# The stability of solution of the obstacle problem

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*Abstract:* In this paper we are discussing the stability problem of the standard obstacle problem. We show that the solution of the obstacle problem is stable under the boundary conditions. One can show this result for general obstacle problem.

## 1 Introduction

Let $\Omega \subset R^n$ be a bounded domain and let $W^{m,p}(\Omega)$ be a usual Sobolev space. When $p = 2$ we'll write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$. The set of functions which have compact support in $\Omega$ we'll denote $C_0(\Omega)$ and the closer of $C_0^\infty(\Omega)$ by the norm of $H^m(\Omega)$ we'll denote $H_0^m(\Omega)$.
Suppose $g, \varphi$ are functions from $H^1(\Omega)$ and define

$$K = \{u \in H^1(\Omega) \mid u - g \in H_0^1(\Omega),\ u \geq \varphi \text{ a.e. in } \Omega\}.$$

We always will assume that $g \geq \varphi$, so the set $K$ is not empty.
One can easily show that

$$\text{the set } K \text{ is closed and convex} \tag{1}$$

(see for example [1], [2]).
Define

$$G(u) = \int_\Omega \mid Du \mid^2 dx - 2 \cdot \int_\Omega f u dx$$

We conceder the following problem: for given $f \in L^2(\Omega),\ g, \varphi \in H^1(\Omega)$ find

$$u \in K \text{ such that } G(u) = \min_{v \in K} G(v) \tag{2}$$

It is easy to see that this problem is equivalent to the following problem:

$$\text{find } u \in K \text{ such that } \int_\Omega Du \cdot D(v-u)dx \geq \int_\Omega f \cdot (v-u),\ \forall\ v \in K \tag{3}$$

The problem (2) or (3) is called obstacle problem (standard obstacle problem, obstacle problem for the Laplace operator), the function $\varphi$ is called obstacle.
Define

$$a(u,v) = \int_\Omega Du \cdot Dv dx$$

$u, v \in H^1(\Omega)$.
Since there exists an $\alpha > 0$ such that

$$\int_\Omega | Du |^2 \geq \alpha \cdot \int_\Omega | u |^2, \ \forall \ u \in H_0^1(\Omega)$$

(Poincaré's inequality) (see [2]), we can write

$$a(u, u) \geq \beta \cdot \| u \|_{H^1(\Omega)}^2, \ \forall \ u \in H_0^1(\Omega) \tag{4}$$

$\beta > 0$ is a constant. This means that the bilinear form $a(u, v)$ is coercive on $H_0^1(\Omega)$.
It is well known that the Sobolev space $H^1(\Omega)$ is a Hilbert space with the scalar product

$$(f, g)_{H^1} = \int_\Omega f \cdot g + Df \cdot Dg dx$$

The dual space of $H_0^1(\Omega)$ we'll denote by $H^{-1}(\Omega)$.

Elements of $H^{-1}(\Omega)$ may be characterized as the derivatives of functions $f_\imath \in L^2(\Omega)$ in the distributional sense, namely, for $f \in H^{-1}(\Omega)$ there exists $f_0, f_1, \ldots f_n \in L^2(\Omega)$ such that

$$\langle f, h \rangle = \int_\Omega \{ f_0 - \sum_{\imath=1}^n f_\imath h_{x_\imath} \} dx, \ h \in H_0^1(\Omega),$$

$\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ (see [2], [3]).
If $f = f_0$, the problem (3) could be rewritten with the help of this terminology in the following way: find $u \in K$ such that

$$a(u, v - u) \geq (f, v - u), \ \forall \ v \in K,$$

where $(\cdot, \cdot)$ is the scalar product in $L^2(\Omega)$.

In general we will reformulate the problem (3) in the following way: for given $f \in H^{-1}(\Omega)$ find $u \in K$ such that

$$a(u, v - u) \geq \langle f, v - u \rangle, \ \forall \ v \in K \tag{5}$$

***Proposition 1.*** $| a(u, v) | \leq \| u \|_{H^1} \cdot \| v \|_{H^1}$.

*Proof.* First let assume that $\| u \|_{H^1} = \| v \|_{H^1} = 1$.

In this case

$$| a(u, v) | = | \int Du \cdot Dv | \leq \int | Du \cdot Dv | \leq$$

$$\leq \frac{1}{2} \cdot \int (| u |^2 + | Du |^2 + | v |^2 + | Dv |^2) =$$

$$= \frac{1}{2} \cdot (\| u \|_{H^1} + \| v \|_{H^1}) = 1 = \| u \|_{H^1(\Omega)} \cdot \| v \|_{H^1(\Omega)} \; .$$

In general case use the previous fact with functions $u_0 = \frac{u}{\|u\|_{H^1}}, \; v_0 = \frac{v}{\|v\|_{H^1}}$

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Since $a(u, v)$ satisfies (4) and proposition 1 is right, we know that in this case:

***Theorem 2.*** There exists a unique solution to problem (5). And also

$$\| u_1 - u_2 \|_{H^1} \leq \frac{1}{\beta} \cdot \| f_1 - f_2 \|_{H^{-1}},$$

where $u_\imath$ is solution corresponding to $f_\imath$.

For proof of this theorem see [1] or [2].

## 2 Some important facts

Let $X$ be a reflexive Banach space with dual $X^*$. Let $\langle \cdot, \cdot \rangle$ denote a pairing between $X^*$ and $X$.

***Definition 3.*** A mapping $A : D(A) \to X^*$ ($D(A) \subset X$ is the domain where $A$ is defined)is called *monotone*, if

$$\langle Au - Av, u - v \rangle \geq 0, \; \forall \; u, v \in D(A)$$

If $D(A)$ is convex, a mapping $A$ is called *semicontinuous*, if for all $u, v \in D(A)$ the mapping

$$[0, 1] \ni t \to \langle A(tu + (1 - t)v), u - v \rangle$$

is continuous.

***Lemma 4 (Minty).*** Let $K$ be a closed convex subset of $X$, and let a mapping $A : K \to X^*$ is monotone and continuous. Then $u$ satisfies

$$u \in K : \; \langle Au, v - u \rangle \geq 0 \text{ for all } v \in K$$

if and only if it satisfies

$$u \in K : \; \langle Av, v - u \rangle \geq 0 \text{ for all } v \in K.$$

For proof of this lemma see [1] or [2].

***Theorem 5 (Mazur).*** A convex, closed subset of $X$ is weakly closed.

(see [2], [4]).

## 3 Stability of the solution

Suppose $\varphi, g_n \in H^1(\Omega)$ $(n = 1, 2, \ldots)$ and define

$$K_n = \{u \in H^1(\Omega) \mid u - g_n \in H_0^1(\Omega),\ u \geq \varphi \text{ a.e. in } \Omega\}.$$

We always will assume that $g_n \geq \varphi$, so the set $K_n$ is not empty for every $n$. The sets $K_n$ are closed and convex.

***Theorem 6 (Stability).*** Suppose $f \in H^{-1}(\Omega)$. The solution of the obstacle problem is stabile in the following sense:
let $u_n$ be the solution to problem

$$u_n \in K_n,\ a(u_n, v - u_n) \geq \langle f, v - u_n\rangle,\ \forall\ v \in K_n$$

and let $u$ be the solution to problem

$$u \in K,\ a(u, v - u) \geq \langle f, v - u\rangle,\ \forall\ v \in K.$$

If $g_n \to g$ in $H^1(\Omega)$, then $u_n \to u$ weakly in $H^1(\Omega)$.

*Proof.* If $v_n \in K_n$, then using (4) and proposition 1, we have:

$$\beta \cdot \| u_n - v_n \|_{H^1}^2 \leq a(u_n - v_n, u_n - v_n) = a(u_n, u_n - v_n) - a(v_n, u_n - v_n) \leq$$

$$\leq \langle f, u_n - v_n\rangle + \| v_n \|_{H^1} \cdot \| u_n - v_n \|_{H^1} \leq$$

$$\leq (\| v_n \|_{H^1} + \| f \|_{H^{-1}}) \cdot \| u_n - v_n \|_{H^1}\ .$$

So

$$\| u_n - v_n \|_{H^1} \leq \frac{1}{\beta} \cdot (\| v_n \|_{H^1} + \| f \|_{H^{-1}}).$$

$\| u_n \|_{H^1} \leq \| u_n - v_n \|_{H^1} + \| v_n \|_{H^1}$. Therefore, if $v_n \to v \in K$ in $H^1(\Omega)$, we obtain that $\| u_n \|_{H^1} \leq C$. Hence $u_n$ has a weakly convergent subsequence (see [3], [4]). If we show, that from $u_n \to w$ (weakly) follows that $w$ is the unique solution to problem (5), then the proof of the theorem will be complete.
Since the mapping $u \to a(v, u)$ is continuous, we have

$$a(v, u_n) \to a(v, w). \tag{6}$$

According to Minty's lemma

$$a(v_n, v_n - u_n) \geq \langle f, v_n - u_n\rangle,\ \forall\ v_n \in K_n.$$

Choose $v \in K$ and $v_n \in K_n$ such that $\| v_n - v \|_{H^1} \to 0$ (for example take $v_n = v + g_n - g$). Then

$$\mid a(v, v - u_n) - a(v_n, v_n - u_n) \mid = \mid a(v - v_n, v - u_n) + a(v_n, v - v_n) \mid \leq$$

$$\leq \| v-v_n \|_{H^1} \cdot \| v-u_n \|_{H^1} + \| v_n \|_{H^1} \cdot \| v-v_n \|_{H^1} \leq C \| v-v_n \|_{H^1} \to 0.$$

So

$$a(v, v-u_n) \geq \langle f, v_n - u_n \rangle + \varepsilon_n, \ \varepsilon_n \to 0.$$

Since

$$\langle f, v_n - u_n \rangle \to \langle f, v - w \rangle,$$

from (6) we get

$$a(v, v-w) \geq \langle f, v - w \rangle.$$

Now, if we show that $w \in K$, we can insist that $w$ is the unique solution to problem (5), and the proof will be complete.
Since $g_n \to g$ in $H^1(\Omega)$, then $g_n \to g$ weakly in $H^1(\Omega)$ (see, for example, [3], [4]). So

$$u_n - g_n \to w - g, \text{ weakly in } H^1(\Omega)$$

But $u_n - g_n \in H_0^1(\Omega)$ and $H_0^1(\Omega)$ is a closed convex subset of $H^1(\Omega)$. So, according to Mazur's theorem, $w - g \in H_0^1(\Omega)$.
Therefore $w \in K$.■

# 4 References

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