DYNAMICAL LOWER BOUNDS FOR 1D DIRAC OPERATORS

ROBERTO A. PRADO AND CÉSAR R. DE OLIVEIRA

Abstract. Quantum dynamical lower bounds for continuous and discrete one-dimensional Dirac operators are established in terms of transfer matrices. Then such results are applied to various models, including the Bernoulli-Dirac one and, in contrast to the discrete case, critical energies are also found for the continuous Dirac case with positive mass.

1. Introduction

We consider discrete, resp. continuous, Dirac operators

$$\mathbf{D}(m,c) := \mathbf{D}_0(m,c) + V\mathrm{I}_2 = \begin{pmatrix} mc^2 & cD^* \\ cD & -mc^2 \end{pmatrix} + V\mathrm{I}_2, \tag{1}$$

with Dirichlet boundary conditions, acting on $\ell^2(\mathbb{N},\mathbb{C}^2)$, resp. $L^2([0,\infty),\mathbb{C}^2)$, where $c>0$ represents the speed of light, $m \geq 0$ the mass of a particle, I_2 is the 2×2 identity matrix and V is a bounded real potential. In the discrete case D is the finite difference operator defined by $(D\varphi)(n) = \varphi(n+1)-\varphi(n)$, with adjoint $(D^*\varphi)(n) = \varphi(n-1) - \varphi(n)$, and in the continuous case $D = D^* = -i\frac{d}{dx}$.

Model (1) in the continuous case is well known in relativistic quantum mechanics [1, 13], and the discrete version was introduced and studied in [6, 7].

The goal of this paper is to establish lower bounds on the dynamics associated to $\mathbf{D}(m,c)$ through the behaviour of the corresponding transfer matrices. To this end we will consider the time averaged q-th moments A_ψ of the position operator

$$\left[X\begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}\right](x) = \begin{pmatrix} x\ \varphi_+(x) \\ x\ \varphi_-(x) \end{pmatrix}$$

acting in $\ell^2(\mathbb{N},\mathbb{C}^2)$, resp. $L^2([0,\infty),\mathbb{C}^2)$, defined by $(T>0)$

$$A_\psi(m,T,q) := \frac{2}{T}\int_0^\infty e^{-2t/T}\left\| |X|^{q/2}\, e^{-it\mathbf{D}(m,c)}\psi \right\|^2\, dt, \tag{2}$$

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E-mail addresses: rap@dm.ufscar.br (RAP), oliveira@dm.ufscar.br (CRdeO).
Fax: +55 16 33518218.

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with initial state $\psi = \delta_1^+$ in $\ell^2(\mathbb{N}, \mathbb{C}^2)$, resp. $\psi = f$ in $L^2([0,\infty), \mathbb{C}^2)$, where δ_1^+ is an element of the canonical basis of $\ell^2(\mathbb{N}, \mathbb{C}^2)$ and f is an element of $L^2([0,\infty), \mathbb{C}^2)$ with compact support which satisfies a suitable technical condition.

To investigate the polynomial behaviour in time T of $A_\psi(m, T, q)$, one usually considers the lower growth exponents

$$\beta_\psi^-(m, q) := \liminf_{T\to\infty} \frac{\log A_\psi(m, T, q)}{\log T} \ . \tag{3}$$

In the Schrödinger setting, dynamical lower bounds was found for random polymer models [11] and for random palindrome models [2], due to existence of critical energies [11]. For discrete Schrödinger operators in $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$, in [5] a general method was developed which allows one to derive dynamical lower bounds from upper bounds on the growth of norms of transfer matrices. Damanik, Lenz and Stolz [4] have presented an extension of this method to continuous Schrödinger operators in $L^2([0,\infty))$ and $L^2(\mathbb{R})$, with application to the continuous Bernoulli-Anderson model.

In this paper we adapt the above mentioned methods to the Dirac model (1) for both discrete and continuous cases. One important consequence of Theorem 1 ahead is the following: suppose that there is an energy $E_0 \in \mathbb{R}$ such that the transfer matrices $\Phi_m(E_0, x, y)$ (defined in Section 2) satisfies $\|\Phi_m(E_0, x, y)\| \le CN^\alpha$ for all N large enough, $\alpha \ge 0$, $C > 0$ and $0 \le x, y \le N$, then it follows that

$$A_\psi(m, T, q) \ge \tilde{C} T^{\frac{q-1-4\alpha}{1+\alpha}} \ ,$$

for ψ as in (2) and $\tilde{C} > 0$. We then apply such result to the continuous Bernoulli-Dirac model, the discrete Dirac model with zero mass ($m = 0$) and any two-valued potential, the Thue-Morse Dirac model and discrete Dirac model with Sturmian potentials.

There are some reasons justifying the adaptation of known results in the Schrödinger setting to the Dirac one. First of all, although expected, it is not immediately clear (nor trivial) which and how such adaptations work. Second, although we have found the abstract results have similar statements, in applications usually different conditions on the potentials appear in case of Dirac operators (see, e.g., Theorem 3). Third, and this was our main motivation for considering dynamical lower bounds for model (1), is that for the continuous Bernoulli-Dirac model it is possible to construct examples (see Subsection 3.1) which have critical energies for $m = 0$ and also for $m > 0$, in contrast with the discrete case which have critical energies only for $m = 0$ [6, 7]. Fourth, with respect to transfer matrices, the discrete Dirac operator has some kind of "built-in dimerization" [7] (implying transport) which motivates the study of the corresponding continuous case. Finally, we have found that the upper and lower components of some initial conditions in the Dirac setting produce interferences so that the technique in the Schrödinger case does not apply (so leaving an interesting open problem); see the remark at the end of Subsection 3.1.

We anticipate that the presence of critical energies in continuum Bernoulli-Dirac models produces dynamical lower bounds in the sense that almost surely

$$\beta_f^-(m,q) \geq q - \frac{1}{2},$$

for all $q > 0$, for any mass $m \geq 0$ and suitable initial conditions f.

Another method to obtain dynamical lower bounds from upper bounds on transfer matrices was lately developed in [9], with application to Schrödinger operators with random decaying potentials and sparse potentials. Their method is suitable for models that admit upper bounds on transfer matrix norms for large sets of energies (i.e., sets with positive Lebesgue measure), while with the method used here (based on [4, 5]) it is possible to get dynamical bounds for models with large or small (e.g., finite) sets of such energies. An approach for quasi-ballistic dynamics for discrete Schrödinger as well Dirac operators with potentials along some dynamical systems have recently been obtained in [8].

This paper is organized as follows: In Section 2 the result about dynamical lower bounds (Theorem 1) for the Dirac model (1) is presented, whose proof appears in Section 4. In Section 3 applications of Theorem 1 are discussed, including the continuous Bernoulli-Dirac model.

2. Dynamical Bounds

In this section we will present results about dynamical lower bounds for the operators $\mathbf{D}(m,c)$ defined by (1) in both the discrete and continuous cases.

For a given operator $\mathbf{D}(m,c)$ on $\ell^2(\mathbb{N},\mathbb{C}^2)$, resp. $L^2([0,\infty),\mathbb{C}^2)$, the transfer matrices $\Phi_m(E,x,y)$ between sites y and x are defined as

$$\Phi_m(E,x,y) = \begin{pmatrix} u_+^N(x+1) & u_+^D(x+1) \\ u_-^N(x) & u_-^D(x) \end{pmatrix}, \text{ resp. } \begin{pmatrix} u_+^N(x) & u_+^D(x) \\ u_-^N(x) & u_-^D(x) \end{pmatrix},$$

where $u^N = \begin{pmatrix} u_+^N \\ u_-^N \end{pmatrix}$ and $u^D = \begin{pmatrix} u_+^D \\ u_-^D \end{pmatrix}$ denote the solutions of equation $\mathbf{D}(m,c)u = Eu$, $E \in \mathbb{R}$, satisfying

$$\begin{pmatrix} u_+^N(y+1) \\ u_-^N(y) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_+^D(y+1) \\ u_-^D(y) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

resp.

$$u^N(y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; u^D(y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It follows from the definitions that if $u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ is a solution of the eigenvalue equation $\mathbf{D}(m,c)u = Eu$, then

$$\begin{pmatrix} u_+(x+1) \\ u_-(x) \end{pmatrix} = \Phi_m(E,x,y) \begin{pmatrix} u_+(y+1) \\ u_-(y) \end{pmatrix},$$

resp.

$$\begin{pmatrix} u_+(x) \\ u_-(x) \end{pmatrix} = \Phi_m(E,x,y) \begin{pmatrix} u_+(y) \\ u_-(y) \end{pmatrix}.$$

Note that in the discrete case, the matrix $\Phi_m(E,x,y)$, $x > y \geq 0$, can be written as

$$\Phi_m(E,x,y) = T_m(E,V(x)) \cdots T_m(E,V(y+1)),$$

with

$$T_m(E,V(k)) = \begin{pmatrix} 1 + \dfrac{m^2c^4 - (E-V(k))^2}{c^2} & \dfrac{mc^2 + E - V(k)}{c} \\ \\ \dfrac{mc^2 - E + V(k)}{c} & 1 \end{pmatrix}.$$

We denote by $\delta_n^{\pm}$ the elements of the canonical position basis of $\ell^2(\mathbb{N}, \mathbb{C}^2)$, for which all entries are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ except the nth one, which is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the superscript indices $+$ and $-$, respectively.

In the continuous case, consider the measurable locally bounded vector-valued functions w_E, v_E defined by

$$w_E(x) = u_+^N(0) \begin{pmatrix} -u_+^D(x) \\ \\ u_-^D(x) \end{pmatrix} + u_+^D(0) \begin{pmatrix} u_+^N(x) \\ \\ -u_-^N(x) \end{pmatrix}$$

and

$$v_E(x) = u_-^N(0) \begin{pmatrix} -u_+^D(x) \\ \\ u_-^D(x) \end{pmatrix} + u_-^D(0) \begin{pmatrix} u_+^N(x) \\ \\ -u_-^N(x) \end{pmatrix}.$$

For $g = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$, with g_+, g_- measurable and locally bounded functions, and $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in L^2([0,\infty), \mathbb{C}^2)$ of compact support, define

$$[g,f] := \int_0^\infty \left(\overline{g_+(t)}\, f_+(t) + \overline{g_-(t)}\, f_-(t) \right) dt.$$

Note that in case all involved functions are square integrable $[\cdot,\cdot]$ coincides with their inner product.

For fixed parameters m and c, let $\mathcal{H}_E$ be the set of the vectors $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in L^2([0,\infty), \mathbb{C}^2)$ with compact support, which satisfies one of the following conditions:

(i) $f_+ \neq 0,\ f_- = 0$ and $[\overline{u}, f] = \int_0^\infty u_+(t) f_+(t) dt \neq 0$ for some solution $u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ of $\mathbf{D}(m,c)u = Eu$;

(ii) $f_+ = 0,\ f_- \neq 0$ and $[\overline{u}, f] = \int_0^\infty u_-(t) f_-(t) dt \neq 0$ for some solution $u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ of $\mathbf{D}(m,c)u = Eu$;

(iii) $f_+ \neq 0,\ f_- \neq 0$ and $[\overline{w_E}, f] \neq 0$ or $[\overline{v_E}, f] \neq 0$ (or both).

For $\alpha, m \geq 0,\ C > 0$ and $N > 1$ define the set

$$P_m(\alpha, C, N) = \Big\{ E \in \mathbb{R} : \|\Phi_m(E,x,y)\| \leq C N^\alpha \text{ for all } 0 \leq x, y \leq N \Big\}.$$

Now we are in position to state the main result about dynamical lower bounds.

Theorem 1. *Let $\boldsymbol{D}(m,c)$ be the operator defined by (1). Suppose $E_0 \in \mathbb{R}$ is such that there exist $C > 0$ and $\alpha \geq 0$ with $E_0 \in P_m(\alpha, C, N)$ for all sufficiently large N.*
(**i**) (discrete case) Let $A(N)$ be a uniformly bounded sequence of subset of $P_m(\alpha, C, N)$ containing E_0 and μ_+^m the spectral measure for $\mathbf{D}(m,c)$ associated to δ_1^+. Then, there exists $\tilde{C} > 0$ such that for $T > 0$ large enough

$$A_{\delta_1^+}(m,T,q) \geq \tilde{C} \left(|B_2(T)| + \mu_+^m(B_1(T)) \right) T^{\frac{q-3\alpha}{1+\alpha}},$$

where $B_j(T), j = 1, 2$, is the j/T neighborhood of $A(T^{\frac{1}{1+\alpha}})$.
(**ii**) (continuous case) Let $A(N)$ be a subset of $P_m(\alpha, C, N)$ containing E_0 such that $\operatorname{diam}(A(N)) \to 0$ as $N \to \infty$. Then, for every $f \in \mathcal{H}_{E_0}$ there exists $\tilde{C} > 0$ such that for $T > 0$ large enough

$$A_f(m,T,q) \geq \tilde{C}\, |B_1(T)|\, T^{\frac{q-3\alpha}{1+\alpha}}.$$

Remarks. **1**. Theorem 1 can be adapted to the operator $\mathbf{D}(m,c)$ on $\ell^2(\mathbb{Z}, \mathbb{C}^2)$ and $L^2(\mathbb{R}, \mathbb{C}^2)$, and always with similar statements.
2. The dynamical lower bounds obtained in Theorem 1 are stable under suitable power-decaying perturbations of the potential V as in [5], because the power-law bounds of the transfer matrices keep unchanged.

The proof of Theorem 1 will be given in Section 4. As in [4, 5], Theorem 1 have the following immediate consequences.

Corollary 1. *Let A be a nonempty bounded subset of $P_m(\alpha, C, N)$ for some $C > 0$, $\alpha \geq 0$ and for all N large enough, such that $\mu_+^m(A) > 0$. Then*

$$\beta^-_{\delta_1^+}(m, q) \geq \frac{q - 3\alpha}{1 + \alpha}.$$

Proof. Take $A(N) = A$ for every N. Since $\mu_+^m(B_1(T)) \geq \mu_+^m(A) > 0$, by Theorem 1(i) there exists $\tilde{C} > 0$ such that for $T > 0$ large enough

$$A_{\delta_1^+}(m, T, q) \geq \tilde{C}\; T^{\frac{q-3\alpha}{1+\alpha}}.$$

Hence the result follows. □

Corollary 2. *Suppose there is an energy $E_0 \in \mathbb{R}$ such that $\|\Phi_m(E_0, x, y)\| \leq CN^\alpha$ for all N large enough and $0 \leq x, y \leq N$. Then,*

$$\beta^-_\psi(m, q) \geq \frac{q - 1 - 4\alpha}{1 + \alpha},$$

for every $\psi = f \in \mathcal{H}_{E_0}$ in the continuous case and $\psi = \delta_1^+$ in the discrete case.

Proof. Take $A(N) = \{E_0\}$ for every N. Then $B_1(T) = \left[E_0 - \frac{1}{T}, E_0 + \frac{1}{T}\right]$ and by Theorem 1 there exists $\tilde{C} > 0$ such that for T large enough

$$A_\psi(m, T, q) \geq \frac{\tilde{C}}{T}\; T^{\frac{q-3\alpha}{1+\alpha}} = \tilde{C}\; T^{\frac{q-1-4\alpha}{1+\alpha}},$$

for ψ as in the hypothesis. Hence the result follows. □

3. Applications

This section is devoted to applications of Theorem 1 and its corollaries.

3.1. **The continuous Bernoulli-Dirac model.** Let g_0 and g_1 be two real-valued potentials with support in $[0, 1]$. Consider the family of Dirac operators in $L^2([0, \infty), \mathbb{C}^2)$,

$$\mathbf{D}_\omega(m, c) := \mathbf{D}_0(m, c) + V_\omega \mathrm{I}_2, \quad \omega \in \Omega := \{0, 1\}^{\mathbb{N}}, \tag{4}$$

with potential $V_\omega(x) = \sum_n g_{\omega_n}(x - n)$, where $\omega_n \in \{0, 1\}$ are i.i.d. Bernoulli random variables with common probability measure μ satisfying $\mu(\{0\}) = p$, $\mu(\{1\}) = 1 - p$, for some $0 < p < 1$, and product measure $\mathbf{P} = \prod_n \mu(\omega_n))$ on Ω.

Let $T_m^{(j)}(E)$ be the transfer matrix for $\mathbf{D}_\omega(m, c)$ with potential $V_j(x) = \sum_n g_j(x - n)$, $j = 0, 1$, at energy E from 0 to 1.

Definition 1 ([11]). *$E_0 \in \mathbb{R}$ is a critical energy for $\boldsymbol{D}_\omega(m,c)$ if the matrices $T_m^{(j)}(E_0)$, $j=0,1$, are elliptic (i.e., $|\text{trace } T_m^{(j)}(E_0)| < 2$) or equal to $\pm \mathrm{I}_2$, and commute.*

If E_0 is a critical energy for $\mathbf{D}_\omega(m,c)$, it follows from Definition 1 that there exists a real invertible matrix Q such that

$$Q\, T_m^{(j)}(E_0)\, Q^{-1} = \begin{pmatrix} \cos(\eta_j) & -\sin(\eta_j) \\ \sin(\eta_j) & \cos(\eta_j) \end{pmatrix}, \quad \text{for } j=0,1.$$

Adapting the arguments used in [11, 4] for the Bernoulli-Dirac model (4), we obtain the following (details omitted).

Lemma 1. *Assume that $\eta_0 - \eta_1$ is not an integer multiple of π. Let $\lambda > 0$ be arbitrary. Then there are $b > 0$ and $C < \infty$ such that for every $N \in \mathbb{N}$, there exists a set $\Omega_N(\lambda) \subset \Omega$ with $\boldsymbol{P}(\Omega_N(\lambda)) \le Ce^{-bN^\lambda}$ and*

$$\left\| \Phi_m^\omega(E,x,y) \right\| \le C$$

for all $\omega \in \Omega \backslash \Omega_N(\lambda)$, $0 \le x, y \le N$ and $E \in [E_0 - N^{-\lambda-1/2}, E_0 + N^{-\lambda-1/2}]$.

We can now state our main result for model (4).

Theorem 2. *Assume that $\eta_0 - \eta_1$ is not an integer multiple of π. For every $f \in \mathcal{H}_{E_0}$ one has*

$$\beta_f^-(m,q) \ge q - \frac{1}{2}\ , \quad \omega\ \boldsymbol{P} - a.s.\ .$$

Proof. Due to Lemma 1, for each $\lambda > 0$, $\mathbf{P}(\Omega_N(\lambda))$ is summable over N. Thus, by Lemma 1 and a Borel-Cantelli argument, there exists $0 < C < \infty$ such that $\|\Phi_m^\omega(E,x,y)\| \le C$ for all N, $0 \le x, y \le N$, for almost every ω and $E \in A(N) := [E_0 - N^{-\lambda-1/2}, E_0 + N^{-\lambda-1/2}]$. Note that $|B_1(T)| \ge |A(T)| = 2T^{-\lambda-1/2}$. Applying Theorem 1(ii) with $\alpha = 0$, it follows that almost surely $\beta_f^-(m,q) \ge q - \frac{1}{2} - \lambda$ for every $f \in \mathcal{H}_{E_0}$. Taking $\lambda = \frac{1}{n} \to 0$ and using a countable intersection of full measure sets, we obtain the result. $\square$

It is possible to show, by applying similar arguments of [4, 11] for model (4), that if E_0 is a critical energy for $\mathbf{D}_\omega(m,c)$, then for every $f \in \mathcal{H}_{E_0}$ one has $\beta_f^-(m,q) \ge q - 1$, for every ω.

Recently, we have established (see [7]) the same lower bounds obtained above for the discrete Bernoulli-Dirac model with zero mass ($m = 0$), due to existence of critical energies. Now we will present a continuous Bernoulli-Dirac model defined by (4) that have critical energies for both $m = 0$ and $m > 0$ (note that for the latter case critical energies are absent in the discrete case). As a consequence we will obtain lower bounds by Theorem 2.

In fact, consider the Bernoulli-Dirac model (4) with $g_0 = 0$ and $g_1 = \lambda\chi_{[0,1]}$, $\lambda > 0$. By solving the equation $\mathbf{D}_0(m,c)u = Eu$ one finds the following solutions for $E^2 > m^2c^4$: $u^N = \begin{pmatrix} u_+^N \\ u_-^N \end{pmatrix}$ and $u^D = \begin{pmatrix} u_+^D \\ u_-^D \end{pmatrix}$, with

$$u_+^N(x) = \cos(\xi_E\, x), \quad u_-^N(x) = \frac{-i(mc^2 - E)}{c\xi_E}\sin(\xi_E\, x),$$

$$u_+^D(x) = \frac{-ic\xi_E}{mc^2 - E}\sin(\xi_E\, x), \quad u_-^D(x) = \cos(\xi_E\, x),$$

where $\xi_E = \dfrac{\sqrt{E^2 - m^2c^4}}{c}$, and they satisfy

$$u^N(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u^D(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus, the transfer matrices are

$$T_m^{(0)}(E) = \begin{pmatrix} \cos\xi_E & \dfrac{-ic\,\xi_E}{mc^2 - E}\sin\xi_E \\ & \\ \dfrac{-i(mc^2 - E)}{c\,\xi_E}\sin\xi_E & \cos\xi_E \end{pmatrix}$$

for $E^2 > m^2c^4$ and $T_m^{(1)}(E) = T_m^{(0)}(E-\lambda)$ for $(E-\lambda)^2 > m^2c^4$.

If $E = \pm\sqrt{m^2c^4 + n^2\pi^2c^2}$ for $n \in \mathbb{N}^*$ and $m \geq 0$, then $T_m^{(0)}(E) = \pm\mathrm{I}_2$. Moreover, taking

$$0 < \lambda < \sqrt{m^2c^4 + n^2\pi^2c^2} - mc^2 \;\text{ or }\; \lambda > \sqrt{m^2c^4 + n^2\pi^2c^2} + mc^2$$

(this implies $(E-\lambda)^2 > m^2c^4$), it follows that $|\text{trace } T_m^{(1)}(E)| < 2$ (i.e., $T_m^{(1)}(E)$ is elliptic). On the other hand, if $E = \lambda \pm \sqrt{m^2c^4 + n^2\pi^2c^2}$ for $n \in \mathbb{N}^*$, $m \geq 0$ and λ as above, we have $T_m^{(1)}(E) = \pm\mathrm{I}_2$ and $|\text{trace } T_m^{(0)}(E)| < 2$. Thus, for such values of λ we have the following set of critical energies:

$$\left\{\pm\sqrt{m^2c^4 + n^2\pi^2c^2},\ \lambda \pm \sqrt{m^2c^4 + n^2\pi^2c^2} : n \in \mathbb{N}^*, m \geq 0\right\}.$$

For such energies the condition required in Theorem 2 holds, that is, $\eta_0 - \eta_1 \neq k\pi$, $k \in \mathbb{Z}$.

Corollary 3. *Let $\boldsymbol{D}_\omega(m,c)$ be defined by (4) with $g_0 = 0$ and $g_1 = \lambda\chi_{[0,1]}$, $\lambda > 0$. If $\lambda < \sqrt{m^2c^4 + n^2\pi^2c^2} - mc^2$ or $\lambda > \sqrt{m^2c^4 + n^2\pi^2c^2} + mc^2$, then*

$$\beta_f^-(m,q) \geq q - \frac{1}{2}, \quad \omega\ \boldsymbol{P}-a.s.,$$

for all masses $m \geq 0$ and any $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in L^2([0,1],\mathbb{C}^2)$ satisfying one of the following conditions:
(i) $0 \neq f_+ \in L^2([0,1])$ and $f_- = 0$.

(ii) $f_+ = 0$ *and* $0 \neq f_- \in L^2([0,1])$.
(iii) $f_+ \neq 0,\ f_- \neq 0$ *and*

$$[\overline{w_E}, f] = \int_0^1 \left[\left(\frac{-in\pi c}{mc^2 \mp \sqrt{m^2c^4 + n^2\pi^2c^2}} \right) f_+(x)\sin(n\pi x) + f_-(x)\cos(n\pi x) \right] dx \neq 0$$

or

$$[\overline{v_E}, f] = \int_0^1 \left[f_+(x)\cos(n\pi x) - i\left(\frac{mc^2 \mp \sqrt{m^2c^4 + n^2\pi^2c^2}}{n\pi c} \right) f_-(x)\sin(n\pi x) \right] dx \neq 0.$$

Note that in this case the above conditions on f depends on m.

Proof. We consider two cases:

1. $\omega_0 = 0$, that is, $V_\omega(x) = 0$ on $[0,1]$.
2. $\omega_0 = 1$, that is, $V_\omega(x) = \lambda$ on $[0,1]$.

If $\omega_0 = 0$, then applying Theorem 2 for the critical energies

$$E = \pm\sqrt{m^2c^4 + n^2\pi^2c^2},\ n \in \mathbb{N}^*,\ m \geq 0,$$

we obtain

$$\beta_f^-(m,q) \geq q - \frac{1}{2}\ , \quad \omega\ \mathbf{P} - a.s.,$$

for all mass values $m \geq 0$ and for any $f \in \mathcal{H}_E$ with $\operatorname{supp} f \subset [0,1]$. Note that for such energies

$$u^N(x) = \begin{pmatrix} \cos(n\pi x) \\ \\ -i\dfrac{(mc^2 \mp \sqrt{m^2c^4 + n^2\pi^2c^2})}{n\pi c}\sin(n\pi x) \end{pmatrix}$$

and

$$u^D(x) = \begin{pmatrix} \dfrac{-in\pi c}{mc^2 \mp \sqrt{m^2c^4 + n^2\pi^2c^2}}\sin(n\pi x) \\ \\ \cos(n\pi x) \end{pmatrix}$$

are fundamental solutions of $\mathbf{D}_0(m,c)u = Eu$. By definition we have the vectors $w_E(x) = \begin{pmatrix} -u_+^D(x) \\ u_-^D(x) \end{pmatrix}$ and $v_E(x) = \begin{pmatrix} u_+^N(x) \\ -u_-^N(x) \end{pmatrix}$.

For any $f_+ \in L^2([0,1]),\ f_+ \neq 0$, there is at least one $n \in \mathbb{N}$ such that

$$\int_0^1 f_+(t)\cos(n\pi x)dt \neq 0 \ \text{ or } \ \int_0^1 f_+(t)\sin(n\pi x)dt \neq 0$$

(similarly for $0 \neq f_- \in L^2([0,1])$). This is valid because

$$\{1\} \cup \{\cos(2k\pi x), \sin(2k\pi x) : k \in \mathbb{N}\}$$

form a basis of $L^2([0,1])$. Therefore, by using the definition of the set $\mathcal{H}_E$ the required result is obtained.

If $\omega_0 = 1$, then we conclude the result in the same way, but now based on the critical energies $E = \lambda \pm \sqrt{m^2c^4 + n^2\pi^2c^2}$, $n \in \mathbb{N}^*$ and $m \geq 0$. $\square$

Remark. Note that Corollary 3*(iii)* does not assure $\beta_f^-(m,q) \geq q - \frac{1}{2}$ for any $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in L^2([0,1],\mathbb{C}^2)$, due to some kind of quantum interference. For instance, for any integer $n, \tilde{n}$, by taking

$$f_+(x) = \frac{mc^2 \mp \sqrt{m^2c^4 + n^2\pi^2c^2}}{in\pi c}\sin(\tilde{n}\pi x) \quad \text{and} \quad f_-(x) = -\cos(\tilde{n}\pi x),$$

one obtains $[\overline{w_E}, f] = 0$ and $[\overline{v_E}, f] = 0$. In the corresponding Schrödinger model [4] one has $\beta_f^-(q) \geq q - \frac{1}{2}$ for any $f \in L^2([0,1]), f \neq 0$.

3.2. **The discrete massless Dirac model with two-valued potentials.** Consider the discrete Dirac operator $\mathbf{D}(0,c)$ defined by (1). The following result holds.

Theorem 3. *Let $V : \mathbb{N} \to \{a,b\} \subset \mathbb{R}$ be a potential for $\boldsymbol{D}(0,c)$.*

(i) *If $|a-b| < 2c$, then for every $q > 0$, $\beta^-_{\delta_1^+}(0,q) \geq q - 1$.*

(ii) *If $|a-b| = 2c$, then for every $q > 0$, $\beta^-_{\delta_1^+}(0,q) \geq \frac{q-5}{2}$.*

Proof. We shall find upper bounds for the transfer matrices $\Phi_0(E_0,x,y)$ for a suitable energy E_0. Let $E_0 = a$. Then

$$T_0(E_0,a) = \mathrm{I}_2 \quad \text{and} \quad T_0(E_0,b) = \begin{pmatrix} 1 - \dfrac{(a-b)^2}{c^2} & \dfrac{a-b}{c} \\[2ex] \dfrac{-a+b}{c} & 1 \end{pmatrix}.$$

This implies that $\Phi_0(E_0,x,y) = (T_0(E_0,b))^{n_b}$, where n_b is the number of times that b occurs in the product. If $|a-b| < 2c$, then $T_0(E_0,b)$ is elliptic ($|\text{trace}\, T_0(E_0,b)| < 2$) and hence

$$\|\Phi_0(E_0,x,y)\| \leq C(E_0), \quad \forall\, x,y \in \mathbb{N}.$$

Thus, by Corollary 2 with $\alpha = 0$, we obtain

$$\beta^-_{\delta_1^+}(0,q) \geq q - 1, \quad \forall\, q > 0.$$

On the other hand, if $|a-b| = 2c$, then $T_0(E_0,b)$ is parabolic ($|\text{trace}\, T_0(E_0,b)| = 2$) and hence $T_0(E_0,b)$ can be written as $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$ with $d \neq 0$. Because

$$\left\| \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}^{n_b} \right\| = \left\| \begin{pmatrix} 1 & n_b d \\ 0 & 1 \end{pmatrix} \right\| \leq C_d\, n_b,$$

it follows that

$$\|\Phi_0(E_0,x,y)\| \leq C(E_0)n_b \leq C(E_0)|x-y|, \quad \forall\, x,y \in \mathbb{N}.$$

Therefore, by Corollary 2 with $\alpha = 1$, we obtain

$$\beta^{-}_{\delta_1^+}(0,q) \geq \frac{q-5}{2}, \quad \forall\, q > 0.$$

□

3.3. **The Thue-Morse Dirac model.** This model is defined as in (1) by

$$\mathbf{D}_\omega(m,c) := \mathbf{D}_0(m,c) + V_\omega \mathrm{I}_2,$$

acting on $\ell^2(\mathbb{N},\mathbb{C}^2)$ or $L^2([0,\infty),\mathbb{C}^2)$, where V_ω is generated by the Thue-Morse substitution on the alphabet $\{a,b\}$ given by $S(a) = ab, S(b) = ba$. For more details see [4, 5]. Let Ω_{TM} be the associated subshift.

Since the boundedness of the transfer matrices in this case depends only on the structure of the potential and it is independent on the explicit form of these matrices, by adapting a similar model [4, 5] in the Schrödinger setting we obtain the following result (details omitted).

Lemma 2. *There are $E_0 \in \mathbb{R}$ and $C > 0$ such that for every $\omega \in \Omega_{\mathrm{TM}}$ and every $m \geq 0$,*

$$\|\Phi^\omega_m(E_0,x,y)\| \leq C, \ \forall\, x,y \in \mathbb{N} \quad \textit{or} \quad \forall\, x,y \in [0,\infty).$$

Thus, by Corollary 2 with $\alpha = 0$, it follows that

$$\beta^{-}_{\omega,\psi}(m,q) \geq q-1,$$

for every $\omega \in \Omega_{\mathrm{TM}},\ q > 0,\ m \geq 0$ and for every $\psi = f \in \mathcal{H}_{E_0}$ in the continuous case and $\psi = \delta_1^+$ in the discrete case. This should be compared with Theorem 3.

3.4. **The discrete Dirac model with Sturmian Potentials.** We discuss dynamical lower bounds for the model

$$\mathbf{D}_{\lambda,\omega,\theta}(m,c) := \mathbf{D}_0(m,c) + V_{\lambda,\omega,\theta}\mathrm{I}_2$$

defined by (1) on $\ell^2(\mathbb{N},\mathbb{C}^2)$, whose potential is given by

$$V_{\lambda,\omega,\theta}(x) = \lambda\chi_{[1-\omega,1)}(x\omega + \theta \mod 1),$$

where $\lambda \neq 0$ is the coupling constant, $\omega \in (0,1)$ irrational is the rotation number and $\theta \in [0,1)$ is the phase. For more details on this potential in the corresponding Schrödinger case see [3, 10].

Since the boundedness of the transfer matrices in this case depends only on the structure of the potential, again a direct adaptation of results in the Schrödinger setting shows that

Lemma 3. *Suppose ω is a number of bounded density. For every λ, there are a constant $C > 0$ and $\alpha = \alpha(\lambda, \omega) > 0$ such that for every θ and every $E \in \sigma(\boldsymbol{D}_{\lambda,\omega,\theta})$ we have*

$$\|\Phi^{\omega}_{m,\lambda,\theta}(E, x, y)\| \leq C \; |x - y|^{\alpha},$$

for every $x, y \in \mathbb{N}$ and any $m \geq 0$.

Therefore, by Corollary 1 with $A = \sigma(\mathbf{D}_{\lambda,\omega,\theta})$ (so $\mu^m_+(A) = 1$), it is found that for every λ, θ, the operator $\mathbf{D}_{\lambda,\omega,\theta}$ satisfies

$$\beta^-_{\delta_1^+}(m, q) \geq \frac{q - 3\alpha}{1 + \alpha} \; ,$$

for every $q > 0$ and any $m \geq 0$.

4. Proof of Dynamical Bounds

In this section the proof of Theorem 1 will be presented. We first gather some preliminary results that we will used in the proof.

For the operator $\mathbf{D}(m, c)$, $m \geq 0$, on $\ell^2(\mathbb{N}, \mathbb{C}^2)$, we introduce the two-components Green's function

$$\begin{pmatrix} G^+_m(z, n) \\ G^-_m(z, n) \end{pmatrix} = \begin{pmatrix} \left\langle \delta^+_n, (\mathbf{D}(m, c) - z)^{-1} \, \delta^+_1 \right\rangle \\ \left\langle \delta^-_n, (\mathbf{D}(m, c) - z)^{-1} \, \delta^+_1 \right\rangle \end{pmatrix}, \; z \in \mathbb{C} \backslash \mathbb{R},$$

so that

$$(\mathbf{D}(m, c) - z) \begin{pmatrix} G^+_m(z, n) \\ G^-_m(z, n) \end{pmatrix} = \delta^+_1(n) \; . \tag{5}$$

By using transfer matrices, one obtains for $n \geq 1$,

$$\begin{pmatrix} G^+_m(z, n) \\ G^-_m(z, n - 1) \end{pmatrix} = \Phi_m(z, n, 1) \begin{pmatrix} G^+_m(z, 1) \\ G^-_m(z, 0) \end{pmatrix}. \tag{6}$$

Lemma 4. *Let $\boldsymbol{D}(m, c)$ be the operator (1). For $z = E + i/T$ $(T > 0)$ and $m \geq 0$, one has*

$$(i) \quad A_{\delta_1^+}(m, T, q) = \frac{1}{\pi T} \sum_{n \in \mathbb{N}} n^q \int_{\mathbb{R}} \left(|G^+_m(z, n)|^2 + |G^-_m(z, n)|^2 \right) \, dE,$$

in the discrete case and

$$(ii) \quad A_f(m, T, q) = \frac{1}{\pi T} \int_0^{\infty} x^q \int_{\mathbb{R}} \left\| (\boldsymbol{D}(m, c) - z)^{-1} \, f(x) \right\|^2 \, dE \; dx,$$

for every $f \in L^2([0, \infty), \mathbb{C}^2)$, in the continuous case.

Proof. The identity *(i)* follows by Lemma 3.2 in [12] adapted for the operator $\mathbf{D}(m, c)$ on $\ell^2(\mathbb{N}, \mathbb{C}^2)$, and the identity *(ii)* follows by Lemma 2.3 in [4] applied to $\mathbf{D}(m, c)$ in $L^2([0, \infty), \mathbb{C}^2)$. □

Lemma 5. *Let $E \in \mathbb{R}$, $N > 0$, $m \geq 0$ and consider*

$$L_m(N) := \sup_{0 \leq x,y \leq N} \left\| \Phi_m(E,x,y) \right\| .$$

Then, there is $0 < C_1 < \infty$ such that for every $\delta \in \mathbb{C}$ and $0 \leq x, y \leq N$, one has

$$\left\| \Phi_m(E+\delta,x,y) \right\| \leq L_m(N) \ \exp\left[\frac{|\delta|}{c} \left(\frac{|\delta|}{c} + C_1 \right) L_m(N)|x-y| \right].$$

Proof. We consider the discrete case with $x, y \in \mathbb{N}$, $x > y$ (the continuous case is similar). An inductive argument shows that, for $\delta \in \mathbb{C}$ and $m \geq 0$, we can write the identity

$$\Phi_m(E+\delta,x,y) = \Phi_m(E,x,y) - \delta \sum_{j=y}^{x-1} \Phi_m(E+\delta,x,j+1)\ B_\delta(E,j)\ \Phi_m(E,j,y)\ ,$$

with

$$B_\delta(E,j) = \frac{\delta}{c^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{c} \begin{pmatrix} \frac{2}{c}(E - V(j)) & -1 \\ 1 & 0 \end{pmatrix}.$$

By iteration, using the hypothesis and the above identity, we obtain

$$\begin{aligned} \left\| \Phi_m(E+\delta,x,y) \right\| &\leq L_m(N) \left[1 + \frac{|\delta|}{c} \left(\frac{|\delta|}{c} + C_1 \right) L_m(N) \right]^{x-y} \\ &\leq L_m(N)\ \exp\left[\frac{|\delta|}{c} \left(\frac{|\delta|}{c} + C_1 \right) L_m(N)(x-y) \right], \end{aligned}$$

for some $0 < C_1 < \infty$ and for $1 \leq y < x \leq N$. $\square$

The following result will be important for the proof of Theorem 1 in the continuous case; it is based on Lemmas 2.6 and 2.7 of [4].

Lemma 6. *Let $\boldsymbol{D}(m,c)$ be the operator defined by (1) on $L^2([0,\infty),\mathbb{C}^2)$. For $z \in \mathbb{C}\backslash\mathbb{R}$, define $u^m_{f,z} = (\boldsymbol{D}(m,c) - z)^{-1} f$. Suppose $E \in \mathbb{R}$ and $0 \neq f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \in L^2([0,\infty),\mathbb{C}^2)$ with $\operatorname{supp} f \subset [0,s]$ are such that*

$$\liminf_{\delta \to 0^+} \left\{ \|u^m_{f,z}(s)\| : z \in \mathbb{C}_+,\ |z - E| \leq \delta \right\} = 0. \tag{7}$$

Then $f \notin \mathcal{H}_E$.

Proof. By (7) there exists a sequence $(z_n) \subset \mathbb{C}_+$ with $z_n \to E$ and $u^m_{f,z_n}(s) \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $n \to \infty$. Since $u^m_{f,z_n}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for all n and by continuity, the inhomogeneous equation

$$(\mathbf{D}(m,c) - E) \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \begin{pmatrix} f_+ \\ f_- \end{pmatrix} \tag{8}$$

has a solution $v = \begin{pmatrix} v_+ \\ v_- \end{pmatrix}$ with $v(0) = v(s) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Let $Y(t)$ be the fundamental matrix of the homogeneous equation at $x = s$, i.e.,

$$Y(t) = \begin{pmatrix} v_+^N(t) & v_+^D(t) \\ \\ v_-^N(t) & v_-^D(t) \end{pmatrix},$$

where $v^N = \begin{pmatrix} v_+^N \\ v_-^N \end{pmatrix}$ and $v^D = \begin{pmatrix} v_+^D \\ v_-^D \end{pmatrix}$ are solutions of the homogeneous equation which satisfy $v^N(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v^D(s) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By writing equation (8) as

$$\begin{pmatrix} u_+'(x) \\ u_-'(x) \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{c}(mc^2 - V(x) + E) \\ \frac{i}{c}(-mc^2 - V(x) + E) & 0 \end{pmatrix} \begin{pmatrix} u_+(x) \\ u_-(x) \end{pmatrix} + \frac{i}{c} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix},$$

we have the variation of parameters formula

$$\begin{pmatrix} v_+(x) \\ v_-(x) \end{pmatrix} = Y(x) \int_s^x Y(t)^{-1} \, \frac{i}{c} \begin{pmatrix} f_-(t) \\ f_+(t) \end{pmatrix} \, dt.$$

Replacing $Y(t)$ in the above equation and considering $x = 0$, we obtain

$$0 = v_+(0) = \frac{i}{c} \, [\overline{w_E}, f] \quad \text{and} \quad 0 = v_-(0) = \frac{i}{c} \, [\overline{v_E}, f], \tag{9}$$

where

$$w_E(t) = v_+^N(0) \begin{pmatrix} -v_+^D(t) \\ \\ v_-^D(t) \end{pmatrix} + v_+^D(0) \begin{pmatrix} v_+^N(t) \\ \\ -v_-^N(t) \end{pmatrix},$$

$$v_E(t) = v_-^N(0) \begin{pmatrix} -v_+^D(t) \\ \\ v_-^D(t) \end{pmatrix} + v_-^D(0) \begin{pmatrix} v_+^N(t) \\ \\ -v_-^N(t) \end{pmatrix}$$

and $f = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}$, with $f_+, f_- \neq 0$.

Now,

$$u^1(t) := -v_+^N(0) \begin{pmatrix} v_+^D(t) \\ \\ v_-^D(t) \end{pmatrix} + v_+^D(0) \begin{pmatrix} v_+^N(t) \\ \\ v_-^N(t) \end{pmatrix}$$

and

$$u^2(t) := -v_-^N(0) \begin{pmatrix} v_+^D(t) \\ \\ v_-^D(t) \end{pmatrix} + v_-^D(0) \begin{pmatrix} v_+^N(t) \\ \\ v_-^N(t) \end{pmatrix}$$

are solutions of equation $\mathbf{D}(m,c)\begin{pmatrix} u_+ \\ u_- \end{pmatrix} = E\begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ satisfying $u^1(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $u^2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus, u^1, u^2 form a fundamental system of solutions of $\mathbf{D}(m,c)u = Eu$ and it follows from (9) that

$$\left[\overline{u^i}, \begin{pmatrix} f_+ \\ 0 \end{pmatrix}\right] = 0 = \left[\overline{u^i}, \begin{pmatrix} 0 \\ f_- \end{pmatrix}\right], \ i = 1, 2.$$

Therefore, if $f = \begin{pmatrix} f_+ \\ 0 \end{pmatrix}$, $f_+ \neq 0$, resp. $f = \begin{pmatrix} 0 \\ f_- \end{pmatrix}$, $f_- \neq 0$, one has

$$\int_0^s u_+(t) f_+(t) dt = 0, \text{ resp. } \int_0^s u_-(t) f_-(t) dt = 0,$$

for every solution $u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ of $\mathbf{D}(m,c)u = Eu$. Hence, we conclude that $f \notin \mathcal{H}_E$. □

Proof. **(Theorem 1)**
(i) By Lemma 4, we have for $T > 0$,

$$A_{\delta_1^+}(m,T,q) = \frac{1}{\pi T}\sum_{n\in\mathbb{N}} n^q \int_{\mathbb{R}} \left(|G_m^+(E+i/T,n)|^2 + |G_m^-(E+i/T,n)|^2\right)\, dE.$$

Define $N(T) := T^{\frac{1}{1+\alpha}}$. By hypothesis,

$$L_m(N(T)) := \sup_{0\le n,k\le N(T)} \left\|\Phi_m(E', n, k)\right\| \le C\ (N(T))^\alpha, \quad \forall\ E' \in A(N(T)).$$

By Lemma 5, we obtain for every $E \in B_2(T)$ and $1 \le n \le N(T)$,

$$\left\|\Phi_m(E+i/T, n, 1)\right\| \le B\ (N(T))^\alpha,$$

with $B = C\ e^{\frac{3}{c}\left(\frac{3}{c}+C_1\right)C}$. For every $E \in B_2(T)$ and T sufficiently large, it follows from (6) and the above estimate that

$$\begin{aligned}
(10) \qquad & \sum_{n\ge \frac{N(T)}{2}} \left(|G_m^+(E+i/T,n)|^2 + |G_m^-(E+i/T,n)|^2\right) \\
\ge \ & \sum_{n=\frac{N(T)}{2}+1}^{N(T)} \left(|G_m^+(E+i/T,n)|^2 + |G_m^-(E+i/T,n-1)|^2\right) \\
\ge \ & \frac{B^{-2}}{4}(N(T))^{1-2\alpha}(|G_m^+(E+i/T,2)|^2 + |G_m^-(E+i/T,1)|^2 \\
& + |G_m^+(E+i/T,1)|^2 + |G_m^-(E+i/T,0)|^2).
\end{aligned}$$

Observe that

$$G_m^+(E+i/T,1) = \left\langle \delta_1^+, (\mathbf{D}(m,c) - E - i/T)^{-1}\,\delta_1^+ \right\rangle = F_m(E+i/T),$$

where $F_m(z)$ is the Borel transform of the spectral measure corresponding to the pair $(\mathbf{D}(m,c),\delta_1^+)$. Using equation (5) one shows that

$$|G_m^+(E+i/T,2)|^2+|G_m^-(E+i/T,1)|^2+|G_m^-(E+i/T,0)|^2\geq a>0$$

for some uniform constant a. Therefore, it follows from (10) that for T sufficiently large,

$$\begin{aligned}
&\frac{1}{\pi T}\int_{\mathbb{R}}\sum_{n\geq\frac{N(T)}{2}}\left(|G_m^+(E+i/T,n)|^2+|G_m^-(E+i/T,n)|^2\right)\,dE\\
\geq\;&\frac{\tilde{B}}{T}(N(T))^{1-2\alpha}\int_{B_2(T)}\left(1+\Im m^2F_m(E+i/T)\right)\,dE\\
\geq\;&\frac{\tilde{B}}{T}(N(T))^{1-2\alpha}\int_{B_2(T)}\left(\frac{1}{2}+\Im mF_m(E+i/T)\right)\,dE,
\end{aligned}$$

for some constant $\tilde{B}>0$. In the last step it was used that $1+\Im m^2F_m(z)\geq 2\,\Im mF_m(z)$.

For any set $S\subset\mathbb{R}$, denote by S_ϵ the ϵ-neighborhood of S. It was shown in [5, 12] that

$$\int_{S_\epsilon}\Im\, mF_m(E+i/T)\,dE\geq\frac{\pi}{2}\,\mu_+^m(S).$$

Thus, taking $S=B_1(T)$ we conclude that for T large enough,

$$\begin{aligned}
&A_{\delta_1^+}(m,T,q)\;\geq\\
&\geq\frac{1}{\pi T}\left(\frac{N(T)}{2}\right)^q\int_{\mathbb{R}}\sum_{n\geq\frac{N(T)}{2}}\left(|G_m^+(E+i/T,n)|^2+|G_m^-(E+i/T,n)|^2\right)\,dE\\
&\geq\frac{\tilde{C}}{T}(N(T))^{q+1-2\alpha}\int_{B_2(T)}\left(1+\Im mF_m(E+i/T)\right)\,dE\\
&\geq\tilde{C}\,T^{\frac{q-3\alpha}{1+\alpha}}\,\left(|B_2(T)|+\mu_+^m(B_1(T))\right).
\end{aligned}$$

(**ii**) As in Lemma 6 we write $u_{f,z}^m=(\mathbf{D}(m,c)-z)^{-1}f$. Let $s>0$ with $\operatorname{supp}f\subset[0,s]$ and define $N(T):=T^{\frac{1}{1+\alpha}}$. By Lemma 4, we have for $T>0$,

$$\begin{aligned}
A_f(m,T,q)&=\frac{1}{\pi T}\int_0^\infty x^q\int_{\mathbb{R}}\|u_{f,E+i/T}^m(x)\|^2\,dE\,dx\\
&\geq\frac{1}{2\pi T}\sum_{n=s+1}^{\infty}(n-1)^q\int_{n-1}^{n+1}\int_{\mathbb{R}}\|u_{f,E+i/T}^m(x)\|^2\,dE\,dx\\
&\geq\frac{1}{2\pi T}\sum_{n=s+1}^{\infty}(n-1)^q\int_{B_1(T)}\int_{n-1}^{n+1}\|u_{f,E+i/T}^m(x)\|^2\,dx\,dE.
\end{aligned}\tag{11}$$

Using the fact that $u^m_{f,E+i/T}$ is a solution of $\mathbf{D}(m,c)u = \left(E+\frac{i}{T}\right)u$ on $[n-1,n+1]$ and the transfer matrices satisfy $\|\Phi_m^{-1}\| = \|\Phi_m\|$, we obtain from (11) that

$$\begin{aligned} &A_f(m,T,q) \;\geq \\ &\frac{1}{2\pi T}\sum_{n=s+1}^{\infty}(n-1)^q\int_{B_1(T)}\int_{n-1}^{n+1}\|\Phi_m(E+i/T,x,s)\|^{-2}\|u^m_{f,E+i/T}(s)\|^2 dx\; dE. \end{aligned}$$

By hypothesis and Lemma 5, it follows that for T large enough,

$$\begin{aligned} A_f(m,T,q) &\geq \frac{1}{\pi T}\sum_{n=\frac{N(T)}{2}+1}^{N(T)}\left(\frac{N(T)}{2}\right)^q\int_{B_1(T)} C_0\, N(T)^{-2\alpha}\|u^m_{f,E+i/T}(s)\|^2\; dE \\ &\geq \frac{1}{\pi T}\left(\frac{N(T)}{2}\right)^{q+1}|B_1(T)|C_0\, N(T)^{-2\alpha}\inf_{\mathrm{dist}(z,B_1(T))\leq\frac{1}{T}}\|u^m_{f,z}(s)\|^2, \end{aligned}$$

for some constant $C_0>0$.

For every $f\in\mathcal{H}_{E_0}$ with supp $f\subset[0,s]$, Lemma 6 implies that there exists $\kappa>0$ and $\delta>0$ satisfying

$$\inf\left\{\|u^m_{f,z}(s)\|^2 : z\in\mathbb{C}_+,\; |z-E_0|\leq\delta\right\}\geq\kappa.$$

By hypothesis, $\mathrm{diam}(A(N))\longrightarrow 0$ as $N\to\infty$ and $E_0\in A(N)$ for all N. Hence,

$$\inf\left\{\|u^m_{f,z}(s)\|^2 : \mathrm{dist}(z,B_1(T))\leq\frac{1}{T}\right\}\geq\kappa>0$$

for T sufficiently large.

Therefore, for T large enough we obtain

$$A_f(m,T,q)\geq\frac{\tilde{C}}{T}\,N(T)^{q+1-2\alpha}\,|B_1(T)| = \tilde{C}\;T^{\frac{q-3\alpha}{1+\alpha}}\,|B_1(T)|.$$

The proof is complete. □

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Departamento de Matemática – UFSCar, São Carlos, SP, 13560-970 Brazil,
E-mail address: `rap@dm.ufscar.br`

Departamento de Matemática – UFSCar, São Carlos, SP, 13560-970 Brazil
E-mail address: `oliveira@dm.ufscar.br`