INFRAPARTICLE SCATTERING STATES IN NON-RELATIVISTIC QED: II. MASS SHELL PROPERTIES

THOMAS CHEN, JÜRG FRÖHLICH, AND ALESSANDRO PIZZO

ABSTRACT. We study the infrared problem in the usual model of QED with non-relativistic matter. We prove spectral and regularity properties characterizing the mass shell of an electron and one-electron infraparticle states of this model. Our results are crucial for the construction of infraparticle scattering states, which are treated in a separate paper.

I. INTRODUCTION

We study the dynamics of an electron interacting with the quantized electromagnetic field in the framework of non-relativistic Quantum Electrodynamics (QED). In a theory describing a massive particle (the electron) interacting with a field of *massless* bosons (the photons), massive one-particle states do, in general, not exist in the physical Hilbert space of the theory. This fact was first observed by Schroer [24], who also coined the term "infraparticle", a notion that generalizes that of a particle. In relativistic QED, charged infraparticles were shown to occur, using arguments from general quantum field theory; see [16, 3]. For the spectrum of $(H, \vec{P})$ in Nelson's model, a simplified variant of non-relativistic QED, with H denoting the Hamiltonian, and $\vec{P}$ the total, conserved momentum of the massive particle and the massless bosons, it was proven in [14, 15] that the bottom of the spectrum of the fiber Hamiltonian $H_{\vec{P}}$ at a fixed total momentum $\vec{P} \in \mathbb{R}^3$ is not an eigenvalue of $H_{\vec{P}}$, for any value of $\vec{P}$ with $\frac{|\vec{P}|}{m} < \rho_0(\lambda) < 1$ where λ is the coupling constant and m is the electron mass. To prove this result, one introduces an infrared cutoff $\sigma > 0$ in the Hamiltonian $H_{\vec{P}}$ turning off all interactions of the non-relativistic, massive particle with the soft modes (with frequencies $< \sigma$) of the relativistic, massless boson field. One then aims to establish spectral properties of the model in the limit $\sigma \to 0$.

Extending results of [14, 15], an iterative algorithm for constructing the ground state vector $\Psi_{\vec{P}}^{\sigma}$ of the infrared regularized Hamiltonian $H_{\vec{P}}^{\sigma}$ in Nelson's model has been developed in [22] using a novel multiscale analysis technique. In [22], important regularity properties have been derived, which are crucial for the analysis of the asymptotic dynamics of the electron. Similarly as in [15], the strategy in [22] is to apply a specific *Bogoliubov transformation*

to the photon variables in $H_{\vec{P}}^{\sigma}$, in order to obtain a Hamiltonian $K_{\vec{P}}^{\sigma}$ whose ground state $\Phi_{\vec{P}}^{\sigma}$ remains in Fock space, as $\sigma \to 0$. Subsequently, one derives properties of the ground state vector of the physical Hamiltonian $H_{\vec{P}}$ in the singular limit $\sigma \to 0$ by inverting the Bogoliubov transformation. In the limit $\sigma \to 0$, the latter gives rise to a coherent representation of the observable algebra of the boson field *unitarily inequivalent* to the Fock representation and to the coherent representations associated to different values of the total momentum.

The identification of the correct Bogoliubov transformation is crucial for the constructions in [22, 23]. For Nelson's model, this Bogoliubov transformation has been found in [14] by a method that exploits the linearity of the interaction in the Nelson Hamiltonian with respect to the creation- and annihilation operators. Due to the more complicated structure of the interaction Hamiltonian in non-relativistic QED, this argument cannot be applied, and the correct Bogoliubov transformation for non-relativistic QED has only recently been identified in [9], based on uniform bounds on the renormalized electron mass established in [8]. This makes it possible to extend the constructions and methods of [22, 23] to non-relativistic QED.

By a generalization of the multiscale methods based on *recursive analytic perturbation theory* introduced in [22], we present a new construction of the correct Bogoliubov transformation, and we prove the following main results:

- The ground state vectors $\Phi_{\vec{P}}^{\sigma}$ of the Bogoliubov-transformed Hamiltonians $K_{\vec{P}}^{\sigma}$ converge strongly to a vector in Fock space, in the limit $\sigma \to 0$. The convergence rate is estimated by $\mathcal{O}(\sigma^{\eta})$, for some explicit $\eta > 0$.
- The vectors $\Phi_{\vec{P}}^{\sigma}$ in Fock space are strongly Hölder continuous in $\vec{P}$, uniformly in σ.

These properties are key ingredients for the construction of infraparticle scattering states, which we present in [10]. A key difficulty in this analysis is the fact that the infrared behavior of the interaction in QED is, in the terminology of renormalization group theory, of *marginal* type (see also [8]).

II. Definition of the Model

The Hilbert space of pure state vectors of the system consisting of one non-relativistic electron interacting with the quantized electromagnetic field is given by

$$\mathcal{H} := \mathcal{H}_{el} \otimes \mathcal{F}, \tag{II.1}$$

where $\mathcal{H}_{el} = L^2(\mathbb{R}^3)$ is the Hilbert space for a single Schrödinger electron (for expository convenience, we neglect the spin of the electron). The Fock space used to describe the states of the transverse modes of the quantized electromagnetic field (the *photons*) in the Coulomb gauge is given by

$$\mathcal{F} := \bigoplus_{N=0}^{\infty} \mathcal{F}^{(N)}, \quad \mathcal{F}^{(0)} = \mathbb{C}\,\Omega, \tag{II.2}$$

where Ω is the vacuum vector (the state of the electromagnetic field without any excited modes), and

$$\mathcal{F}^{(N)} := \mathcal{S}_N \bigotimes_{j=1}^{N} \mathfrak{h}, \quad N \geq 1, \tag{II.3}$$

where the Hilbert space $\mathfrak{h}$ of a single photon is

$$\mathfrak{h} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2). \tag{II.4}$$

Here, $\mathbb{R}^3$ is momentum space, and $\mathbb{Z}_2$ accounts for the two independent transverse polarizations (or helicities) of a photon. In (II.3), $\mathcal{S}_N$ denotes the orthogonal projection onto the subspace of $\bigotimes_{j=1}^{N} \mathfrak{h}$ of totally symmetric N-photon wave functions, to account for the fact that photons satisfy Bose-Einstein statistics. Thus, $\mathcal{F}^{(N)}$ is the subspace of $\mathcal{F}$ of state vectors for configurations of exactly N photons. It is convenient to represent the Hilbert space $\mathcal{H}$ as the space of square-integrable wave functions on the electron position space $\mathbb{R}^3$ with values in the photon Fock space $\mathcal{F}$, i.e.,

$$\mathcal{H} \cong L^2(\mathbb{R}^3\,;\mathcal{F}). \tag{II.5}$$

In this paper, we use units such that Planck's constant $\hbar$, the speed of light c, and the mass of the electron are equal to unity. The dynamics of the system is generated by the Hamiltonian

$$H := \frac{\left(-i\vec{\nabla}_{\vec{x}} + \alpha^{1/2}\vec{A}(\vec{x})\right)^2}{2} + H^f. \tag{II.6}$$

The multiplication operator $\vec{x} \in \mathbb{R}^3$ accounts for the position of the electron. The electron momentum operator is given by $\vec{p} = -i\vec{\nabla}_{\vec{x}}$. $\alpha \cong 1/137$ is the feinstructure constant (which,

in this paper, plays the rôle of a small parameter), $\vec{A}(\vec{x})$ denotes the vector potential of the transverse modes of the quantized electromagnetic field in the *Coulomb gauge*,

$$\vec{\nabla}_{\vec{x}} \cdot \vec{A}(\vec{x}) \; = \; 0 \,. \tag{II.7}$$

The operator H^f is the Hamiltonian of the quantized, free electromagnetic field,

$$H^f \; := \; \sum_{\lambda=\pm} \int d^3k \, |\vec{k}| \, a^*_{\vec{k},\lambda} \, a_{\vec{k},\lambda} \,, \tag{II.8}$$

where $a^*_{\vec{k},\lambda}$ and $a_{\vec{k},\lambda}$ are the usual photon creation- and annihilation operators, satisfying the canonical commutation relations

$$[a_{\vec{k},\lambda} \,, \, a^*_{\vec{k}',\lambda'}] \;\; = \;\; \delta_{\lambda\lambda'} \, \delta(\vec{k} - \vec{k}') \,, \tag{II.9}$$

$$[a^{\#}_{\vec{k},\lambda} \,, \, a^{\#}_{\vec{k}',\lambda'}] \;\; = \;\; 0 \tag{II.10}$$

(where $a^{\#} = a$ or a^*). The vacuum vector Ω is characterized by the condition

$$a_{\vec{k},\lambda} \Omega \; = \; 0 \,, \tag{II.11}$$

for all $\vec{k}, \vec{k}' \in \mathbb{R}^3$ and $\lambda, \lambda' \in \mathbb{Z}_2 \equiv \{\pm\}$.

The quantized electromagnetic vector potential is given by

$$\vec{A}(\vec{x}) \; := \; \sum_{\lambda=\pm} \int_{\mathcal{B}_\Lambda} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{ \vec{\varepsilon}_{\vec{k},\lambda} e^{-i\vec{k}\cdot\vec{x}} a^*_{\vec{k},\lambda} \; + \; \vec{\varepsilon}^{\,*}_{\vec{k},\lambda} e^{i\vec{k}\cdot\vec{x}} a_{\vec{k},\lambda} \right\} \,, \tag{II.12}$$

where $\vec{\varepsilon}_{\vec{k},-}$, $\vec{\varepsilon}_{\vec{k},+}$ are photon polarization vectors, i.e., two unit vectors in $\mathbb{R}^3 \otimes \mathbb{C}$ satisfying

$$\vec{\varepsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\varepsilon}_{\vec{k},\mu} \; = \; \delta_{\lambda\mu} \,, \qquad \vec{k} \cdot \vec{\varepsilon}_{\vec{k},\lambda} \; = \; 0 \,, \tag{II.13}$$

for $\lambda, \mu = \pm$. The equation $\vec{k} \cdot \vec{\varepsilon}_{\vec{k},\lambda} = 0$ expresses the Coulomb gauge condition. Moreover, $\mathcal{B}_\Lambda$ is a ball of radius Λ centered at the origin in momentum space. Λ represents an *ultraviolet cutoff* that will be kept fixed throughout our analysis. The vector potential defined in (II.12) is thus cut off in the ultraviolet.

Throughout this paper, it will be assumed that $\Lambda \approx 1$ (the rest energy of an electron), and that α is sufficiently small. Under these assumptions, the Hamitonian H is selfadjoint on $D(H^0)$, i.e., on the domain of definition of the operator

$$H^0 \; := \; \frac{(-i\vec{\nabla}_{\vec{x}})^2}{2} \; + \; H^f \,. \tag{II.14}$$

The perturbation $H - H^0$ is small in the sense of Kato; see, e.g., [25].

The operator measuring the total momentum of the system consisting of the electron and the electromagnetic radiation field is given by

$$\vec{P} := \vec{p} + \vec{P}^f , \tag{II.15}$$

where $\vec{p} = -i\vec{\nabla}_{\vec{x}}$ is the momentum operator for the electron, and

$$\vec{P}^f := \sum_{\lambda=\pm} \int d^3k \, \vec{k} \, a^*_{\vec{k},\lambda} \, a_{\vec{k},\lambda} \tag{II.16}$$

is the momentum operator associated with the photon field.

The operators H and $\vec{P}$ are essentially selfadjoint on the domain $D(H_0)$, and since the dynamics is invariant under translations, they commute, $[H, \vec{P}] = \vec{0}$. The Hilbert space $\mathcal{H}$ can be decomposed on the joint spectrum, $\mathbb{R}^3$, of the component-operators of $\vec{P}$. Their spectral measure is absolutely continuous with respect to Lebesque measure. Thus,

$$\mathcal{H} := \int^{\oplus} \mathcal{H}_{\vec{P}} \, d^3P , \tag{II.17}$$

where each fiber space $\mathcal{H}_{\vec{P}}$ is a copy of Fock space $\mathcal{F}$.

Remark *Throughout this paper, the symbol $\vec{P}$ stands both for a variable in $\mathbb{R}^3$ and for a vector operator in $\mathcal{H}$, depending on the context. Similarly, a double meaning is also associated with functions of the total momentum operator.*

To each fiber space $\mathcal{H}_{\vec{P}}$ there corresponds an isomorphism

$$I_{\vec{P}} : \mathcal{H}_{\vec{P}} \longrightarrow \mathcal{F}^b , \tag{II.18}$$

where $\mathcal{F}^b$ is the Fock space corresponding to the annihilation- and creation operators $b_{\vec{k},\lambda}$, $b^*_{\vec{k},\lambda}$, where $b_{\vec{k},\lambda}$ is given by $e^{i\vec{k}\cdot\vec{x}}a_{\vec{k},\lambda}$, and $b^*_{\vec{k},\lambda}$ by $e^{-i\vec{k}\cdot\vec{x}}a^*_{\vec{k},\lambda}$, with vacuum $\Omega_f = I_{\vec{P}}(e^{i\vec{P}\cdot\vec{x}})$, where $\vec{x}$ is the electron position. To define $I_{\vec{P}}$ more precisely, we consider an (improper) vector $\psi_{(f_1,\ldots,f_n;\vec{P})} \in \mathcal{H}_{\vec{P}}$ with a definite total momentum, which describes an electron and n photons in a product state. Its wave function, in the variables $(\vec{x}; \vec{k}_1, \ldots, \vec{k}_n; \lambda_1, \ldots, \lambda_n)$, is given by

$$e^{i(\vec{P}-\vec{k}_1-\cdots-\vec{k}_n)\cdot\vec{x}} \frac{1}{n!} \sum_{p\in\mathcal{P}_n} f_{p(1)}(\vec{k}_1; \lambda_{p(1)}) \, \cdots \, f_{p(n)}(\vec{k}_n; \lambda_{p(n)}) , \tag{II.19}$$

$\mathcal{P}_n$ being the group of permutations of n elements. The isomorphism $I_{\vec{P}}$ acts by way of

$$I_{\vec{P}}\big(e^{i(\vec{P}-\vec{k}_1-\cdots-\vec{k}_n)\cdot\vec{x}}\frac{1}{n!}\sum_{\mathcal{P}_n}f_{p(1)}(\vec{k}_1;\lambda_{p(1)})\,\cdots\,f_{p(n)}(\vec{k}_n;\lambda_{p(n)})\big) = \tag{II.20}$$

$$= \frac{1}{\sqrt{n!}}\int d^3k'_1\ldots d^3k'_n\,f_1(\vec{k}'_1;\lambda_1)\,\cdots\,f_n(\vec{k}'_n;\lambda_n)\,b^*_{\vec{k}'_1,\lambda_1}\cdots b^*_{\vec{k}'_n,\lambda_n}\,\Omega_f\,. \tag{II.21}$$

The Hamiltonian H maps each $\mathcal{H}_{\vec{P}}$ into itself, i.e., it can be written as

$$H = \int H_{\vec{P}}\,d^3P\,, \tag{II.22}$$

where

$$H_{\vec{P}} : \mathcal{H}_{\vec{P}} \longrightarrow \mathcal{H}_{\vec{P}}\,. \tag{II.23}$$

Written in terms of the operators $b_{\vec{k},\lambda}$, $b^*_{\vec{k},\lambda}$, and of the variable $\vec{P}$, the fiber Hamiltonian $H_{\vec{P}}$ has the form

$$H_{\vec{P}} := \frac{\big(\vec{P}-\vec{P}^f+\alpha^{1/2}\vec{A}\big)^2}{2} + H^f\,, \tag{II.24}$$

where

$$\vec{P}^f := \sum_{\lambda}\int d^3k\,\vec{k}\,b^*_{\vec{k},\lambda}\,b_{\vec{k},\lambda}\,, \tag{II.25}$$

$$H^f := \sum_{\lambda}\int d^3k\,|\vec{k}|\,b^*_{\vec{k},\lambda}\,b_{\vec{k},\lambda}\,, \tag{II.26}$$

and

$$\vec{A} := \sum_{\lambda}\int_{\mathcal{B}_\Lambda}\frac{d^3k}{\sqrt{|\vec{k}|}}\,\big\{b^*_{\vec{k},\lambda}\vec{\varepsilon}_{\vec{k},\lambda}\,+\,\vec{\varepsilon}^{\,*}_{\vec{k},\lambda}b_{\vec{k},\lambda}\big\}\,. \tag{II.27}$$

In the following, we will only construct infraparticle states of momentum $\vec{P}\in\mathcal{S}$, where

$$\mathcal{S} := \big\{\,\vec{P}\in\mathbb{R}^3\,:\,|\vec{P}|<\frac{1}{3}\,\big\}\,. \tag{II.28}$$

In order to give a well-defined meaning to the operations we use in the sequel, we introduce an infrared cut-off at energy $\sigma>0$ in the interaction term

$$H_{I,\vec{P}} := \alpha^{1/2}\,\vec{A}\cdot\frac{(\vec{P}-\vec{P}^f)}{2} + \alpha\,\frac{\vec{A}^2}{2} \tag{II.29}$$

of the Hamiltonian $H_{\vec{P}}$, which is imposed on the vector potential $\vec{A}$. Its removal is the main problem solved in this paper. Our results are crucial ingredients for infraparticle scattering theory; see [10]. We will start by studying the regularized fiber Hamiltonian

$$H^\sigma_{\vec{P}} := \frac{\big(\vec{P}-\vec{P}^f+\alpha^{1/2}\vec{A}^\sigma\big)^2}{2} + H^f \tag{II.30}$$

acting on the fiber space $\mathcal{H}_{\vec{P}}$, for $\vec{P} \in \mathcal{S}$, where

$$\vec{A}^\sigma := \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma} \frac{d^3k}{\sqrt{2|\vec{k}|}} \left\{ b^*_{\vec{k},\lambda} \vec{\varepsilon}_{\vec{k},\lambda} + \vec{\varepsilon}^*_{\vec{k},\lambda} b_{\vec{k},\lambda} \right\} \tag{II.31}$$

and where $\mathcal{B}_\sigma$ is a ball of radius σ. We will consider a sequence $(\sigma_j)_{j=0}^\infty$ of infrared cutoffs given by $\sigma_j := \Lambda \epsilon^j$, with $0 < \epsilon < 1$ and $j \in \mathbb{N}$.

In Section IV, we construct the ground state vector $(\Psi_{\vec{P}}^{\sigma_j})$ of the Hamiltionan $(H_{\vec{P}}^{\sigma_j})$, and we compare ground state vectors $\Psi_{\vec{P}}^{\sigma_j}$, $\Psi_{\vec{P}'}^{\sigma_{j'}}$ corresponding to different fiber Hamiltonians $H_{\vec{P}}^{\sigma_j}$, $H_{\vec{P}'}^{\sigma_{j'}}$ with $\vec{P} \neq \vec{P}'$. We compare the vectors $\Psi_{\vec{P}}^{\sigma_j}$, $\Psi_{\vec{P}'}^{\sigma_{j'}}$ as elements of the space $\mathcal{F}^b$. More precisely, we use the expression

$$\|\Psi_{\vec{P}}^{\sigma_j} - \Psi_{\vec{P}'}^{\sigma_j}\|_{\mathcal{F}} \tag{II.32}$$

as an abbreviation for

$$\|I_{\vec{P}}(\Psi_{\vec{P}}^{\sigma_j}) - I_{\vec{P}'}(\Psi_{\vec{P}'}^{\sigma_j})\|_{\mathcal{F}}. \tag{II.33}$$

II.1. **Background.** In a companion paper [10], we construct a vector $\psi_{h,\Lambda_1}(t)$ converging to a scattering state $\psi_{h,\Lambda_1}^{out/in}$, as time t tends to infinity, applying and extending mathematical techniques developed in [23] for Nelson's model. The vector $\psi_{h,\Lambda_1}^{out/in}$ represents an electron with a wave function h in the momentum variable with support contained in $\mathcal{S} = \{\vec{P} : |\vec{P}| < \frac{1}{3}\}$, accompanied by a cloud of real photons described by a Bloch-Nordsieck factor, and with an upper photon frequency cutoff Λ_1.

In [10] we also construct the scattering subspaces $\mathcal{H}^{out/in}$, starting from certain sub-spaces, $\mathcal{H}^{1\ out/in}$, and applying "hard" asymptotic photon creation operators. These spaces carry representations of the algebras, $\mathcal{A}_{ph}^{out/in}$ and $\mathcal{A}_{el}^{out/in}$, of asymptotic photon- and electron observables, respectively, and the fact that their actions commute proves, mathematically, asymptotic decoupling of the electron and photon dynamics, as time $t \to \pm\infty$. Properties of the representations of $\mathcal{A}_{ph}^{out/in}$ in the infrared expected on the basis of the Bloch-Nordsieck paradigm are rigorously established; see [10].

III. Statement of the Main Results

The main results of our paper are summarized in Theorem III.1 below. They are fundamental for the construction of scattering states in [10] and are very similar to those used in the analysis of Nelson's model in [23].

We define the energy of a dressed one-electron state of momentum $\vec{P}$ by

$$E_{\vec{P}}^\sigma = \inf \operatorname{spec} H_{\vec{P}}^\sigma \quad , \qquad E_{\vec{P}} = \inf \operatorname{spec} H_{\vec{P}} = E_{\vec{P}}^{\sigma=0} . \tag{III.1}$$

We refer to $E_{\vec{P}}^\sigma$ as the *ground state energy* of the fiber Hamiltonian $H_{\vec{P}}^\sigma$. If it exists the corresponding ground state is denoted by $\Psi_{\vec{P}}^\sigma$. We always assume that $\vec{P} \in \mathcal{S} := \{\vec{P} \in \mathbb{R}^3 : |\vec{P}| < \frac{1}{3}\}$ and that α is so small that, for all $\vec{P} \in \mathcal{S}$,

$$|\vec{\nabla} E_{\vec{P}}^\sigma| < \nu_{max} < 1 \tag{III.2}$$

for some constant ν_{max}, uniformly in σ.

Let $\delta_{\vec{P}}^\sigma(\widehat{k})$ be given by

$$\delta_{\vec{P}}^\sigma(\widehat{k}) := 1 - \frac{\vec{k} \cdot \vec{\nabla} E_{\vec{P}}^\sigma}{|\vec{k}|} . \tag{III.3}$$

We introduce an operator

$$W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) := \exp\left(\alpha^{\frac{1}{2}} \sum_\lambda \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma} d^3k \frac{\vec{\nabla} E_{\vec{P}}^\sigma}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^\sigma(\widehat{k})} \cdot (\vec{\varepsilon}_{\vec{k},\lambda} b_{\vec{k},\lambda}^* - h.c.) \right), \tag{III.4}$$

on $\mathcal{H}_{\vec{P}}$, which is unitary for $\sigma > 0$, and consider the transformed fiber Hamiltonian

$$K_{\vec{P}}^\sigma := W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) H_{\vec{P}}^\sigma W_\sigma^*(\vec{\nabla} E_{\vec{P}}^\sigma) . \tag{III.5}$$

Conjugation by $W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma)$ acts on the creation- and annhilation operators by a (Bogoliubov) translation

$$W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) b_{\vec{k},\lambda}^\# W_\sigma^*(\vec{\nabla} E_{\vec{P}}^\sigma) = b_{\vec{k},\lambda}^\# - \frac{\mathbf{1}_{\sigma,\Lambda}(\vec{k})}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^\sigma(\widehat{k})} \vec{\nabla} E_{\vec{P}}^\sigma \cdot \vec{\varepsilon}_{\vec{k},\lambda}^\# , \tag{III.6}$$

where $\mathbf{1}_{\sigma,\Lambda}(\vec{k})$ stands for the characteristic function of the set $\mathcal{B}_\Lambda \setminus \mathcal{B}_\sigma$. Our methods exploit regularity properties in σ and $\vec{P}$ of the ground state vector, $\Phi_{\vec{P}}^\sigma$, and of the ground state energy, $E_{\vec{P}}^\sigma$, of $K_{\vec{P}}^\sigma$. These properties are formulated in the following theorem, which is the main result of this paper.

Theorem III.1. *For $\vec{P} \in \mathcal{S}$ and for $\alpha > 0$ sufficiently small, the following statements hold.*

($\mathscr{I}1$) *The energy $E_{\vec{P}}^\sigma$ is a simple eigenvalue of the operator $K_{\vec{P}}^\sigma$ on $\mathcal{F}^b$. Let $\mathcal{B}_\sigma := \{\vec{k} \in \mathbb{R}^3 \,|\, |\vec{k}| \leq \sigma\}$, and let $\mathcal{F}_\sigma$ denote the Fock space over $L^2((\mathbb{R}^3 \setminus \mathcal{B}_\sigma) \times \mathbb{Z}_2)$. Likewise,*

we define $\mathcal{F}_0^\sigma$ to be the Fock space over $L^2(\mathcal{B}_\sigma \times \mathbb{Z}_2)$; hence $\mathcal{F}^b = \mathcal{F}_\sigma \otimes \mathcal{F}_0^\sigma$. On $\mathcal{F}_\sigma$, the operator $K_{\vec{P}}^\sigma$ has a spectral gap of size $\rho^- \sigma$ or larger, separating $E_{\vec{P}}^\sigma$ from the rest of its spectrum, for some constant ρ^- (depending on α), with $0 < \rho^- < 1$.

The contour

$$\gamma := \left\{ z \in \mathbb{C} \,\middle|\, |z - E_{\vec{P}}^\sigma| = \frac{\rho^- \sigma}{2} \right\} \ , \ \sigma > 0 \tag{III.7}$$

bounds a disc which intersects the spectrum of $K_{\vec{P}}^\sigma$ in only one point, $\{E_{\vec{P}}^\sigma\}$. The ground state vectors,

$$\Phi_{\vec{P}}^\sigma = \zeta \, W_\sigma(\vec{\nabla} E_{\vec{P}}^\sigma) \frac{\Psi_{\vec{P}}^\sigma}{\|\Psi_{\vec{P}}^\sigma\|} \ , \ \zeta \in \mathbb{C} \ , \ |\zeta| = 1 \,, \tag{III.8}$$

of the operators $K_{\vec{P}}^\sigma$ are given by

$$\Phi_{\vec{P}}^\sigma := \frac{\frac{1}{2\pi i} \int_\gamma \frac{1}{K_{\vec{P}}^\sigma - z} \, dz \, \Omega_f}{\left\| \frac{1}{2\pi i} \int_\gamma \frac{1}{K_{\vec{P}}^\sigma - z} \, dz \, \Omega_f \right\|_{\mathcal{F}}} \tag{III.9}$$

and converge strongly to a non-zero vector $\Phi_{\vec{P}} \in \mathcal{F}^b$, in the limit $\sigma \to 0$. The rate of convergence is of order $\sigma^{\frac{1}{2}(1-\delta)}$, for any $0 < \delta < 1$.

The dependence of the ground state energies $E_{\vec{P}}^\sigma$ of the fiber Hamiltonians $K_{\vec{P}}^\sigma$ on the infrared cutoff σ is characterized by the following estimates.

$$| E_{\vec{P}}^\sigma - E_{\vec{P}}^{\sigma'} | \leq \mathcal{O}(\sigma) \,, \tag{III.10}$$

and

$$| \vec{\nabla} E_{\vec{P}}^\sigma - \vec{\nabla} E_{\vec{P}}^{\sigma'} | \leq \mathcal{O}(\sigma^{\frac{1}{2}(1-\delta)}) \,, \tag{III.11}$$

for any $0 < \delta < 1$, with $\sigma > \sigma' > 0$.

$(\mathscr{I}2)$ The following Hölder regularity properties in $\vec{P} \in \mathcal{S}$ hold uniformly in $\sigma \geq 0$:

$$\|\Phi_{\vec{P}}^\sigma - \Phi_{\vec{P}+\Delta\vec{P}}^\sigma\|_{\mathcal{F}} \leq C_{\delta'} |\Delta\vec{P}|^{\frac{1}{4}-\delta'} \tag{III.12}$$

and

$$|\vec{\nabla} E_{\vec{P}}^\sigma - \vec{\nabla} E_{\vec{P}+\Delta\vec{P}}^\sigma| \leq C_{\delta''} |\Delta\vec{P}|^{\frac{1}{4}-\delta''} \,, \tag{III.13}$$

for $0 < \delta'' < \delta' < \frac{1}{4}$, with $\vec{P}, \vec{P} + \Delta\vec{P} \in \mathcal{S}$, where $C_{\delta'}$ and $C_{\delta''}$ are finite constants depending on δ' and δ'', respectively.

$(\mathscr{I}3)$ Given a positive number ν_{min}, there are numbers $r_\alpha = \nu_{min} + \mathcal{O}(\alpha) > 0$ and $\nu_{max} < 1$ such that, for $\vec{P} \in \mathcal{S} \setminus \mathcal{B}_{r_\alpha}$ and for α sufficiently small,

$$1 > \nu_{max} \geq |\vec{\nabla} E_{\vec{P}}^\sigma| \geq \nu_{min} > 0 \,, \tag{III.14}$$

uniformly in σ.

($\mathscr{I}$4) *For $\vec{P} \in \mathcal{S}$ and for any $\vec{k} \neq 0$, the following inequality holds uniformly in σ, for α small enough:*

$$E^\sigma_{\vec{P}-\vec{k}} > E^\sigma_{\vec{P}} - C_\alpha |\vec{k}| , \qquad (\text{III.15})$$

where $E^\sigma_{\vec{P}-\vec{k}} := \inf \mathrm{spec}\, H_{\vec{P}-\vec{k}}$ and $\frac{1}{3} < C_\alpha < 1$, with $C_\alpha \to \frac{1}{3}$ as $\alpha \to 0$.

($\mathscr{I}$5) *For $\vec{P} \in \mathcal{S}$, one has that*

$$\| b_{\vec{k},\lambda} \Psi^\sigma_{\vec{P}} \| \leq C\, \alpha^{1/2} \frac{\mathbf{1}_{\sigma,\Lambda}(\vec{k})}{|\vec{k}|^{3/2}} , \qquad (\text{III.16})$$

see Lemma 6.1 of [9] which can be extended to $\vec{k} \in \mathbb{R}^3$ using ($\mathscr{I}$4).
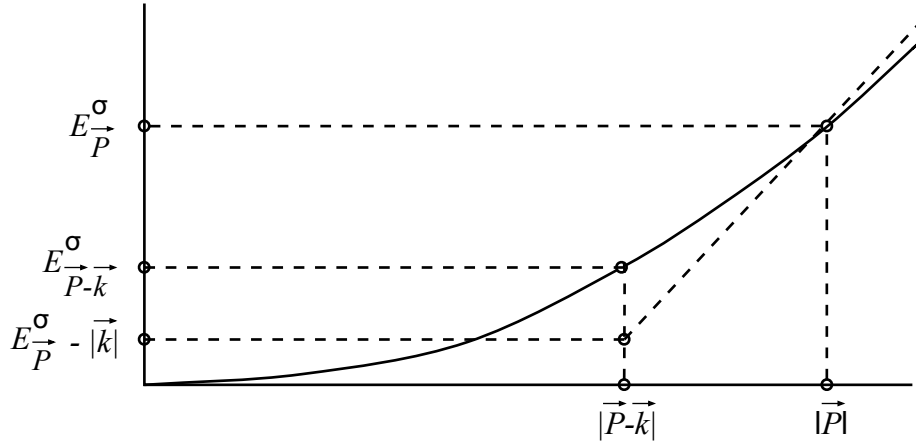

Figure 1. The condition ($\mathscr{I}$4).

The proof of statement ($\mathscr{I}$1) is given in Section IV; the proofs of statements ($\mathscr{I}$2) and ($\mathscr{I}$3) are presented in Section V. Statement ($\mathscr{I}$4) is proven in Section VI. We note that condition ($\mathscr{I}$4) plays an important rôle also in atomic and molecular bound state problems, see for instance [19].

III.1. **Remark about infrared representations.** The statement ($\mathscr{I}$5), which states that

$$\| b_{\vec{k},\lambda} \Psi^\sigma_{\vec{P}} \| \leq C\, \alpha^{1/2} \frac{\mathbf{1}_{\sigma,\Lambda}(\vec{k})}{|\vec{k}|^{3/2}} , \qquad (\text{III.17})$$

follows from the identity

$$b_{\vec{k},\lambda} \Psi^\sigma_{\vec{P}} = -\alpha^{\frac{1}{2}} \frac{\mathbf{1}_{\sigma,\Lambda}(\vec{k})}{|\vec{k}|^{\frac{1}{2}}} \frac{1}{H_{\vec{P}-\vec{k},\sigma} + |\vec{k}| - E^\sigma_{\vec{P}}} \vec{\varepsilon}_{\vec{k},\lambda} \cdot \vec{\nabla}_{\vec{P}} H^\sigma_{\vec{P}} \Psi^\sigma_{\vec{P}} \qquad (\text{III.18})$$

which is derived by using a standard "pull-through argument". Combined with the *uniform bounds on the renormalized mass* of the electron established in [8], it is used in [9] to prove the bound

$$\left\langle \Psi_{\vec{P}}^{\sigma}, \, N_f \, \Psi_{\vec{P}}^{\sigma} \right\rangle \; := \; \int d^3k \left\langle \Psi_{\vec{P}}^{\sigma}, \, b_{\vec{k},\lambda}^* b_{\vec{k},\lambda} \Psi_{\vec{P}}^{\sigma} \right\rangle \; \leq \; C\alpha(1 + |\vec{P}|^2 |\ln(\sigma)|) \qquad \text{(III.19)}$$

on the expected number of photons in the ground state $\Psi_{\vec{P}}^{\sigma}$. Without using the uniform bounds on the renormalized mass, one obtains the weaker upper bound (III.17). Important implications of this result, analyzed in [9] and used in [10] can be summarized as follows.

Let $\mathfrak{A}_\rho$ denote the C^*-algebra of bounded operators on the Fock space $\mathcal{F}(L^2((\mathbb{R}^3 \setminus B_\rho) \times \mathbb{Z}_2))$, where $B_\rho = \{\vec{k} \in \mathbb{R}^3 \, | \, |\vec{k}| \leq \rho\}$, and let $\mathfrak{A}$ denote the C^*-algebra $\mathfrak{A} := \overline{\bigvee_{\rho>0} \mathfrak{A}_\rho}^{\|\cdot\|_{op}}$, where the closure is taken in the operator norm. We define the state $\omega_{\vec{P}}^{\sigma} := \left\langle \Psi_{\vec{P}}^{\sigma}, \, (\,\cdot\,) \Psi_{\vec{P}}^{\sigma} \right\rangle$ on $\mathfrak{A}$. In [9], it is proven that there exists a well-defined state $\omega_{\vec{P}}$ on $\mathfrak{A}$ corresponding to the weak-* limit of $\omega_{\vec{P}}^{\sigma}$ as $\sigma \to 0$; i.e., any sequence $(\sigma_n)_{n\in\mathbb{N}_0}$ converging to zero contains a subsequence $(\sigma_{n_j})_{j\in\mathbb{N}_0}$ converging to zero such that

$$\omega_{\vec{P}}(A) \; = \; \lim_{j\to\infty} \omega_{\vec{P},\sigma_{n_j}}(A), \qquad \text{(III.20)}$$

for all $A \in \mathfrak{A}$. The proof of this statement in [9] is based on arguments combining (III.16) with the uniform bounds on the renormalized mass of the electron established in [8].

The representation of $\mathfrak{A}$ corresponding to $\omega_{\vec{P}}$ obtained by the GNS construction can be characterized as follows. Let $\alpha_{\vec{P}} : \mathfrak{A} \to \mathfrak{A}$ denote the *Bogoliubov automorphism* defined by

$$\alpha_{\vec{P}}(A) \; = \; \lim_{\sigma\to 0} W_\sigma(\vec{\nabla} E_{\vec{P}}^{\sigma}) \, A \, W_\sigma^*(\vec{\nabla} E_{\vec{P}}^{\sigma}) \qquad \text{(III.21)}$$

with $W_\sigma(\vec{\nabla} E_{\vec{P}}^{\sigma})$ defined in (III.4), and $A \in \mathfrak{A}$. Then the GNS representation $\pi_{\vec{P}}$ of $\mathfrak{A}$ is quasi-equivalent to $\pi_{Fock} \circ \alpha_{\vec{P}}$, where π_{Fock} denotes the Fock representation. In particular, $\pi_{\vec{P}}$ is a *coherent* infrared representation unitarily inequivalent to π_{Fock}, for $\vec{P} \neq \vec{0}$, and unitarily equivalent to π_{Fock} if $\vec{P} = \vec{0}$. For proofs see [9].

IV. PROOF OF ($\mathscr{I}1$) IN THEOREM III.1

In this section, we prove the statements ($\mathscr{I}1$) in Theorem III.1. This is the most involved part of our analysis.

IV.1. **Construction of the sequence $\{\Psi_{\vec{P}}^{\sigma_j}\}$ of ground states.** We recall the definition of the fiber Hamiltonian from (II.24),

$$H_{\vec{P}}^{\sigma_j} = \frac{\left(\vec{P} - \vec{P}^f + \alpha^{1/2}\vec{A}^{\sigma_j}\right)^2}{2} + H^f. \tag{IV.1}$$

It acts on a fixed fiber space $\mathcal{H}_{\vec{P}}$, with $\vec{P} \in \mathcal{S}$, where

$$\vec{A}^{\sigma_j} = \sum_{\lambda=\pm} \int_{\mathcal{B}_\Lambda \setminus \mathcal{B}_{\sigma_j}} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{\vec{\varepsilon}_{\vec{k},\lambda} b_{\vec{k},\lambda}^* + \vec{\varepsilon}_{\vec{k},\lambda}^* b_{\vec{k},\lambda}\right\} \tag{IV.2}$$

contains an infrared cutoff at

$$\sigma_j := \Lambda \epsilon^j \quad, \quad j \in \mathbb{N}, \tag{IV.3}$$

with $0 < \epsilon < 1$ to be fixed later (we recall that $\Lambda \approx 1$). As we will see, the Hamiltonian $H_{\vec{P}}^{\sigma_j}$ has a *unique* ground state $\Psi_{\vec{P}}^{\sigma_j}$, which we construct below using an approach developed in [22].

We define the Fock spaces

$$\mathcal{F}_{\sigma_j} := \mathcal{F}^b(L^2((\mathbb{R}^3 \setminus \mathcal{B}_{\sigma_j}) \times \mathbb{Z}_2)) \quad \text{and} \quad \mathcal{F}_{\sigma_{j+1}}^{\sigma_j} := \mathcal{F}^b(L^2((\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}) \times \mathbb{Z}_2)).$$

It is clear that

$$\mathcal{F}_{\sigma_{j+1}} = \mathcal{F}_{\sigma_j} \otimes \mathcal{F}_{\sigma_{j+1}}^{\sigma_j}, \tag{IV.4}$$

and that the Hamiltonians $\{H_{\vec{P}}^{\sigma_j} \,|\, j \in \mathbb{N}\}$ are related to one another by

$$H_{\vec{P}}^{\sigma_{j+1}} = H_{\vec{P}}^{\sigma_j} + \Delta H_{\vec{P}}|_{\sigma_{j+1}}^{\sigma_j}, \tag{IV.5}$$

where

$$\Delta H_{\vec{P}}|_{\sigma_{j+1}}^{\sigma_j} := \alpha^{\frac{1}{2}} \vec{\nabla}_{\vec{P}} H_{\vec{P}}^{\sigma_j} \cdot \vec{A}|_{\sigma_{j+1}}^{\sigma_j} + \frac{\alpha}{2} (\vec{A}|_{\sigma_{j+1}}^{\sigma_j})^2 \tag{IV.6}$$

and

$$\vec{A}|_{\sigma_{j+1}}^{\sigma_j} := \sum_{\lambda=\pm} \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} \frac{d^3k}{\sqrt{|\vec{k}|}} \left\{\vec{\varepsilon}_{\vec{k},\lambda} b_{\vec{k},\lambda}^* + \vec{\varepsilon}_{\vec{k},\lambda}^* b_{\vec{k},\lambda}\right\}. \tag{IV.7}$$

For α sufficiently small and $\vec{P} \in \mathcal{S}$, we construct ground state vectors $\{\Psi_{\vec{P}}^{\sigma_j}\}$ of the Hamiltonians $\{H_{\vec{P}}^{\sigma_j}\}$, $j \in \mathbb{N}$. We will prove the following results, adapting recursive arguments developed in [22].

We introduce four parameters ϵ, ρ^+, ρ^-, μ with the properties that

$$0 < \rho^- < \mu < \rho^+ < 1 - C_\alpha < \frac{2}{3} \tag{IV.8}$$

$$0 < \epsilon < \frac{\rho^-}{\rho^+} \tag{IV.9}$$

where C_α is defined in (III.15). Then, for α small enough depending on Λ, ϵ, ρ^-, μ, ρ^+, we prove:

- The infimum of the spectrum of $H_{\vec{P}}^{\sigma_j}$ on $\mathcal{F}_{\sigma_j}$, which we denote by $E_{\vec{P}}^{\sigma_j}$, is an isolated, simple eigenvalue which is separated from the rest of the spectrum by a gap $\rho^- \sigma_j$ or larger.

- $E_{\vec{P}}^{\sigma_j}$ is also the ground state energy of the operators $H_{\vec{P}}^{\sigma_j}$ and $H_{\vec{P}}^{\sigma_j} - H^f|_{\sigma_{j+1}}^{\sigma_j}$ on $\mathcal{F}_{\sigma_{j+1}}$, where $H^f|_{\sigma_{j+1}}^{\sigma_j}$ is defined in Eq (IV.21). $E_{\vec{P}}^{\sigma_j}$ is, in this case, a simple eigenvalue which is separated from the rest of the spectrum by a gap $\rho^+ \sigma_{j+1}$ or larger.

- The ground state energies $E_{\vec{P}}^{\sigma_j}$ and $E_{\vec{P}}^{\sigma_{j+1}}$ of the Hamiltonians $H_{\vec{P}}^{\sigma_j}$ and $H_{\vec{P}}^{\sigma_{j+1}}$, respectively, acting on the same space $\mathcal{F}_{\sigma_{j+1}}$ satisfy

$$0 \leq E_{\vec{P}}^{\sigma_{j+1}} \leq E_{\vec{P}}^{\sigma_j} + c\,\alpha\,\sigma_j^2 \,, \tag{IV.10}$$

where c is a constant independent of j and α but Λ-dependent.

We recursively construct the ground state vector, $\Psi_{\vec{P}}^\sigma$ (which is at this stage not normalized), of $H_{\vec{P}}^\sigma$ on $\mathcal{F}_\sigma$, as follows. In the initial step, we set $\Psi_{\vec{P}}^{\sigma_0} = \Omega_f$.

Let $\Psi_{\vec{P}}^{\sigma_j}$ denote the ground state of the Hamiltonian $H_{\vec{P}}^{\sigma_j}$ on $\mathcal{F}_{\sigma_j}$ with non-degenerate eigenvalue $E_{\vec{P}}^{\sigma_j}$ and a spectral gap at least as large as $\rho^- \sigma_j$. We note that $E_{\vec{P}}^{\sigma_0} \equiv \frac{\vec{P}^2}{2}$ is a non-degenerate eigenvalue of $H_{\vec{P}}^{\sigma_0}$ on $\mathcal{F}_{\sigma_0}$, and that

$$\mathrm{gap}(H_{\vec{P}}^{\sigma_0}|_{\mathcal{F}_{\sigma_0}}) \geq \frac{2}{3} \sigma_0 \geq \rho^- \sigma_0 \,. \tag{IV.11}$$

We observe that

$$\Psi_{\vec{P}}^{\sigma_j} \otimes \Omega_f \ \in \ \mathcal{F}_{\sigma_{j+1}} \ = \ \mathcal{F}_{\sigma_j} \otimes \mathcal{F}_{\sigma_{j+1}}^{\sigma_j} \,, \tag{IV.12}$$

where

$$\| \Psi_{\vec{P}}^{\sigma_j} \otimes \Omega_f \| \ = \ \| \Psi_{\vec{P}}^{\sigma_j} \| \,, \tag{IV.13}$$

is an eigenvector of $H_{\vec{P}}^{\sigma_j}|_{\mathcal{F}_{\sigma_{j+1}}}$. In (IV.12), Ω_f stands for the vacuum state in $\mathcal{F}_{\sigma_{j+1}}^{\sigma_j}$ (if not further specified otherwise, Ω_f denotes the vacuum state in any of the photon Fock spaces).

Moreover, we note that (IV.12) is the ground state of $H_{\vec{P}}^{\sigma_j}$ restricted to $\mathcal{F}_{\sigma_{j+1}}$, because

$$
\begin{aligned}
\inf \operatorname{spec}&\Big(H_{\vec{P}}^{\sigma_j}\big|_{\mathcal{F}_{\sigma_{j+1}} \ominus \{\mathbb{C}\Psi_{\vec{P}}^{\sigma_j} \otimes \Omega_f\}} - E_{\vec{P}}^{\sigma_j} \Big)\\
&\geq \; \min\Big\{ \rho^- \sigma_j \,, \; \inf_{\vec{k} \in \mathbb{R}^3 \setminus \mathcal{B}_{\sigma_{j+1}}} \big\{ E_{\vec{P}+\vec{k}}^{\sigma_j} + |\vec{k}| - E_{\vec{P}}^{\sigma_j} \big\} \Big\}\\
&\geq \; \min\Big\{ \rho^- \sigma_j \,, \; (1 - C_\alpha)\, \sigma_{j+1} \Big\}\\
&\geq \; \rho^+ \sigma_{j+1} > 0 \,,
\end{aligned}
\tag{IV.14}
$$

where $\mathcal{F}_{\sigma_{j+1}} \ominus \{\mathbb{C}\Psi_{\vec{P}}^{\sigma_j} \otimes \Omega_f\}$ is the orthogonal complement in $\mathcal{F}_{\sigma_{j+1}}$ of the one-dimensional subspace $\{\mathbb{C}\Psi_{\vec{P}}^{\sigma_j} \otimes \Omega_f\}$. We use property $(\mathscr{I}4)$ to pass from the second to the third line in (IV.14); for a proof of property $(\mathscr{I}4)$ see Section VI.

Consequently, the spectral gap of $H_{\vec{P}}^{\sigma_j}$ restricted to $\mathcal{F}_{\sigma_{j+1}}$ is bounded from below by

$$
\operatorname{gap}(H_{\vec{P}}^{\sigma_j}\big|_{\mathcal{F}_{\sigma_{j+1}}}) \geq \rho^+ \sigma_{j+1} \,,
\tag{IV.15}
$$

where

$$
\operatorname{gap}(H) := \inf\{\operatorname{spec}(H) \setminus \{\inf \operatorname{spec}(H)\}\} - \inf \operatorname{spec}(H) \,.
\tag{IV.16}
$$

We define the contour $\gamma_{\sigma_{j+1}} := \{z_{j+1} \in \mathbb{C} \,|\, |z_{j+1} - E_{\vec{P}}^{\sigma_j}| = \mu \sigma_{j+1}\}$ which is the boundary of a closed disc that contains the non-degenerate ground state eigenvalue $E_{\vec{P}}^{\sigma_j}$ of $H_{\vec{P}}^{\sigma_j}$, but no other elements of the spectrum of $H_{\vec{P}}^{\sigma_j}$; see also Figure 2 below.

Then we define

$$
\begin{aligned}
\Psi_{\vec{P}}^{\sigma_{j+1}} \;:=\; & \frac{1}{2\pi i} \oint_{\gamma_{j+1}} dz_{j+1} \frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \Psi_{\vec{P}}^{\sigma_j} \otimes \Omega_f\\
=\; & \sum_{n \geq 0} \frac{1}{2\pi i} \oint_{\gamma_{j+1}} dz_{j+1} \frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}}\\
& \Big(-\Delta H_{\vec{P}}\big|_{\sigma_{j+1}}^{\sigma_j} \frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big)^n \Psi_{\vec{P}}^{\sigma_j} \otimes \Omega_f \,,
\end{aligned}
\tag{IV.17}
$$

which is, by construction, the ground state eigenvector of $H_{\vec{P}}^{\sigma_{j+1}}\big|_{\mathcal{F}_{\sigma_{j+1}}}$. The associated ground state eigenvalue $E_{\vec{P}}^{\sigma_{j+1}}$, with $H_{\vec{P}}^{\sigma_{j+1}} \Psi_{\vec{P}}^{\sigma_{j+1}} = E_{\vec{P}}^{\sigma_{j+1}} \Psi_{\vec{P}}^{\sigma_{j+1}}$, is non-degenerate. To control the expansion in (IV.17) for sufficiently small α, we show that, for $z_{j+1} \in \gamma_{j+1}$,

$$
\begin{aligned}
\sup_{z_{j+1} \in \gamma_{j+1}} &\Big\| \Big(\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big)^{\frac{1}{2}} \Delta H_{\vec{P}}\big|_{\sigma_{j+1}}^{\sigma_j} \Big(\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big)^{\frac{1}{2}} \Big\|_{\mathcal{F}_{\sigma_{j+1}}}\\
&\leq C \, \frac{\alpha^{1/2}}{\epsilon^{1/2}\, [\min\{(\rho^+ - \mu), \mu\}]^{1/2}} \,,
\end{aligned}
\tag{IV.18}
$$

where the constant on the r.h.s. depends on $\vec{P}$ and Λ. The largest value of α such that $(IV.18) < 1$ may depend on ϵ and μ. The estimate (IV.18) is obtained from the following bounds, which depend critically on the spectral gap (as in the model treated in [22]):

i) For $z_{j+1} \in \gamma_{j+1}$,

$$\sup_{z_{j+1} \in \gamma_{j+1}} \left\| \left(\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \right)^{\frac{1}{2}} (\vec{\nabla}_{\vec{P}} H_{\vec{P}}^{\sigma_j})^2 \left(\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \right)^{\frac{1}{2}} \right\|_{\mathcal{F}_{\sigma_{j+1}}}$$
$$\leq \mathcal{O}\left(\frac{1}{\epsilon^{j+1} \min\{(\rho^+ - \mu), \mu\}} \right) \tag{IV.19}$$

where the implicit constant depends on $\vec{P}$ and Λ.

ii) Writing $(\vec{A}|_{\sigma_{j+1}}^{\sigma_j})^-$ and $(\vec{A}|_{\sigma_{j+1}}^{\sigma_j})^+$ for the parts in $\vec{A}|_{\sigma_{j+1}}^{\sigma_j}$ which contain annihilation- and creation operators, respectively, we have that

$$\| (\vec{A}|_{\sigma_{j+1}}^{\sigma_j})^- \psi \| \leq \left(\int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} \frac{d^3k}{|\vec{k}|^2} \right)^{1/2} \| (H^f|_{\sigma_{j+1}}^{\sigma_j})^{1/2} \psi \|$$
$$\leq c \, \epsilon^{\frac{j}{2}} \| (H^f|_{\sigma_{j+1}}^{\sigma_j})^{1/2} \psi \| , \tag{IV.20}$$

where

$$H^f|_{\sigma_{j+1}}^{\sigma_j} := \sum_{\lambda} \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} d^3k \, |\vec{k}| \, b_{\vec{k},\lambda}^* \, b_{\vec{k},\lambda} , \tag{IV.21}$$

with ψ in the domain of $(H^f|_{\sigma_{j+1}}^{\sigma_j})^{1/2}$. Moreover,

$$0 < [(\vec{A}|_{\sigma_{j+1}}^{\sigma_j})^-, (\vec{A}|_{\sigma_{j+1}}^{\sigma_j})^+] \leq c' \, \epsilon^{2j} , \tag{IV.22}$$

where the constants c, c' are proportional to $\Lambda^{1/2}$ and Λ, respectively.

iii) For $z_{j+1} \in \gamma_{j+1}$,

$$\sup_{z_{j+1} \in \gamma_{j+1}} \left\| \left(\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \right)^{\frac{1}{2}} H^f|_{\sigma_{j+1}}^{\sigma_j} \left(\frac{1}{H_{\vec{P}}^{\sigma_j} - z_{j+1}} \right)^{\frac{1}{2}} \right\|_{\mathcal{F}_{\sigma_{j+1}}} \leq \mathcal{O}\left(\frac{1}{\rho^+ - \mu} \right) , \tag{IV.23}$$

which follows from the spectral theorem for the commuting operators $H^f|_{\sigma_{j+1}}^{\sigma_j}$ and $H_{\vec{P}}^{\sigma_j}$.

Using (IV.18), one concludes that

$$\| \Psi_{\vec{P}}^{\sigma_{j+1}} - \Psi_{\vec{P}}^{\sigma_j} \|_{\mathcal{F}} \leq C \, \alpha^{\frac{1}{2}} \| \Psi_{\vec{P}}^{\sigma_j} \|_{\mathcal{F}} , \tag{IV.24}$$

with C uniform in j, such that, for α small enough,

$$\| \Psi_{\vec{P}}^{\sigma_{j+1}} \|_{\mathcal{F}} \geq C' \| \Psi_{\vec{P}}^{\sigma_j} \|_{\mathcal{F}} , \tag{IV.25}$$

for a constant $C' > 0$ independent of j. In particular, the vector constructed in (IV.17) is indeed non-zero.

Because of (IV.10), which follows from a variational argument, we find that, for α small enough and Λ-dependent, but independent of j,

$$\text{gap}(H_{\vec{P}}^{\sigma_{j+1}}|_{\mathcal{F}_{\sigma_{j+1}}}) \geq \mu\sigma_{j+1} - c\,\alpha\,\sigma_j^2 \geq \rho^-\sigma_{j+1}. \qquad (\text{IV}.26)$$

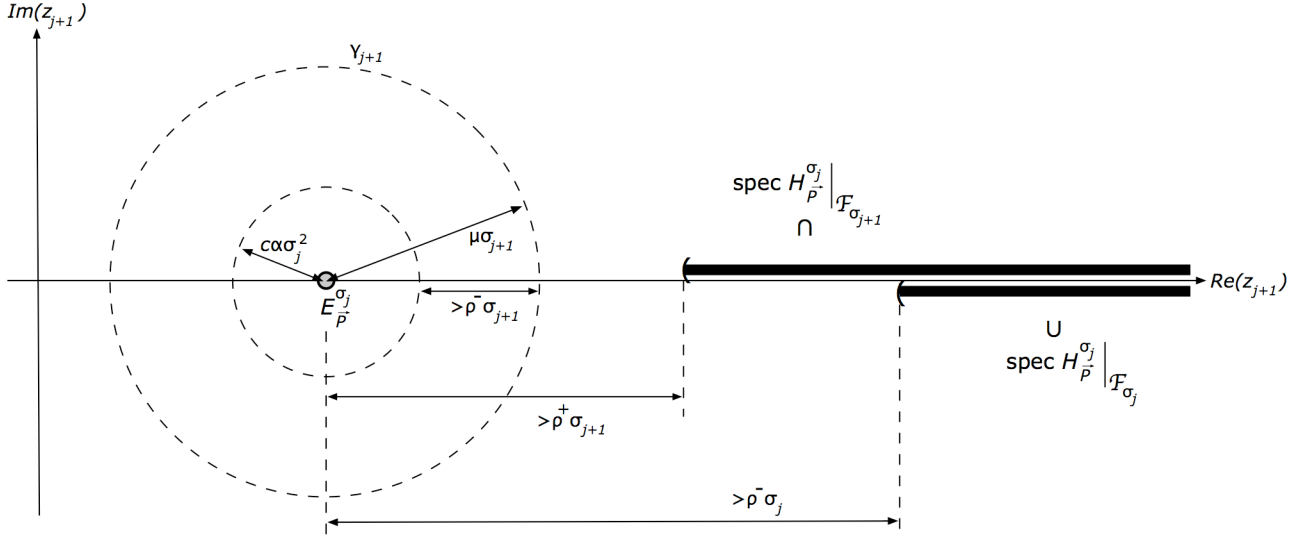This estimate allows us to proceed to the next scale.

Figure 2. The contour integral in the energy plane.

It easily follows from the previous results that $E_{\vec{P}}^{\sigma_j}$ is simple and isolated, and $(H_{\vec{P}}^{\sigma_j})_{\vec{P}\in\mathcal{S}}$ is an analytic family of type A. In particular, this allows us to express $\vec{\nabla}E_{\vec{P}}^{\sigma_j}$ as a function of $\vec{P}$ by using the Feynman-Hellman formula; see (IV.27) below.

IV.2. **Transformed Hamiltonians and the sequence of ground states $\{\Phi_{\vec{P}}^{\sigma_j}\}$.** In this section, we consider the Hamiltonians obtained from $\{H_{\vec{P}}^{\sigma_j}\}$ after a $j-$dependent Bogoliubov transformation of the photon variables. In the limit $j \to \infty$, this transformation coincides with the one identified in [9], which provides the correct representation of the photon degrees of freedom for which the Hamiltonian $H_{\vec{P}}$ has a ground state.

IV.2.1. *Bogoliubov transformation and canonical form of the Hamiltonian.* The Feynman-Hellman formula yields

$$\vec{\nabla}E_{\vec{P}}^{\sigma_j} = \vec{P} - \big\langle \vec{P}^f - \alpha^{1/2}\vec{A}^{\sigma_j} \big\rangle_{\Psi_{\vec{P}}^{\sigma_j}}, \qquad (\text{IV}.27)$$

where

$$\big\langle \vec{P}^f - \alpha^{1/2}\vec{A}^{\sigma_j} \big\rangle_{\Psi_{\vec{P}}^{\sigma_j}} := \frac{\big\langle \Psi_{\vec{P}}^{\sigma_j} \, , \, (\vec{P}^f - \alpha^{1/2}\vec{A}^{\sigma_j})\,\Psi_{\vec{P}}^{\sigma_j} \big\rangle}{\big\langle \Psi_{\vec{P}}^{\sigma_j} \, , \, \Psi_{\vec{P}}^{\sigma_j} \big\rangle} \, . \tag{IV.28}$$

We define

$$\begin{aligned}
\vec{\beta}^{\sigma_j} &:= \vec{P}^f - \alpha^{1/2}\vec{A}^{\sigma_j} \\[4pt]
\delta_{\vec{P}}^{\sigma_j}(\widehat{k}) &:= 1 - \widehat{k}\cdot\vec{\nabla}E_{\vec{P}}^{\sigma_j} \\[4pt]
c_{\vec{k},\lambda}^* &:= b_{\vec{k},\lambda}^* + \alpha^{\frac{1}{2}}\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} \\[4pt]
c_{\vec{k},\lambda} &:= b_{\vec{k},\lambda} + \alpha^{\frac{1}{2}}\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} \, .
\end{aligned} \tag{IV.29}$$

We then rewrite $H_{\vec{P}}^{\sigma_j}$ as

$$H_{\vec{P}}^{\sigma_j} = \frac{\big(\vec{P} - \vec{\beta}^{\sigma_j}\big)^2}{2} + H^f \, , \tag{IV.30}$$

and

$$\vec{P} = \vec{\nabla}E_{\vec{P}}^{\sigma_j} + \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}} \, , \tag{IV.31}$$

thus obtaining

$$\begin{aligned}
H_{\vec{P}}^{\sigma_j} &= \frac{\vec{P}^2}{2} - (\vec{\nabla}E_{\vec{P}}^{\sigma_j} + \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}})\cdot\vec{\beta}^{\sigma_j} + \frac{(\vec{\beta}^{\sigma_j})^2}{2} + H^f \\[6pt]
&= \frac{\vec{P}^2}{2} + \frac{(\vec{\beta}^{\sigma_j})^2}{2} - \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}}\cdot\vec{\beta}^{\sigma_j} \\[4pt]
&\quad + \sum_\lambda \int_{\mathbb{R}^3\setminus(\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j})} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\, b_{\vec{k},\lambda}^* b_{\vec{k},\lambda}\, d^3k \\[4pt]
&\quad + \sum_\lambda \int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\, c_{\vec{k},\lambda}^* c_{\vec{k},\lambda}\, d^3k \\[4pt]
&\quad - \alpha\sum_\lambda \int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})}\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})}\, d^3k \, .
\end{aligned} \tag{IV.32}$$

Adding and subtracting $\frac{1}{2}\langle\vec{\beta}^{\sigma_j}\rangle^2_{\Psi^{\sigma_j}_{\vec{P}}}$, one gets

$$
\begin{aligned}
H^{\sigma_j}_{\vec{P}} \;=\; & \frac{\vec{P}^2}{2} - \frac{\langle\vec{\beta}^{\sigma_j}\rangle^2_{\Psi^{\sigma_j}_{\vec{P}}}}{2} + \frac{\big(\vec{\beta}^{\sigma_j} - \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi^{\sigma_j}_{\vec{P}}}\big)^2}{2} \\
& + \sum_\lambda \int_{\mathbb{R}^3\setminus(\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j})} |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k})\, b^*_{\vec{k},\lambda} b_{\vec{k},\lambda} d^3k \\
& + \sum_\lambda \int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}} |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k})\, c^*_{\vec{k},\lambda} c_{\vec{k},\lambda} d^3k \\
& -\alpha \sum_\lambda \int_{\mathcal{B}_k\setminus\mathcal{B}_{\sigma_j}} |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) \frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\varepsilon}^*_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\widehat{k})} \frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\widehat{k})}\, d^3k\,.
\end{aligned}
\tag{IV.33}
$$

Next, we apply the Bogoliubov transformation

$$
\begin{aligned}
b^*_{\vec{k},\lambda} \;\longrightarrow\; & W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})b^*_{\vec{k},\lambda}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) = b^*_{\vec{k},\lambda} - \alpha^{\frac{1}{2}}\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\varepsilon}^*_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\widehat{k})} \\
b_{\vec{k},\lambda} \;\longrightarrow\; & W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})b_{\vec{k},\lambda}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) = b_{\vec{k},\lambda} - \alpha^{\frac{1}{2}}\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\widehat{k})}
\end{aligned}
\tag{IV.34}
$$

for $\vec{k}\in\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}$, where

$$
W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) := \exp\Big(\alpha^{\frac{1}{2}}\sum_\lambda\int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_j}} d^3k\,\frac{\vec{\nabla}E^{\sigma_j}_{\vec{P}}}{|\vec{k}|^{\frac{3}{2}}\delta^{\sigma_j}_{\vec{P}}(\widehat{k})}\cdot(\vec{\varepsilon}_{\vec{k},\lambda}b^*_{\vec{k},\lambda} - h.c.)\Big)\,.
\tag{IV.35}
$$

It is evident that W_{σ_j} acts as the identity on $\mathcal{F}^b(L^2(\mathcal{B}_{\sigma_j}\times\mathbb{Z}_2))$ and on $\mathcal{F}^b(L^2((\mathbb{R}^3\setminus\mathcal{B}_\Lambda)\times\mathbb{Z}_2))$. Moreover, we define the vector operators

$$
\begin{aligned}
\vec{\Pi}^{\sigma_j}_{\vec{P}} \;:=\; & W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\,\vec{\beta}^{\sigma_j}\,W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}}) \\
& - \langle W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\,\vec{\beta}^{\sigma_j}\,W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\rangle_{\Omega_f}\,,
\end{aligned}
\tag{IV.36}
$$

noting that

$$
\begin{aligned}
\langle\vec{\beta}^{\sigma_j}\rangle_{\Psi^{\sigma_j}_{\vec{P}}} \;=\; & \vec{P} - \vec{\nabla}E^{\sigma_j}_{\vec{P}} \\
\;=\; & \frac{\big\langle\Phi^{\sigma_j}_{\vec{P}}\,,\,\vec{\Pi}^{\sigma_j}_{\vec{P}}\Phi^{\sigma_j}_{\vec{P}}\big\rangle}{\big\langle\Phi^{\sigma_j}_{\vec{P}}\,,\,\Phi^{\sigma_j}_{\vec{P}}\big\rangle} + \langle W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\,\vec{\beta}^{\sigma_j}\,W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\rangle_{\Omega_f}\,,
\end{aligned}
$$

$$
\tag{IV.37}
$$

where $\Phi^{\sigma_j}_{\vec{P}}$ is the ground state of the Bogoliubov-transformed Hamiltonian

$$
K^{\sigma_j}_{\vec{P}} := W_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})H^{\sigma_j}_{\vec{P}}W^*_{\sigma_j}(\vec{\nabla}E^{\sigma_j}_{\vec{P}})\,.
\tag{IV.38}
$$

It thus follows that

$$W_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})\,\vec{\beta}^{\sigma_j}\,W_{\sigma_j}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_j}) - \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}} \;=\; \vec{\Pi}_{\vec{P}}^{\sigma_j} - \langle\vec{\Pi}_{\vec{P}}^{\sigma_j}\rangle_{\Phi_{\vec{P}}^{\sigma_j}}\,. \tag{IV.39}$$

As in [22], it is convenient to write $K_{\vec{P}}^{\sigma_j}$ in the "canonical" form

$$K_{\vec{P}}^{\sigma_j} \;=\; \frac{(\vec{\Gamma}_{\vec{P}}^{\sigma_j})^2}{2} + \sum_\lambda \int_{\mathbb{R}^3} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\, b_{\vec{k},\lambda}^* b_{\vec{k},\lambda}\, d^3k + \mathcal{E}_{\vec{P}}^{\sigma_j}\,, \tag{IV.40}$$

where

$$\vec{\Gamma}_{\vec{P}}^{\sigma_j} \;:=\; \vec{\Pi}_{\vec{P}}^{\sigma_j} - \langle\vec{\Pi}_{\vec{P}}^{\sigma_j}\rangle_{\Phi_{\vec{P}}^{\sigma_j}}\,, \tag{IV.41}$$

so that

$$\langle\vec{\Gamma}_{\vec{P}}^{\sigma_j}\rangle_{\Phi_{\vec{P}}^{\sigma_j}} \;=\; 0\,, \tag{IV.42}$$

and

$$\mathcal{E}_{\vec{P}}^{\sigma_j} \;:=\; \frac{\vec{P}^2}{2} - \frac{(\vec{P}-\vec{\nabla}E_{\vec{P}}^{\sigma_j})^2}{2} \tag{IV.43}$$
$$- \alpha \sum_\lambda \int_{\mathcal{B}_\Lambda\backslash\mathcal{B}_{\sigma_j}} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k}) \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})}\, d^3k\,.$$

One arrives at (IV.40) using

$$W_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})\, c_{\vec{k},\lambda}^*\, W_{\sigma_j}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_j}) \;=\; b_{\vec{k},\lambda}^*\,,$$
$$W_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})\, c_{\vec{k},\lambda}\, W_{\sigma_j}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_j}) \;=\; b_{\vec{k},\lambda}\,, \tag{IV.44}$$

for $\vec{k}\in\mathcal{B}_\Lambda\backslash\mathcal{B}_{\sigma_j}$. The Hamiltonian $K_{\vec{P}}^{\sigma_j}$ has a structure similar to the Bogoliubov-transformed Nelson Hamiltonian in [22].

Following ideas of [22], we define the intermediate Hamiltonian

$$\widehat{K}_{\vec{P}}^{\sigma_{j+1}} \;:=\; W_{\sigma_{j+1}}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})H_{\vec{P}}^{\sigma_{j+1}}W_{\sigma_{j+1}}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_j})\,, \tag{IV.45}$$

where

$$W_{\sigma_{j+1}}(\vec{\nabla}E_{\vec{P}}^{\sigma_j}) \;:=\; \exp\Big(\alpha^{\frac{1}{2}}\sum_\lambda \int_{\mathcal{B}_\Lambda\backslash\mathcal{B}_{\sigma_{j+1}}} d^3k\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})}\cdot(\vec{\varepsilon}_{\vec{k},\lambda}b_\lambda^*(\vec{k})-h.c.)\Big)\,, \tag{IV.46}$$

and split it into different terms similarly as for $K_{\vec{P}}^{\sigma_j}$. We write

$$H_{\vec{P}}^{\sigma_{j+1}} \;=\; \frac{\vec{P}^2}{2} - \vec{P}\cdot\vec{\beta}^{\sigma_{j+1}} + \frac{(\vec{\beta}^{\sigma_{j+1}})^2}{2} + H^f\,, \tag{IV.47}$$

and replace $\vec{P}$ by $\vec{\nabla}E_{\vec{P}}^{\sigma_j} + \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}}$, thus obtaining

$$
\begin{aligned}
H_{\vec{P}}^{\sigma_{j+1}} \;=\; & \frac{\vec{P}^2}{2} - \big(\vec{\nabla}E_{\vec{P}}^{\sigma_j} + \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}}\big)\cdot\vec{\beta}^{\sigma_{j+1}} + \frac{(\vec{\beta}^{\sigma_{j+1}})^2}{2} + H^f \\
\;=\; & \frac{\vec{P}^2}{2} + \frac{(\vec{\beta}^{\sigma_{j+1}})^2}{2} - \langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}}\cdot\vec{\beta}^{\sigma_{j+1}} \\
& + \sum_\lambda \int_{\mathbb{R}^3\setminus(\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_{j+1}})} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\, b_{\vec{k},\lambda}^* b_{\vec{k},\lambda} d^3k \\
& + \sum_\lambda \int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_{j+1}}} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\, c_{\vec{k},\lambda}^* c_{\vec{k},\lambda} d^3k \\
& -\alpha \sum_\lambda \int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_{j+1}}} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k}) \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} d^3k\,.
\end{aligned}
\tag{IV.48}
$$

We add and subtract $\frac{1}{2}\langle\vec{\beta}^{\sigma_j}\rangle_{\Psi_{\vec{P}}^{\sigma_j}}^2$, and apply a Bogoliubov transformation by conjugating with the unitary operator $W_{\sigma_{j+1}}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})$. Formally, we find that

$$
\widehat{K}_{\vec{P}}^{\sigma_{j+1}} \;=\; \frac{\big(\vec{\Gamma}_{\vec{P}}^{\sigma_j} + \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} + \vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j}\big)^2}{2}
\tag{IV.49}
$$
$$
+ \sum_\lambda \int_{\mathbb{R}^3} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\, b_{\vec{k},\lambda}^* b_{\vec{k},\lambda} d^3k \;+\; \widehat{\mathcal{E}}_{\vec{P}}^{\sigma_j}
$$

where

$$
\begin{aligned}
\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} \;:=\; & -\alpha^{\frac{1}{2}} \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\setminus\mathcal{B}_{\sigma_{j+1}}} \vec{k}\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^* b_{\vec{k},\lambda} + h.c.}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} d^3k \\
& -\alpha^{\frac{1}{2}} \vec{A}|_{\sigma_{j+1}}^{\sigma_j}
\end{aligned}
\tag{IV.50}
$$

$$
\begin{aligned}
\vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j} \;:=\; & \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\setminus\mathcal{B}_{\sigma_{j+1}}} \vec{k}\,\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^* \; \vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3(\delta_{\vec{P}}^{\sigma_j}(\widehat{k}))^2} d^3k \\
& +\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\setminus\mathcal{B}_{\sigma_{j+1}}} \big[\vec{\varepsilon}_{\vec{k},\lambda}\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + h.c.\big]\frac{d^3k}{\sqrt{|\vec{k}|}}
\end{aligned}
\tag{IV.51}
$$

$$
\begin{aligned}
\widehat{\mathcal{E}}_{\vec{P}}^{\sigma_j} \;:=\; & \frac{\vec{P}^2}{2} - \frac{(\vec{P} - \vec{\nabla}E_{\vec{P}}^{\sigma_j})^2}{2} \\
& -\alpha \sum_\lambda \int_{\mathcal{B}_\Lambda\setminus\mathcal{B}_{\sigma_{j+1}}} |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k})\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})}\frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j}\cdot\vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} d^3k\,.
\end{aligned}
\tag{IV.52}
$$

For details on the derivation of (IV.49) and for the proof that (IV.40) and (IV.49) hold in the operator sense (and not only formally), we refer to Lemmata A.1 and A.2 in the Appendix.

We also define the operators

$$\widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_j} \;:=\; W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}}) W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \vec{\Pi}_{\vec{P}}^{\sigma_j} W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}}) \tag{IV.53}$$

and

$$\widehat{\vec{\Gamma}}_{\vec{P}}^{\sigma_j} \;:=\; \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_j} \;-\; \langle \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_j} \rangle_{\widehat{\Phi}_{\vec{P}}^{\sigma_j}} \,, \tag{IV.54}$$

which are used in the proofs in the next section. Here, $\widehat{\Phi}_{\vec{P}}^{\sigma_j}$ denotes the ground state vector of the Hamiltonian $\widehat{K}_{\vec{P}}^{\sigma_j} := W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}}) K_{\vec{P}}^{\sigma_j} W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}})$.

IV.3. Construction and convergence of $\{\Phi_{\vec{P}}^{\sigma_j}\}$.

In this section, we construct a sequence $\{\Phi_{\vec{P}}^{\sigma_j} \,|\, j \in \mathbb{N}\}$ of ground state vectors of the (Bogoliubov-transformed) Hamiltonians $K_{\vec{P}}^{\sigma_j}$, and establish the existence of

$$\Phi_{\vec{P}} \;:=\; s - \lim_{j \to \infty} \Phi_{\vec{P}}^{\sigma_j} \,. \tag{IV.55}$$

We estimate the rate of convergence, and establish regularity properties with respect to $\vec{P}$. Our results are similarly to those in [22] for the Nelson model.

In the initial step of the construction corresponding to $j = 0$, we define $\Phi_{\vec{P}}^{\sigma_0} := \Omega_f$, with $\|\Omega_f\| = 1$.

To pass from scale j to $j+1$, we proceed in two steps. First, we construct an intermediate vector $\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}$

$$\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \;=\; \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma_{j+1}} dz_{j+1} \, \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big[- \Delta K_{\vec{P}}|_{\sigma_{j+1}}^{\sigma_j} \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big]^n \Phi_{\vec{P}}^{\sigma_j} \,, \tag{IV.56}$$

where

$$\begin{aligned} \Delta K_{\vec{P}}|_{\sigma_{j+1}}^{\sigma_j} \;&:=\; \widehat{K}_{\vec{P}}^{\sigma_{j+1}} - \widehat{\mathcal{E}}_{\vec{P}}^{\sigma_{j+1}} + \mathcal{E}_{\vec{P}}^{\sigma_j} - K_{\vec{P}}^{\sigma_j} \\ &=\; \vec{\Gamma}_{\vec{P}}^{\sigma_j} \cdot \big(\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} + \vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j} \big) + \big(\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} + \vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j} \big)^2 \,. \end{aligned} \tag{IV.57}$$

Then, we define

$$\Phi_{\vec{P}}^{\sigma_{j+1}} \;:=\; W_{\sigma_{j+1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}}) W_{\sigma_{j+1}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \,. \tag{IV.58}$$

The series in (IV.56) is termwise well-defined and converges strongly to a non-zero vector, provided α is small enough (independently of j). This follows from operator-norm estimates of the type used for (IV.18).

To prove the convergence of the sequence $\{\Phi_{\vec{P}}^{\sigma_j}\}$, we proceed as follows. The key point is to show that the term

$$\vec{\Gamma}_{\vec{P}}^{\sigma_j} \cdot \left(\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} + \vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j}\right) \tag{IV.59}$$

contained in (IV.57), which is superficially marginal in the infrared by power counting (using the terminology of renormalization group theory), is in fact irrelevant. This is a consequence of the orthogonality relation

$$\left\langle \Phi_{\vec{P}}^{\sigma_j} \,,\, \vec{\Gamma}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \right\rangle \,=\, 0 \,, \tag{IV.60}$$

as we will show. We then proceed to showing that

$$\left\| \left(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}}\right)^{\frac{1}{2}} \left[\vec{\Gamma}_{\vec{P}}^{\sigma_j} \cdot \left(\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j\,(+)} + \vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j}\right)\right] \left(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}}\right)^{\frac{1}{2}} \Phi_{\vec{P}}^{\sigma_j} \right\|_{\mathcal{F}} \tag{IV.61}$$

(where $\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j\,(+)}$ stands for the part which contains only photon creation operators) is of order $\mathcal{O}(\epsilon^{\eta j})$, for some $\eta > 0$, and we consequently deduce that

$$\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} - \Phi_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}} \tag{IV.62}$$

tends to 0, as $j \to \infty$.

Theorem IV.1. *The strong limit*

$$\Phi_{\vec{P}} \,=\, s - \lim_{j\to\infty} \Phi_{\vec{P}}^{\sigma_j} \tag{IV.63}$$

exists and is non-zero, and the rate of convergence is $\mathcal{O}(\sigma_j^{\frac{1}{2}(1-\delta)})$, for any $0 < \delta < 1$.

In the proof, we can import results from [22] at various places. Thus, we will be sketchy in part of our presentation.

IV.4. **Key ingredients of the proof of proof of Theorem IV.1.**

• *Constraints on ϵ, μ and α*

In addition to the conditions on α, ϵ and μ imposed in our discussion so far, the analysis in this part will require an upper bound on μ and an upper bound on ϵ strictly smaller than the ones imposed by the inequalities (IV.8), (IV.9); see Lemma A.3 and (IV.90) below. We note that the more restrictive conditions on μ and ϵ imply new bounds on ρ^-, ρ^+. Moreover, ϵ must satisfy a lower bound $\epsilon > \mathcal{O}(\alpha^{\frac{1}{2}})$. We will point out below where these constraints are needed.

- *Estimates on the shift of the ground state energy and its gradient*

    There are constants C_1, C_2 such that the following hold.

($\mathscr{A}1$)

$$| E_{\vec{P}}^{\sigma_j} - E_{\vec{P}}^{\sigma_{j+1}} | \leq C_1 \, \alpha \, \epsilon^j \tag{IV.64}$$

This estimate can be proved as inequality (II.19) in [4].

($\mathscr{A}2$)

$$| \vec{\nabla} E_{\vec{P}}^{\sigma_j} - \vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}} | \leq C_2 \Big(\Big\| \frac{\widehat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}}} - \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|_{\mathcal{F}}} \Big\|_{\mathcal{F}} + \epsilon^{\frac{j}{2}} \Big) \tag{IV.65}$$

For the proof, see Lemma A.2 in the Appendix.

- *Bounds relating expectations of operators to those of their absolute values*

    There are constants C_3, $C_4 > 1$ such that the following hold.

($\mathscr{A}3$) For $z_{j+1} \in \gamma_{j+1}$,

$$\Big\langle (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \, , \, \Big| \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big| (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \Big\rangle$$

$$\leq \; C_3 \Big| \Big\langle (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \, , \, \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \Big\rangle \Big|. \tag{IV.66}$$

($\mathscr{A}4$) For $z_{j+1} \in \gamma_{j+1}$,

$$\Big\langle \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j \, (+)} (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \, , \, \Big| \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \Big| \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j \, (+)} (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \Big\rangle$$

$$\leq \; C_4 \Big| \Big\langle \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j \, (+)} (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \, , \, \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j \, (+)} (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \Big\rangle \Big|. \tag{IV.67}$$

To obtain these two bounds, it suffices to exploit the fact that the spectral support (with respect to $K_{\vec{P}}^{\sigma_j}$) of the two vectors $(\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j}$ and $\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j \, (+)}(\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j}$ is strictly above the ground state energy, since they are both orthogonal to the ground state $\Phi_{\vec{P}}^{\sigma_j}$.

Remark: The constants $C_1, \dots, C_5$ are independent of α, ϵ, μ, and $j \in \mathbb{N}$, provided that α, ϵ, and μ are sufficiently small.

IV.5. **Proof of the convergence of** $(\Phi_{\vec{P}}^{\sigma_j})_{j=0}^{\infty}$**.** The proof of Theorem IV.1 consists of four main steps.

Step (1).

(i) Assuming the bound

$$\left| \left\langle (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \, , \, \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \right\rangle \right| \leq \frac{R_0}{\alpha \, \epsilon^{j\delta}} \qquad 1 > \delta > 0, \qquad \text{(IV.68)}$$

where R_0 is a constant uniform in $j \in \mathbb{N}$, for α, ϵ, μ sufficiently small, we prove that

$$\left\| \left(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \right)^{\frac{1}{2}} \left[\vec{\Gamma}_{\vec{P}}^{\sigma_j} \cdot \left(\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j \, (+)} + \vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j} \right) \right] \left(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \right)^{\frac{1}{2}} \Phi_{\vec{P}}^{\sigma_j} \right\|_{\mathcal{F}} \leq \mathcal{O}(\epsilon^{\frac{j}{2}(1-\delta)}) \, ; \qquad \text{(IV.69)}$$

(see (IV.61)). (ii) For α and R_0 small enough independently of j, we prove that

$$\| \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} - \Phi_{\vec{P}}^{\sigma_j} \|_{\mathcal{F}} \leq \epsilon^{\frac{j+1}{2}(1-\delta)}. \qquad \text{(IV.70)}$$

For the term on the l.h.s. of (IV.69) proportional to to $\vec{\mathcal{I}}_{\sigma_{j+1}}^{\sigma_j}$, the asserted upper bound is readily obtained from estimate $(\mathscr{A}3)$ combined with (IV.68). For the term proportional to $\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j \, (+)}$, we prove (IV.69) following arguments developed in [22]; see Lemma A.3 of the Appendix for details. This involves the application of a "pull-through formula", a resolvent expansion, and the bounds $(\mathscr{A}3)$, $(\mathscr{A}4)$.

Step (2).

We relate the l.h.s. of (IV.68) to the corresponding quantity with j replaced by $j-1$, and to the norm difference

$$\| \widehat{\Phi}_{\vec{P}}^{\sigma_j} - \Phi_{\vec{P}}^{\sigma_{j-1}} \|_{\mathcal{F}} \qquad \text{(IV.71)}$$

(see (IV.80) – (IV.83) below).

By unitarity of $W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}}) W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j})$, the l.h.s. of (IV.68) equals

$$\left| \left\langle (\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j} \, , \, \frac{1}{\widehat{K}_{\vec{P}}^{\sigma_j} - z_{j+1}} (\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j} \right\rangle \right| . \qquad \text{(IV.72)}$$

Assuming that α is small enough and $\epsilon > \mathcal{O}(\alpha^{\frac{1}{2}})$, we may use $(\mathscr{A}1)$ to re-expand the resolvent and find

$$\left| \left\langle (\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j} \, , \, \frac{1}{\widehat{K}_{\vec{P}}^{\sigma_j} - z_{j+1}} (\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j} \right\rangle \right|$$

$$\leq \; 2 \left| \left\langle (\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j} \, , \, \left| \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \right| (\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j} \right\rangle \right| . \qquad \text{(IV.73)}$$

We then readily obtain that

$$2\left|\left\langle (\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i\widehat{\Phi}_{\vec{P}}^{\sigma_j}\,,\,\left|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\right|(\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i\widehat{\Phi}_{\vec{P}}^{\sigma_j}\right\rangle\right|$$

$$\leq\quad 4\left\|\left(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\right)^{\frac{1}{2}}((\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i\widehat{\Phi}_{\vec{P}}^{\sigma_j}-(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}})\right\|_{\mathcal{F}}^2 \tag{IV.74}$$

$$+4\left|\left\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\,,\,\left|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\right|(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\right\rangle\right|. \tag{IV.75}$$

Our strategy is to construct a recursion that relates (IV.75) to the initial expression (IV.72) with j replaced by $j-1$, while (IV.74) is a remainder term.

We bound the remainder term (IV.74) by

$$4\left\|\left(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\right)^{\frac{1}{2}}((\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i\widehat{\Phi}_{\vec{P}}^{\sigma_j}-(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}})\right\|_{\mathcal{F}}^2$$

$$\leq\quad 8\left\|\left(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\right)^{\frac{1}{2}}((\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i\widehat{\Phi}_{\vec{P}}^{\sigma_j}-(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\widehat{\Phi}_{\vec{P}}^{\sigma_j})\right\|_{\mathcal{F}}^2 \tag{IV.76}$$

$$+8\left\|\left(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\right)^{\frac{1}{2}}(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i(\widehat{\Phi}_{\vec{P}}^{\sigma_j}-\Phi_{\vec{P}}^{\sigma_{j-1}})\right\|_{\mathcal{F}}^2$$

$$\leq\quad \frac{R_1}{\epsilon^{\frac{j}{2}}}\left(\frac{\|\widehat{\Phi}_{\vec{P}}^{\sigma_j}-\Phi_{\vec{P}}^{\sigma_{j-1}}\|_{\mathcal{F}}+\epsilon^{\frac{j}{2}}}{\epsilon^{\frac{j}{4}}}\right)^2 \tag{IV.77}$$

$$+\frac{R_2}{\epsilon^{\frac{j}{2}}}\left(\frac{\left\|\frac{\widehat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}}}-\frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|}\right\|_{\mathcal{F}}+\epsilon^{\frac{j}{2}}}{4\epsilon^{\frac{j}{4}}}\right)^2,$$

where R_1 and R_2 are constants independent of α, ϵ, μ, and $j\in\mathbb{N}$, provided that α, ϵ, and μ are sufficiently small, and $\epsilon>\mathcal{O}(\alpha^{\frac{1}{2}})$. For details on the step from (IV.76) to (IV.77), we refer to Lemma A.4 of the Appendix.

To bound the term (IV.75), we use $(\mathcal{A}3)$ and the orthogonality property expressed in (IV.60). We find that, for any $z_j\in\gamma_j$,

$$4\left|\left\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\,,\,\left|\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}\right|(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\right\rangle\right|$$

$$\leq\quad 4C_3\left|\left\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\,,\,\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_{j+1}}(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\right\rangle\right| \tag{IV.78}$$

$$\leq\quad 8C_3^2\left|\left\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\,,\,\frac{1}{K_{\vec{P}}^{\sigma_{j-1}}-z_j}(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}\right\rangle\right|. \tag{IV.79}$$

In passing from (IV.78) to (IV.79), we have used the constraint on the spectral support (with respect to $K_{\vec{P}}^{\sigma_{j-1}}$) of the vector $(\Gamma_{\vec{P}}^{\sigma_{j-1}})^i\Phi_{\vec{P}}^{\sigma_{j-1}}$.

Therefore, for sufficiently small values of the parameters ϵ and α, we conclude that

$$\left| \left\langle (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \, , \, \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} (\Gamma_{\vec{P}}^{\sigma_j})^i \Phi_{\vec{P}}^{\sigma_j} \right\rangle \right| \tag{IV.80}$$

$$\leq \quad \frac{R_1}{\epsilon^{\frac{j}{2}}} \Big(\frac{\|\widehat{\Phi}_{\vec{P}}^{\sigma_j} - \Phi_{\vec{P}}^{\sigma_{j-1}}\| + \epsilon^{\frac{j}{2}}}{\epsilon^{\frac{j}{4}}} \Big)^2 \tag{IV.81}$$

$$+ \frac{R_2}{\epsilon^{\frac{j}{2}}} \Big(\frac{\big\| \frac{\widehat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_j}\|} - \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \big\| + \epsilon^{\frac{j}{2}}}{4\epsilon^{\frac{j}{4}}} \Big)^2 \tag{IV.82}$$

$$+ 8C_3^2 \left| \left\langle (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \Phi_{\vec{P}}^{\sigma_{j-1}} \, , \, \frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_j} (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \Phi_{\vec{P}}^{\sigma_{j-1}} \right\rangle \right|. \tag{IV.83}$$

Step (3).

We prove that

$$\|\Phi_{\vec{P}}^{\sigma_j} - \widehat{\Phi}_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}} \leq C_5 \, \alpha^{\frac{1}{2}} \, |\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}} - \vec{\nabla} E_{\vec{P}}^{\sigma_j}| \, |\ln(\epsilon^j)| \, . \tag{IV.84}$$

From the definition

$$\Phi_{\vec{P}}^{\sigma_j} := W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}}) \widehat{\Phi}_{\vec{P}}^{\sigma_j} \, , \tag{IV.85}$$

we get that

$$\| \Phi_{\vec{P}}^{\sigma_j} - \widehat{\Phi}_{\vec{P}}^{\sigma_j} \|_{\mathcal{F}} = \|W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_{j-1}}) W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \Psi_{\vec{P}}^{\sigma_j} - \Psi_{\vec{P}}^{\sigma_j} \|_{\mathcal{F}} \tag{IV.86}$$

where (with an abuse of notation) we have denoted by $\Psi_{\vec{P}}^{\sigma_j}$ the ground state eigenvector

$$W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \Phi_{\vec{P}}^{\sigma_j} \, , \tag{IV.87}$$

$\|W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \Phi_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}} \leq 1$, of the Hamiltonian $H_{\vec{P}}^{\sigma_j}$. Then, we apply formula (III.16) (which was derived in [9]), and obtain the logarithmic bound $\langle N_f \rangle_{\Psi_{\vec{P}}^{\sigma_j}} \leq \mathcal{O}(|\ln \sigma_j|)$ for the expectation value of the photon number operator N_f in $\Psi_{\vec{P}}^{\sigma_j}$, where $\sigma_j = \Lambda \epsilon^j$, and $\Lambda \approx 1$. Hence, the estimate (IV.84) follows.

Step (4).

We prove the bound (IV.68) assumed in step (1) by an inductive argument (see (IV.95) below).

We assume α, ϵ, and μ to be sufficiently small for all our previous results to hold, and such that:

i)

$$S_1^j := \sum_{m=1}^{j} \left[\epsilon^{\frac{m}{2}(1-\delta)} + 4\, C_5\, C_2\, \alpha^{\frac{1}{2}}\, \epsilon^{\frac{m}{2}(1-\delta)} |\ln(\epsilon^m)| \right] \le \frac{1}{3}\,, \tag{IV.88}$$

uniformly in j.

ii)

$$\|\widehat{\Phi}_{\vec{P}}^{\sigma_1} - \Phi_{\vec{P}}^{\sigma_0}\|_{\mathcal{F}} \le \epsilon^{\frac{1}{2}(1-\delta)}\,. \tag{IV.89}$$

iii) The bound (IV.68) holds for $j = 1$, and

$$0 < R_1 + R_2 \le (1 - 8\, C_3\, \epsilon^\delta)\frac{R_0}{\alpha}\,. \tag{IV.90}$$

Notably, (IV.90) imposes a more restrictive upper bound on the admissible values of ϵ. Then, we proceed with the induction in j.

- *Inductive hypotheses* We assume that, for $j - 1$

($\mathcal{H}1$) we have an estimate

$$\|\Phi_{\vec{P}}^{\sigma_{j-1}} - \Phi_{\vec{P}}^{\sigma_0}\|_{\mathcal{F}} \le S_1^{j-1} = \sum_{m=1}^{j-1} \left[\epsilon^{\frac{m}{2}(1-\delta)} + 4\, C_5\, C_2\, \alpha^{\frac{1}{2}}\, \epsilon^{\frac{m}{2}(1-\delta)} |\ln(\epsilon^m)| \right];$$

($\mathcal{H}2$) the bound (IV.68) holds for $j - 1$.

- *Induction step from $j - 1$ to j*

From ($\mathcal{H}2$), we get that

$$\|\widehat{\Phi}_{\vec{P}}^{\sigma_j} - \Phi_{\vec{P}}^{\sigma_{j-1}}\|_{\mathcal{F}} \le \epsilon^{\frac{j}{2}(1-\delta)}\,. \tag{IV.91}$$

From ($\mathcal{H}1$), ($\mathcal{H}2$) and ($\mathscr{A}2$), we can conclude that

$$\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|_{\mathcal{F}} \ge \|\Phi_{\vec{P}}^{\sigma_0}\|_{\mathcal{F}} - \|\Phi_{\vec{P}}^{\sigma_{j-1}} - \Phi_{\vec{P}}^{\sigma_0}\|_{\mathcal{F}} \ge \frac{2}{3} \tag{IV.92}$$

$$|\vec{\nabla}E^{\sigma_j}(\vec{P}) - \vec{\nabla}E^{\sigma_{j-1}}(\vec{P})| \le 4C_2\epsilon^{\frac{j}{2}(1-\delta)}\,. \tag{IV.93}$$

and then, by combining (IV.70), (IV.84) and (IV.65), that

$$\|\Phi_{\vec{P}}^{\sigma_j} - \Phi_{\vec{P}}^{\sigma_0}\|_{\mathcal{F}} \le S_1^j\,. \tag{IV.94}$$

Finally, we obtain from (IV.81) – (IV.83) that

$$
\left| \left\langle (\Gamma^{\sigma_j}_{\vec{P}})^i \Phi^{\sigma_j}_{\vec{P}} \,,\, \frac{1}{K^{\sigma_j}_{\vec{P}} - z_{j+1}} (\Gamma^{\sigma_j}_{\vec{P}})^i \Phi^{\sigma_j}_{\vec{P}} \right\rangle \right|
$$

$$
\leq \quad \frac{R_1}{\epsilon^{\frac{j}{2}}} \Big(\frac{\| \widehat{\Phi}^{\sigma_j}_{\vec{P}} - \Phi^{\sigma_{j-1}}_{\vec{P}} \|_{\mathcal{F}} + \epsilon^{\frac{j}{2}(1-\delta)}}{2\epsilon^{\frac{j}{4}}} \Big)^2
$$

$$
+ \frac{R_2}{\epsilon^{\frac{j}{2}}} \Big(\frac{\big\| \frac{\widehat{\Phi}^{\sigma_j}_{\vec{P}}}{\|\widehat{\Phi}^{\sigma_j}_{\vec{P}}\|_{\mathcal{F}}} - \frac{\Phi^{\sigma_{j-1}}_{\vec{P}}}{\|\Phi^{\sigma_{j-1}}_{\vec{P}}\|_{\mathcal{F}}} \big\|_{\mathcal{F}} + \epsilon^{\frac{j}{2}(1-\delta)}}{4\epsilon^{\frac{j}{4}}} \Big)^2
$$

$$
+ 8C_3^2 \left| \left\langle (\Gamma^{\sigma_{j-1}}_{\vec{P}})^i \Phi^{\sigma_{j-1}}_{\vec{P}} \,,\, \frac{1}{K^{\sigma_{j-1}}_{\vec{P}} - z_j} (\Gamma^{\sigma_{j-1}}_{\vec{P}})^i \Phi^{\sigma_{j-1}}_{\vec{P}} \right\rangle \right|
$$

$$
\leq \quad \frac{R_1}{\epsilon^{j\delta}} + \frac{R_2}{\epsilon^{j\delta}} + 8C_3^2 \frac{R_0}{\alpha \epsilon^{(j-1)\delta}} \leq \frac{R_0}{\alpha \epsilon^{j\delta}} \,. \tag{IV.95}
$$

This proves (IV.69) and implies that the sequence $\{\Phi^{\sigma_j}_{\vec{P}}\}$ converges (as can be derived from (IV.94)). The limit is a non-zero vector because of (IV.92).

This concludes the proof of statement ($\mathscr{I}$1) in Theorem III.1. $\qquad\square$

V. Proof of Statements ($\mathscr{I}$2) and ($\mathscr{I}$3) in Theorem III.1

Statement ($\mathscr{I}$2) expresses Hölder regularity of $\Phi^{\sigma}_{\vec{P}}$ and $\vec{\nabla} E^{\sigma}_{\vec{P}}$ with respect to $\vec{P} \in \mathcal{S}$, uniformly in $\sigma \geq 0$. That is,

$$
\| \Phi^{\sigma}_{\vec{P}} - \Phi^{\sigma}_{\vec{P}+\Delta\vec{P}} \|_{\mathcal{F}} \leq C_{\delta'} |\Delta\vec{P}|^{\frac{1}{4}-\delta'} \tag{V.1}
$$

and

$$
| \vec{\nabla} E^{\sigma}_{\vec{P}} - \vec{\nabla} E^{\sigma}_{\vec{P}+\Delta\vec{P}} | \leq C_{\delta''} |\Delta\vec{P}|^{\frac{1}{4}-\delta''} \,, \tag{V.2}
$$

for any $0 < \delta'' < \delta' < \frac{1}{4}$, where $\vec{P}, \vec{P}+\Delta\vec{P} \in \mathcal{S}$. The constants $C_{\delta'}$ and $C_{\delta''}$ depend on δ' and δ'', respectively. This result can be taken over from [22].

Statement ($\mathscr{I}$3) follows easily from ($\mathscr{I}$5). In fact, we recall from the beginning of Section IV.2.1 that

$$
\vec{P} - \vec{\nabla} E^{\sigma}_{\vec{P}} = \left\langle \vec{P}^f - \alpha^{1/2} \vec{A}_\sigma \right\rangle_{\Psi^{\sigma}_{\vec{P}}} \,. \tag{V.3}
$$

We then find that

$$
| \left\langle \vec{P}^f \right\rangle_{\Psi^{\sigma}_{\vec{P}}} | \leq \sum_{\lambda} \int d^3k \, |\vec{k}| \, \| b_{\vec{k},\lambda} \Psi^{\sigma}_{\vec{P}} \|^2
$$

$$
\leq C' \alpha \int_{\mathcal{B}_\Lambda} \frac{d^3k}{|\vec{k}|^2} \leq C \alpha \,, \tag{V.4}
$$

and

$$\left| \left\langle \alpha^{1/2} \vec{A}_\sigma \right\rangle_{\Psi_{\vec{P}}^\sigma} \right| \leq \alpha^{1/2} \sum_\lambda \int \frac{d^3 k}{|\vec{k}|^{1/2}} \, \| \, b_{\vec{k},\lambda} \Psi_{\vec{P}}^\sigma \, \|$$

$$\leq C' \alpha \int_{\mathcal{B}_\Lambda} \frac{d^3 k}{|\vec{k}|^2} \leq C \alpha \,, \tag{V.5}$$

where we used $(\mathscr{I}5)$ in (V.5). Therefore,

$$| \, \vec{P} - \vec{\nabla} E_{\vec{P}}^\sigma \, | \leq C \alpha \,, \tag{V.6}$$

for a constant C independent of $\vec{P} \in \mathcal{S}$ and σ. Statement $(\mathscr{I}3)$ then follows immediately.

VI. Proof of Statement ($\mathscr{I}4$) in Theorem III.1

To prove statement ($\mathscr{I}4$) in Theorem III.1, we must show that, for $\vec{P} \in \mathcal{S}$, α small enough, $\vec{k} \neq 0$ and $\sigma \geq 0$,

$$E^\sigma_{\vec{P}-\vec{k}} > E^\sigma_{\vec{P}} - C_\alpha |\vec{k}| \tag{VI.1}$$

holds, where $E^\sigma_{\vec{P}-\vec{k}} := \inf \operatorname{spec} H^\sigma_{\vec{P}-\vec{k}}$, and $\frac{1}{3} < C_\alpha < 1$, with $C_\alpha \to \frac{1}{3}$ as $\alpha \to 0$.

To prove (VI.1), we first note that

$$H^\sigma_{\vec{P}+\vec{k}} = H^\sigma_{\vec{P}} + \vec{k} \cdot \vec{\nabla} H^\sigma_{\vec{P}} + \frac{|\vec{k}|^2}{2}, \tag{VI.2}$$

and that

$$\begin{aligned}
\left\langle \phi, H^\sigma_{\vec{P}+\vec{k}} \phi \right\rangle &\geq \left\langle \phi, H^\sigma_{\vec{P}} \phi \right\rangle - |\vec{k}| \left\langle \phi, (\vec{\nabla} H^\sigma_{\vec{P}})^2 \phi \right\rangle^{1/2} + \frac{|\vec{k}|^2}{2} \\
&\geq \left\langle \phi, H^\sigma_{\vec{P}} \phi \right\rangle - \sqrt{2} |\vec{k}| \left\langle \phi, H^\sigma_{\vec{P}} \phi \right\rangle^{1/2} + \frac{|\vec{k}|^2}{2}
\end{aligned} \tag{VI.3}$$

for $\phi \in D(H^\sigma_{\vec{P}+\vec{k}}) \cap \mathcal{F}_\sigma$, with $\|\phi\| = 1$. Thus, we obtain the inequality

$$\begin{aligned}
&\left\langle \phi, H^\sigma_{\vec{P}+\vec{k}} \phi \right\rangle - E^\sigma_{\vec{P}} \\
&\geq \inf_{z \geq 0} \left\{ (z + E^\sigma_{\vec{P}}) - \sqrt{2}|\vec{k}| (z + E^\sigma_{\vec{P}})^{1/2} + \frac{|\vec{k}|^2}{2} - E^\sigma_{\vec{P}} \right\} \\
&= \inf_{x \geq (E^\sigma_{\vec{P}})^{1/2}} \left\{ x^2 - \sqrt{2}|\vec{k}| x + \frac{|\vec{k}|^2}{2} - E^\sigma_{\vec{P}} \right\},
\end{aligned} \tag{VI.4}$$

where $z := \left\langle \phi, H^\sigma_{\vec{P}} \phi \right\rangle - E^\sigma_{\vec{P}} \geq 0$ in the expression on the second line.

Setting $\partial_x(\cdots) = 0$ in the expression on the last line of (VI.4), we find

$$2x - \sqrt{2}|\vec{k}| = 0. \tag{VI.5}$$

The minimum is therefore attained at $x = \frac{\sqrt{2}}{2}|\vec{k}|$, if $\frac{\sqrt{2}}{2}|\vec{k}| \geq (E^\sigma_{\vec{P}})^{1/2}$, and at $x = (E^\sigma_{\vec{P}})^{1/2}$, corresponding to $z = 0$, otherwise. That is,

$$x_{min} = \max\{\frac{\sqrt{2}}{2}|\vec{k}|, (E^\sigma_{\vec{P}})^{1/2}\}. \tag{VI.6}$$

Now, for $\frac{\sqrt{2}}{2}|\vec{k}| \geq (E^\sigma_{\vec{P}})^{1/2}$, so that $x_{min} = \frac{\sqrt{2}}{2}|\vec{k}|$, we evaluate the lower bound and get

$$\frac{|\vec{k}|^2}{2} - |\vec{k}|^2 + \frac{|\vec{k}|^2}{2} - E^\sigma_{\vec{P}}, \tag{VI.7}$$

and we observe that

$$-E^\sigma_{\vec{P}} \geq -\frac{1}{3}|\vec{k}|, \tag{VI.8}$$

because

$$E_{\vec{P}}^{\sigma} \leq \frac{1}{\sqrt{2}}(\frac{1}{3} + c\alpha)\,(E_{\vec{P}}^{\sigma})^{1/2} \leq \frac{1}{3}\,|\vec{k}| \qquad (\text{VI.9})$$

for $|\vec{P}| < \frac{1}{3}$. This follows from

$$0 < E_{\vec{P}}^{\sigma} = \mathrm{infspec} H_{\vec{P}}^{\sigma} \leq \langle \Omega_f, H_{\vec{P}}^{\sigma}\Omega_f \rangle = \frac{1}{2}|\vec{P}|^2 + \frac{\alpha}{2}\langle (\vec{A}^{\sigma})^2 \rangle \qquad (\text{VI.10})$$

by Rayleigh-Ritz, so that $(E_{\vec{P}}^{\sigma})^{1/2} \leq \frac{1}{\sqrt{2}}(\frac{1}{3} + c\alpha)$ for $|\vec{P}| < \frac{1}{3}$.

If, however, $\frac{\sqrt{2}}{2}|\vec{k}| \leq (E_{\vec{P}}^{\sigma})^{1/2}$, so that $x_{min} = (E_{\vec{P}}^{\sigma})^{1/2}$, evaluation of the lower bound yields

$$-\sqrt{2}|\vec{k}|\,(E_{\vec{P}}^{\sigma})^{1/2} + \frac{|\vec{k}|^2}{2}, \qquad (\text{VI.11})$$

and we observe that

$$-\sqrt{2}|\vec{k}|\,(E_{\vec{P}}^{\sigma})^{1/2} + \frac{|\vec{k}|^2}{2} \geq -(|\vec{P}| + c\alpha)|\vec{k}| \geq -(\frac{1}{3} + c\alpha)|\vec{k}| \qquad (\text{VI.12})$$

for $|\vec{P}| < \frac{1}{3}$.

Therefore, we conclude that

$$E_{\vec{P}+\vec{k}}^{\sigma} > E_{\vec{P}}^{\sigma} - C_{\alpha}\,|\vec{k}| \qquad (\text{VI.13})$$

for

$$C_{\alpha} = \frac{1}{3} + c\alpha, \qquad (\text{VI.14})$$

and all $\vec{k} \neq 0$.

This establishes statement $(\mathscr{I}4)$ in Theorem III.1. $\qquad\qquad\square$

Thus, we have proven our main result, up to auxiliary results proven in the Appendix.

Appendix A

A.1. **Well-definedness of the operators $K_{\vec{P}}^{\sigma_j}$ and $\widehat{K}_{\vec{P}}^{\sigma_j}$.** We need to verify that the canonical form of the Hamiltonians $K_{\vec{P}}^{\sigma_j}$ and $\widehat{K}_{\vec{P}}^{\sigma_j}$ in (IV.40) and (IV.49) are not only formal. This can be achieved by adapting an argument in the work [21] of E. Nelson, Lemma 3. We shall only outline the proof for $K_{\vec{P}}^{\sigma_j}$; the case of $\widehat{K}_{\vec{P}}^{\sigma_j}$ is similar.

To this end, we write $(K_{\vec{P}}^{\sigma_{j+1}})'$ for the operator on the right hand side of (IV.40), in order to distinguish it from (IV.38). We let $\mathcal{H}_{\vec{P}}(\infty)$ denote the linear span of vectors in $\mathcal{H}_{\vec{P}}$ with a finite number of photons. For the values of α and of Λ assumed in Section II, we know that $H_{\vec{P}}^{\sigma_j}$ is selfadjoint in $D(H_{\vec{P}}^0)$, where

$$H_{\vec{P}}^0 := \frac{(\vec{P} - \vec{P}^f)^2}{2} + H^f . \tag{A.1}$$

Then, we conclude the following:

1) The equality (IV.40) trivially holds on $\mathcal{H}_{\vec{P}}(\infty) \bigcap D(H_{\vec{P}}^0)$, because vectors in this space are analytic for the generator of $W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})$, and since $H_{\vec{P}}^{\sigma_j}$, $H_{\vec{P}}^0$ and the generator of $W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})$ map $\mathcal{H}_{\vec{P}}(\infty) \bigcap D(H_{\vec{P}}^0)$ into itself.

2) By standard arguments, one shows that

$$\|H_{\vec{P}}^0 W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})\psi\| \leq b \left(\|H_{\vec{P}}^0 \psi\| + \|\psi\|\right), \tag{A.2}$$

where $\psi \in \mathcal{H}_{\vec{P}}(\infty) \bigcap D(H_{\vec{P}}^0)$, for some $b > 0$.

Because $\mathcal{H}_{\vec{P}}(\infty) \bigcap D(H_{\vec{P}}^0)$ is dense in $D(H_{\vec{P}}^0)$ with respect to the norm $\|H_{\vec{P}}^0 \psi\| + \|\psi\|$, it follows that $W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})$ and $W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j})$ map $D(H_{\vec{P}}^0)$ into itself.

Consequently,

$$D(H_{\vec{P}}^0) \equiv D(K_{\vec{P}}^{\sigma_j}) . \tag{A.3}$$

3) The equality (IV.40) holds on $D(K_{\vec{P}}^{\sigma_j})$ because $\mathcal{H}_{\vec{P}}(\infty) \bigcap D(H_{\vec{P}}^0)$ is dense in $D(H_{\vec{P}}^0)$ in the norm $\|H_{\vec{P}}^0 \psi\| + \|\psi\|$, and because of (A.3). Since $(K_{\vec{P}}^{\sigma_j})' \equiv K_{\vec{P}}^{\sigma_j}$ on the domain of selfadjointness of $K_{\vec{P}}^{\sigma_j}$, we can therefore conclude that $D((K_{\vec{P}}^{\sigma_j})') \equiv D(K_{\vec{P}}^{\sigma_j})$. Consequently, we have proven that $(K_{\vec{P}}^{\sigma_j})' \equiv K_{\vec{P}}^{\sigma_j}$. This is what we intended to prove.

A.2. **Technical lemmata for the proof of $(\mathscr{I}1)$ in Theorem III.1.**

Lemma A.1. *The Hamiltonian $\widehat{K}_{\vec{P}}^{\sigma_{j+1}}$ has the form (IV.49), with (IV.50), (IV.51), and (IV.52).*

Proof.

Recalling the definitions of Section IV.2.1, we have

$$W_{\sigma_{j+1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})\vec{\beta}^{\sigma_{j+1}} W_{\sigma_{j+1}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) - \langle \vec{\beta}^{\sigma_j} \rangle_{\Psi_{\vec{P}}^{\sigma_j}} \tag{A.4}$$

$$= \; W_{\sigma_{j+1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})\vec{\beta}^{\sigma_j} W_{\sigma_{j+1}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) - \langle \vec{\beta}^{\sigma_j} \rangle_{\Psi_{\vec{P}}^{\sigma_j}} \tag{A.5}$$

$$- \alpha^{\frac{1}{2}} W_{\sigma_{j+1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})\vec{A}_{\sigma_{j+1}}^{\sigma_j} W_{\sigma_{j+1}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \tag{A.6}$$

$$= \; W_{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})\vec{\beta}^{\sigma_j} W_{\sigma_j}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) - \langle \vec{\beta}^{\sigma_j} \rangle_{\Psi_{\vec{P}}^{\sigma_j}} \tag{A.7}$$

$$+ W_{\sigma_{j+1}}^{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})\vec{P}^f W_{\sigma_{j+1}}^{\sigma_j\,*}(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) - \vec{P}^f \tag{A.8}$$

$$- \alpha^{\frac{1}{2}} W_{\sigma_{j+1}}(\vec{\nabla} E_{\vec{P}}^{\sigma_j})\vec{A}_{\sigma_{j+1}}^{\sigma_j} W_{\sigma_{j+1}}^*(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \tag{A.9}$$

$$= \; \vec{\Pi}_{\vec{P}}^{\sigma_j} - \langle \vec{\Pi}_{\vec{P}}^{\sigma_j} \rangle_{\Phi_{\vec{P}}^{\sigma_j}} \tag{A.10}$$

$$- \alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\sigma_j}\setminus\mathcal{B}_{\sigma_{j+1}}} \vec{k} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \, b_{\vec{k},\lambda} + h.c.}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\widehat{k})} \, d^3k - \alpha^{\frac{1}{2}} \vec{A}_{\sigma_{j+1}}^{\sigma_j} \tag{A.11}$$

$$+ \alpha \sum_{\lambda} \int_{\mathcal{B}_{\sigma_j}\setminus\mathcal{B}_{\sigma_{j+1}}} \vec{k} \; \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \, \vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_j}(\widehat{k}))^2} \, d^3k \tag{A.12}$$

$$+ \alpha \sum_{\lambda} \int_{\mathcal{B}_{\sigma_j}\setminus\mathcal{B}_{\sigma_{j+1}}} \left[\vec{\varepsilon}_{\vec{k},\lambda} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + h.c.\right] \frac{d^3k}{\sqrt{|\vec{k}|}} \, , \tag{A.13}$$

where

$$W_{\sigma_{j+1}}^{\sigma_j}(\vec{\nabla} E_{\vec{P}}^{\sigma_j}) := \exp\left(\alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\mathcal{B}_{\sigma_j}\setminus\mathcal{B}_{\sigma_{j+1}}} d^3k \, \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\widehat{k})} \cdot (\vec{\varepsilon}_{\vec{k},\lambda} b_{\vec{k},\lambda}^* - h.c.) \right). \tag{A.14}$$

This establishes (IV.50) and (IV.51). $\qquad\square$

Lemma A.2. *For $\vec{P} \in \mathcal{S}$, there exists $C_2 > 0$ such that, uniformly in $j \in \mathbb{N}$, the inequality*

$$|\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} - \vec{\nabla}E_{\vec{P}}^{\sigma_j}| \leq C_2 \Big(\Big\| \frac{\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\|_{\mathcal{F}}} - \frac{\Phi_{\vec{P}}^{\sigma_j}}{\|\Phi_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}}} \Big\|_{\mathcal{F}} + \epsilon^{\frac{j+1}{2}} \Big) \tag{A.15}$$

holds.

Proof.

Using (IV.37) and (IV.53), we write $\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}}$ and $\vec{\nabla}E_{\vec{P}}^{\sigma_j}$ in the form

$$\vec{\nabla}E_{\vec{P}}^{\sigma_j} = \vec{P} - \frac{\langle \Phi_{\vec{P}}^{\sigma_j}, \vec{\Pi}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \rangle}{\langle \Phi_{\vec{P}}^{\sigma_j}, \Phi_{\vec{P}}^{\sigma_j} \rangle} - \langle W_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})\, \vec{\beta}^{\sigma_j}\, W_{\sigma_j}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_j}) \rangle_{\Omega_f} \tag{A.16}$$

$$\begin{aligned}
\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} = \vec{P} &- \frac{\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \rangle}{\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \rangle} \\
&- \langle W_{\sigma_{j+1}}(\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}})\, \vec{\beta}^{\sigma_{j+1}}\, W_{\sigma_{j+1}}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}}) \rangle_{\Omega_f}.
\end{aligned} \tag{A.17}$$

By a simple, but slightly lengthy calculation, one can check that

$$\langle W_{\sigma_j}(\vec{\nabla}E_{\vec{P}}^{\sigma_j})\, \vec{\beta}^{\sigma_j}\, W_{\sigma_j}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_j}) \rangle_{\Omega_f} - \tag{A.18}$$

$$- \langle W_{\sigma_{j+1}}(\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}})\, \vec{\beta}^{\sigma_{j+1}}\, W_{\sigma_{j+1}}^*(\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}}) \rangle_{\Omega_f} \tag{A.19}$$

$$= \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}} \vec{k}\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \; \vec{\nabla}E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_j}(\widehat{k}))^2}\, d^3k \tag{A.20}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}} \vec{k}\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \; \vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k}))^2}\, d^3k \tag{A.21}$$

$$+\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}} \Big[\vec{\varepsilon}_{\vec{k},\lambda}\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + h.c. \Big] \frac{d^3k}{\sqrt{|\vec{k}|}} \tag{A.22}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}} \Big[\vec{\varepsilon}_{\vec{k},\lambda}\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k})} + h.c. \Big] \frac{d^3k}{\sqrt{|\vec{k}|}} \tag{A.23}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} \vec{k}\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \; \vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k}))^2}\, d^3k \tag{A.24}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}} \Big[\vec{\varepsilon}_{\vec{k},\lambda}\, \frac{\vec{\nabla}E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k})} + h.c. \Big] \frac{d^3k}{\sqrt{|\vec{k}|}}. \tag{A.25}$$

On the other hand, using definition (IV.53), we can calculate

$$\widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} - \vec{\Pi}_{\vec{P}}^{\sigma_j} \tag{A.26}$$

$$= \quad \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} \tag{A.27}$$

$$+\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_{j+1}}} \vec{k} \, \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \, \vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_j}(\widehat{k}))^2} \, d^3 k \tag{A.28}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_{j+1}}} \vec{k} \, \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \, \vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k}))^2} \, d^3 k \tag{A.29}$$

$$+\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_{j+1}}} \left[\vec{\varepsilon}_{\vec{k},\lambda} \, \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + h.c. \right] \frac{d^3 k}{\sqrt{|\vec{k}|}} \tag{A.30}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_{j+1}}} \left[\vec{\varepsilon}_{\vec{k},\lambda} \, \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k})} + h.c. \right] \frac{d^3 k}{\sqrt{|\vec{k}|}} \, . \tag{A.31}$$

In order to shorten our notations, we define

$$F_{j+1}^j(\mathcal{B}_{\sigma_j}) \quad := \quad (A.20) + (A.21) + (A.22) + (A.23) \tag{A.32}$$

$$F_{j+1}^j(\mathcal{B}_{\sigma_{j+1}}) \quad := \quad (A.28) + (A.29) + (A.30) + (A.31) \tag{A.33}$$

$$G_{j+1}(\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}) \quad := \quad (A.24) + (A.25) \tag{A.34}$$

Returning to (A.16), (A.18), we can write

$$\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} - \vec{\nabla} E_{\vec{P}}^{\sigma_j} - F_{j+1}^j(\mathcal{B}_{\sigma_j}) \tag{A.35}$$

$$= \quad -\frac{1}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\|} \left\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \, , \, \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} \left(\frac{\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\|} - \frac{\Phi_{\vec{P}}^{\sigma_j}}{\|\Phi_{\vec{P}}^{\sigma_j}\|} \right) \right\rangle \tag{A.36}$$

$$-\frac{\left\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \, , \, \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} \Phi_{\vec{P}}^{\sigma_j} \right\rangle}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\| \, \|\Phi_{\vec{P}}^{\sigma_j}\|} + \frac{\left\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \, , \, \vec{\Pi}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \right\rangle}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\| \, \|\Phi_{\vec{P}}^{\sigma_j}\|} \tag{A.37}$$

$$-\frac{\left\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \, , \, \vec{\Pi}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \right\rangle}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\| \, \|\Phi_{\vec{P}}^{\sigma_j}\|} + \frac{\left\langle \Phi_{\vec{P}}^{\sigma_j} \, , \, \vec{\Pi}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \right\rangle}{\|\Phi_{\vec{P}}^{\sigma_j}\|^2} + G_{j+1}(\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}) \, . \tag{A.38}$$

Using (A.26) – (A.31), this can be rewritten into

$$\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} - \vec{\nabla} E_{\vec{P}}^{\sigma_j} - F_{j+1}^{j}(\mathcal{B}_{\sigma_j}) + \frac{\left\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \Phi_{\vec{P}}^{\sigma_j} \right\rangle}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\| \, \|\Phi_{\vec{P}}^{\sigma_j}\|} F_{j+1}^{j}(\mathcal{B}_{\sigma_{j+1}}) \tag{A.39}$$

$$= \; -\frac{1}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\|} \left\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} \big(\frac{\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\|} - \frac{\Phi_{\vec{P}}^{\sigma_j}}{\|\Phi_{\vec{P}}^{\sigma_j}\|} \big) \right\rangle \tag{A.40}$$

$$-\frac{1}{\|\Phi_{\vec{P}}^{\sigma_j}\|} \left\langle \big(\frac{\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\|} - \frac{\Phi_{\vec{P}}^{\sigma_j}}{\|\Phi_{\vec{P}}^{\sigma_j}\|} \big), \vec{\Pi}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \right\rangle \tag{A.41}$$

$$-\frac{\left\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \right\rangle}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\| \, \|\Phi_{\vec{P}}^{\sigma_j}\|} + G_{j+1}(\mathcal{B}_{\sigma_j} \setminus \mathcal{B}_{\sigma_{j+1}}). \tag{A.42}$$

We deduce from the definitions (A.32) and (A.33) that

$$|F_{j+1}^{j}(\mathcal{B}_{\sigma_j})|, \; |F_{j+1}^{j}(\mathcal{B}_{\sigma_{j+1}})| \; < \; c' \, |\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} - \vec{\nabla} E_{\vec{P}}^{\sigma_j}| \tag{A.43}$$

where c' is α-dependent but j-independent. Then, it suffices to check that, for α small enough, there are positive constants c, C uniform in j, such that

$$C\big(\big\| \frac{\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}\|_{\mathcal{F}}} - \frac{\Phi_{\vec{P}}^{\sigma_j}}{\|\Phi_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}}} \big\|_{\mathcal{F}} + \epsilon^{\frac{j+1}{2}} \big) \tag{A.44}$$

$$\geq \; \big| (A.40) + (A.41) + (A.42) \big| \; \geq \; c \, |\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} - \vec{\nabla} E_{\vec{P}}^{\sigma_j}| \tag{A.45}$$

is satisfied. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Lemma A.3. *Assume $\vec{P} \in \mathcal{S}$, and α, μ, and ϵ small enough. Then, uniformly in $j \in \mathbb{N}$, the bound*

$$\left\|\left(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}}\right)^{\frac{1}{2}} \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(+)\,l} (\Gamma_{\vec{P}}^{\sigma_j})^l \left(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}}\right)^{\frac{1}{2}} \Phi_{\vec{P}}^{\sigma_j}\right\|_{\mathcal{F}}^2 \tag{A.46}$$

$$\leq \frac{2}{1-c} C_3 C_4 Z_{j+1}^j \frac{1}{|E_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}|} \left|\left\langle (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} , \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} \right\rangle\right|$$

holds for each $l = 1, 2, 3$, where $\gamma_{\sigma_{j+1}} := \{z_{j+1} \in \mathbb{C} \,|\, |z_{j+1} - E_{\vec{P}}^{\sigma_j}| = \mu\sigma_{j+1}\}$, and $c < 1$. C_3 and C_4 are defined in (IV.66), (IV.67) (($\mathscr{A}$3) and ($\mathscr{A}$4) from Section IV.4), and

$$Z_{j+1}^j := \langle \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(-)\,l} \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(+)\,l} \rangle_{\Omega_f} \tag{A.47}$$

$$= \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j} \backslash \mathcal{B}_{\sigma_{j+1}}} d^3k \left|k^l \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\hat{k})} + \frac{(\hat{l} \cdot \vec{\varepsilon}_{\vec{k},\lambda})}{\sqrt{|\vec{k}|}}\right|^2.$$

Proof.

We first use Eq. (IV.67) to estimate

$$\left\|\left(\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}}\right)^{\frac{1}{2}} \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(+)\,l} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j}\right\|_{\mathcal{F}}^2 \tag{A.48}$$

$$\leq \left\langle \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(+)\,l} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} , \left|\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}}\right| \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(+)\,l} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} \right\rangle$$

$$\leq C_4 \left|\left\langle \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(+)\,l} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} , \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} \mathcal{L}_{\sigma_{j+1}}^{\sigma_j(+)\,l} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} \right\rangle\right|. \tag{A.49}$$

Then we use pull-through formula to derive the following equality which holds in the sense of distributions for $\vec{k} \in \mathcal{B}_{\sigma_j}$

$$\frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} b(\vec{k}) = \tag{A.50}$$

$$= b(\vec{k}) \frac{1}{\frac{(\vec{\Gamma}_{\vec{P}}^{\sigma_j} + \vec{k})^2}{2} + \sum_\lambda \int_{\mathbb{R}^3} |\vec{q}| \delta_{\vec{P}}^{\sigma_j}(\hat{q}) b_{\vec{q},\lambda}^* b_{\vec{q},\lambda} d^3q + \mathcal{E}_{\vec{P}}^{\sigma_j} + |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) - z_{j+1}}.$$

Moreover, for $\sigma_{j+1} \leq |\vec{k}| \leq \sigma_j$, $j \geq 1$, and for α, μ, and ϵ small enough but uniform in j, we can control the series expansion in the space $\mathcal{F}_{\sigma_j}$

$$\frac{1}{\frac{(\vec{\Gamma}_{\vec{P}}^{\sigma_j})^2}{2} + H_{\delta_{\vec{P}}^{\sigma_j}}^f + \mathcal{E}_{\vec{P}}^{\sigma_j} + |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) - z_{j+1}} \times \tag{A.51}$$

$$\times \sum_{n=0}^{+\infty} \left[-\left(\vec{\Gamma}_{\vec{P}}^{\sigma_j} \cdot \vec{k} + \frac{|k|^2}{2}\right) \frac{1}{\frac{(\vec{\Gamma}_{\vec{P}}^{\sigma_j})^2}{2} + H_{\delta_{\vec{P}}^{\sigma_j}}^f + \mathcal{E}_{\vec{P}}^{\sigma_j} + |\vec{k}| \delta_{\vec{P}}^{\sigma_j}(\hat{k}) - z_{j+1}}\right]^n$$

where

$$H^f_{\delta^{\sigma_j}_{\vec{P}}} := \sum_\lambda \int_{\mathbb{R}^3} |\vec{q}| \delta^{\sigma_j}_{\vec{P}}(\widehat{q})\, b^*_{\vec{q},\lambda} b_{\vec{q},\lambda} d^3 q\,,$$

the key estimate being

$$\left\| \left(\frac{1}{\frac{(\vec{\Gamma}^{\sigma_j}_{\vec{P}})^2}{2} + H^f_{\delta^{\sigma_j}_{\vec{P}}} + \mathcal{E}^{\sigma_j}_{\vec{P}} + |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) - z_{j+1}} \right)^{1/2} \times \right. \tag{A.52}$$

$$\left. \times\ (\vec{\Gamma}^{\sigma_j}_{\vec{P}} \cdot \vec{k} + \frac{|k|^2}{2}) \left(\frac{1}{\frac{(\vec{\Gamma}^{\sigma_j}_{\vec{P}})^2}{2} + H^f_{\delta^{\sigma_j}_{\vec{P}}} + \mathcal{E}^{\sigma_j}_{\vec{P}} + |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) - z_{j+1}} \right)^{1/2} \right\|_{\mathcal{F}_{\sigma_j}} \le c < 1\,.$$

In order to control the term proportional to $\vec{\Gamma}^{\sigma_j}_{\vec{P}} \cdot \vec{k}$, we note that, for α sufficiently small but uniform in j,

$$\left\| \left(\frac{1}{K^{\sigma_j}_{\vec{P}} + |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) - z_{j+1}} \right)^{1/2} \frac{(\vec{\Gamma}^{\sigma_j}_{\vec{P}})^i}{\sqrt{2}} \right\|^2 \le \frac{1}{3} \left\| K^{\sigma_j}_{\vec{P}} \left| \frac{1}{K^{\sigma_j}_{\vec{P}} + |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) - z_{j+1}} \right| \right\|\,. \tag{A.53}$$

Then, we observe that

$$\left(\frac{1}{|\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k})} \right)^{1/2} \left\| K^{\sigma_j}_{\vec{P}} \left| \frac{1}{K^{\sigma_j}_{\vec{P}} + |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) - z_{j+1}} \right| \right\|^{1/2} \sqrt{6}\,|\vec{k}| \longrightarrow \frac{|\vec{P}|\sqrt{3}}{1 - |\vec{P}|} \le \frac{\sqrt{3}}{2} \tag{A.54}$$

as $\mu, \alpha \to 0$; therefore, the estimate (A.52) also holds true for the term proportional to $\vec{\Gamma}^{\sigma_j}_{\vec{P}} \cdot \vec{k}$ if $\mu > 0$ and $\alpha > 0$ are small enough, but uniform in j. For the last estimate, we used that by assumption, $\vec{P} \in \mathcal{S}$. To estimate of the term proportional to $\frac{|\vec{k}|^2}{2}$, we use

$$\frac{|\vec{k}|^2}{2} \left\| \frac{1}{K^{\sigma_j}_{\vec{P}} + |\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) - z_{j+1}} \right\| \le \frac{|\vec{k}|^2}{2(|\vec{k}|\delta^{\sigma_j}_{\vec{P}}(\widehat{k}) - \mu\sigma_{j+1})} \ll 1\,, \tag{A.55}$$

for α, ϵ, μ small enough but unifor in j. Therefore, recalling that $b_{\vec{k},\lambda}\Phi_{\vec{P}}^{\sigma_j} = 0$ for $|\vec{k}| \leq \sigma_j$, we find

$$(A.49) \hspace{10cm} (A.56)$$

$$\leq C_4 \left\{ \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} d^3k \left| k^l \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + \frac{(\widehat{l}\cdot\vec{\varepsilon}_{\vec{k},\lambda})}{\sqrt{|\vec{k}|}} \right|^2 \times \right. \hspace{2cm} (A.57)$$

$$\left. \times \left\| \left(\frac{1}{K_{\vec{P}}^{\sigma_j} + |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k}) - z_{j+1}} \right)^{\frac{1}{2}} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} \right\|^2 \right\} \sum_{n=0}^{+\infty} (\frac{1}{2})^n$$

$$\leq \frac{1}{1-c} C_4 \left\{ \alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} d^3k \left| k^l \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + \frac{(\widehat{l}\cdot\vec{\varepsilon}_{\vec{k},\lambda})}{\sqrt{|\vec{k}|}} \right|^2 \times \right. \hspace{1cm} (A.58)$$

$$\left. \times \left\| \left(\frac{1}{K_{\vec{P}}^{\sigma_j} + |\vec{k}|\delta_{\vec{P}}^{\sigma_j}(\widehat{k}) - z_{j+1}} \right)^{\frac{1}{2}} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} \right\|^2 \right\}$$

$$\leq \frac{1}{1-c} C_3 C_4 \left(\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} d^3k \left| k^l \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^{\frac{3}{2}}\delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + \frac{(\widehat{l}\cdot\vec{\varepsilon}_{\vec{k},\lambda})}{\sqrt{|\vec{k}|}} \right| \right) \times \hspace{1cm} (A.59)$$

$$\times \left| \left\langle (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} , \frac{1}{K_{\vec{P}}^{\sigma_j} - z_{j+1}} (\Gamma_{\vec{P}}^{\sigma_j})^l \Phi_{\vec{P}}^{\sigma_j} \right\rangle \right|$$

where, in passing from (A.58) to (A.59), we use (IV.67), and property ($\mathscr{A}$3) from Section IV.4. For $\sigma_1 \leq |\vec{k}| \leq \sigma_0$, a similar argument yields (A.59).

This proves the lemma. $\hspace{1cm}\square$

Lemma A.4. *For α and ϵ small enough, with $\epsilon > \mathcal{O}(\alpha)$, there exist constants R_1, $R_2 \leq \mathcal{O}(\epsilon^{-1})$, uniformly in $j \in \mathbb{N}$ and $\vec{P} \in \mathcal{S}$, for which*

$$8 \left\| \left(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \right)^{\frac{1}{2}} ((\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j} - (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \widehat{\Phi}_{\vec{P}}^{\sigma_j}) \right\|_{\mathcal{F}}^2 \tag{A.60}$$

$$+ 8 \left\| \left(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \right)^{\frac{1}{2}} (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i (\widehat{\Phi}_{\vec{P}}^{\sigma_j} - \Phi_{\vec{P}}^{\sigma_{j-1}}) \right\|_{\mathcal{F}}^2$$

$$\leq \quad \frac{R_1}{\epsilon^{\frac{j}{2}}} \left(\frac{\|\widehat{\Phi}_{\vec{P}}^{\sigma_j} - \Phi_{\vec{P}}^{\sigma_{j-1}}\|_{\mathcal{F}} + \epsilon^{\frac{j}{2}}}{\epsilon^{\frac{j}{4}}} \right)^2 \tag{A.61}$$

$$+ \frac{R_2}{\epsilon^{\frac{j}{2}}} \left(\frac{\left\| \frac{\widehat{\Phi}_{\vec{P}}^{\sigma_j}}{\|\widehat{\Phi}_{\vec{P}}^{\sigma_j}\|_{\mathcal{F}}} - \frac{\Phi_{\vec{P}}^{\sigma_{j-1}}}{\|\Phi_{\vec{P}}^{\sigma_{j-1}}\|} \right\|_{\mathcal{F}} + \epsilon^{\frac{j}{2}}}{4\epsilon^{\frac{j}{4}}} \right)^2 .$$

Proof.

In order to justify the estimate in the statement, it is enough to make the difference

$$(\widehat{\Gamma}_{\vec{P}}^{\sigma_j})^i - (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \tag{A.62}$$

explicit. The definitions are given in (IV.41) and (IV.54).

From (A.16), (A.18), we get

$$-\frac{\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \rangle}{\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \rangle} + \frac{\langle \Phi_{\vec{P}}^{\sigma_j}, \vec{\Pi}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \rangle}{\langle \Phi_{\vec{P}}^{\sigma_j}, \Phi_{\vec{P}}^{\sigma_j} \rangle} \tag{A.63}$$

$$= \quad \vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} - \vec{\nabla} E_{\vec{P}}^{\sigma_j} \tag{A.64}$$

$$+ \langle W_{\sigma_{j+1}} (\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}}) \, \vec{\beta}^{\sigma_{j+1}} \, W_{\sigma_{j+1}}^* (\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}}) \rangle_{\Omega_f} \tag{A.65}$$

$$- \langle W_{\sigma_j} (\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \, \vec{\beta}^{\sigma_j} \, W_{\sigma_j}^* (\vec{\nabla} E_{\vec{P}}^{\sigma_j}) \rangle_{\Omega_f} .$$

From (A.26) – (A.31), we obtain

$$\widehat{\vec{\Gamma}}_{\vec{P}}^{\sigma_j} - \vec{\Gamma}_{\vec{P}}^{\sigma_{j-1}} \quad := \quad \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} - \vec{\Pi}_{\vec{P}}^{\sigma_j} \tag{A.66}$$

$$-\frac{\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \widehat{\vec{\Pi}}_{\vec{P}}^{\sigma_{j+1}} \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \rangle}{\langle \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}}, \widehat{\Phi}_{\vec{P}}^{\sigma_{j+1}} \rangle} + \frac{\langle \Phi_{\vec{P}}^{\sigma_j}, \vec{\Pi}_{\vec{P}}^{\sigma_j} \Phi_{\vec{P}}^{\sigma_j} \rangle}{\langle \Phi_{\vec{P}}^{\sigma_j}, \Phi_{\vec{P}}^{\sigma_j} \rangle} \tag{A.67}$$

$$= \ \vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \ - \ \vec{\nabla} E_{\vec{P}}^{\sigma_j} \ + \ \vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j} \tag{A.68}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} \vec{k} \ \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \ \vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_j}(\widehat{k}))^2} \ d^3k \tag{A.69}$$

$$+\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} \vec{k} \ \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \ \vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k}))^2} \ d^3k \tag{A.70}$$

$$-\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} \Big[\vec{\varepsilon}_{\vec{k},\lambda} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_j} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_j}(\widehat{k})} + h.c. \Big] \frac{d^3k}{\sqrt{|\vec{k}|}} \tag{A.71}$$

$$+\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} \Big[\vec{\varepsilon}_{\vec{k},\lambda} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k})} + h.c. \Big] \frac{d^3k}{\sqrt{|\vec{k}|}} \tag{A.72}$$

$$+\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} \vec{k} \ \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^* \ \vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}}{|\vec{k}|^3 (\delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k}))^2} \ d^3k \tag{A.73}$$

$$+\alpha \sum_\lambda \int_{\mathcal{B}_{\sigma_j}\backslash\mathcal{B}_{\sigma_{j+1}}} \Big[\vec{\varepsilon}_{\vec{k},\lambda} \frac{\vec{\nabla} E_{\vec{P}}^{\sigma_{j+1}} \cdot \vec{\varepsilon}_{\vec{k},\lambda}^*}{|\vec{k}|^{\frac{3}{2}} \delta_{\vec{P}}^{\sigma_{j+1}}(\widehat{k})} + h.c. \Big] \frac{d^3k}{\sqrt{|\vec{k}|}} \ . \tag{A.74}$$

Now, we simply combine the result in (A.15) with the bounds

$$\Big\| \Big(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \Big)^{\frac{1}{2}} (\Gamma_{\vec{P}}^{\sigma_{j-1}})^i \Big\| \ \leq \ \mathcal{O}(\epsilon^{-\frac{j+1}{2}}) \tag{A.75}$$

$$\Big\| \Big(\frac{1}{K_{\vec{P}}^{\sigma_{j-1}} - z_{j+1}} \Big)^{\frac{1}{2}} \vec{A}_{\sigma_{j+1}}^{\sigma_j} \Big\| \ \leq \ \mathcal{O}(\epsilon^{-\frac{j+1}{2}}) \, , \tag{A.76}$$

and similarly for $\vec{\mathcal{L}}_{\sigma_{j+1}}^{\sigma_j}$. The size of all other expressions (A.69) – (A.74) can trivially be seen to be of order $\mathcal{O}(\epsilon^j)$. The assertion of the lemma follows. $\square$

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DEPARTMENT OF MATHEMATICS, FINE HALL, WASHINGTON ROAD, PRINCETON, NJ 08544, USA
E-mail address: tc@math.princeton.edu

INSTITUT FÜR THEORETISCHE PHYSIK, ETH HÖNGGERBERG, CH-8093 ZÜRICH, SWITZERLAND, AND IHÉS, BURES SUR YVETTE, FRANCE
E-mail address: juerg@itp.phys.ethz.ch

INSTITUT FÜR THEORETISCHE PHYSIK, ETH HÖNGGERBERG, CH-8093 ZÜRICH, SWITZERLAND
E-mail address: pizzo@itp.phys.ethz.ch