\lambda(\beta,\theta)^+\right)^2\varphi_1 &= U_\lambda(\beta,\theta)^+U_\lambda(\beta,\theta)^+\varphi_1 \\ &= U_\lambda(\beta,\theta)^+(a_1e^{-i(2\pi\beta)}\varphi_1+a_2e^{-i(2\pi\beta)}\varphi_2) \\ &= e^{i\lambda}a_1e^{-i(2\pi\beta)}U(\beta,\theta)^+\varphi_1+a_2e^{-i(2\pi\beta)}U(\beta,\theta)^+\varphi_2 \\ &= e^{i\lambda}a_1e^{-i(2\pi\beta)}\left(b_1e^{-i(2\pi\beta)}\varphi_1+b_2e^{-i(2\pi\beta)}\varphi_2\right) \\ &\quad +a_2e^{-i(2\pi\beta)}\Big(c_1e^{-i(3.(2\pi\beta))}\varphi_1+c_2e^{-i(3.(2\pi\beta))}\varphi_2 \\ &\quad +c_3e^{-i(5.(2\pi\beta))}\varphi_3+c_4e^{-i(5.(2\pi\beta))}\varphi_4\Big) \end{aligned}$$

Since $|a_j|<1$, $|b_j|<1$, $|c_j|<1$ and there are $2+4<4\times 4$ terms in the expansion of $\left(U_\lambda(\beta,\theta)^+\right)^2\varphi_1$ and the largest exponent (which provides the largest contribution by (30)) is obtained from the product of the exponentials $e^{-i(2\pi\beta)}e^{-i((2+3)2\pi\beta)} = e^{-i((1+2+3)2\pi\beta)}$, we obtain

$$\begin{aligned} \left\|\left(U_\lambda(\beta,\theta)^+\right)^2\varphi_1-\left(U_\lambda(\beta',\theta)^+\right)^2\varphi_1\right\| &\le 4\times 4\times 2(1+2+3)2\pi\left|\beta-\beta'\right| \\ &= 2\times 4^2(1+2+3)2\pi\left|\beta-\beta'\right|, \end{aligned}$$

and the lemma is proved for $n=2$. In a similar way by the structure of $U_\lambda(\beta,\theta)^+$ we conclude that $\left(U_\lambda(\beta,\theta)^+\right)^3\varphi_1$ has at most $4^2\times 4$ terms where the largest exponent is in $e^{-i(1+2+3)2\pi\beta}e^{-i((4+5)2\pi\beta)} = e^{-i((1+2+3+4+5)2\pi\beta)}$

and so

$$\begin{aligned} &\left\| \left(U_\lambda(\beta,\theta)^+\right)^3 \varphi_1 - \left(U_\lambda(\beta',\theta)^+\right)^3 \varphi_1 \right\| \le \\ \le\ & 4\times 4\times 4\times 2(1+2+3+4+5)2\pi \left|\beta-\beta'\right| \\ =\ & 2\times 4^3(1+2+3+4+5)2\pi \left|\beta-\beta'\right| . \end{aligned}$$

Inductively one finds that $\left(U_\lambda(\beta,\theta)^+\right)^n \varphi_1$ has at the most 4^n terms, and according to (30) the largest contribution comes from the product

$$e^{-i(1+2+\ldots+2n-3)2\pi\beta} e^{-i(((2n-2)+(2n-1))2\pi\beta)} = e^{-i((1+2+\ldots+2n-1)2\pi\beta)}$$

and then

$$\begin{aligned} &\left\| \left(U_\lambda(\beta,\theta)^+\right)^n \varphi_1 - \left(U_\lambda(\beta',\theta)^+\right)^n \varphi_1 \right\| \\ &\le 2\times 4^n(1+2+\ldots+2n-1)2\pi \left|\beta-\beta'\right| ; \end{aligned}$$

since $2n^2-n = 1+2+\ldots+2n-1$, the result follows. □

4.4. **Proof of Theorem 2(ii).** Finally, using this preparatory set of results, we finish the proof of our main result.

Let $f(n) = (\ln(2+|n|))^{\frac{1}{5}}$. Sequences β_m, T_m, Δ_m will be built inductively, starting with $\beta_1 = 1$, so that

(i) $\beta_{m+1}-\beta_m = 2^{-\kappa_m!}$ for some $\kappa_m \in \mathbb{N}$;

(ii) $\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n\ge \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta,\theta)^+\right)^j \varphi_1\|^2 \ge \frac{1}{f(T_m)^2}$ for all $\theta\in[0,2\pi)$, $\lambda\in[\frac{\pi}{6},\frac{\pi}{2}]$ and β with $|\beta-\beta_m|\le\Delta_m$;

(iii) $|\beta_{m+1}-\beta_k| < \Delta_k$ for $k=1,2,\ldots,m$.

If (i), (ii) and (iii) are satisfied then we conclude by (i) that $\beta_\infty = \lim\beta_m$ is irrational, by (iii) that $|\beta_\infty-\beta_m|\le\Delta_m$ and then by (ii) that

$$\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n\ge \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_\infty,\theta)^+\right)^j \varphi_1\|^2 \ge \frac{1}{f(T_m)^2}$$

for $\theta\in[0,2\pi)$ and $\lambda\in[\frac{\pi}{6},\frac{\pi}{2}]$. So by Lemma 2

$$\limsup_{n\to\infty} \|X\left(U_\lambda(\beta,\theta)^+\right)^n \varphi_1\|^2 \frac{f(n)^5}{n^2} = \infty$$

for $\beta=\beta_\infty$ and the result is proved.

Then we shall construct β_m, T_m, Δ_m such that (i), (ii) and (iii) hold. Start with $\beta_1 = 1$. Given $\beta_1,\ldots,\beta_m$, $T_1,\ldots,T_{m-1}$ and $\Delta_1,\ldots,\Delta_{m-1}$ we shall show how to choose T_m, Δ_m and β_{m+1}.

Given β_m, let $\varphi_1 = \eta+\psi$ be the decomposition given by Lemma 6 and let C_1, C_2 be the corresponding constants. Choose $T_m \ge 2T_{m-1}$ (and $T_1\ge 2$) so that

$$C_1^2 - 3\sqrt{2\pi}C_2(2f(T_m)^{-1}+T_m^{-1})^{\frac{1}{2}} \ge 2f(T_m)^{-1}. \tag{31}$$

This is possible since C_1 and C_2 are fixed (given β_m) and $f(n) \to \infty$.

Note that

(32)
$$\frac{1}{T+1}\sum_{j=T}^{2T} \|P_{n<\frac{T}{f(T)}} \left(U_\lambda(\beta,\theta)^+\right)^j \psi\|^2 \le \frac{2\pi}{T+1} \#\Big\{n : n < \frac{T}{f(T)}\Big\} |||\psi|||^2;$$

in fact

$$\begin{aligned}
&\frac{1}{T+1}\sum_{j=T}^{2T} \|P_{n<\frac{T}{f(T)}} \left(U_\lambda(\beta,\theta)^+\right)^j \psi\|^2 = \\
= \; & \frac{1}{T+1}\sum_{j=T}^{2T} \sum_{n<\frac{T}{f(T)}} \Big|\Big(\left(U_\lambda(\beta,\theta)^+\right)^j \psi\Big)(n)\Big|^2 \\
= \; & \frac{1}{T+1}\sum_{n<\frac{T}{f(T)}} \sum_{j=T}^{2T} \Big|\Big(\left(U_\lambda(\beta,\theta)^+\right)^j \psi\Big)(n)\Big|^2 \\
\le \; & \frac{1}{T+1}\sum_{n<\frac{T}{f(T)}} \sum_{j=-\infty}^{\infty} \Big|\langle \varphi_n, \left(U_\lambda(\beta,\theta)^+\right)^j \psi\rangle\Big|^2,
\end{aligned}$$

then by Lemma 4

$$\frac{1}{T+1}\sum_{j=T}^{2T} \|P_{n<\frac{T}{f(T)}} \left(U_\lambda(\beta,\theta)^+\right)^j \psi\|^2 \le \frac{1}{T+1}\sum_{n<\frac{T}{f(T)}} 2\pi |||\psi|||^2,$$

and (32) follows.

By Lemma 3 and (32)

$$\begin{aligned}
&\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n\ge\frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_m,\theta)^+\right)^j \varphi_1\|^2 \ge \\
\ge \; & \|\psi\|^2 - 3\Big(\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n<\frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_m,\theta)^+\right)^j \psi\|^2\Big)^{\frac{1}{2}} \\
\ge \; & \|\psi\|^2 - 3\Big(\frac{2\pi}{T_m+1}\#\Big\{n : n < \frac{T_m}{f(T_m)}\Big\} |||\psi|||^2\Big)^{\frac{1}{2}} \\
\ge \; & C_1^2 - 3\Big(\frac{2\pi}{T_m+1}\#\Big\{n : n < \frac{T_m}{f(T_m)}\Big\} C_2^2\Big)^{\frac{1}{2}} \\
= \; & C_1^2 - 3\sqrt{2\pi}C_2\Big(\frac{1}{T_m+1}\#\Big\{n : n < \frac{T_m}{f(T_m)}\Big\}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $\#\Big\{n : n < \frac{T_m}{f(T_m)}\Big\} \leq 2\frac{T_m}{f(T_m)} + 1$ it follows that

$$\begin{aligned} & \frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_m,\theta)^+\right)^j \varphi_1\|^2 \\ \geq\ & C_1^2 - 3\sqrt{2\pi}C_2\Big(\frac{1}{T_m+1}\Big(\frac{2T_m}{f(T_m)}+1\Big)\Big)^{\frac{1}{2}} \\ \geq\ & C_1^2 - 3\sqrt{2\pi}C_2\Big(\frac{2}{f(T_m)}+\frac{1}{T_m}\Big)^{\frac{1}{2}} \end{aligned}$$

for $\theta \in [0,2\pi)$ and $\lambda \in [\frac{\pi}{6},\frac{\pi}{2}]$. Thus by (31), we obtain

$$\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_m,\theta)^+\right)^j \varphi_1\|^2 \geq \frac{2}{f(T_m)}$$

for $\theta \in [0,2\pi)$ and $\lambda \in [\frac{\pi}{6},\frac{\pi}{2}]$.

So, by Lemma 7, for $\beta \in \mathbb{R}$, $\theta \in [0,2\pi)$ and $\lambda \in [\frac{\pi}{6},\frac{\pi}{2}]$

$$\begin{aligned} & \frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta,\theta)^+\right)^j \varphi_1\|^2 \geq \\ \geq\ & \Big(\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \|P_{|n|\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta,\theta)^+\right)^j \varphi_1\|\Big)^2 \\ =\ & \Bigg(\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \Big\|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_m,\theta)^+\right)^j \varphi_1 \\ & +P_{n\geq \frac{T_m}{f(T_m)}}\Big(\left(U_\lambda(\beta,\theta)^+\right)^j \varphi_1 - \left(U_\lambda(\beta_m,\theta)^+\right)^j \varphi_1\Big)\Big\|\Bigg)^2 \\ \geq\ & \Bigg(\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} \Big\|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_m,\theta)^+\right)^j \varphi_1\Big\| \\ & -\Big\|\Big(\left(U_\lambda(\beta,\theta)^+\right)^j - \left(U_\lambda(\beta_m,\theta)^+\right)^j\Big)\varphi_1\Big\|\Bigg)^2 \\ \geq\ & \Big(\frac{1}{T_m+1}\sum_{j=T_m}^{2T_m} (\|P_{n\geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta_m,\theta)^+\right)^j \varphi_1\|^2 \\ & -4^{j+1}(2j^2-j)\pi|\beta-\beta_m|)\Big)^2 \\ \geq\ & \Big(\frac{2}{f(T_m)} - \frac{1}{T_m+1}\Big(\sum_{j=T_m}^{2T_m} 4^{j+1}(2j^2-j)\pi\Big)|\beta-\beta_m|\Big)^2. \end{aligned}$$

Taking

$$\Delta_m = \frac{T_m + 1}{f(T_m) \sum_{j=T_m}^{2T_m} 4^{j+1}(2j^2 - j)\pi}$$

we obtain that, if $|\beta - \beta_m| < \Delta_m$,

$$\frac{1}{T_m + 1} \sum_{j=T_m}^{2T_m} \|P_{n \geq \frac{T_m}{f(T_m)}} \left(U_\lambda(\beta, \theta)^+\right)^j \varphi_1\|^2 \geq \frac{1}{f(T_m)^2}.$$

Finally, pick β_{m+1} rational so that

$$|\beta_n - \beta_{m+1}| < \Delta_n \qquad n = 1, \ldots, m,$$

and $\beta_{m+1} = \beta_m + 2^{-\kappa_m!}$ for some $\kappa_m \in \mathbb{N}$. This finishes the proof of Theorem 2(ii).

Remark. For the operator $U_\lambda(\beta, \theta) := U(\beta, \theta)(\mathrm{I_d} + (e^{i\lambda} - 1)P_{\varphi_1})$ on $l^2(\mathbb{Z})$ we can similarly prove an analogous result. The proof of dynamical instability for some irrational β is essentially unchanged except for Lemma 6 which is simplified since $U(\beta, \theta)$ is purely absolutely continuous for β rational. On the other hand, about pure point spectrum, the main difference in this case is that φ_1 might not be cyclic, an thus, we don't get pure point spectrum for $U_\lambda(\beta, \theta)$ for a.e. θ and λ as obtained on $l^2(\mathbb{N}^*)$, but we get that φ_1 is in the point spectral subspace corresponding to $U_\lambda(\beta, \theta)$ for a.e. θ and λ.

References

[1] Blatter G., Browne D.: Zener Tunneling and Localization in small Conducting Rings. Phys. Rev. B **37**, 3856–3880 (1988)

[2] Bourget O.: Singular Continuous Floquet Operator for Periodic Quantum Systems. J. Math. Anal. Appl. **301**, 65–83 (2005)

[3] Bourget O., Howland J. S., Joye A.: Spectral Analysis of Unitary Band Matrices. Commun. Math. Phys. **234**, 191–227 (2003)

[4] Combescure M.: Spectral Properties of a Periodically Kicked Quantum Hamiltonian. J. Stat. Phys. **59**, 679–690 (1990)

[5] Cycon H. L., Froese R. G., Kirsch W., Simon B.: Schrödinger Operators. Berlin: Springer-Verlag, 1987

[6] de Oliveira C. R., Prado R. A.: Spectral and Localization Properties for the One-Dimensional Bernoulli Discrete Dirac Operator. J. Math. Phys. **46**, 072105 (2005)

[7] del Rio R., Jitomirskaya S., Last Y., Simon B.: Operators with Singular Continuous Spectrum IV: Hausdorff Dimensions, Rank One Perturbations and Localization. J. d'Analyse Math. **69**, 153–200 (1996)

[8] Enss V., Veselic K.: Bound States and Propagating States for Time-Dependent Hamiltonians. Ann. Inst. H. Poincaré Sect. A **39**, 159–191 (1983)

[9] Hamza E., Joye A., Stolz G.: Localization for Random Unitary Operators. Lett. Math. Phys. **75**, 255–272 (2006)

[10] Jitomirskaya S., Schulz-Baldes H., Stolz G.: Delocalization in Random Polymer Models. Commun. Math. Phys. **233**, 27–48 (2003)

[11] Joye A.: Density of States and Thouless Formula for Random Unitary Band Matrices. Ann. Henri Poincaré **5**, 347–379 (2004)

[12] Joye A.: Fractional Moment Estimates for Random Unitary Band Matrices. Lett. Math. Phys. **72**, 51–64 (2005)
[13] Kato T.: Perturbation Theory for Linear Operators—Second Edition. Berlin: Springer-Verlag, 1980
[14] Katznelson Y.: An Introduction to Harmonic Analysis. New York: John Wiley, 1968
[15] Reed M., Simon B.: Methods of Modern Mathematical Physics III Scattering Theory. New York: Acad. Press 1979
[16] Simon B.: Absence of Ballistic Motion. Commun. Math. Phys. **134**, 209–212 (1990)
[17] Simon B.: Spectral Analysis of rank one Perturbations and Applications. CRM Lecture Notes Vol. **8**, 109–149 (1995) (J. Feldman, R. Froese, L. Rosen, eds.)
[18] Simon B.: Analogs of the M-Function in the Theory of Orthogonal Polynomials on the Unit Circle. J. Comput. Appl. Math. **171**, 411–424 (2004)
[19] Simon B.: Aizenman's Theorem for Orthogonal Polynomials On the Unit Circle. Constr. Approx. **23**, 229–240 (2006)
[20] Simon B., Wolf T.: Singular Continuous Spectrum under rank one Perturbations and Localization for Random Hamiltonians. Commun. Pure Appl. Math. **39**, 75–90 (1986)

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