Metastability for reversible probabilistic cellular automata with self–interaction

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Abstract

The problem of metastability for a stochastic dynamics with a parallel updating rule is addressed in the Freidlin–Wentzel regime namely, finite volume, small magnetic field, and small temperature. The model is characterized by the existence of many fixed points and cyclic pairs of the zero temperature dynamics, in which the system can be trapped in its way to the stable phase. Nevertheless, the main features of metastability can be proven by using recent powerful approaches, which do not need a complete description of such fixed points but rely on few model dependent results such as a recurrence property to the metastable states and the determination of all the saddles between the metastable and the stable state.

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1. Introduction

Metastable states are very common in nature and are typical of systems close to a first order phase transition. It is often observed that a system can persist for a long period of time into a phase which is not the one favored by the thermodynamic parameters; classical examples are the super–saturated vapor and the magnetic hysteresis. The rigorous description of this phenomenon in the framework of well defined mathematical models is quite recent, it comes back to the pioneering paper [CGOV] and has experienced great progresses in the last decade, see [OV] for a complete review of the most important papers appeared on this subject.

A natural setup in which the phenomenon of metastability can be studied is that of Markov chains, or processes, describing the time evolution of a Statistical Mechanics system, think for instance to a stochastic lattice spin system. In this context powerful theories, see [BEGM, MNOS, OS], have been developed with the aim to find answers valid in general and to reduce to the minimum the number of model dependent inputs necessary to describe the metastable behavior of the system. Whatever approach is chosen, the key model dependent question is the computation of the minimal energy barrier namely, the communication energy, that must be bypassed by a path connecting the metastable to the stable state. Such a problem is in general quite complicated and becomes particularly difficult when the dynamics has a parallel character. Indeed, if as many simultaneous updates as possible are allowed on the lattice, no constraint to the structure of the trajectories in the configuration space is imposed; therefore, to compute the communication energy one must examine all the possible direct jumps from any configuration to any other.

The problem of the computation of the communication energy in a parallel dynamics setup has been faced in [C, CN]; in particular in [CN] the typical questions of metastability namely, the determination of the *exit time* and of the *exit tube*, have been answered for a reversible Probabilistic Cellular Automaton, see [GLD, R, St, To, D, CNP], in which each spin is coupled with its nearest neighbors. In that paper it has been shown that during the transition from the minus metastable state to the stable plus one, the system visits an intermediate chessboard–like phase. In the present paper we study the reversible PCA in which each spin interacts both with itself and with its nearest neighbors; the metastable behavior of such model has been investigated on heuristic and numerical grounds in [BCLS]. The addition of the self–interaction changes completely the metastability scenario, in particular we show that the chessboard phase plays no role in the exit from the metastable phase.

Another very interesting feature of this model is the presence of a huge variety of fixed points of the zero–temperature dynamics in which the system can be trapped. By using the powerful approach of [MNOS] we can compute the exit time and identify the saddle configurations avoiding a complete description of such trapping states, but we cannot describe the exit tube namely, the tube of trajectories followed by the system during its exit from the metastable to the stable phase. By using this approach the model dependent ingredients that must be provided are essentially two: the solution of the global variational problem for all the paths connecting the metastable and the stable state namely, the computation of the communication energy and the characterization of the saddle configuration; a sort of recurrence property stating that starting from each configuration different from the metastable and the stable state, it is possible to reach a configuration at lower energy following a path with an energy cost strictly smaller that the communication energy.

To solve the global variational problem, see item 2 and 3 in Theorem 2.2, we give an upper bound to the communication energy by exhibiting a path connecting the metastable state to the stable state with maximal energy along the path equal to the communication energy. To find the lower bound we perform a partition of the configuration space, study the direct jumps between configurations chosen in those subsets of the space and reduce the computation to the optimal jump, see Figure 7. To prove the recurrence property, see item 1 in Theorem 2.2, we have to face the problem of the existence of a huge variety of fixed point of the dynamics; we solve the problem showing that for each configuration different from the metastable state, it is possible to find a path connecting it to the stable state namely, the unique global minimum of the energy, such that the energy along such a path is strictly smaller than the communication energy.

We finally give a brief description of the content of the paper. In Section 2 we define the model and state the Theorem 2.1; in particular such a theorem is proven in Subsection 2.7. The Section 3 is devoted to the proof of the estimates on the energy landscape in Theorem 2.2 namely, the global variational problem and the recurrence property. Finally, in Section 4 we give the proof of the model dependent result stated in Proposition 3.2 on which is based the proof of Theorem 2.2.

2. Model and results

In this section we introduce the basic notation, define the reversible Probabilistic Cellular Automaton which will be studied in the sequel, and state the main results. First of all we recall that for any x positive real $[x]$ denotes the *integer part* of x namely, the largest integer smaller than or equal to x; moreover, for $x = (x_1, x_2) \in \mathbb{R}^2$ we set $|x| := |x_1| + |x_2|$.

2.1. The lattice

The spatial structure is modeled by the two–dimensional finite square $\Lambda \subset \mathbb{Z}^2$ of side length L with periodic boundary condition namely, by the torus Λ, in which we let e_1 and e_2 be the coordinate unit vectors. We use $X^{\mathrm{c}} := \Lambda \setminus X$ to denote the complement of $X \subset \Lambda$.

We consider Λ endowed with the distance $\mathrm{d}(x, y) := |x - y|$. As usual for $X, Y \subset \Lambda$ we set $\mathrm{d}(X, Y) := \inf\{\mathrm{d}(x, y),\ x \in X,\ y \in Y\}$ and $\mathrm{diam}(X) := \sup\{\mathrm{d}(x, y),\ x, y \in X\}$. We say that $x, y \in \Lambda$ are *nearest neighbors* iff $\mathrm{d}(x, y) = 1$; we say that $X \subset \Lambda$ is *connected* iff for each $x, y \in X$ there exists a path of pairwise nearest neighbor sites of X joining x and y. For $X \subset \Lambda$ we let $\partial X := \{x \in X^{\mathrm{c}} : \mathrm{d}(x, X) = 1\}$ be the *external boundary* of X and $\overline{X} := X \cup \partial X$ be the *closure* of X. Two sets $X, Y \subset \Lambda$ are said to be *not interacting* if and only if $\mathrm{d}(X, Y) \geq 3$.

Let $x \in \Lambda$; for ℓ_1, ℓ_2 strictly positive reals we let $Q_{\ell_1,\ell_2}(x) := \{y \in \Lambda : x_1 \leq y_1 \leq x_1 + (\ell_1 - 1),\ x_2 \leq y_2 \leq x_2 + (\ell_2 - 1)\}$ be the rectangle of side lengths $[\ell_1]$ and $[\ell_2]$ with

x the site with smallest coordinates. For ℓ a positive real we let $Q_\ell(x) := Q_{\ell,\ell}(x)$; note that $Q_\ell(x)$ is the square of side length $[\ell]$ with x the site with smallest coordinates. For $X \subset \Lambda$ and $\ell > 0$ we set $B_\ell(X) := \{y \in \Lambda :\, \mathrm{d}(X,y) \le \ell\}$; if $\ell = 1$ we shall write $B(X)$ for $B_1(X)$, note that $B(X) = \overline{X}$. If $x \in \Lambda$ we write $B_\ell(x)$ for $B_\ell(\{x\})$, note that $B_\ell(x)$ is the ball of radius $[\ell]$ centered at x. Finally, we remark that $B(x)$ is the five site cross centered at $x \in \Lambda$.

2.2. The configuration space

The single spin state space is given by the finite set $\mathcal{S}_0 := \{-1, +1\}$ which we consider endowed with the discrete topology; the associated Borel σ–algebra is denoted by $\mathcal{F}_0$. The configuration space in $X \subset \Lambda$ is defined as $\mathcal{S}_X := \mathcal{S}_0^X$ and considered equipped with the product topology and the corresponding Borel σ algebra $\mathcal{F}_X$. The model and the related quantities that will be introduced later on will all depend on Λ, but since Λ is fixed it will be dropped from the notation; in this spirit we let $\mathcal{S}_\Lambda =: \mathcal{S}$ and $\mathcal{F}_\Lambda =: \mathcal{F}$.

Given $Y \subset X \subset \Lambda$ and $\sigma := \{\sigma_x \in \mathcal{S}_{\{x\}},\, x \in X\} \in \mathcal{S}_X$, we denote by σ_Y the *restriction* of σ to Y namely, $\sigma_Y := \{\sigma_x,\, x \in Y\}$. Let m be a positive integer and let $X_1, \dots, X_m \subset \Lambda$ be pairwise disjoint subsets of Λ; for each $\sigma_k \in \mathcal{S}_{X_k}$, with $k = 1, \dots, m$, we denote by $\sigma_1\sigma_2 \cdots \sigma_m$ the configuration in $\mathcal{S}_{X_1 \cup \dots \cup X_m}$ such that $(\sigma_1\sigma_2 \cdots \sigma_m)_{X_k} = \sigma_k$ for all $k \in \{1, \dots, m\}$. Moreover, given $\sigma \in \mathcal{S}$ and $x \in \Lambda$, we denote by σ^x the configuration such that $\sigma^x(x) = -\sigma(x)$ and $\sigma^x(y) = \sigma(y)$ for $y \neq x$. For $x \in \Lambda$ we define the shift Θ_x acting on $\mathcal{S}$ by setting $(\Theta_x\sigma)_y := \sigma_{y+x}$, for all $y \in \Lambda$ and $\sigma \in \mathcal{S}$.

For any function $f : \mathcal{S} \to \mathbb{R}$ we denote by $\mathrm{supp}(f)$ the so–called support of f namely, the smallest $X \subset \Lambda$ such that $f \in \mathcal{F}_X$. If $f \in \mathcal{F}_X$ we shall sometimes write $f(\sigma_X)$ for $f(\sigma)$.

Let $F, G : \mathcal{S} \to \mathcal{S}$ be two functions, we consider the *product* or *composed* function $FG : \mathcal{S} \to \mathcal{S}$ such that $FG(\sigma) := F(G(\sigma))$ for any $\sigma \in \mathcal{S}$. We also let $F^2 := FF$ and, for n a positive integer, $F^n := FF^{n-1}$. We say that a configuration $\sigma \in \mathcal{S}$ is a *fixed point* for the map $F : \mathcal{S} \to \mathcal{S}$ if and only if $F(\sigma) = \sigma$. Let $\sigma \in \mathcal{S}$, consider the sequence $F^n(\sigma)$ with $n \ge 1$, if there exists n' such that $F^n(\sigma) = F^{n'}(\sigma)$ for any $n \ge n'$, we then let $\bar{F}\sigma := F^{n'}\sigma$.

2.3. The model

Let $\beta > 0$ and $h \in \mathbb{R}$ such that $|h| < 1$ and $2/h$ is not integer, we consider the Markov chain on $\mathcal{S}$ with *transition matrix*

$$p(\sigma, \eta) := \prod_{x \in \Lambda} p_{x,\sigma}(\eta(x)) \quad \forall \sigma, \eta \in \mathcal{S} \tag{2.1}$$

where, for each $x \in \Lambda$ and $\sigma \in \mathcal{S}$, $p_{x,\sigma}(\cdot)$ is the probability measure on $\mathcal{S}_{\{x\}}$ defined as follows

$$p_{x,\sigma}(s) := \frac{1}{1 + \exp\{-2\beta s(S_\sigma(x) + h)\}} = \frac{1}{2}\left[1 + s \tanh \beta \left(S_\sigma(x) + h\right)\right] \tag{2.2}$$

with $s \in \{-1, +1\}$ and

$$S_\sigma(x) := \sum_{y \in B(x)} \sigma(y) \tag{2.3}$$

The normalization condition $p_{x,\sigma}(s)+p_{x,\sigma}(-s)=1$ is trivially satisfied. Note that for x and s fixed $p_{x,\cdot}(s)\in\mathcal{F}_{B(x)}$ namely, the probability $p_{x,\sigma}(s)$ for the spin at site x to be equal to s depends only on the values of the five spins of σ inside the cross $B(x)$ centered at x.

Such a Markov chain on the finite space $\mathcal{S}$ is an example of *reversible probabilistic cellular automata* (PCA), see [D, CNP]. Let $n\in\mathbb{N}$ be the discrete time variable and $\sigma_n\in\mathcal{S}$ denote the state of the chain at time n, the configuration at time $n+1$ is chosen according to the law $p(\sigma_n,\cdot)$, see (2.1), hence all the spins are updated simultaneously and independently at any time. Finally, given $\sigma\in\mathcal{S}$ we consider the chain with initial configuration $\sigma_0=\sigma$, we denote with $\mathbb{P}_\sigma$ the probability measure on the space of trajectories, by $\mathbb{E}_\sigma$ the corresponding expectation value, and by

$$\tau_A^\sigma := \inf\{t>0:\ \sigma_t\in A\} \tag{2.4}$$

the *first hitting time on* $A\subset\mathcal{S}$; we shall drop the initial configuration from the notation (2.4) whenever it is equal to $-\underline{1}$, we shall write τ_A for $\tau_A^{-\underline{1}}$, namely.

2.4. The stationary measure and the phase diagram

The model (2.1) has been studied numerically in [BCLS], we refer to it for a detailed discussion about its stationary properties. Here we simply recall the main features. It is straightforward, see for instance [CNP,D], that the PCA (2.1) is reversible with respect to the finite volume Gibbs measure $\mu(\sigma):=\exp\{-H(\sigma)\}/Z$ with $Z:=\sum_{\eta\in\mathcal{S}}\exp\{-H(\eta)\}$ and

$$H(\sigma):=H_{\beta,h}(\sigma):=-\beta h\sum_{x\in\Lambda}\sigma(x)-\sum_{x\in\Lambda}\log\cosh\left[\beta\left(S_\sigma(x)+h\right)\right] \tag{2.5}$$

in other words the *detailed balance* condition

$$p(\sigma,\eta)\,e^{-H(\sigma)}=p(\eta,\sigma)\,e^{-H(\eta)} \tag{2.6}$$

is satisfied for any $\sigma,\eta\in\mathcal{S}$. Hence, the measure μ is stationary for the PCA (2.1); to understand its most important features it is useful to study the related Hamiltonian. Since the Hamiltonian has the form (2.5) we shall often refer to $1/\beta$ as to the *temperature* and to h as to the *magnetic field.*

The interaction is short range and it is possible to extract the potentials; following [BCLS] we rewrite the Hamiltonian as

$$H_{\beta,h}(\sigma)=\sum_{x\in\Lambda}U_{x,\beta,h}(\sigma)-\beta h\sum_{x\in\Lambda}\sigma(x) \tag{2.7}$$

where $U_{x,\beta,h}(\sigma)=U_{0,\beta,h}(\Theta_x\sigma)$, recall the shift operator Θ_x has been defined in Subsection 2.2 and that periodic boundary are considered on Λ, and

$$U_{0,\beta,h}(\sigma)=-\sum_{X\subset B(0)}J_{|X|,\beta,h}\prod_{x\in X}\sigma(x) \tag{2.8}$$

The six coefficients $J_{0,\beta,h},\dots,J_{5,\beta,h}$ are determined by using (2.5), (2.7), and (2.8). In the case $h=0$ only even values of $|X|$ occur and we find that the pair interactions are ferromagnetic while the four–spin interactions are not. For a more detailed discussion see [BCLS].

The definition of ground states is not completely trivial in our model, indeed the Hamiltonian H depends on β. The ground states are those configurations on which the Gibbs measure μ is concentrated when the limit $\beta \to \infty$ is considered, so they can be defined as the minima of the *energy*

$$E(\sigma) := \lim_{\beta\to\infty} \frac{H(\sigma)}{\beta} = -h \sum_{x\in\Lambda} \sigma(x) - \sum_{x\in\Lambda} |S_\sigma(x) + h| \tag{2.9}$$

uniformly in $\sigma \in \mathcal{S}$. Let $\mathcal{X} \subset \mathcal{S}$ if the energy E is constant on $\mathcal{X}$ namely, if all the configurations in $\mathcal{X}$ have the same energy, we shall misuse the notation by writing $E(\mathcal{X})$ for $E(\sigma)$ with $\sigma \in \mathcal{X}$.

We first consider the case $h = 0$. Since $E(\sigma) = -\sum_{x\in\Lambda} |S_\sigma(x)|$, it is obvious that there exist the two coexisting minima $+\underline{1}, -\underline{1} \in \mathcal{S}$, with $\pm\underline{1}(x) = \pm 1$ for each $x \in \Lambda$, such that $E(+\underline{1}) = E(-\underline{1}) = -5|\Lambda|$. For $h \neq 0$ we have $E(+\underline{1}) = -|\Lambda|(h + |5 + h|)$ and $E(-\underline{1}) = -|\Lambda|(-h + |-5 + h|)$; it is immediate to verify that $E(+\underline{1}) < E(-\underline{1})$ for $h > 0$ and $E(-\underline{1}) < E(+\underline{1})$ for $h < 0$. We conclude that at $h = 0$ there exist the two coexisting ground states $-\underline{1}$ and $+\underline{1}$, while at $h > 0$ the unique ground state is given by $+\underline{1}$ and at $h < 0$ the unique ground state is given by $-\underline{1}$.

We give, now, a heuristic argument showing that at finite, but very low, temperature the structure of the phase diagram is not changed. More precisely the argument suggests that at $h = 0$ the two states $+\underline{1}$ and $-\underline{1}$ still coexist, see also [KV, V] and [DLR]. At finite temperature ground states are perturbed because small droplets of different phases show up. We compute the energetic cost of a square droplet perturbation at $h = 0$. By flipping to plus one the spins at sites inside a square of side length $\ell \geq 3$ in the configuration $-\underline{1}$ the energy increases of the amount 16ℓ; similarly, by flipping to minus one the spins at sites inside a square of side length $\ell \geq 3$ in the configuration $+\underline{1}$ the energy increases of the same amount. None of the two ground states is more easily perturbed, this suggests that even at β finite, but very large, the two phases should coexist.

2.5. Metastable behavior

We pose now the problem of metastability and state the related theorem on the exit time. In this context configurations with all the spins equal to minus one excepted those in rectangular subsets of the lattice will play a key role. We then let

$$\Lambda^\pm(\sigma) := \{x \in \Lambda \colon \sigma(x) = \pm 1\} \tag{2.10}$$

for any $\sigma \in \mathcal{S}$; the set $\Lambda^+(\sigma)$ will be called the *support* of σ. We say that $\sigma \in \mathcal{S}$ is a *rectangular droplet with side lengths* ℓ *and* m, with ℓ, m integers such that $2 \leq \ell, m \leq L-2$, if and only if there exist $x \in \Lambda$ such that either $\Lambda^+(\sigma) = Q_{\ell,m}(x)$ or $\Lambda^+(\sigma) = Q_{m,\ell}(x)$. We say that $\sigma \in \mathcal{S}$ is a *n–rectangular droplet with side lengths* $\ell_1, m_1, \ldots, \ell_n, m_n$, with $n \geq 1$ an integer and ℓ_i, m_i integers such that $2 \leq \ell_i, m_i \leq L-2$ for $i = 1, \ldots, n$, if and only if $\Lambda^+(\sigma)$ is the union of n pairwise not interacting rectangles, see Subsection 2.1, with side lengths ℓ_i and m_i for $i = 1, \dots, n$. We say, finally, that $\sigma \in \mathcal{S}$ is a *multi–rectangular droplet* if and only if σ is a n–rectangular droplet for some integer $n \geq 1$. Note that a 1–rectangular droplet is indeed a rectangular droplet.

Consider, now, the model (2.1) with $0 < h < 1$ and suppose that the system is prepared in the state $\sigma_0 = -\underline{1}$; in the infinite time limit the systems tends to the phase with positive magnetization. We shall show that the minus one state is metastable in the sense that the system spends a huge amount of time close to $-\underline{1}$ before visiting $+\underline{1}$, more precisely we shall show that the first hitting time $\tau_{+\underline{1}}$, recall (2.4) and the remark below, to $+\underline{1}$ is an exponential random variable with mean exponentially large in β.

Moreover, we give some information on the exit path that the system follows during the escape from minus one to plus one. More precisely we show that there exists a class of configuration $\mathcal{C} \subset \mathcal{S}$, which we shall call set of *critical droplets*, which is visited with high probability by the system during its escape from $-\underline{1}$ to $+\underline{1}$. The set $\mathcal{C}$ is defined as the collection of configurations with all the spins equal to -1 excepted those in a rectangle of sides $\lambda - 1$ and λ and in a pair of neighboring sites adjacent to one of the longest side of the rectangle, with the *critical length* defined as

$$\lambda := \left[\frac{2}{h}\right] + 1 \tag{2.11}$$

Since h has been chosen such that $2/h$ is not integer, see Subsection 2.3, we have that $\lambda = 2/h + \delta_h$ with $\delta_h \in (0, 1)$. As it has been explained in the introduction the energy of the configurations in $\mathcal{C}$ is strictly connected to the typical exit time from the metastable state. Indeed, given $\gamma \in \mathcal{C}$ we let

$$\Gamma := E(\gamma) - E(-\underline{1}) + 2(1 + h) = -4h\lambda^2 + 16\lambda + 4h(\lambda - 2) + 2(1 + h) \tag{2.12}$$

Note that by (2.12) and (2.11) it follows

$$\Gamma < 8\lambda + 10 - 2h \tag{2.13}$$

We finally state the following theorem.

Theorem 2.1 *Consider the probabilistic cellular automaton (2.1); for $h > 0$ small enough and $L = L(h)$ large enough, we have that*

1. *the random variable $(1/\beta) \log \tau_{+\underline{1}}$ converges in probability to Γ as $\beta \to \infty$ namely, for any $\varepsilon > 0$*
$$\lim_{\beta \to \infty} \mathbb{P}_{-\underline{1}}(e^{\beta\Gamma - \beta\varepsilon} < \tau_{+\underline{1}} < e^{\beta\Gamma + \beta\varepsilon}) = 1 \tag{2.14}$$

2. *the expectation value of the first hitting time to $+\underline{1}$ converges to Γ in the sense*
$$\lim_{\beta \to \infty} \frac{1}{\beta} \log \mathbb{E}_{-\underline{1}}[\tau_{+\underline{1}}] = \Gamma \tag{2.15}$$

3. *the system visits $\mathcal{C}$ before hitting $+\underline{1}$ namely,*
$$\lim_{\beta \to \infty} \mathbb{P}_{-\underline{1}}(\tau_{\mathcal{C}} < \tau_{+\underline{1}}) = 1 \tag{2.16}$$

The proof of Theorem 2.1 will be given in Subsection 2.7. We note that as usual in Probabilistic Cellular Automata, see also [CN], the highest energy Γ reached along the exit path is not achieved in a configuration, which is the typical situation in Glauber dynamics, but is the communication energy of the jump from the "largest subcritical" configuration to the "smallest supercritical" one, see the heuristic discussion in the following Subsection 2.6.

2.6. Stable states and stable pairs

The proof of Theorem 2.1, although mathematically complicated, relies on a physical argument very straightforward based on a careful description of the low temperature namely, large β, dynamics of the system. In this subsection we give an heuristic explanation of the exponential estimate (2.14) for the exit time $\tau_{+\underline{1}}$.

Consider two configurations $\sigma, \eta \in \mathcal{S}$, the probability that the systems perform the jump from σ to η is given by $p(\sigma, \eta)$, see (2.1)–(2.3); in the limit $\beta \to \infty$, since $|h| < 1$, such a probability tends either to 0 or to 1. Moreover, due to the product structure of (2.1), given σ there exists a unique configuration η such that $p(\sigma, \eta) \to 1$ in the limit $\beta \to \infty$; this configuration is the one such that each spin $\eta(x)$ is chosen so that $p_{x,\sigma}(\eta(x)) \to 1$ in the same zero temperature limit. We consider the map $T : \mathcal{S} \to \mathcal{S}$, called the *zero temperature dynamics*, such that for each $\sigma \in \mathcal{S}$ the configuration $T\sigma$ is the unique configuration such that $p(\sigma, T\sigma) \to 1$ in the limit $\beta \to \infty$. If $T\sigma = \sigma$ the configuration σ is called *stable*; equivalently, we say that $\sigma \in \mathcal{S}$ is stable if and only if for any $\eta \in \mathcal{S} \setminus \{\sigma\}$ one has $p(\sigma, \eta) \to 0$ in the limit $\beta \to \infty$. If σ is not stable, but $T^2\sigma = \sigma$ we say that $(\sigma, T\sigma)$ is the *stable pair* associated to σ, equivalently we say that $(\sigma, T\sigma)$ is a stable pair if and only if $p(\sigma, T\sigma) \to 1$ and $p(T\sigma, \sigma) \to 1$ in the limit $\beta \to \infty$.

Suppose the initial condition is $\sigma_0 = \sigma \in \mathcal{S}$, at low temperature with high probability the system will follow the unique *zero temperature trajectory*

$$\sigma_0 = \sigma,\, \sigma_1 = T\sigma,\, \sigma_2 = T\sigma_1 = T^2\sigma,\, \dots,\, \sigma_t = T(T^{t-1}\sigma) = T^t\sigma, \dots$$

Once the zero temperature trajectory ends up in a stable configuration it remains there forever. Different trajectories will be observed with probability exponentially small in β; we report in Figure 1 the table in [BCLS, FIG. 1] where the probabilities of the single site events are enumerated. In the figure the large β behavior of the probability associated to the flip of the spin at the center is reported.

We can now depict the typical behavior of the system at very low temperature. Starting from σ the system will reach in a time of order one either the stable configuration $\bar{T}\sigma$ or the stable pair associated to $\overline{T^2}\sigma$; note that $\bar{T}\sigma$ and $\overline{T^2}\sigma$ are unique. After a time exponentially large in β, the chain will depart from the stable configuration, or from the stable pair, and possibly reach a different stable configuration where it will remain for another exponentially large time. And so on. It is then clear that in the study of the low temperature dynamics a key role is played by stable configurations and stable pairs, indeed a large amount of the time of each trajectory is spent there. In our model only those configurations in which there exists at least one spin with a majority of opposite spins among its neighbors are not stable, see Figure 1.

$\begin{array}{ccc} & + & \\ + & + & + \\ & + & \end{array}$	$e^{-2\beta(5+h)}$	$\begin{array}{ccc} & - & \\ - & - & - \\ & - & \end{array}$	$e^{-2\beta(5-h)}$
$\begin{array}{ccc} & + & \\ + & + & + \\ & - & \end{array}$	$e^{-2\beta(3+h)}$	$\begin{array}{ccc} & - & \\ - & - & - \\ & + & \end{array}$	$e^{-2\beta(3-h)}$
$\begin{array}{ccc} & + & \\ + & + & - \\ & - & \end{array}$	$e^{-2\beta(1+h)}$	$\begin{array}{ccc} & - & \\ - & - & + \\ & + & \end{array}$	$e^{-2\beta(1-h)}$
$\begin{array}{ccc} & - & \\ + & + & - \\ & - & \end{array}$	$1-e^{-2\beta(1-h)}$	$\begin{array}{ccc} & + & \\ - & - & + \\ & + & \end{array}$	$1-e^{-2\beta(1+h)}$
$\begin{array}{ccc} & - & \\ - & + & - \\ & - & \end{array}$	$1-e^{-2\beta(3-h)}$	$\begin{array}{ccc} & + & \\ + & - & + \\ & + & \end{array}$	$1-e^{-2\beta(3+h)}$

Figure 1: Large β behavior of the probabilities for the flip of the central spin for all possible configurations in the 5–spin neighborhood.

Among the huge variety of possible stable states, there are those configurations in which the plus spins fill a rectangular region. In [BCLS] it has been conjectured that those rectangular stable configurations are the relevant ones for metastability. Moreover, it has been developed an heuristic argument to show that λ, see (2.11), is the *critical length* in the following sense: rectangular droplets with smallest side length smaller or equal to $\lambda-1$ are *subcritical* namely, starting from such a configuration the system visits $-\underline{1}$ before $+\underline{1}$ with large probability that is with probability one in the limit $\beta \to \infty$. Moreover, rectangular droplets with smallest side length larger or equal to λ are *supercritical* namely, starting from such a configuration the system visits $+\underline{1}$ before $-\underline{1}$ with large probability.

We consider, then, a rectangular droplet with sides λ and $\lambda - 1$; from the table in Figure 1 it follows immediately that the configuration π obtained by attaching a single site protuberance to one of the two longest sides of the rectangle is not stable and starting from it the system goes back to the original rectangular droplet in a typical time 1. Moreover, the configuration γ obtained by flipping one of the two minuses on the longest side of the rectangle and neighboring the single site protuberance in π is stable and starting from it the systems, in a typical time $\exp\{2\beta(1-h)\}$, reaches the configuration in which the square of side length λ surrounding the initial cluster of plus spins is completely filled by pluses. The discussion above suggests that the highest energy level attained during the escape from $-\underline{1}$ to $+\underline{1}$ is the one involved in the direct jump from π to γ; more precisely, the typical exit time is $\exp\{\beta(E(\pi,\gamma)-E(-\underline{1}))\}$, see (2.19) below. It is an easy exercise to show that $E(\pi,\gamma)-E(-\underline{1})=\Gamma$, see (2.12), starting from the expression

$$E(\psi)-E(-\underline{1}) = -4h\ell_1\ell_2 + 8(\ell_1+\ell_2) \tag{2.17}$$

for a rectangular droplet $\psi \in \mathcal{S}$ of side lengths ℓ_1 and ℓ_2, recall that in such a configuration

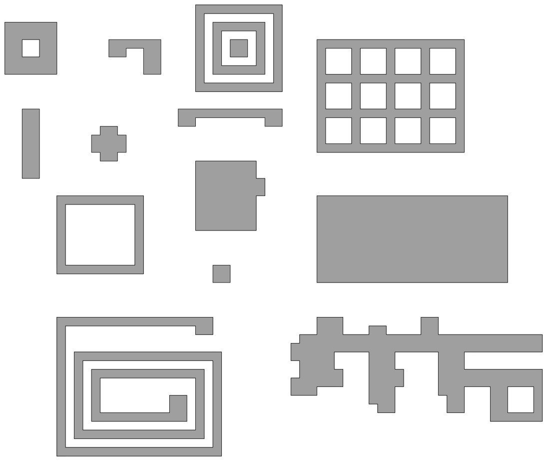

Figure 2: Examples of stable states, pluses and minuses are represented respectively by grey and white regions.

$2 \le \ell_1, \ell_2 \le L - 2$.

As mentioned above our model is characterized by the presence of a huge variety of stable configurations, to prove Theorem 2.1 we shall use the powerful technique introduced in [MNOS] which will allow us to avoid a precise and boring description of the full set of stable configurations. All the configurations in which each spin is surrounded by at least two spins of the same sign are stable, some of the possible situations are shown in Figure 2, in particular notice that plus squared rings plunged into the sea of minuses are stable states. This scenario is complicated by the presence of stable pairs, some of them are depicted in Figure 3, in particular notice the chessboards leant to stable pluses regions. As we shall see in the sequel, the stable pairs do not play any important role in the study of metastability in model (2.1); we also recall that in the case of a similar model studied in [CN], due to the presence of such pairs, the system was forced to visit an intermediate chessboard phase in its way from the minus metastable phase to the stable plus phase.

2.7. Escape time

In this section we prove the Theorem 2.1, the main ingredients will be the results in [MNOS], the solution of the model dependent variational problem (2.28) namely, the computation of the energy barrier between $-\underline{1}$ and $+\underline{1}$, and the recurrence estimate (2.27).

In our problem, see also [CN], the energy difference between two configurations σ and η is not sufficient to say if the system prefers to jump from σ to η or vice versa. Indeed, there are pair of configurations such that the system sees an energetic barrier in both directions. For this reason we associate a sort of *communication height* $H(\sigma, \eta)$ to each pair of configurations $\sigma, \eta \in \mathcal{S}$. More precisely we extend the Hamiltonian (2.5) to

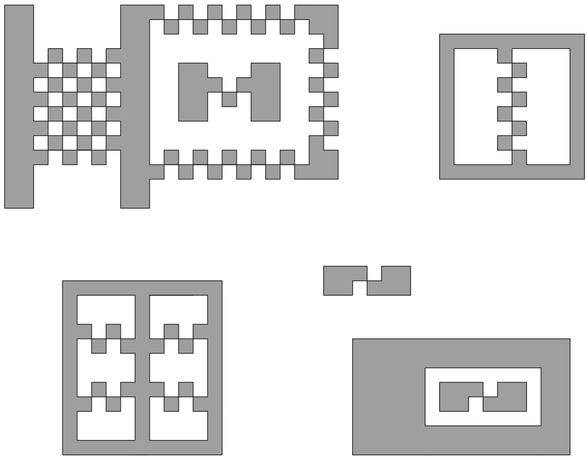

Figure 3: Examples of stable pairs, pluses and minuses are represented respectively by grey and white regions.

$H : \mathcal{S} \cup \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ so that

$$H(\sigma, \eta) := H(\sigma) - \log p(\sigma, \eta) \tag{2.18}$$

We consider the *communication energy*

$$E(\sigma, \eta) := \lim_{\beta \to \infty} \frac{H(\sigma, \eta)}{\beta} \geq \max\{E(\sigma), E(\eta)\} \tag{2.19}$$

and the *transition rate*

$$\Delta(\sigma, \eta) := E(\sigma, \eta) - E(\sigma) = \sum_{\substack{x \in \Lambda: \\ \eta(x)(S_\sigma(x)+h)<0}} 2|S_\sigma(x) + h| \geq 0 \tag{2.20}$$

where in the last equality we have used the definition (2.9) of $E(\sigma)$, (2.18), (2.1), and (2.2). Note that by (2.19), the definition of T in Subsection 2.6 and the dynamics in (2.2), we also have

$$E(\sigma, T\sigma) = E(\sigma) \quad \text{and} \quad E(\sigma) \geq E(T\sigma) \tag{2.21}$$

for any $\sigma \in \mathcal{S}$. Finally, by using the detailed principle (2.6) we get

$$H(\sigma, \eta) = H(\eta, \sigma) \quad \text{and} \quad E(\sigma, \eta) = E(\eta, \sigma) \tag{2.22}$$

for any $\sigma, \eta \in \mathcal{S}$. We recall that the dynamics (2.1) is reversible w.r.t. the Hamiltonian (2.5), see (2.6), however the relevant quantity for studying the behavior of the model in the low temperature regime ($\beta \to \infty$) is the communication energy. Indeed we have that for $\beta > 0$ large enough

$$e^{-\beta[E(\sigma,\eta)-E(\sigma)]-\beta\gamma(\beta)} \leq p(\sigma, \eta) \leq e^{-\beta[E(\sigma,\eta)-E(\sigma)]+\beta\gamma(\beta)} \tag{2.23}$$

for any $\sigma, \eta \in \mathcal{S}$, where $\gamma(\beta)$ does not depend on σ, η and tends to zero in the limit $\beta \to \infty$. To get (2.23) we first prove that for β large enough

$$\Big| -\frac{1}{\beta}[H(\sigma,\eta) - H(\sigma)] + [E(\sigma,\eta) - E(\sigma)]\Big| \le e^{-\beta(1-h)} \tag{2.24}$$

To prove (2.24) we note that by using (2.18), (2.1), (2.2), and (2.20) we get

$$\begin{aligned}
&\frac{1}{\beta}[H(\sigma,\eta) - H(\sigma)] - [E(\sigma,\eta) - E(\sigma)] = \\
&\quad = \frac{1}{\beta}\sum_{x\in\Lambda}\log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h]}) + \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} 2\eta(x)[S_\sigma(x)+h] \\
&\quad = \frac{1}{\beta}\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)>0}} \log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h]}) + \frac{1}{\beta}\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} \log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h]}) \\
&\qquad + \sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} 2\eta(x)[S_\sigma(x)+h] \\
&\quad = \frac{1}{\beta}\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)>0}} \log(1+e^{-2\beta\eta(x)[S_\sigma(x)+h]}) + \frac{1}{\beta}\sum_{\substack{x\in\Lambda:\\ \eta(x)(S_\sigma(x)+h)<0}} \log(e^{+2\beta\eta(x)[S_\sigma(x)+h]}+1) \\
&\quad = \frac{1}{\beta}\sum_{x\in\Lambda}\log(1+e^{-2\beta|S_\sigma(x)+h|})
\end{aligned}$$

The bound (2.24) follows once we note that $\log(1+\exp\{-2\beta|S_\sigma(x)+h|\}) \ge 0$ for any $x \in \Lambda$ and $|S_\sigma(x)+h| \ge 1-h$ uniformly in $\sigma \in \mathcal{S}$ and $x \in \Lambda$, and choose $\beta \ge (\log|\Lambda|)/(1-h)$. Finally, (2.23) follows from (2.24) and (2.18).

As noted above to prove Theorem 2.1 we shall use the model dependent estimates in Theorem 2.2 below and the general results in [MNOS, Theorem 4.1, 4.9 and 5.4]. In that paper the theory has been developed with quite strict hypotheses on the dynamics, see [MNOS, equation (1.3)], nevertheless it is easy to show that the same results hold in the more general setup introduced in [OS, Property $\mathcal{P}$] which, by virtue of the inequalities (2.23), includes our model (2.1).

To state the estimates on the energy landscape we need few more definitions. A finite sequence of configurations $\omega = \{\omega_1, \dots, \omega_n\}$ is called the *path* with starting configuration ω_1 and ending configuration ω_n; we let $|\omega| := n$. Given two paths ω and ω' such that $\omega_{|\omega|} = \omega'_1$ we let $\omega + \omega' := \{\omega_1, \dots, \omega_{|\omega|}, \omega'_2, \dots, \omega'_{|\omega'|}\}$; note that $|\omega + \omega'| = |\omega| + |\omega'| - 1$. Given a path ω we define the *height* along ω as

$$\Phi_\omega := \begin{cases} E(\omega_1) & \text{if } |\omega| = 1 \\ \max_{i=1,\dots,|\omega|-1} E(\omega_i, \omega_{i+1}) & \text{otherwise} \end{cases} \tag{2.25}$$

Given two configurations $\sigma, \eta \in \mathcal{S}$, we denote by $\Theta(\sigma,\eta)$ the set of all the paths ω starting from σ and ending in η. The minimax between σ and η is defined as

$$\Phi(\sigma,\eta) := \min_{\omega\in\Theta(\sigma,\eta)} \Phi_\omega \tag{2.26}$$

Theorem 2.2 *Recall Γ has been defined in (2.12) and suppose $h > 0$ is chosen small enough; then*

1. *for any $\sigma \in \mathcal{S} \setminus \{-\underline{1}\}$ we have*

$$\Phi(\sigma, +\underline{1}) - E(\sigma) < \Gamma \tag{2.27}$$

2. *we have*

$$\Phi(-\underline{1}, +\underline{1}) - E(-\underline{1}) = \Gamma \tag{2.28}$$

3. *for each path $\omega = \{\omega_1, \dots, \omega_n\} \in \Theta(-\underline{1}, +\underline{1})$ such that $\Phi_\omega = \Gamma$, there exists $i \in \{2, \dots, n\}$ such that $\omega_i \in \mathcal{C}$ and $E(\omega_{i-1}, \omega_i) = \Gamma$.*

The Theorem 2.2 will be proven in Section 3. Now we show that this theorem and the already quoted results in [MNOS] yield Theorem 2.1.

Proof of Theorem 2.1. Following [MNOS] we let $\mathcal{S}^{\mathrm{s}}$ be the set of global minima of the energy (2.9), we have $\mathcal{S}^{\mathrm{s}} = \{+\underline{1}\}$. For any $\sigma \in \mathcal{S}$ we let $\mathcal{I}_\sigma := \{\eta \in \mathcal{S} : E(\eta) < E(\sigma)\}$ and $V_\sigma := \Phi(\sigma, \mathcal{I}_\sigma) - E(\sigma)$; we define the set of *metastable states* $\mathcal{S}^{\mathrm{m}} := \{\eta \in \mathcal{S} : V_\eta = \max_{\sigma \in \mathcal{S} \setminus \mathcal{S}^{\mathrm{s}}} V_\sigma\}$. We remark that since $\mathcal{S}^{\mathrm{s}} = \{+\underline{1}\}$, then for any $\sigma \in \mathcal{S} \setminus \mathcal{S}^{\mathrm{s}}$ we have $E(+\underline{1}) < E(\sigma)$; this implies, together with (2.28) and (2.27), that $\mathcal{S}^{\mathrm{m}} = \{-\underline{1}\}$. Finally, items 1 and 2 follow from Theorem 4.1 and 4.9 in [MNOS], respectively.

To prove item 3, given $\eta, \zeta \in \mathcal{S}$, we introduce $\mathcal{W}(\eta, \zeta)$ the *gate* between η and ζ as the set of configurations $\sigma \in \mathcal{S}$ such that for each path $\omega = \{\omega_1, \dots, \omega_n\}$, such that $\omega_1 = \eta$, $\omega_n = \zeta$, and $\Phi_\omega = \Phi(\eta, \zeta)$, there exists $i \in \{2, \dots, n\}$ such that $\omega_i \in \mathcal{W}(\eta, \zeta)$ and $E(\omega_{i-1}, \omega_i) = \Phi(\eta, \zeta)$. By using item 3 in Theorem 2.2 we get $\mathcal{W}(-\underline{1}, +\underline{1}) = \mathcal{C}$; the item 3 finally follows from [MNOS, Theorem 5.4]. $\square$

3. The variational problem and the recurrence property

In this section we prove the energy landscape estimates in Theorem 2.2; in particular the recurrence property (2.27) is proven in Subsection 3.1 and the variational problem (2.28) is solved in Subsection 3.2. We state in advance few more definitions. Let $\sigma \in \mathcal{S}$ and $x \in \Lambda$, we say that the site x is *stable* (resp. *unstable*) w.r.t. σ if and only if $\sigma(x) S_\sigma(x) > 0$ (resp. $\sigma(x) S_\sigma(x) < 0$). Note that the stable sites are those that are not changed by the zero temperature dynamics, more precisely $T\sigma(x) = \sigma(x)$ if and only if x is stable w.r.t. σ. Given $\sigma \in \mathcal{S}$ and $k \in \{-5, -3, -1, +1, +3, +5\}$ we denote by $\Lambda_k^\pm(\sigma)$ the collection of the sites $x \in \Lambda^\pm(\sigma)$ such that $S_\sigma(x) = k$ namely,

$$\Lambda_k^\pm(\sigma) := \{x \in \Lambda^\pm(\sigma) \colon S_\sigma(x) = k\} \tag{3.1}$$

note that $\Lambda_{-5}^+(\sigma) = \emptyset$ and $\Lambda_{+5}^-(\sigma) = \emptyset$; moreover we set

$$\Lambda_{\le k}^\pm(\sigma) := \Lambda_{-5}^\pm(\sigma) \cup \dots \cup \Lambda_k^\pm(\sigma) \quad \text{and} \quad \Lambda_{\ge k}^\pm(\sigma) := \Lambda_k^\pm(\sigma) \cup \dots \cup \Lambda_{+5}^\pm(\sigma) \tag{3.2}$$

Finally, given $\sigma \in \mathcal{S}$ we denote by $\Lambda_{\rm s}^+(\sigma)$ (resp. $\Lambda_{\rm u}^+(\sigma)$) the collection of the sites $x \in \Lambda$ such that $\sigma(x) = +1$ and x is stable (resp. unstable) w..r.t. σ; similarly we define $\Lambda_{\rm s}^-(\sigma)$ and $\Lambda_{\rm u}^-(\sigma)$. By definition of stable and unstable sites we get that for any $\sigma \in \mathcal{S}$

$$\Lambda_{\rm u}^+(\sigma) = \Lambda_{\le -1}^+(\sigma),\ \Lambda_{\rm u}^-(\sigma) = \Lambda_{\ge +1}^-(\sigma),\ \Lambda_{\rm s}^+(\sigma) = \Lambda_{\ge +1}^+(\sigma),\ \text{and } \Lambda_{\rm s}^-(\sigma) = \Lambda_{\le -1}^-(\sigma) \quad (3.3)$$

3.1. The recurrence property

Equation (2.27) in Theorem 2.2 states that for any configuration σ different from the metastable state $-\underline{1}$ it is possible to exhibit a path ω joining σ to the stable state $+\underline{1}$ i.e., the absolute minimum of the energy, such that $\Phi_\omega < E(\sigma) + \Gamma$. On the heuristic ground, given $\sigma \in \mathcal{S} \setminus \{-\underline{1}\}$ there exists at least a plus spin; starting from such a plus it is possible to build a supercritical $\lambda \times \lambda$ droplet of pluses paying an energy cost strictly smaller than $E(\sigma) + \Gamma$. Indeed, by virtue of (2.28) starting from $-\underline{1}$ the cost would be exactly Γ, on the other hand starting from σ no energy must be paid to get the first plus spin and the other pluses of σ, if any, help the production of the supercritical droplet.

A rigorous proof needs the explicit construction of the path; such a path will firstly realize the growth of a supercritical $\lambda \times \lambda$ square with σ as a background, and then its growth towards $+\underline{1}$. More precisely, recall Λ is a squared torus, let L be its side length and $0 = (0, 0)$ the origin; recall the zero temperature dynamics mapping T defined in Subsection 2.6 and let $\sigma \in \mathcal{S}$ be such that $\sigma(x) = +1$ for any $x \in Q_{2,2}(0)$. We define the path

$$\Omega_\sigma := \Xi^2 + \sum_{n=3}^{L} [\Psi^n + \Xi^n] \quad (3.4)$$

where the paths Ξ^n for $n = 2, \ldots, L$ are defined algorithmically as follows: $\xi^2 := \sigma$, let $n \in \{2, \ldots, L-1\}$ and suppose ξ^n is such that $\xi^n(x) = +1$ for $x \in Q_{n,n}(0)$, then

1. `set` $i = 1$, $\xi_i^n = \xi^n$;
2. `if` $T^2\xi_i^n = \xi_i^n$ `then goto` 3 `else set` $i = i + 1$ `and` $\xi_i^n = T\xi_{i-1}^n$ `and goto` 2;
3. `if` $\xi_i^n(x) = +1$ for all $x \in Q_{1,n}(n, 0)$ `then set` $\psi^{n+1} = \xi_i^n$ `and goto` 7;
4. `if` $Q_{1,n}(n, 0) \cap \Lambda_{\rm s}^+(\xi_i^n) \neq \emptyset$, `then pick` $y, y' \in Q_{1,n}(n, 0)$ such that $\mathrm{d}(y, y') = 1$, $\xi_i^n(y) = -1$, and $y' \in \Lambda_{\rm s}^+(\xi_i^n)$, `set` $i = i + 1$, $\xi_i^n(y) = +1$, $\xi_i^n(x) = T\xi_{i-1}^n(x)$ $\forall x \in \Lambda \setminus \{y\}$ `and goto` 3;
5. `if` $Q_{1,n}(n, 0) \cap \Lambda_{\rm u}^+(\xi_i^n) \neq \emptyset$, `then pick` $y, y' \in Q_{1,n}(n, 0)$ such that $\mathrm{d}(y, y') = 1$, $\xi_i^n(y) = -1$, and $y' \in \Lambda_{\rm u}^+(\xi_i^n)$, `set` $i = i + 1$, $\xi_i^n(y) = +1$, $\xi_i^n(y') = +1$, $\xi_i^n(x) = T\xi_{i-1}^n(x)$ for any $x \in \Lambda \setminus \{y, y'\}$ `and goto` 3;
6. `set` $i = i + 1$, $y = (n, 0)$, $\xi_i^n(y) = +1$, $\xi_i^n(x) = T\xi_{i-1}^n(x)$ for any $x \in \Lambda \setminus \{y\}$ `and goto` 3;
7. `set` $h_n = i$, $\Xi^n = \{\xi_1^n, \ldots, \xi_{h_n}^n\}$ `and exit`.

The paths Ψ^n for $n = 3, \ldots, L$ are defined via a very similar algorithm: let $n \in \{3, \ldots, L\}$ and suppose ψ^n is such that $\psi^n(x) = +1$ for $x \in Q_{n,n-1}(0)$, then

1. `set` $i = 1$, $\psi^n_i = \psi^n$;
2. `if` $T^2\psi^n_i = \psi^n_i$ `then goto` 3 `else set` $i = i + 1$ `and` $\psi^n_i = T\psi^n_{i-1}$ `and goto` 2;
3. `if` $\psi^n_i(x) = +1$ for all $x \in Q_{n,1}(0, n)$ `then set` $\xi^n = \psi^n_i$ `and goto` 7;
4. `if` $Q_{n,1}(0, n) \cap \Lambda^+_{\mathrm{s}}(\psi^n_i) \neq \emptyset$, `then pick` $y, y' \in Q_{n,1}(0, n)$ such that $\mathrm{d}(y, y') = 1$, $\psi^n_i(y) = -1$, and $y' \in \Lambda^+_{\mathrm{s}}(\psi^n_i)$, `set` $i = i + 1$, $\psi^n_i(y) = +1$, $\psi^n_i(x) = T\psi^n_{i-1}(x)$ $\forall x \in \Lambda \setminus \{y\}$ `and goto` 3;
5. `if` $Q_{n,1}(0, n) \cap \Lambda^+_{\mathrm{u}}(\psi^n_i) \neq \emptyset$, `then pick` $y, y' \in Q_{n,1}(0, n)$ such that $\mathrm{d}(y, y') = 1$, $\psi^n_i(y) = -1$, and $y' \in \Lambda^+_{\mathrm{u}}(\psi^n_i)$, `set` $i = i + 1$, $\psi^n_i(y) = +1$, $\psi^n_i(y') = +1$, $\psi^n_i(x) = T\psi^n_{i-1}(x)$ for any $x \in \Lambda \setminus \{y, y'\}$ `and goto` 3
6. `set` $i = i + 1$, $y = (0, n)$, $\psi^n_i(y) = +1$, $\psi^n_i(x) = T\psi^n_{i-1}(x)$ for any $x \in \Lambda \setminus \{y\}$ `and goto` 3;
7. `set` $k_n = i$, $\Psi^n = \{\psi^n_1, \ldots, \psi^n_{k_n}\}$ `and exit`.

Note that in the first two steps of the algorithm constructing Ξ^n (resp. Ψ^n) the path Ξ^n (resp. Ψ^n) starting at ξ^n (resp. ψ^n) follows the zero temperature dynamics down to the stable pair or to the stable state associated to ξ^n (resp. ψ^n).

Lemma 3.1 *Let $\sigma \in \mathcal{S}$ be such that $\sigma(x) = +1$ for any $x \in Q_{2,2}(0)$, consider the path Ω_σ defined by (3.4). Then*

1. *for any $n = 3, \ldots, L$ the configuration ψ^n is such that $\psi^n(x) = +1$ for all $x \in Q_{n,n-1}(0)$, for any $n = 3, \ldots, L$ the configuration ξ^n is such that $\xi^n(x) = +1$ for all $x \in Q_{n,n}(0)$, in particular $\xi^L = +\underline{1}$ and $\Xi^L = \{\xi^L\}$;*
2. *for any $n = 2, \ldots, L$ we have*
$$E(\psi^{n+1}) - E(\xi^n) \leq (8 - 4hn) \vee 0 \quad \text{and} \quad \Phi_{\Xi^n} \leq E(\xi^n) + 10 - 6h \tag{3.5}$$
3. *for any $n = 3, \ldots, L$ we have*
$$E(\xi^n) - E(\psi^n) \leq (8 - 4hn) \vee 0 \quad \text{and} \quad \Phi_{\Psi^n} \leq E(\psi^n) + 10 - 6h \tag{3.6}$$
4. *we have*
$$\Phi_{\Omega_\sigma} - E(\sigma) \leq \Gamma - 16(2 - h) \tag{3.7}$$
where we recall Γ has been defined in (2.12).

Proof of Lemma 3.1. Item 1 is an immediate consequence of the algorithmic definition of Ω_σ. The proof of item 2 is similar to the proof of item 3.

Item 3. Let $\Psi^n := \{\psi_1^n, \dots, \psi_k^n, \dots, \psi_{k_n}^n\}$ with $k_n \ge k \ge 1$ such that $\psi_i^n = T\psi_{i-1}^n$ for $i = 2, \dots, k$ and $\psi_k^n = T^2\psi_k^n$; note that by construction $\psi_1^n = \psi^n$, $\psi_{k_n}^n = \xi^n$, and $k_n - k \le n$. By using (2.21) we get

$$\Phi_{\{\psi_1^n,\dots,\psi_k^n\}} \le E(\psi_1^n) \quad \text{and} \quad E(\psi_i^n) \ge E(\psi_{i+1}^n) \tag{3.8}$$

for $i = 1 \dots, k-1$. If $k_n = k$ then (3.6) follows immediately from (3.8). In the case $k_n \ge k+1$ we shall prove that

$$\Phi_{\{\psi_k^n,\psi_{k+1}^n,\dots,\psi_{k_n}^n\}} \le E(\psi_k^n) + 10 - 6h \quad \text{and} \quad E(\psi_{k_n}^n) - E(\psi_k^n) \le (8 - 4hn) \vee 0 \tag{3.9}$$

The bounds (3.6) will then follow from (3.8) and (3.9).

We are then left with the proof (3.9) which can be achieved by discussing the following three cases.

Case 1. There exist $y, y' \in Q_{n,1}(0,n)$ such that $\psi_k^n(y) = -1$, $y' \in \Lambda_{\mathrm{s}}^+(\psi_k^n)$. The configuration ψ_{k+1}^n is defined at the step 4 of the algorithm; it is immediate to remark that all the configurations ψ_i^n with $i = k+1, \dots, k_n$ are indeed defined at the step 4. Then by using (2.19), see also figure 1, we get the following bounds on the communication energies:

$$E(\psi_i^n, \psi_{i+1}^n) \le E(\psi_i^n) + 2(1-h) \;\text{ and }\; E(\psi_{i+1}^n, \psi_i^n) \ge E(\psi_{i+1}^n) + 2(1+h) \tag{3.10}$$

for any $i = k, \dots, k_n - 1$. By using (3.10), (2.25), and (2.22) we get

$$\Phi_{\{\psi_k^n,\psi_{k+1}^n,\dots,\psi_{k_n}^n\}} \le E(\psi_k^n) + 2(1-h) \;\text{ and }\; E(\psi_{k_n}^n) - E(\psi_k^n) \le -4h(k_n - k) \tag{3.11}$$

which, recalling $k_n \ge k+1$, imply (3.9).

Case 2. There exist $y, y' \in Q_{n,1}(0,n)$ such that $\psi_k^n(y) = -1$, $\psi_k^n(y') = +1$, and $\Lambda_{\mathrm{s}}^+(\psi_k^n) \cap Q_{n,1}(0,n) = \emptyset$. The configuration ψ_{k+1}^n is defined at the step 5 of the algorithm; it is immediate to remark that all the configurations ψ_i^n with $i = k+1, \dots, k_n$ are instead defined at the step 4.

Let $i = k, \dots, k_n - 1$, let $y, y' \in Q_{n,1}(0,n)$ the two sites which are picked up by the algorithm, let Δ_i the collection of the sites in $Q_{n,1}(0,n)$ different from y, y' and such that they become stable plus sites at this step of the path; more precisely we let $\Delta_i := \Lambda_{\mathrm{s}}^+(\psi_{i+1}^n) \setminus (\Lambda_{\mathrm{s}}^+(\psi_i^n) \cup \{y, y'\})$. Note that the update of the sites in Δ_i has no energy cost since they follow the zero temperature dynamics T.

By using (2.19), see also figure 1, we get the estimates

$$\begin{array}{ll} E(\psi_k^n, \psi_{k+1}^n) \le E(\psi_k^n) + 4(1-h) & E(\psi_{k+1}^n, \psi_k^n) \ge E(\psi_{k+1}^n) + 2(1+h)(1+|\Delta_k|) \\ E(\psi_i^n, \psi_{i+1}^n) \le E(\psi_i^n) + 2(1-h) & E(\psi_{i+1}^n, \psi_i^n) \ge E(\psi_{i+1}^n) + 2(1+h)(1+|\Delta_i|) \end{array} \tag{3.12}$$

for any $i = k+1, \dots, k_n - 1$. If $|\Delta_i| = 0$ for any $i = k, \dots, k_n - 1$, then it must necessarily be $k_n - k = n - 1$. We get

$$\Phi_{\{\psi_k^n,\psi_{k+1}^n,\dots,\psi_{k_n}^n\}} \le E(\psi_k^n) + 4(1-h) \text{ and } E(\psi_{k_n}^n) - E(\psi_k^n) \le 2 - 2h - 4h(n-1) \tag{3.13}$$

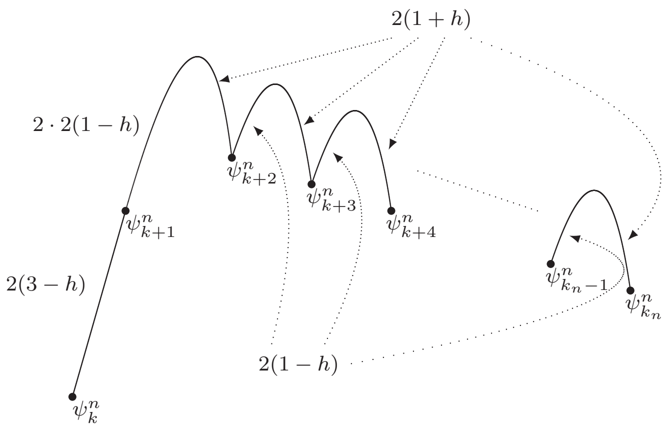

Figure 4: Graphical representation of the estimates (3.15).

The bound (3.9) follows for h small enough since $8-4hn = 2-2h-4h(n-1)+(6-2h)$. Suppose, finally, that there exists $i \in \{k, \dots, k_n-1\}$ such that $|\Delta_i| \neq 0$; then

$$\Phi_{\{\psi^n_k,\psi^n_{k+1},\dots,\psi^n_{k_n}\}} \leq E(\psi^n_k) + 4(1-h) \text{ and } E(\psi^n_{k_n}) - E(\psi^n_k) \leq -4h(k_n-k+1) \tag{3.14}$$

Recall $k_n \geq k+$; the bounds (3.14) imply (3.9) trivially.
Case 3. For each $y \in Q_{n,1}(0,n)$ we have $\psi^n_k(y) = -1$. In this case $k_n - k = n$, ψ^n_{k+1} is defined at the step 6, ψ^n_{k+2} is defined either at the step 4 or at the step 5, and ψ^n_{k+i}, with $i = 3, \dots, k_n$ are defined at the step 4 of the algorithm. By using (2.19), see also figure 1, we get

$$\begin{array}{llll} E(\psi^n_k,\psi^n_{k+1}) & \leq E(\psi^n_k) + 2(3-h) & E(\psi^n_{k+1},\psi^n_k) & \geq E(\psi^n_{k+1}) \\ E(\psi^n_{k+1},\psi^n_{k+2}) & \leq E(\psi^n_{k+1}) + 4(1-h) & E(\psi^n_{k+2},\psi^n_{k+1}) & \geq E(\psi^n_{k+2}) + 2(1+h) \\ E(\psi^n_i,\psi^n_{i+1}) & \leq E(\psi^n_i) + 2(1-h) & E(\psi^n_{i+1},\psi^n_i) & \geq E(\psi^n_{i+1}) + 2(1+h) \end{array} \tag{3.15}$$

for $i = k+2, \dots, k_n-1$; see the Figure 4 for a graphical representation of the estimates (3.15), note that the equalities hold for instance in the case $\psi^n_k(x) = -1$ for any $x \in \partial Q_{n,1}(0.n) \setminus Q_{n,n-1}(0)$. By using (3.15), (2.25), and (2.22) we get

$$\Phi_{\{\psi^n_k,\psi^n_{k+1},\dots,\psi^n_{k_n}\}} \leq E(\psi^n_k) + 10 - 6h \text{ and } E(\psi^n_{k_n}) - E(\psi^n_k) \leq [8-4h(k_n-k)] = 8-4hn \tag{3.16}$$

which imply (3.9).

We remark that in this case 3 the path $\{\psi^n_k, \dots, \psi^n_{k_n}\}$ realizes the standard growth of the rectangular plus droplet ψ^n up to the square droplet $\psi^n_{k_n}$ via the formation of a unit plus protuberance in the slice adjacent to one of the longest sides of the rectangle and the bootstrap percolation plus filling of the same slice.

Item 4. Let η, η' two consecutive configurations of the path Ω_σ, we shall prove that

$$E(\eta,\eta') - E(\sigma) \leq \Gamma - 16(2-h) \tag{3.17}$$

The bound (3.7) will then follow, see (2.25). Recall the critical length λ has been defined in (2.11), and consider the following four cases.

Case 1. The configurations η, η' belong to Ξ^n for some $n \leq \lambda - 1$. This case is similar to the case 2.
Case 2. The configurations η, η' belong to Ψ^n for some $n \leq \lambda$. By using (2.25), (3.5), and (3.6) we have

$$\begin{aligned} E(\eta, \eta') \leq \Phi_{\Psi^n} &\leq E(\psi^n) + 10 - 6h \\ &\leq E(\psi^n) - E(\xi^{n-1}) + E(\xi^{n-1}) - \cdots - E(\psi^3) + E(\psi^3) - E(\xi^2) + E(\xi^2) \\ &\quad + 10 - 6h \\ &\leq E(\sigma) + 18 - 14h + 8\sum_{i=3}^{n-1}[2 - hi] \leq E(\sigma) + 18 - 14h + 8\sum_{i=3}^{\lambda-1}[2 - hi] \end{aligned}$$

where we have used that $2 - hi > 0$ for $i \leq \lambda - 1$ and $\xi^2 = \sigma$. The bound (3.17) follows easily.
Case 3. The configurations η, η' belong to Ξ^n for some $n \geq \lambda$. Note that for $n \geq \lambda$ the bounds (3.5) and (3.6) on the differences of energy become trivial since $8 - 4hn < 0$, hence $E(\xi^n) \leq E(\psi^\lambda)$. Then

$$E(\eta, \eta') \leq \Phi_{\Xi^n} \leq E(\xi^n) + 10 - 6h \leq E(\psi^\lambda) + 10 - 6h$$

where in the first inequality we have used (2.25), in the second the bound (3.5), and in the last the fact that $E(\xi^n) \leq E(\psi^\lambda)$. To get (3.17) we then perform the same computation as in the case 2.
Case 4. The configurations η, η' belong to Ψ^n for some $n \geq \lambda + 1$. This case is similar to the case 3. $\square$

Proof of item 1 of Theorem 2.2. Let $\sigma \in \mathcal{S} \setminus \{-\underline{1}\}$. If $\sigma = +\underline{1}$ the statement of the lemma is trivial; we then suppose $\sigma \neq +\underline{1}$. Since by hypothesis $\sigma \neq -\underline{1}$, there exists $x \in \Lambda$ such that $\sigma(x) = +1$; without loss of generality we suppose $\sigma(0) = +1$. Consider the path $\omega := \{\sigma, \sigma^1, \sigma^2, \sigma^3\}$ with

- σ^1 is such that $\sigma^1(x) = +1$ for all $x \in Q_{2,1}(0)$ and $\sigma^1(x) = T\sigma(x)$ for all $x \in \Lambda \setminus Q_{2,1}(0)$;
- σ^2 is such that $\sigma^2(x) = +1$ for all $x \in Q_{2,1}(0) \cup Q_1(0,1)$ and $\sigma^2(x) = T\sigma^1(x)$ for all $x \in \Lambda \setminus [Q_{2,1}(0) \cup Q_1(0,1)]$;
- σ^3 is such that $\sigma^3(x) = +1$ for all $x \in Q_{2,2}(0)$ and $\sigma^3(x) = T\sigma^2(x)$ for all $x \in \Lambda \setminus Q_{2,2}(0)$.

By definition the path $\omega + \Omega_{\sigma^3}$ starts at σ and ends in $+\underline{1}$ namely, $\omega + \Omega_{\sigma^3} \in \Theta(\sigma, +\underline{1})$, moreover we shall prove that

$$\Phi_{\omega + \Omega_{\sigma^3}} < E(\sigma) + \Gamma \tag{3.18}$$

the item 1 of Theorem 2.2 will then follow.

To prove (3.18) we first consider the path ω; by using (2.19), see also figure 1, we get the following bounds on the communication energies:

$$\begin{array}{ll} E(\sigma,\sigma^1) \le E(\sigma)+2\cdot 2(3-h) & E(\sigma^1,\sigma) \ge E(\sigma^1) \\ E(\sigma^1,\sigma^2) \le E(\sigma^1)+2\cdot 2(1-h)+2(3-h) & E(\sigma^2,\sigma^1) \ge E(\sigma^2) \\ E(\sigma^2,\sigma^3) \le E(\sigma^2)+3\cdot 2(1-h) & E(\sigma^3,\sigma^2) \ge E(\sigma^3)+2(1+h) \end{array} \tag{3.19}$$

By using (3.19), (2.25), (2.22), (2.12), and the definition (2.11) of the critical length λ it is easy to show that, provided h is chosen small enough,

$$\Phi_\omega - E(\sigma) \le 28 - 16h < \Gamma \tag{3.20}$$

and

$$E(\sigma^3) - E(\sigma) \le 26 - 18h \tag{3.21}$$

We consider, now, the path Ω_{σ^3}; by using (3.7) and (3.21) we get

$$\Phi_{\Omega_{\sigma^3}} - E(\sigma) = \Phi_{\Omega_{\sigma^3}} - E(\sigma^3) + E(\sigma^3) - E(\sigma) \le \Gamma - 16(2-h) + 26 - 18h = \Gamma - 2(3+h) \tag{3.22}$$

The inequality (3.18) finally follows from (3.20) and (3.22). $\square$

3.2. The variational problem

The item 2 of Theorem 2.2 deals with the determination of the minimal energy barrier between the metastable state $-\underline{1}$ and the stable one $+\underline{1}$, more precisely with the computation of $\Phi(-\underline{1},+\underline{1})$. In the context of serial Glauber dynamics this problem has been faced with different approaches each suited to the model under exam, see [OV] and [MNOS, Section 4.2] where a quite general technique is described. All these methods rely on the continuity of the dynamics namely, on the property that at each step only one spin is updated.

In the case of parallel dynamics, see [CN], the lacking of continuity increases the difficulty of the computation of the minimax between the metastable and the stable state. We follow, here, the method proposed in [CN] which is based on the construction of a set $\mathcal{G} \subset \mathcal{S}$ containing $-\underline{1}$, but not $+\underline{1}$, and on the evaluation of the communication energy for all the possible direct jumps from the interior to the exterior of such a set $\mathcal{G}$.

To define the set $\mathcal{G}$ we need to introduce the two mappings $A, B : \mathcal{S} \to \mathcal{S}$. Let $\sigma \in \mathcal{S}$, we set $A\sigma := \sigma$ if $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda^+_{\mathrm{u}}(\sigma)$, otherwise $A\sigma := \sigma^x$ where x is the first element of $\Lambda^+_{\mathrm{u}}(\sigma)$ in lexicographic order. The map A flips the first, in lexicographic order, unstable plus spin of σ to which corresponds a decrease of the energy; under the effect of the map A the number of pluses decreases, but only unstable pluses are flipped. Let $\sigma \in \mathcal{S}$, the configuration $B\sigma \in \mathcal{S}$ is such that for each $x \in \Lambda$

$$B\sigma(x) := \begin{cases} -\sigma(x) & \text{if } x \in \Lambda^-_{\ge -1}(\sigma) \\ \sigma(x) & \text{otherwise} \end{cases} \tag{3.23}$$

Note that the operator B perform a single step of *bootstrap percolation* namely, relatively to σ, flips all the minus unstable spins and, among the stable minus spins, only those with two neighboring minuses.

In the sequel a relevant role will be played by the configuration $\bar{B}\bar{A}\sigma$, for any $\sigma \in \mathcal{S}$. The sole unstable positive spins in $\bar{A}\sigma$ are those corresponding to energy increasing flips; starting from $\bar{A}\sigma$ the map B, which flips the minus spins with at least two plus spins among the nearest neighbors, is applied iteratively until a fixed point is reached. It is easy to convince oneself that the pluses in such a fixed point form well separated rectangles or stripes winding around the torus; more precisely the pluses in $\bar{B}\bar{A}\sigma$ occupy the region $\bigcup_{i=1}^{n} Q_{\ell_{i,1},\ell_{i,2}}(x_i) \subset \Lambda$, where $n, \ell_{1,1}, \ell_{1,2}, \dots, \ell_{n,1}, \ell_{n,2}$ are positive integers and $x_i \in \Lambda$ for any $i = 1, \dots, n$, with $Q_{\ell_{i,1},\ell_{i,2}}(x_i)$ being pairwise not interacting, see Subsection 2.1. Note that, depending on the values of $\ell_{i,1}, \ell_{i,2}$, the set $Q_{\ell_{i,1},\ell_{i,2}}(x_i)$ can be either a rectangle or a stripe winding around the torus.

We can now define the set $\mathcal{G}$. Let $\sigma \in \mathcal{S}$, consider $\bar{B}\bar{A}\sigma$, and, provided $\bar{B}\bar{A}\sigma \neq -\underline{1}$, denote by $Q_{\ell_{i,1},\ell_{i,2}}(x_i)$ the collection of pairwise not interacting rectangles (or stripes) obtained by collecting all the sites $y \in \Lambda$ such that $\bar{B}\bar{A}\sigma(y) = +1$. We say that $\sigma \in \mathcal{G}$ if and only if $\bar{B}\bar{A}\sigma = -\underline{1}$ or $\min\{\ell_{i,1}, \ell_{i,2}\} \leq \lambda - 1$ and $\max\{\ell_{i,1}, \ell_{i,2}\} \leq L - 2$ for any $i = 1, \dots, n$. Note that configurations σ such that $\bar{B}\bar{A}\sigma$ contains plus stripes winding around the torus Λ do not belong to $\mathcal{G}$.

In general $\bar{T}\sigma \neq \bar{B}\bar{A}\sigma$, this means that $\bar{B}\bar{A}\sigma$ is not necessarily the result of the zero temperature dynamics started at σ. This is not a problem, when one is looking for the minimal energy barrier between $-\underline{1}$ and $+\underline{1}$, provided the energy of such configurations is larger than Γ. The definition of $\mathcal{G}$ is indeed satisfactory because we can prove the following Proposition 3.2 on which the proof of items 2 and 3 of Theorem 2.2 is mostly based. To state the lemma we need one more definition: recall the set $\mathcal{C}$ is defined as the collection of configurations with all the spins equal to -1 excepted those in a rectangle of sides $\lambda - 1$ and λ and in a pair of neighboring sites adjacent to one of the longest side of the rectangle. Then given $\gamma \in \mathcal{C}$ we let $\pi(\gamma) \subset \mathcal{S}$ the set whose elements are the two configurations that can be obtained from γ by flipping one of the two plus spins in the pair attached to one of the longest side of the plus spin $\lambda \times (\lambda - 1)$ rectangle. We also let $\mathcal{P}$ be the collection of all the configurations with all the spins equal to -1 excepted those in a rectangle of sides $\lambda - 1$ and λ and in a single site adjacent to one of the longest side of the rectangle; finally we let $\mathcal{R}$ be the collection of rectangular droplets with sides $\lambda - 1$ and λ. We have

$$E(\mathcal{R}) - E(-\underline{1}) = -4h\lambda^2 + 4h\lambda + 16\lambda - 8 = \Gamma - 10 + 6h \tag{3.24}$$

where we have used in the last equality the definition (2.12) of Γ; by using (2.13) we have also the easy bound

$$E(\mathcal{R}) - E(-\underline{1}) < 8\lambda + 4h \tag{3.25}$$

Proposition 3.2 *With the definitions above, for $h > 0$ small enough and $L = L(h)$ large enough, we have*

1. $-\underline{1} \in \mathcal{G}$, $+\underline{1} \in \mathcal{S} \setminus \mathcal{G}$, *and* $\mathcal{C} \subset \mathcal{S} \setminus \mathcal{G}$;

2. *for each* $\eta \in \mathcal{G}$ *and* $\zeta \in \mathcal{S} \setminus \mathcal{G}$ *we have* $E(\eta, \zeta) \geq E(-\underline{1}) + \Gamma$;

3. *for each $\eta \in \mathcal{G}$ and $\zeta \in \mathcal{S} \setminus \mathcal{G}$ we have $E(\eta, \zeta) = E(-\underline{1}) + \Gamma$ if and only if $\zeta \in \mathcal{C}$ and $\eta \in \pi(\zeta)$.*

Proof of item 2 of Theorem 2.2. Since $-\underline{1} \in \mathcal{G}$ and $+\underline{1} \in \mathcal{S}\setminus\mathcal{G}$, see item 1 in Proposition 3.2, we have that any path $\omega = \{\omega_1, \dots, \omega_n\}$ such that $\omega_1 = -\underline{1}$ and $\omega_n = +\underline{1}$ must necessarily contain a direct jump from $\mathcal{G}$ to $\mathcal{S} \setminus \mathcal{G}$ namely, there must be $i \in \{2, \dots, n\}$ such that $\omega_{i-1} \in \mathcal{G}$ and $\omega_i \in \mathcal{S} \setminus \mathcal{G}$. Thus item 2 in Proposition 3.2 implies that $\Phi_\omega \geq E(-\underline{1}) + \Gamma$; from the arbitrarity of the path ω it follows that

$$\Phi(-\underline{1}, +\underline{1}) \geq E(-\underline{1}) + \Gamma \tag{3.26}$$

To complete the proof of (2.28) we need to exhibit a path connecting $-\underline{1}$ to $+\underline{1}$ such that the height along such a path is less than or equal to $E(-\underline{1}) + \Gamma$. Consider the path $\omega := \{-\underline{1}, \sigma^1, \sigma^2, \sigma^3, \sigma^4\}$ with σ^1 the configuration with all the spins equal to minus one excepted the one at the origin, σ^2 the configuration with all the spins equal to minus one excepted the ones associated to the sites in the rectangle $Q_{2,1}(0)$, σ^3 the configuration with all the spins equal to minus one excepted the ones associated to the sites in $Q_{2,1}(0) \cup Q_1(0,1)$, and σ^4 the configuration with all the spins equal to minus one excepted the ones associated to the sites in the square $Q_2(0)$.

By definition the path $\omega + \Omega_{\sigma^4}$ starts at $-\underline{1}$ and ends in $+\underline{1}$ namely, $\omega + \Omega_{\sigma^4} \in \Theta(-\underline{1}, +\underline{1})$, moreover we shall prove that

$$\Phi_{\omega + \Omega_{\sigma^4}} - E(-\underline{1}) \leq \Gamma \tag{3.27}$$

The inequality (3.27), together with (3.26), implies (2.28).

We are then left with the proof of (3.27). We first consider the path ω; by using (2.19), see also figure 1, we get

$$\begin{array}{llll} E(-\underline{1}, \sigma^1) = E(-\underline{1}) + 2(5-h) & & E(\sigma^1, -\underline{1}) = E(\sigma^1) \\ E(\sigma^1, \sigma^2) \;= E(\sigma^1) + 2 \cdot 2(3-h) & & E(\sigma^2, \sigma^1) \;= E(\sigma^2) + 2(1-h) \\ E(\sigma^2, \sigma^3) \;= E(\sigma^2) + 2 \cdot 2(1-h) + 2(3-h) & & E(\sigma^3, \sigma^2) \;= E(\sigma^3) + 2(1-h) \\ E(\sigma^3, \sigma^4) \;= E(\sigma^3) + 3 \cdot 2(1-h) & & E(\sigma^4, \sigma^3) \;= E(\sigma^4) + 2(1+h) \end{array} \tag{3.28}$$

see figure 5 for a graphical representation.

By using (3.28), (2.25), (2.22), (2.12), and the definition (2.11) of the critical length λ it is easy to show that, provided h is chosen small enough,

$$\Phi_\omega - E(-\underline{1}) \leq 34 - 14h < \Gamma \tag{3.29}$$

and

$$E(\sigma^4) - E(-\underline{1}) \leq 32 - 16h \tag{3.30}$$

We consider, now, the path Ω_{σ^4}; by using (3.7) and (3.30) we get

$$\Phi_{\Omega_{\sigma^4}} - E(-\underline{1}) = \Phi_{\Omega_{\sigma^4}} - E(\sigma^4) + E(\sigma^4) - E(-\underline{1}) \leq \Gamma - 16(2-h) + 32 - 16h = \Gamma \tag{3.31}$$

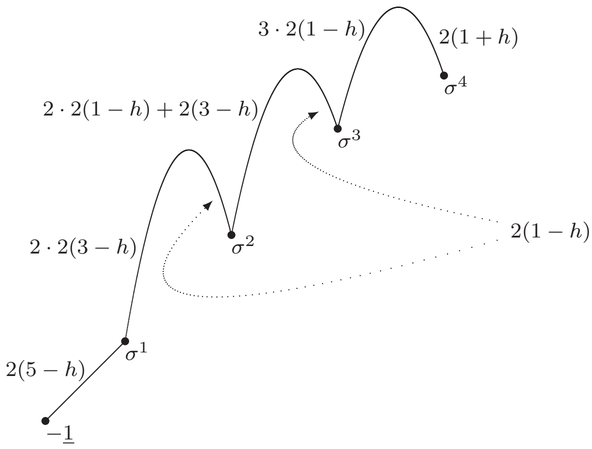

Figure 5: Energy landscape for the path ω.

The inequality (3.27) finally follows from (3.29) and (3.31). This completes the proof of item 2 of Theorem 2.2. □

Proof of item 3 of Theorem 2.2. The item follows from item 2 of Theorem 2.2 and item 3 of Proposition 3.2. □

4. The direct jump proposition

In Subsection 4.4 we shall prove Proposition 3.2 concerning the solution of the *minmax* problem. We state in advance some preliminary Lemmata.

4.1. Energy estimates for the maps A and B

In Lemma 4.1 we give estimates on the energy of the configurations obtained by applying the maps A and B. For any $\sigma \in \mathcal{S}$ we let

$$N_A(\sigma) := \sum_{x\in\Lambda}[1 - \delta_{\sigma(x),\bar{A}\sigma(x)}] \quad \text{and} \quad N_B(\sigma) := \sum_{x\in\Lambda}[1 - \delta_{\bar{A}\sigma(x),\bar{B}\bar{A}\sigma(x)}] \tag{4.1}$$

with δ the Kronecker δ. Note that $N_A(\sigma)$ is the number of plus spins which are flipped by the iterative application of the map A to σ, while $N_B(\sigma)$ is the number of minus spins which are flipped by the iterative application of the bootstrap percolation map B to $\bar{A}\sigma$.

Lemma 4.1 *Let $\sigma \in \mathcal{S}$ and $h > 0$ small enough. Then*

1. *we have*
$$E(\sigma) \geq E(\bar{A}\sigma) + (2 - 10h)N_A(\sigma) \tag{4.2}$$

2. *we have*
$$E(\bar{A}\sigma) \geq E(\bar{B}\bar{A}\sigma) + 4hN_B(\sigma) \tag{4.3}$$

In order to prove Lemma 4.1 we state Lemma 4.2 on some properties of unstable plus spins and Lemma 4.3 concerning the energy estimate for a single application of the bootstrap percolation map B. Recall (3.1), (3.2), (3.3), and that for $\sigma \in \mathcal{S}$ and $x \in \Lambda$ the configuration σ^x has been defined in Subsection 2.2.

Lemma 4.2 *Let $\sigma \in \mathcal{S}$; for $h > 0$ small enough we have that the following statements hold true:*

1. *if there exists $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) > E(\sigma)$, then $|\partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)| \leq 1$ namely, there exists at most one nearest neighbor of x which is stable w.r.t. σ and such that the associated spin is minus one;*

2. *if there exists $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) \leq E(\sigma)$, then $E(\sigma) \geq E(\sigma^x) + 2 - 10h$;*

3. *if $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda_{\mathrm{u}}^+(\sigma)$, then*

$$2|\Lambda_{-1}^+(\sigma)| + 3|\Lambda_{-3}^+(\sigma)| \leq 3|\Lambda_{+1}^-(\sigma)| + 4|\Lambda_{+3}^-(\sigma)| \tag{4.4}$$

Proof of Lemma 4.2. Let $x \in \Lambda_{\mathrm{u}}^+(\sigma)$, then $\sigma(x) = +1$, $\sigma^x(x) = -1$, and $S_\sigma(x) < 0$; by using (2.9) we get

$$E(\sigma^x) - E(\sigma) = 2h - 2 + \sum_{y \in \partial\{x\}} (|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|) \tag{4.5}$$

Note that since $\sigma(x) = +1$, we have that $S_\sigma(y)$, with $y \in \partial\{x\}$, can assume the values $-3, -1, +1, +3, +5$; by performing the direct computations one shows that

$$|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h| \in \{-2, 2h, +2\} \tag{4.6}$$

for $y \in \partial\{x\}$.

Item 1. Let $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) > E(\sigma)$; since $S_\sigma(y) < 0$ for $y \in \partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)$, by using (4.5) we get

$$E(\sigma^x) - E(\sigma) = 2h - 2(1 + |\partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)|) + \sum_{y \in \partial\{x\} \setminus \Lambda_{\mathrm{s}}^-(\sigma)} (|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|)$$

Suppose, by absurdity, that $|\partial\{x\} \cap \Lambda_{\mathrm{s}}^-(\sigma)| \geq 2$, then we have

$$E(\sigma^x) - E(\sigma) \leq 2h - 6 + \sum_{y \in \partial\{x\} \setminus \Lambda_{\mathrm{s}}^-(\sigma)} (|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|)$$

By (4.6) we obtain $|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h| \leq 2$ for $y \in \partial\{x\}$, and, noting that $|\partial\{x\} \setminus \Lambda_{\mathrm{s}}^-(\sigma)| \leq 2$, we finally get $E(\sigma^x) - E(\sigma) \leq 2h - 6 + 4 = 2h - 2 < 0$, which is in contradiction with the hypothesis.

Item 2. Let $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ such that $E(\sigma^x) \leq E(\sigma)$. Since the only allowed negative value for $|S_\sigma(y) + h| - |S_\sigma(y) - 2 + h|$, with $y \in \partial\{x\}$, is -2, we have that, provided

$E(\sigma^x) - E(\sigma) \le 0$, (4.5) necessarily implies $E(\sigma^x) - E(\sigma) \le -2 + 10h$. Note that the equality holds when x is surrounded by four unstable neighboring minus sites such that $S_\sigma(y) = +1$ for all $y \in \partial\{x\}$.

Item 3. Consider $\sigma \in \mathcal{S}$ such that $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda_{\mathrm{u}}^+(\sigma)$ and let $r_\sigma(y) = 1$ if $y \in \Lambda_{\mathrm{u}}^-$ and $r_\sigma(y) = 0$ otherwise. Recall that $\Lambda_{-5}^+(\sigma) = \emptyset$ and (3.3); by exploiting the first part of this lemma we get

$$\sum_{x\in\Lambda_{\mathrm{u}}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) = \sum_{x\in\Lambda_{-1}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) + \sum_{x\in\Lambda_{-3}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) \ge 2|\Lambda_{-1}^+(\sigma)| + 3|\Lambda_{-3}^+(\sigma)|$$

On the other hand, a site in $\Lambda_{+1}^-(\sigma)$ is nearest neighbor of at most three sites in Λ_{u}^+, indeed the number of unstable pluses neighboring such a site can be less than three since some of the pluses can be stable ones, and a site in $\Lambda_{+3}^-(\sigma)$ is nearest neighbor of at most four sites in Λ_{u}^+; then we have

$$\sum_{x\in\Lambda_{\mathrm{u}}^+(\sigma)} \sum_{y\in\partial\{x\}} r_\sigma(y) \le 3|\Lambda_{+1}^-(\sigma)| + 4|\Lambda_{+3}^-(\sigma)|$$

The inequality (4.4) follows trivially from the two bounds above. $\square$

Lemma 4.3 *Suppose $h > 0$ small enough. Let $\sigma \in \mathcal{S}$, suppose $E(\sigma^x) > E(\sigma)$ for any $x \in \Lambda_{\mathrm{u}}^+(\sigma)$. Then*

$$E(\sigma) \ge E(B\sigma) + 4h|\Lambda_{\ge -1}^-(\sigma)| \tag{4.7}$$

Recall that $\Lambda_{\ge -1}^-(\sigma)$ is exactly the set of sites whose associated spin flips under the action of the bootstrap percolation map B, see (3.23).

Proof of Lemma 4.3. To compare $E(\sigma)$ and $E(B\sigma)$ we shall use (2.22) and suitable bounds on $E(\sigma, B\sigma)$ and $E(B\sigma, \sigma)$. Recall (2.19), see also figure 1, and the definition (3.23) of the bootstrap percolation map B; we have that in the forward jump from σ to $B\sigma$ the energy costs are those associated to the flip of the stable minuses with two neighboring pluses and those associated to the permanence of the unstable pluses, more precisely we have

$$E(\sigma, B\sigma) \le E(\sigma) + 2(1-h)|\Lambda_{-1}^-(\sigma)| + 2(1-h)|\Lambda_{-1}^+(\sigma)| + 2(3-h)|\Lambda_{-3}^+(\sigma)| \tag{4.8}$$

On the other hand in the backward jump from $B\sigma$ to σ the energy costs that must be surely paid are those associated to the reverse flipping of the pluses that have been created in the forward jump, more precisely we have

$$E(B\sigma, \sigma) \ge E(B\sigma) + 2(1+h)|\Lambda_{-1}^-(\sigma)| + 2(3+h)|\Lambda_{+1}^-(\sigma)| + 2(5+h)|\Lambda_{+3}^-(\sigma)| \tag{4.9}$$

Note that in (4.9) it is not possible to take advantage from the permanence of the eventual unstable pluses in $B\sigma$, because, as we shall see in the proof of item 2 of the Lemma 4.1, we have $\Lambda_{\mathrm{u}}^+(B\sigma) = \emptyset$.

To complete the proof we have to distinguish two cases. Suppose, first, that $\Lambda^+_{-1}(\sigma) = \Lambda^-_{+3}(\sigma) = \emptyset$; by using (4.8), (4.9), and (2.22) we get

$$E(\sigma) \geq E(B\sigma) + 4h|\Lambda^-_{-1}(\sigma)| - 2(3-h)|\Lambda^+_{-3}(\sigma)| + 2(3+h)|\Lambda^-_{+1}(\sigma)|$$

The bound (4.7) follows noting that in this case $\Lambda^-_{\geq -1}(\sigma) = \Lambda^-_{-1}(\sigma) \cup \Lambda^-_{+1}(\sigma)$ and that (4.4) reduces to $|\Lambda^+_{-3}(\sigma)| \leq |\Lambda^-_{+1}(\sigma)|$. Suppose, now, that either $\Lambda^+_{-1}(\sigma) \neq \emptyset$ or $\Lambda^-_{+3}(\sigma) \neq \emptyset$; the inequality (4.4) implies $|\Lambda^+_{-1}(\sigma)| + 3|\Lambda^+_{-3}(\sigma)| < 3|\Lambda^-_{+1}(\sigma)| + 5|\Lambda^-_{+3}(\sigma)|$. Thus, provided h is small enough, we get also

$$(1-h)|\Lambda^+_{-1}(\sigma)| + (3-h)|\Lambda^+_{-3}(\sigma)| < (3-h)|\Lambda^-_{+1}(\sigma)| + (5-h)|\Lambda^-_{+3}(\sigma)| \tag{4.10}$$

Finally, the bound (4.7) follows easily by using (2.22), (4.8), (4.9), and (4.10). □

Proof of Lemma 4.1. Item 1. The bound (4.2) is proven easily by applying iteratively item 2 of Lemma 4.2.

Item 2. Suppose $\bar{B}\bar{A}\sigma = B^n\bar{A}\sigma$ for some integer n. We first note that by Lemma 4.2 each site $x \in \Lambda^+_{\mathrm{u}}(\bar{A}\sigma)$ has at least two neighboring minuses which are unstable w.r.t. $\bar{A}\sigma$, more precisely $|\partial\{x\} \cap \Lambda^-_{\mathrm{u}}(\bar{A}\sigma)| \geq 2$. Since $\Lambda^-_{\mathrm{u}}(\bar{A}\sigma) \subset \Lambda^-_{\geq -1}(\bar{A}\sigma)$, the minuses in $\partial\{x\} \cap \Lambda^-_{\mathrm{u}}(\bar{A}\sigma)$ flip under the action of B, recall the definition (3.23) of the bootstrap percolation map B; hence $|\partial\{x\} \cap \Lambda^+(B\bar{A}\sigma)| \geq 2$. We then have $\Lambda^+_{\mathrm{u}}(B\bar{A}\sigma) = \emptyset$; in other words all the unstable pluses in $\bar{A}\sigma$ become stable after the application of a single step of the bootstrap percolation.

By definition of the bootstrap percolation map we also have that $\Lambda^+_{\mathrm{u}}(B^i\bar{A}\sigma) = \emptyset$ for any $i = 2, \dots, n$ namely, no site in $\Lambda^+(B^i\bar{A}\sigma)$ is unstable w.r.t. $B^i\bar{A}\sigma$. Note, finally, that for any $x \in \Lambda^+_{\mathrm{u}}(\bar{A}\sigma)$ we have $E(\sigma^x) > E(\sigma)$. The thesis then follows by applying iteratively Lemma 4.3. □

Let $\sigma \in \mathcal{S}$, we refine the estimate (4.2) by considering the plus spins that are flipped by the iterative application of the map A and are associated with sites outside the support of the configuration $\bar{B}\bar{A}\sigma$. More precisely, we define the *branch*

$$L(\sigma) := |\Lambda^+(\sigma) \setminus \Lambda^+(\bar{B}\bar{A}\sigma)| \tag{4.11}$$

namely, the number of pluses outside the rectangles of $\bar{B}\bar{A}\sigma$ which are flipped by the map A, note $L(\sigma) \leq N_A(\sigma)$, see (4.1), and state the following lemma.

Lemma 4.4 *For any $\sigma \in \mathcal{S}$ such that $L(\sigma) \geq 1$, we have that*

$$E(\sigma) - E(\bar{A}\sigma) \geq \begin{cases} 6-2h & \text{if } L(\sigma) = 1 \\ 10-6h+(2-10h)(L(\sigma)-2) & \text{if } L(\sigma) \geq 2 \end{cases} \tag{4.12}$$

Proof of Lemma 4.4. Let $\sigma \in \mathcal{S}$ such that $L(\sigma) = 1$, the set $\Lambda^+(\sigma) \setminus \Lambda^+(\bar{B}\bar{A}\sigma)$ has a unique element x. There exists a natural number j such that $A^{j-1}\sigma(x) = +1$ and $A^j\sigma(x) = -1$. For $y \in \partial\{x\} \cap (\Lambda^+(\bar{B}\bar{A}\sigma))^{\mathrm{c}}$ we have $|S_{A^j\sigma}(y) + h| - |S_{A^{j-1}\sigma}(y) + h| = 2$, while for

$y \in \partial\{x\} \cap \Lambda^+(\bar{B}\bar{A}\sigma)$ we have the trivial bound $|S_{A^j\sigma}(y) + h| - |S_{A^{j-1}\sigma}(y) + h| \geq -2$. Since $|\partial\{x\} \cap (\Lambda^+(\bar{B}\bar{A}\sigma))^{\mathrm{c}}| \geq 3$ and $|\partial\{x\} \cap \Lambda^+(\bar{B}\bar{A}\sigma)| \leq 1$, by using (2.9) we get

$$\begin{aligned} E(A^{j-1}\sigma) - E(A^j\sigma) = 2 - 2h \quad &+ \sum_{y \in \partial\{x\} \cap (\Lambda^+(\bar{B}\bar{A}\sigma))^{\mathrm{c}}} (|S_{A^j\sigma}(y) + h| - |S_{A^{j-1}\sigma}(y) + h|) \\ &+ \sum_{y \in \partial\{x\} \cap \Lambda^+(\bar{B}\bar{A}\sigma)} (|S_{A^j\sigma}(y) + h| - |S_{A^{j-1}\sigma}(y) + h|) \geq 6 - 2h \end{aligned} \tag{4.13}$$

Recall, finally, that by definition the map A decreases the energy, then by (4.13) we have

$$E(\sigma) \geq E(A^{j-1}\sigma) \geq E(A^j\sigma) - 2h + 6 \geq E(\bar{A}\sigma) - 2h + 6$$

and the bound (4.12) follows.

Let now $\sigma \in \mathcal{S}$ such that $L(\sigma) = 2$; the set $\Lambda^+(\sigma) \setminus \Lambda^+(\bar{B}\bar{A}\sigma)$ has two elements x, y. Since $\bar{B}\bar{A}\sigma = \bar{B}\bar{A}\sigma^y$ and $L(\sigma) = 2$, we have $L(\sigma^y) = 1$; by using $\bar{A}\sigma^y = \bar{A}\sigma$ and (4.12) in the already proven case we have that

$$E(\sigma) - E(\bar{A}\sigma) = E(\sigma) - E(\sigma^y) + E(\sigma^y) - E(\bar{A}\sigma) \geq E(\sigma) - E(\sigma^y) + 6 - 2h \tag{4.14}$$

To bound $E(\sigma) - E(\sigma^y)$ we first note that by (2.9) we get

$$E(\sigma) - E(\sigma^y) = -2h - \sum_{z \in B(y)} (|S_\sigma(z) + h| - |S_{\sigma^y}(z) + h|) \tag{4.15}$$

We distinguish, now, two cases. We first suppose that $x \notin B(y)$ namely, the two sites x and y are not nearest neighbors. It is easy to prove that $-(|S_\sigma(y)+h| - |S_{\sigma^y}(y)+h|) = +2$. Moreover, note that the contribution to the sum (4.15) of all the sites in $\partial\{y\} \cap (\Lambda^+(\bar{B}\bar{A}\sigma))^{\mathrm{c}}$ is equal to $+2$ excepted for at most one site whose contribution is equal to $-2h$. Note also that $|\partial\{y\} \cap (\Lambda^+(\bar{B}\bar{A}\sigma))^{\mathrm{c}}| \geq 3$, hence we have that $E(\sigma) - E(\sigma^y) \geq -2h + (2 - 2h) + 2 + 2 - 2h - 2$, where the contribution of the site $\partial\{y\} \cap \Lambda^+(\bar{B}\bar{A}\sigma)$, which possibly exists, has been bounded trivially by -2. The bound (4.12) follows immediately.

Suppose, now, that $x \in B(y)$ namely, the two sites x and y are adjacent. The only not trivial case, see Figure 6, is the one in which both the sites x and y are at distance one from the set $\Lambda^+(\bar{B}\bar{A}\sigma)$. Since the plus spins associated to x and y are flipped by the iterative application of the map A to σ, the spin associated to at least one of the two sites in $\partial\{x, y\} \cap \Lambda^+(\bar{B}\bar{A}\sigma)$ is equal to -1, see Figure 6. Without loss of generality we let $\partial\{y\} \cap \Lambda^+(\bar{B}\bar{A}\sigma) = \{y'\}$ and $\sigma(y') = -1$. It is easy to prove that $-(|S_\sigma(y)+h| - |S_{\sigma^y}(y)+h|) = +2$, $-(|S_\sigma(x) + h| - |S_{\sigma^y}(x) + h|) \geq -2h$, $-(|S_\sigma(y') + h| - |S_{\sigma^y}(y') + h|) \geq -2$, and $-(|S_\sigma(z) + h| - |S_{\sigma^y}(z) + h|) = 2$ for each $z \in \partial\{y\} \setminus \{x, y'\}$. Hence, by using (4.15) we get

$$E(\sigma) - E(\sigma^y) \geq -2h + 2 - 2h - 2 + 2 + 2 = 4 - 4h \tag{4.16}$$

The bound (4.12) follows by (4.16) and (4.14).

Let, finally, $\sigma \in \mathcal{S}$ such that $L(\sigma) \geq 3$. Let i a suitable integer such that $L(A^i\sigma) = 2$. The bound (4.12) follows easily by using the Lemma 4.1 and (4.12) applied to $A^i\sigma$. $\square$

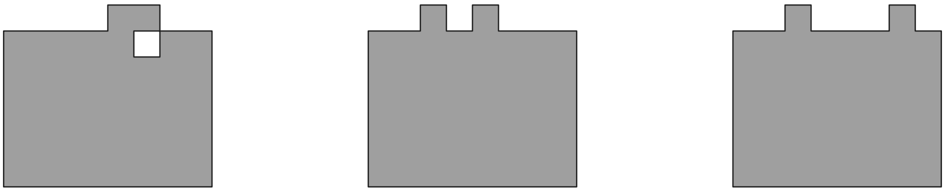

Figure 6: The three cases studied in the proof of the Lemma 4.4; on the left the not trivial one.

4.2. Energy estimates for rectangular droplets

We first state and prove the following Lemma on some simple geometrical properties of rectangles on the lattice.

Lemma 4.5 *Let Q_{l_i,m_i}, for $i = 1,\dots,n$, be pairwise disjoint rectangles with sides $l_i, m_i \in \mathbb{N}\setminus\{0\}$, such that $\ell_i \le m_i$ for $i = 1,\dots,n$, and semi–perimeter $p := \sum_i^n(\ell_i + m_i)$.*

1. *We have*

$$\frac{1}{4}p^2 \ge \sum_{i=1}^{n} l_i\, m_i \tag{4.17}$$

2. *If there exists a positive integer k such that $\ell_i \le k-1$ and $m_i \le k$ for all $i = 1,\dots,n$, we have*

$$\sum_{i=1}^{n} \ell_i m_i \le \frac{1}{2}kp - \frac{1}{2}\sum_{i=1}^{n} m_i \tag{4.18}$$

3. *If $n \ge 2$ and $l_i \ge 2$ then*

$$\frac{1}{4}p^2 \ge \sum_{i=1}^{n} l_i\, m_i + p \tag{4.19}$$

Proof of Lemma 4.5. Item 1: we have

$$\frac{1}{4}p^2 = \frac{1}{4}\Big(\sum_{i=1}^{n}(l_i+m_i)\Big)^2 \ge \frac{1}{4}\sum_{i=1}^{n}(l_i+m_i)^2 = \frac{1}{4}\sum_{i=1}^{n}(l_i-m_i)^2 + \sum_{i=1}^{n} l_i\, m_i \ge \sum_{i=1}^{n} l_i\, m_i$$

Item 2: we have

$$\sum_{i=1}^{n}\ell_i m_i = 2\sum_{i=1}^{n}\frac{1}{2}l_i m_i \le \frac{1}{2}\sum_{i=1}^{n}(k-1)m_i + \frac{1}{2}\sum_{i=1}^{n}\ell_i k \le \frac{1}{2}k\sum_{i=1}^{n}(\ell_i+m_i) - \frac{1}{2}\sum_{i=1}^{n} m_i$$

which implies (4.18). Item 3: note that

$$
\begin{aligned}
\Big(\frac{1}{2}\sum_{i=1}^{n}(l_i+m_i)\Big)^2-\sum_{i}^{n}l_i\,m_i &= \frac{1}{4}\Big(\Big(\sum_{i=1}^{n}(l_i+m_i)\Big)^2-4\sum_{i}^{n}l_i\,m_i\Big)\\
&= \frac{1}{4}\Big(\sum_{i}^{n}(l_i+m_i)^2-4\sum_{i=1}^{n}l_i\,m_i+\sum_{i\neq j}(l_i+m_i)(l_j+m_j)\Big)\\
&= \frac{1}{4}\Big(\sum_{i=1}^{n}(l_i-m_i)^2+\sum_{i\neq j}(l_i+m_i)(l_j+m_j)\Big)\\
&\geq \frac{4}{4}\sum_{j=1}^{n}(l_j+m_j)=p
\end{aligned}
$$

where in the second step we used $n\geq 2$ and in the last step $l_i\wedge m_i\geq 2$. □

We introduce the notion of semi–perimeter of a multi–rectangular droplet. Let $n\geq 1$ and $\ell_1,m_1,\dots,\ell_n,m_n$ integers such that $2\leq \ell_1,m_1,\dots,\ell_n,m_n\leq L-2$, $\sigma\in\mathcal{S}$ a n–rectangular droplet with sides $\ell_1,m_1,\dots,\ell_n,m_n$, we let

$$p(\sigma):=\sum_{i=1}^{n}(\ell_i+m_i) \tag{4.20}$$

be the *semi–perimeter* of the multi–rectangular droplet σ.

Lemma 4.6 *Let ℓ,m two integers such that $2\leq\ell\leq m\leq L-2$ and $\sigma\in\mathcal{S}$ a rectangular droplet with sides ℓ and m. If $\ell\leq\lambda-1$, we have*

$$E(\sigma)-E(-\underline{1})>8\ell>0 \tag{4.21}$$

If $\ell\leq\lambda-1$ and $m\geq\lambda+1$, we have

$$E(\sigma)-E(\mathcal{R})\geq 4h(1-\delta_h)>0 \tag{4.22}$$

where we recall $\mathcal{R}$ has been defined above Proposition 3.2 and δ_h below (2.11).

Moreover, for $n\geq 1$ integer, for any n–rectangular droplet $\eta\in\mathcal{S}$ with sides $2\leq\ell_i\leq m_i$ such that $\ell_i\leq\lambda-1$ and $m_i\leq\lambda$ for $i=1,\dots,n$, we have that

$$E(\eta)-E(-\underline{1})>(4-2h)p(\eta)+\frac{1}{2}\sum_{i=1}^{n}m_i \tag{4.23}$$

Proof of Lemma 4.6. Suppose $\ell\leq\lambda-1$: by using (2.17) we have $E(\sigma)-E(-\underline{1})=-4h\ell m+8(\ell+m)=(8-4h\ell)+8\ell$; since $\ell\leq\lambda-1$ the thesis follows. Suppose $\ell\leq\lambda-1$ and $m\geq\lambda+1$, by using (2.17) we have $E(\sigma)-E(\mathcal{R})=4h(m-\lambda)[(\lambda-\ell)-\delta_h]$, which implies (4.22).

We finally prove (4.23). Recall that by hypothesis $\ell_i \leq \lambda - 1$ and $m_i \leq \lambda$ for any $i = 1, \ldots, n$; by definition of multi–rectangular droplets and by using (4.18) with $k = \lambda$, we have

$$|\Lambda^+(\eta)| \leq \frac{\lambda}{2} p(\eta) - \frac{1}{2} \sum_{i=1}^{n} m_i \tag{4.24}$$

Now, by using (2.17), (4.20), (4.18), and the fact that the support of a multi–rectangular droplet is made of pairwise not interacting rectangles, we have that

$$E(\eta) - E(-\underline{1}) = -4h|\Lambda^+(\eta)| + 8p(\eta) \geq p(\eta)(8 - 2h\lambda) + \frac{1}{2} \sum_{i=1}^{n} m_i$$

which implies (4.23) since $\lambda < (2/h) + 1$. $\square$

4.3. Relations between configurations in $\mathcal{G}$ and in $\mathcal{G}^c$

Consider $\sigma \in \mathcal{G}$ and $\eta \in \mathcal{G}^c$, in Lemma 4.7 we state a property relating the pluses in η to those in $\bar{B}\bar{A}\sigma$ and we bound from below the transition rate $\Delta(\sigma, \eta)$, see(2.20).

Lemma 4.7 *Let $\sigma \in \mathcal{G}$ and $\eta \notin \mathcal{G}$,*

1. *we have*

$$|\Lambda^+(\eta) \setminus \Lambda^+(\bar{B}\bar{A}(\sigma))| \geq 2 \tag{4.25}$$

2. *we have*

$$\Delta(\sigma, \eta) \geq \begin{cases} 12 - 4h & \text{for } L(\sigma) = 0 \\ 4 - 4h & \text{for } L(\sigma) = 1 \end{cases} \tag{4.26}$$

Proof of Lemma 4.7. Item 1: the item follows from the definition of the subcritical set $\mathcal{G}$. Indeed, if $|\Lambda^+(\eta) \setminus \Lambda^+(\bar{B}\bar{A}(\sigma))| \leq 1$, we have that under the map A the positive spin outside $\Lambda^+(\bar{B}\bar{A}\sigma)$ is flipped, so that $\Lambda^+(\bar{B}\bar{A}\eta) \subseteq \Lambda^+(\bar{B}\bar{A}\sigma)$. Hence $\eta \in \mathcal{G}$, that is a contradiction.

Item 2: from (2.20) we get

$$\Delta(\sigma, \eta) = 2 \sum_{\substack{z \in \Lambda: \\ \eta(z)\, S_\sigma(z) < 0}} |S_\sigma(z) + h| \geq 2 \sum_{\substack{z \in \Lambda \setminus \Lambda^+(\bar{B}\bar{A}\sigma): \\ \eta(z)\, S_\sigma(z) < 0}} |S_\sigma(z) + h| \tag{4.27}$$

If $L(\sigma) = 0$, by (4.25),(4.27), the thesis follows. Indeed, in the r.h.s of (4.27) there are at least two terms corresponding to sites x and y such that $\eta(x) = \eta(y) = 1$, and $S_\sigma(x) \leq -3, S_\sigma(y) \leq -3$. If $L(\sigma) = 1$, from (4.25) there exist two sites

$$\{x, y\} \subseteq \Lambda^+(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\sigma) \tag{4.28}$$

Note that $S_\sigma(x) \leq -1$ and $S_\sigma(y) \leq -1$ since $L(\sigma) = 1$. From (4.27) we have

$$\Delta(\sigma, \eta) \geq 2(1 - h) + 2(1 - h) \tag{4.29}$$

and the thesis follows, see also Figure 1. $\square$

4.4. Proof of the Proposition 3.2

Let $\sigma \in \mathcal{S}$ and suppose $\bar{B}\bar{A}\sigma \neq -\underline{1}$, there exist $n(\sigma) \in \mathbb{N} \setminus \{0\}$, $\ell_i(\sigma), m_i(\sigma)$ integers larger than 2, and $x_i(\sigma) \in \Lambda$ for $i = 1, \dots, n(\sigma)$ such that

$$\Lambda^+(\bar{B}\bar{A}\sigma) = \bigcup_{i=1}^{n(\sigma)} Q_{\ell_i(\sigma), m_i(\sigma)}(x_i(\sigma))$$

If $\bar{B}\bar{A}\sigma = -\underline{1}$ we shall understand $n(\sigma) = 1$, $\ell_1(\sigma) = m_1(\sigma) = 0$, and $p(\sigma) = 0$, see also (4.20). Let $\sigma \in \mathcal{S}$, we order the droplets in $\Lambda^+(\bar{B}\bar{A}\sigma)$ so that $\ell_i(\sigma) \wedge m_i(\sigma) \geq \lambda$ for $i = 1, \dots, k(\sigma)$ and $\ell_i(\sigma) \wedge m_i(\sigma) \leq \lambda - 1$ for $i = k(\sigma)+1, \dots, n(\sigma)$; note that for $\sigma \in \mathcal{G}$ we have $k(\sigma) = 0$, while for $\sigma \in \mathcal{G}^c$ we have $k(\sigma) \geq 1$. For the sake of simplicity, for $\sigma \in \mathcal{G}^c$ in the sequel we shall let $r_i(\sigma) := \ell_i(\sigma) - \lambda$ and $q_i(\sigma) := m_i(\sigma) - \lambda$ for $i = 1, \dots, k(\sigma)$.

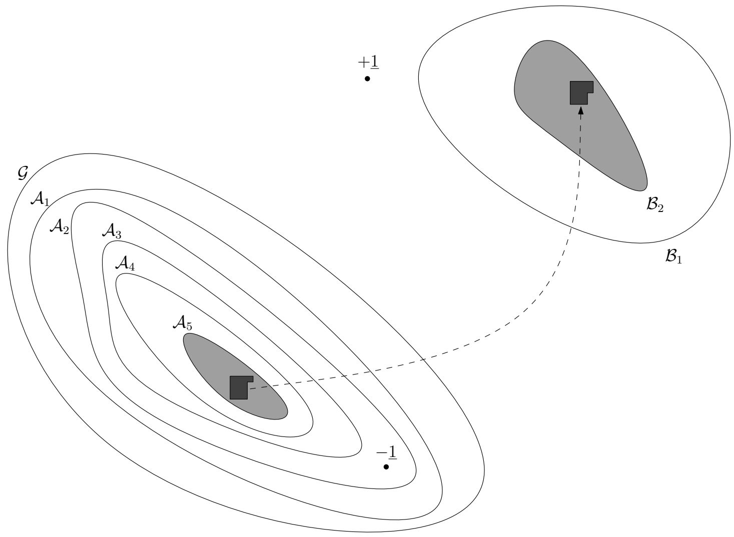

Figure 7: Restricted sets on which we evaluate $E(\eta, \zeta)$ in the proof of item 2 of Proposition 3.2.

Before starting the proof of the Proposition 3.2 we sketch the main idea. We shall define the subsets of the configuration space $\mathcal{A}_5 \subset \mathcal{A}_4 \subset \mathcal{A}_3 \subset \mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{G}$, $\mathcal{B}_2 \subset \mathcal{B}_1 \subset \mathcal{G}^c$, and reduce the proof to the computation of $E(\eta, \zeta)$ for $\eta \in \mathcal{A}_5$ and $\zeta \in \mathcal{B}_2$, as depicted in Figure 7. We recall (4.20), (4.11), (4.1), and let

$$\begin{array}{ll} \mathcal{A}_1 := \{\sigma \in \mathcal{G} :\, \ell_i(\sigma) \vee m_i(\sigma) \leq \lambda \text{ for } i = 1, \dots, n(\sigma)\} & \mathcal{A}_4 := \{\sigma \in \mathcal{A}_3 :\, n(\sigma) = 1\} \\ \mathcal{A}_2 := \{\sigma \in \mathcal{A}_1 :\, p(\sigma) \leq 2\lambda + 4,\, L(\sigma) \leq 4\lambda + 42\} & \mathcal{A}_5 := \{\sigma \in \mathcal{A}_4 :\, p(\sigma) = 2\lambda - 1\} \\ \mathcal{A}_3 := \{\sigma \in \mathcal{A}_2 :\, p(\sigma) \geq 2\lambda - 50\} & \end{array} \tag{4.30}$$

and

$$
\begin{aligned}
&\mathcal{B}_1 := \{\sigma \in \mathcal{G}^{\mathrm{c}} :\ \ell_i(\sigma), m_i(\sigma) \le L-2 \text{ for } i=1,\dots,n(\sigma)\} \\
&\mathcal{B}_2 := \{\sigma \in \mathcal{B}_1 :\ 4hN_B(\sigma) - 4h\sum_{i=1}^{k(\sigma)}(r_i(\sigma)+q_i(\sigma)+r_i(\sigma)q_i(\sigma)) \le 10-2h\}
\end{aligned}
\tag{4.31}
$$

To bound $E(\eta,\zeta)$, for $\eta \in \mathcal{G}$ and $\zeta \in \mathcal{G}^{\mathrm{c}}$ we shall use the following identity, which is a straightforward consequence of the definition (2.20),

$$
\begin{aligned}
E(\eta,\zeta) - E(-\underline{1}) \;=\; & [E(\eta) - E(\bar{A}\eta)] + [E(\bar{A}\eta) - E(\bar{B}\bar{A}\eta)] \\
& + [E(\bar{B}\bar{A}\eta) - E(-\underline{1})] + \Delta(\eta,\zeta)
\end{aligned}
\tag{4.32}
$$

Depending on the choice of η, the different terms in the r.h.s. of the identity (4.32) will be properly bounded in order to get the thesis.

Proof of Proposition 3.2. Item 1. The proof is an immediate application of the definition of the set $\mathcal{G}$, see Subsection 3.2.

Items 2. *Step 1.* Let $\eta \in \mathcal{G} \setminus \mathcal{A}_1$ and $\zeta \in \mathcal{G}^{\mathrm{c}}$. There exists $i \in \{1,\dots,n(\eta)\}$ such that $l_i(\eta) \vee m_i(\eta) \ge \lambda+1$; hence, by using (4.32), (4.3), $N_B(\eta) \ge 0$, (4.21), and (4.22) we get

$$
E(\eta,\zeta) - E(-\underline{1}) \ge [E(\eta) - E(\bar{A}\eta)] + [E(\mathcal{R}) - E(-\underline{1})] + \Delta(\eta,\zeta) \tag{4.33}
$$

Now, if $L(\eta)=0$, by using (4.33), (4.2), $N_A(\eta) \ge 0$, (4.26), and (3.24), we get

$$
E(\eta,\zeta) - E(-\underline{1}) > [E(\mathcal{R}) - E(-\underline{1})] + 12 - 4h > \Gamma
$$

On the other hand, if $L(\eta) \ge 1$ by using (4.33), (4.12), (4.26), and (3.24), we get

$$
E(\eta,\zeta) - E(-\underline{1}) > 6 - 2h + [E(\mathcal{R}) - E(-\underline{1})] + 4 - 4h > \Gamma
$$

Step 2. Let $\eta \in \mathcal{A}_1 \setminus \mathcal{A}_2$ and $\zeta \in \mathcal{G}^{\mathrm{c}}$. By using (4.32), (4.3), and $N_B(\eta) \ge 0$, we get

$$
E(\eta,\zeta) - E(-\underline{1}) \ge [E(\eta) - E(\bar{A}\eta)] + [E(\bar{B}\bar{A}\eta) - E(-\underline{1})] + \Delta(\eta,\zeta) \tag{4.34}
$$

Now, suppose $p(\eta) \ge 2\lambda+5$, by using (4.34), (4.23), $\Delta(\eta,\zeta) \ge 0$, the definition (2.11), and (2.13), for $h>0$ small enough we get

$$
E(\eta,\zeta) - E(-\underline{1}) > (4-2h)(2\lambda+5) > 8\lambda + 12 - 14h > \Gamma
$$

Suppose, finally, $L(\eta) \ge 4\lambda+43$. If $\bar{B}\bar{A}\eta \ne -\underline{1}$, by using (4.21) we get $E(\bar{B}\bar{A}\eta) - E(-\underline{1}) \ge 0$; note that this bound holds trivially also in the case $\bar{B}\bar{A}\eta = -\underline{1}$. Hence, by using this bound, (4.34), (4.12), and (2.13), we get

$$
E(\eta,\zeta) - E(-\underline{1}) > 10 - 6h + (2-10h)(4\lambda+43) > \Gamma
$$

Step 3. Let $\eta \in \mathcal{A}_2$ and $\zeta \in \mathcal{G}^{\mathrm{c}} \setminus \mathcal{B}_1$. There exists $i \in \{1,\dots,k(\zeta)\}$ such that $\ell_i(\zeta) \vee m_i(\zeta) \ge L-2$. Since $\eta \in \mathcal{A}_2$ we have that $p(\eta) \le 2\lambda+4$ and $L(\eta) \le 4\lambda+42$, then by using (4.18) with $k = \lambda$ we have $|\Lambda^+(\eta)| \le |\Lambda^+(\bar{B}\bar{A}\eta)| + L(\eta) \le \lambda p(\eta)/2 + L(\eta) \le \lambda^2 + 6\lambda + 42$. Given the magnetic field $h > 0$, the number of plus spins in η is bounded by a finite number; then we can choose $L = L(h)$ so large that there exist an horizontal and a vertical stripe winding around the torus with arbitrarily large width and such that $\eta(x)$ is equal to -1 for each x in such two stripes. Since in $\bar{B}\bar{A}\zeta$ there exists a rectangular droplet of pluses with one of the two side lengths larger or equal to $L-2$, we then can choose L so large that $\Delta(\eta, \zeta) > \Gamma$. By using, finally, (2.20) we get $E(\eta, \zeta) - E(-\underline{1}) > \Gamma$, once we remark that $E(\eta) - E(-\underline{1}) \ge E(\bar{B}\bar{A}\eta) - E(-\underline{1}) \ge 0$.

Step 4. Let $\eta \in \mathcal{A}_2$ and $\zeta \in \mathcal{B}_1 \setminus \mathcal{B}_2$. By using Lemma 4.1 and $N_A(\zeta) \ge 0$, we have the bound

$$E(\zeta) - E(-\underline{1}) \ge E(\bar{B}\bar{A}\zeta) - E(-\underline{1}) + 4hN_B(\zeta) \tag{4.35}$$

By (2.17) and (2.12) it follows

$$\begin{aligned} E(\bar{B}\bar{A}\zeta) - E(-\underline{1}) &= -4h\sum_{i=1}^{n(\zeta)}(\lambda + r_i(\zeta))(\lambda + q_i(\zeta)) + 8\sum_{i=1}^{n(\zeta)}(2\lambda + r_i(\zeta) + q_i(\zeta)) \\ &= n(\zeta)(\Gamma - 10 + 6h) - \sum_{i=1}^{n(\zeta)}(r_i(\zeta) + q_i(\zeta))(4h\lambda - 8) \\ &\quad -4n(\zeta)(h\lambda - 2) - 4h\sum_{i=1}^{n(\zeta)} r_i(\zeta)q_i(\zeta) \\ &> (\Gamma - 10 + 2h) - 4h\sum_{i=1}^{n(\zeta)}(r_i(\zeta) + q_i(\zeta) + r_i(\zeta)q_i(\zeta)) \end{aligned} \tag{4.36}$$

where in the last inequality we used (2.11) and the fact that $\Gamma > 10 - 6h$. Hence by (4.35) and (4.36) we have

$$E(\zeta) - E(-\underline{1}) \ge \Gamma - (10 - 2h) + 4hN_B(\zeta) - 4h\sum_{i=1}^{n(\zeta)}(r_i(\zeta) + q_i(\zeta) + r_i(\zeta)q_i(\zeta))$$

Hence, since $\zeta \in \mathcal{B}_2 \setminus \mathcal{B}_1$ we get $E(\zeta) - E(-\underline{1}) > \Gamma$; finally, by (2.19) we get $E(\eta, \zeta) - E(-\underline{1}) > \Gamma$.

Step 5. Let $\eta \in \mathcal{A}_2 \setminus \mathcal{A}_3$ and $\zeta \in \mathcal{B}_2$. We note now that $E(\bar{B}\bar{A}\eta) - E(-\underline{1}) \ge 0$, which is trivial if $\Lambda^+(\bar{B}\bar{A}\eta) = -\underline{1}$, otherwise it follows immediately from (4.21). By using this bound, (4.32), Lemma 4.1, $N_A(\eta) \ge 0$, $N_B(\eta) \ge 0$, we get

$$E(\eta, \zeta) - E(-\underline{1}) \ge \Delta(\eta, \zeta) \tag{4.37}$$

We find, now, a lower bound to $\Delta(\eta, \zeta)$ by multiplying the the minimum *quantum* $2(1-h)$, see Figure 1, times the number of flips against the drift in the jump from η to ζ. More precisely,

$$\begin{aligned} \Delta(\eta, \zeta) &\ge 2(1-h)\,|\{x \in \Lambda : \eta(x)S_\eta(x) > 0,\, \eta(x)\zeta(x) < 0\}| \\ &\ge 2(1-h)(|\Lambda^+(\zeta)| - |\bar{\Lambda}(\eta)|) \end{aligned} \tag{4.38}$$

with

$$\bar{\Lambda}(\eta) := \Lambda^+(\bar{B}\bar{A}\eta) \cup \overline{\Lambda^+(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)} \tag{4.39}$$

where we recall the closure of a subset of the lattice has been defined in Subsection 2.1.

Recalling that the application of the map A does not add pluses, the number of plus spins in the configuration ζ can be bounded from below by adding the number of pluses in $\bar{B}\bar{A}\zeta$ to the branch $L(\zeta)$ of ζ and subtracting the number of pluses $N_B(\zeta)$ added by the boostrap map B. Namely, we have

$$|\Lambda^+(\zeta)| \geq n(\zeta)\lambda^2 + \lambda \sum_{i=1}^{n(\zeta)} (r_i(\zeta) + q_i(\zeta)) + \sum_{i=1}^{n(\zeta)} r_i(\zeta) q_i(\zeta) - N_B(\zeta) + L(\zeta)$$

Now, by using that $\zeta \in \mathcal{B}_2$, (2.11), and $L(\zeta) \geq 0$, we get

$$\begin{aligned} |\Lambda^+(\zeta)| &\geq \lambda^2 + \sum_{i=1}^{n(\zeta)} (\lambda(r_i(\zeta) + q_i(\zeta)) - r_i(\zeta) - q_i(\zeta)) - \frac{10 - 2h}{4h} + L(\zeta) \\ &\geq \lambda^2 - \frac{5}{4}\lambda + \sum_{i=1}^{n(\zeta)} (\lambda - 1)(r_i(\zeta) + q_i(\zeta)) \geq \lambda^2 - \frac{5}{4}\lambda \end{aligned} \tag{4.40}$$

where we also used $\lambda - 1 \geq 0$.

We next bound from above $|\bar{\Lambda}(\eta)|$. We first note that by using (4.39) and (4.11) we get

$$|\bar{\Lambda}(\eta)| \leq |\Lambda^+(\bar{B}\bar{A}\eta)| + 5L(\eta) \tag{4.41}$$

Now suppose that $\Lambda^+(\bar{B}\bar{A}\eta) \neq -\underline{1}$, by using (4.18) with $k = \lambda$ and exploiting $\eta \in \mathcal{A}_2$ we conclude

$$|\bar{\Lambda}(\eta)| \leq \frac{1}{2}\lambda p(\eta) + 20\lambda + 210 \tag{4.42}$$

Suppose, on the other hand, that $\Lambda^+(\bar{B}\bar{A}\eta) = -\underline{1}$. By using (4.41) we get $|\bar{\Lambda}(\eta)| \leq 5L(\eta) \leq 20\lambda + 210$, hence the bound (4.42) holds since in this case $p(\eta) = 0$.

We finally bound $\Delta(\eta, \zeta)$ by using the preliminary inequalities (4.38), (4.40), and (4.42); we have

$$\Delta(\eta, \zeta) \geq 2(1 - h)\Big[\lambda^2 - \frac{85}{4}\lambda - \frac{1}{2}\lambda p(\eta) - 210\Big] \tag{4.43}$$

Recall $\eta \in \mathcal{A}_2 \setminus \mathcal{A}_3$, then $p(\eta) \leq 2\lambda - 51$; hence by using (4.37), (4.43), and (2.13) we get

$$E(\eta, \zeta) - E(-\underline{1}) > \Gamma + \frac{1}{h} - \frac{53}{2} + O(h) > \Gamma$$

where in the last inequality we have chosen $h > 0$ small enough.

Step 6. Let $\eta \in \mathcal{A}_3 \setminus \mathcal{A}_4$ and $\zeta \in \mathcal{B}_2$. By using (4.32), Lemma 4.1, $N_A(\eta) \geq 0$, $N_B(\eta) \geq 0$, (2.17), and $\Delta(\eta, \zeta) \geq 0$, we get

$$E(\eta, \zeta) - E(-\underline{1}) \geq E(\bar{B}\bar{A}\eta) - E(-\underline{1}) = -4h|\Lambda^+(\bar{B}\bar{A}\eta)| + p(\eta) \tag{4.44}$$

Now, since $\eta \in \mathcal{A}_3 \setminus \mathcal{A}_4$, we can use (4.19) to obtain

$$E(\eta, \zeta) - E(-\underline{1}) \geq -h(p(\eta))^2 + (4h + 8)p(\eta) \tag{4.45}$$

Finally, by the properties of the parabola on the right–hand side of (4.45) and recalling that for $\eta \in \mathcal{A}_3 \setminus \mathcal{A}_4$ the semi–perimeter satisfies the bounds $2\lambda - 50 \leq p(\eta) \leq 2\lambda + 4$, it is immediate to prove that the parabola attains its minimum at $p(\eta) = 2\lambda - 50$; hence, by using (4.45) and (2.12) we get $E(\eta, \zeta) - E(-\underline{1}) > \Gamma$ for $h > 0$ small enough.

Step 7. Let $\eta \in \mathcal{A}_4 \setminus \mathcal{A}_5$ and $\zeta \in \mathcal{B}_2$. By using (4.32), (4.3), and $N_B(\eta) \geq 0$, we get the bound

$$E(\eta, \zeta) - E(-\underline{1}) \geq [E(\eta) - E(\bar{A}\eta)] + [E(\bar{B}\bar{A}\eta) - E(-\underline{1})] + \Delta(\eta, \zeta) \tag{4.46}$$

Since $\eta \in \mathcal{A}_4 \setminus \mathcal{A}_5$, we have that $n(\eta) = 1$ and then $2\lambda - 50 \leq p(\eta) \leq 2\lambda - 2$. We repeat, now, the same argument used at Step 6, but, since $n(\eta) = 1$, we have to use (4.17) instead of (4.19); we then get

$$E(\eta, \zeta) - E(-\underline{1}) \geq [E(\eta) - E(\bar{A}\eta)] + [\Gamma - 10 + O(h)] + \Delta(\eta, \zeta) \tag{4.47}$$

Moreover, since $n(\eta) = 1$ and $p(\eta) \leq 2\lambda - 2$, by using the same arguments developed in the proof of (4.25) we get

$$|\Lambda^+(\zeta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| \geq 3 \tag{4.48}$$

To complete the proof of the Step 7, we distinguish four cases by means of the parameter $L(\eta)$. Consider, first, the case $L(\eta) \geq 3$; by using (4.47), (4.12), and $\Delta(\eta, \zeta) \geq 0$ it follows immediately $E(\eta, \zeta) - E(-\underline{1}) > \Gamma$.

Consider, now, the case $L(\eta) = 2$. We first note that by using (4.47) and (4.12) we get

$$E(\eta, \zeta) - E(-\underline{1}) \geq 10 - 6h + [\Gamma - 10 + O(h)] + \Delta(\eta, \zeta) \geq \Gamma + \Delta(\eta, \zeta) + O(h) \tag{4.49}$$

The thesis $E(\eta, \zeta) - E(-\underline{1}) > \Gamma$ will then be proven once we shall have obtained the bound $\Delta(\eta, \zeta) \geq 2(1 - h)$.

To prove such a bound we note that there exist $x, y \in \Lambda^+(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)$; since $x, y \in \Lambda \setminus \Lambda^+(\bar{B}\bar{A}\eta)$, by the definition of the two maps A and B it follows that they cannot be both stable w.r.t. η, see Section 3. If one of the two sites x and y, say x, is stable w.r.t. η, it is immediate to prove that $x \in \partial\Lambda^+(\bar{B}\bar{A}\eta)$ and $\{y\} = \partial\{x\} \setminus \Lambda^+(\bar{B}\bar{A}\eta)$. Since x and y are nearest neighbors, it follows that there exist no site in $\Lambda^-(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)$ which is unstable w.r.t. η; hence, by using (4.48) and (2.20) it follows that $\Delta(\eta, \zeta) \geq 2(1 - h)$. We consider now the case when both x and y are unstable w.r.t. η. Suppose either $\zeta(x) = +1$ or $\zeta(y) = +1$, from (2.20) we have $\Delta(\eta, \zeta) \geq 2(1 - h)$; on the other hand if $\zeta(x) = \zeta(y) = -1$, it is easy to see that, since $L(\eta) = 2$, we have $|\Lambda^-_{\mathrm{u}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| \leq 1$, where we recall the definition (3.3). Then, by using (4.48) it follows that $\Delta \geq 2(1 - h)$.

Consider, now, the case $L(\eta) = 1$. We first note that by using (4.47) and (4.12) we get

$$E(\eta, \zeta) - E(-\underline{1}) \geq 6 - 2h + [\Gamma - 10 + O(h)] + \Delta(\eta, \zeta) \geq \Gamma - 4 + \Delta(\eta, \zeta) + O(h) \tag{4.50}$$

The thesis $E(\eta,\zeta) - E(-\underline{1}) > \Gamma$ will then be proven once we shall have obtained the bound $\Delta(\eta,\zeta) \geq 3 \cdot 2(1-h)$.

To prove such a bound we let x the site such that $\{x\} := \Lambda^+(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)$. Suppose $\zeta(x) = +1$, since x is unstable w.r.t. η, by (2.20) and (4.48), we have $\Delta(\eta,\zeta) \geq 2(1-h) + 2(1-h) + 2(1-h)$. On the other hand if $\zeta(x) = -1$, since $|\Lambda^-_{\mathrm{u}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| = 0$, by (4.48) it follows that $\Delta(\eta,\zeta) \geq 2(1-h) + 2(1-h) + 2(3-h)$.

Consider, finally, the case $L(\eta) = 0$. Recall (2.20), the condition (4.48) implies that $\Delta(\eta,\zeta) \geq 3 \cdot 2(3-h)$. Hence by using, also, (4.47), (4.2), and $N_A(\eta) \geq 0$, we get

$$E(\eta,\zeta) - E(-\underline{1}) \geq \Gamma - 10 + O(h) + 3 \cdot 2(3-h) > \Gamma$$

Step 8. Let $\eta \in \mathcal{A}_5$ and $\zeta \in \mathcal{B}_2$. We remark that, since $p(\eta) = 2\lambda - 1$ and $\ell_1 \vee m_1 \leq \lambda$, we have $\bar{B}\bar{A}\eta \in \mathcal{R}$. Hence, by using (4.32), (4.3), and (3.24) we get

$$E(\eta,\zeta) - E(-\underline{1}) \geq [E(\eta) - E(\bar{A}\eta)] + 4hN_B(\eta) + [\Gamma - 10 + 6h] + \Delta(\eta,\zeta) \tag{4.51}$$

To complete the proof of the Step 8, we distinguish four cases by means of the parameter $L(\eta)$. Consider, first, the case $L(\eta) \geq 3$; by using (4.51), (4.12), $N_B(\eta) \geq 0$, and $\Delta(\eta,\zeta) \geq 0$, we get

$$E(\eta,\zeta) - E(-\underline{1}) \geq 12 - 16h + \Gamma - 10 + 6h > \Gamma$$

Consider, now, the case $L(\eta) = 2$; we let x, y be the two sites in $\Lambda^+(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)$. We first note that by using the inequalities (4.51) and (4.12) we get the bound

$$E(\eta,\zeta) - E(-\underline{1}) \geq 4hN_B(\eta) + \Gamma + \Delta(\eta,\zeta) \tag{4.52}$$

Suppose, first, $N_B(\eta) \geq 1$; by (4.52) and $\Delta(\eta,\zeta) \geq 0$ we immediately get $E(\eta,\zeta) - E(-\underline{1}) > \Gamma$. We are then left with the case $N_B(\eta) = 0$ namely, $\Lambda^+(\eta) \supset \Lambda^+(\bar{B}\bar{A}\eta)$; by using (4.52), the thesis $E(\eta,\zeta) - E(-\underline{1}) > \Gamma$ will be proven once we shall have obtained the bound $\Delta(\eta,\zeta) \geq 2(1-h)$.

We note that $|\Lambda^+_{\mathrm{s}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| \leq 1$, indeed if by absurdity x and y belonged both to $\Lambda^+_{\mathrm{s}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)$, then it should necessarily be $x, y \in \partial\Lambda^+(\bar{B}\bar{A}\eta)$ and $\mathrm{d}(x,y) = 1$ namely, there would be a double protuberance added to the $\lambda \times (\lambda - 1)$ rectangle of pluses which is present in η. Hence, we would have $\eta \in \mathcal{C} \subset \mathcal{G}^{\mathrm{c}}$, which is absurd.

Suppose $|\Lambda^+_{\mathrm{s}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| = 1$ and let x be the site in $\Lambda^+_{\mathrm{s}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)$; since x is stable w.r.t. η we must necessarily have $x \in \partial\Lambda^+(\bar{B}\bar{A}\eta)$ and $y \in \partial\{x\} \setminus \overline{\Lambda^+(\bar{B}\bar{A}\eta)}$; note also that this implies $|\Lambda^-_{\mathrm{u}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| = 0$. Thus, for $\zeta(y) = +1$ in the sum in (2.20) there is at least the term corresponding to y, then we have $\Delta(\eta,\zeta) \geq 2(1-h)$. On the other hand, if it were $\zeta(y) = -1$, recalling (4.25) we would have that in the sum in (2.20) there is at least a term corresponding to the flip of the spin associated with a site in $\Lambda^-(\eta)$ which is stable w.r.t. η, hence we would have $\Delta(\eta,\zeta) \geq 2(1-h)$. Suppose, finally, $|\Lambda^+_{\mathrm{s}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| = 0$; it is immediate to prove that $|\Lambda^-_{\mathrm{u}}(\eta) \setminus \Lambda^+(\bar{B}\bar{A}\eta)| \leq 1$. Then we get $\Delta(\eta,\zeta) \geq 2(1-h)$, since from (4.25) it follows that in the sum in (2.20) there is at least a term corresponding to the persistence of the spin associated with a site in $\Lambda^+_{\mathrm{u}}(\eta)$ or to the flip of the spin associated with a site in $\Lambda^-(\eta)$ which is stable w.r.t. η.

Consider, now, the case $L(\eta)=1$. We let x be the site in $\Lambda^+(\eta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)$, note that x in unstable w.r.t η and w is stable w.r.t. η for any $w\in\Lambda^-(\eta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)$. We remark that by using (4.51), (4.12), (4.3), and $N_B(\eta)\geq 0$, we get the bound

$$E(\eta,\zeta)-E(-\underline{1})\geq\Gamma-4+4h+\Delta(\eta,\zeta) \tag{4.53}$$

and distinguish different cases depending on the number of plus spins in the configuration ζ which are associated to sites outside the support of the configuration $\bar{B}\bar{A}\eta$ namely, on $|\Lambda^+(\zeta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)|\geq 2$, see (4.25).

Suppose, first, $|\Lambda^+(\zeta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)|\geq 3$; since $x\in\Lambda^+_{\rm u}(\eta)$ and w is stable w.r.t. η for any $w\in\Lambda^-(\eta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)$, we have $\Delta(\eta,\zeta)\geq 3\cdot 2(1-h)$, for in the sum in (2.20) there are at least three terms.

We are left with the case $|\Lambda^+(\zeta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)|=2$; we let $\{y,z\}:=\Lambda^+(\zeta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)$ and notice that it must be necessarily $y,z\in\partial\Lambda^+(\bar{B}\bar{A}\eta)$ and $\mathrm{d}(y,z)=1$, otherwise it would be $\zeta\in\mathcal{G}$. Suppose, first, $\zeta(x)=-1$; since $x\neq y$, $x\neq z$, and y and z are nearest neighbors, it follows that at most one of the two sites y and z is nearest neighbor of x. Then, since in the sum (2.20) there are at least two terms and one of them is greater or equal to $2(3-h)$, we have $\Delta(\eta,\zeta)\geq 2(1-h)+2(3-h)$. By the previous inequality and (4.53) we get $E(\eta,\zeta)-E(-\underline{1})>\Gamma$. Suppose, finally, $\zeta(x)=+1$; without loss of generality we let $y=x$. Since $z\in\partial\{x\}\cap\partial\Lambda^+(\bar{B}\bar{A}\eta)$, by (2.20) we have $\Delta(\eta,\zeta)\geq 2(1-h)+2(1-h)$, with one of the two terms corresponding to the persistence of the plus spin associated to x in η and the other corresponding to the flip of the minus spin associated to z in η. By the previous inequality and (4.53) we get $E(\eta,\zeta)-E(-\underline{1})\geq\Gamma$.

Consider, finally, the case $L(\eta)=0$. By using (4.51), (4.2), $N_A(\eta)\geq 0$, $N_B(\eta)\geq 0$, and (4.26), we get

$$E(\eta,\zeta)-E(-\underline{1})\geq\Gamma-10+6h+12-4h>\Gamma$$

Item 3. Suppose $\zeta\in\mathcal{C}$ and $\eta\in\pi(\zeta)$, by using (2.9) and (2.20) it follows $E(\eta,\zeta)-E(-\underline{1})=\Gamma$.

Conversely, suppose $\eta\in\mathcal{G}$ and $\zeta\in\mathcal{G}^{\rm c}$ such that $E(\eta,\zeta)-E(-\underline{1})=\Gamma$. By using the results in the proof of the Item 2 above, see in particular the Step 8, we have that it must be necessarily $\eta\in\mathcal{A}_5$, $\zeta\in\mathcal{B}_2$, $L(\eta)=1$, $|\Lambda^+(\zeta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)|=2$, and $\zeta(x)=+1$, with x such that $\{x\}=\Lambda^+(\eta)\setminus\Lambda^+(\bar{B}\bar{A}\eta)$, indeed for any different choice of η and ζ it has be proven $E(\eta,\zeta)-E(-\underline{1})>\Gamma$. For configurations η and ζ as above we have also proven that $\bar{B}\bar{A}\eta\in\mathcal{R}$, that $\Delta(\eta,\zeta)\geq 2\cdot 2(1-h)$, and that there exists $z\in\partial\{x\}\cap\partial\Lambda^+(\bar{B}\bar{A}\eta)$ such that $\zeta(z)=+1$.

Now, by using $\bar{B}\bar{A}\eta\in\mathcal{R}$, $\Delta(\eta,\zeta)\geq 2\cdot 2(1-h)$, (4.32), (4.12), (4.3), and (3.24), we get

$$E(\eta,\zeta)-E(-\underline{1})\geq 6-2h+4hN_B(\eta)+\Gamma-10+6h+2\cdot 2(1-h)=\Gamma+4hN_B(\eta)$$

If it were $N_B(\eta)\geq 1$ it would follow $E(\eta,\zeta)-E(-\underline{1})>\Gamma$, then it must necessarily be $N_B(\eta)=0$.

By the above characterization of η we have that $\eta \in \mathcal{P}$; then, by using (4.32) and the definition of the map A we get the following expression for the coomunication energy $E(\eta, \zeta)$:

$$E(\eta,\zeta) - E(-\underline{1}) = 6 - 2h + \Gamma - 10 + 6h + \Delta(\eta,\zeta) = \Gamma - 4 + 4h + \Delta(\eta,\zeta)$$

Since $\zeta(x) = \zeta(z) = +1$, we have that $\Delta(\eta,\zeta) = 2 \cdot 2(1-h)$ if and only if $\zeta(w) = +1$ for all $w \in \Lambda^+(\bar{B}\bar{A}\eta)$. We then have that $\zeta \in \mathcal{C}$ and $\eta \in \pi(\zeta)$. $\square$

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