On the solutions of generalized discrete Poisson equation

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Abstract

The set of common numerical and analytical problems is introduced in the form of the *generalized multidimensional discrete Poisson equation*. It is shown that its solutions with square-summable discrete derivatives are unique up to a constant. The proof uses the Fourier transform as the main tool. The necessary condition for the existence of the solution is provided.

1 Introduction

The motivation for this paper comes from an attempt to construct (a discrete version of) the quantum field theory interacting with a non-trivial gravitational field. Such a theory would describe a quantum mechanical system with infinitely many degrees of freedom, assigned to the points of an infinite lattice $\mathbb{Z}^d$ [4, 3, 2]. The multidimentional discrete Poisson equation arises as a natural tool of such a theory. In this paper we present a proof of the uniqueness (up to an additive constant) of a class of its solutions. The existence proof will be the subject of further research.

The equation we deal with may be derived from the variational principle:

$$\delta W(\overline{f}, f) = 0 \ , \tag{1}$$

where

$$W(\overline{f}, f) := \sum_{\mathbf{n}\in\mathbb{Z}^d} \sum_{k=1}^{d} \sum_{l=1}^{d} b_{\mathbf{n},kl} \overline{(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_k})}(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l}) \ .$$

Here, f is our unknown function, a complex sequence defined on the lattice $\mathbb{Z}^d$ ($f : \mathbb{Z}^d \mapsto \mathbb{C}$), whereas $b : \mathbb{Z}^d \mapsto \mathbb{C}^{d\times d}$ is a sequence of $d \times d$ positive Hermitian matrices $b_{\mathbf{n}}$ whose spectra $\sigma(b_{\mathbf{n}})$ have common bounds,

$$\forall_{\mathbf{n}\in\mathbb{Z}^d} \qquad \sigma(b_{\mathbf{n}}) \in (b_1, b_2] \ , \qquad 0 \leq b_1 \leq b_2 < \infty \ . \tag{2}$$

We assume that f fulfills the following condition:

$$\forall_{1\le k\le d} \qquad \sum_{\mathbf{n}\in\mathbb{Z}^d} |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_k}|^2 < \infty \ . \tag{3}$$

This makes $W(\overline{f}, f)$ finite, since

$$b_1 \sum_{k=1}^{d} |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_k}|^2 < \sum_{k=1}^{d}\sum_{l=1}^{d} b_{\mathbf{n},kl}\overline{(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_k})}(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l}) \le b_2 \sum_{k=1}^{d} |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_k}|^2 \ .$$

Varying $W(\overline{f}, f)$ over $\overline{f_{\mathbf{n}}}$, we derive the following homogeneous equation for f:

$$\sum_{k=1}^{d}\sum_{l=1}^{d} [b_{\mathbf{n}+\mathbf{e}_k,kl}(f_{\mathbf{n}+\mathbf{e}_k} - f_{\mathbf{n}+\mathbf{e}_k-\mathbf{e}_l}) - b_{\mathbf{n},kl}(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l})] = 0 \ .$$

In the present paper, we consider a general, non-homogeneous case

$$\sum_{k=1}^{d}\sum_{l=1}^{d} [b_{\mathbf{n}+\mathbf{e}_k,kl}(f_{\mathbf{n}+\mathbf{e}_k} - f_{\mathbf{n}+\mathbf{e}_k-\mathbf{e}_l}) - b_{\mathbf{n},kl}(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l})] = g_{\mathbf{n}} \ , \tag{4}$$

which we call the *generalized multidimensional discrete Poisson equation.* The simplest example is provided by the standard multidimensional discrete Poisson equation, corresponding to $b_{\mathbf{n},kl} = \delta_{k,l}$:

$$\sum_{k=1}^{d} (f_{\mathbf{n}+\mathbf{e}_k} + f_{\mathbf{n}-\mathbf{e}_k} - 2f_{\mathbf{n}}) = g_{\mathbf{n}} \ .$$

Of course, adding a constant to a solution f of (4) we again obtain a solution. Within the class of functions fulfilling (3), we prove that any two solutions of equation (4) are equal up to an additive constant.

Contrary to the case of ordinary differential equations, there is no general theorem on the existence and uniqueness of solutions of discrete equations, neither in one nor in many dimensions. Only partial results exist, see for example [1, 5]. In [5], the uniqueness of solutions vanishing at infinity ($\lim_{\|\mathbf{n}\|\to\infty} f_{\mathbf{n}} = 0$) has been proved for a wide class of multidimensional discrete equations. Unfortunately, this is not sufficient for purposes of the quantum field theory. Our result presented here is valid for solutions fulfilling a different condition, namely (3).

Given two solutions of (4), f and f', satisfying condition (3), their difference $x_{\mathbf{n}} := f_{\mathbf{n}} - f'_{\mathbf{n}}$ fulfills (3) and solves the homogeneous equation with $g = 0$:

$$\sum_{k=1}^{d}\sum_{l=1}^{d} [b_{\mathbf{n}+\mathbf{e}_k,kl}(x_{\mathbf{n}+\mathbf{e}_k} - x_{\mathbf{n}+\mathbf{e}_k-\mathbf{e}_l}) - b_{\mathbf{n},kl}(x_{\mathbf{n}} - x_{\mathbf{n}-\mathbf{e}_l})] = 0 \ . \tag{5}$$

It is, therefore, sufficient to prove that, within the class of functions fulfilling (3), any solution of (5) is constant.

For the non-homogeneous equation (4), the existence of a solution depends very much upon the properties of the right-hand side g and will be analyzed elsewhere.

2 Uniqueness theorem

Theorem 1 *Let* $x : \mathbb{Z}^d \mapsto \mathbb{C}$ *be a solution of the homogeneous generalized discrete Poisson equation* (5) *in* d *dimensions. Let us assume that* x *has the property* (3),

$$\forall_{1 \le k \le d} \qquad \sum_{\mathbf{n} \in \mathbb{Z}^d} |x_{\mathbf{n}} - x_{\mathbf{n}-\mathbf{e}_k}|^2 < \infty \ . \tag{6}$$

Then

$$x_{\mathbf{n}} = \text{const} \ .$$

Observe that the uniqueness within the class of square-summable functions:

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} |x_{\mathbf{n}}|^2 < \infty \ ,$$

follows easily from the following, standard argument. We multiply both sides of (5) by $\overline{x_{\mathbf{n}}}$ and sum over $\mathbf{n} \in \mathbb{Z}^d$, obtaining

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} \overline{x_{\mathbf{n}}} \sum_{k=1}^{d} \sum_{l=1}^{d} [b_{\mathbf{n}+\mathbf{e}_k,kl}(x_{\mathbf{n}+\mathbf{e}_k} - x_{\mathbf{n}+\mathbf{e}_k-\mathbf{e}_l}) - b_{\mathbf{n},kl}(x_{\mathbf{n}} - x_{\mathbf{n}-\mathbf{e}_l})] = 0 \ .$$

Changing the order of summation in this expression we get:

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} \sum_{k=1}^{d} \sum_{l=1}^{d} b_{\mathbf{n},kl} \overline{(x_{\mathbf{n}} - x_{\mathbf{n}-\mathbf{e}_k})} \, (x_{\mathbf{n}} - x_{\mathbf{n}-\mathbf{e}_l}) = 0 \ .$$

Due to (2), this implies $x_{\mathbf{n}} = \text{const}$. However, we consider solutions which are not necessarily square-summable, and the above argument does not work.
Proof Consider the following auxiliary quantity $v : \mathbb{Z}^d \otimes [1, d] \mapsto \mathbb{C}$, defined as

$$v_{\mathbf{n},k} := x_{\mathbf{n}} - x_{\mathbf{n}-\mathbf{e}_k} \ . \tag{7}$$

From (6) and (7), we have that for each k, $(v_{\mathbf{n},k})$ is a square-summable sequence,

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} |v_{\mathbf{n},k}|^2 < \infty \ . \tag{8}$$

The fact that v is square-summable allows us to define another auxiliary quantity $\tilde{v} : [1, d] \otimes [-\pi, \pi]^d \mapsto \mathbb{C}$ as the Fourier transform of v,

$$\tilde{v}_k(\mathbf{s}) := \frac{1}{(2\pi)^{d/2}} \sum_{\mathbf{n} \in \mathbb{Z}^d} v_{\mathbf{n},k} e^{-i\mathbf{n}\cdot\mathbf{s}} \ , \qquad \mathbf{s} \in [-\pi, \pi]^d \ . \tag{9}$$

Additionally, (8) leads to $\tilde{v}_k$ being square-integrable,

$$\int_{[-\pi,\pi]^d} |\tilde{v}_k(\mathbf{s})|^2 \, \mathrm{d}^d\mathbf{s} < \infty \ .$$

Due to (7), we have for each pair $1 \le k_1, k_2 \le d$ and each $\mathbf{n} \in \mathbb{Z}^d$

$$v_{\mathbf{n},k_1} - v_{\mathbf{n}-\mathbf{e}_{k_2},k_1} = v_{\mathbf{n},k_2} - v_{\mathbf{n}-\mathbf{e}_{k_1},k_2} \ .$$

The Fourier transform of this equation goes as follows:

$$\tilde{v}_{k_1}(\mathbf{s}) \left(1 - e^{-is_{k_2}}\right) = \tilde{v}_{k_2}(\mathbf{s}) \left(1 - e^{-is_{k_1}}\right) \ .$$

Therefore, the following equality is valid for $s_{k_2} \neq 0$: $\tilde{v}_{k_1}(\mathbf{s}) = \tilde{v}_{k_2}(\mathbf{s}) \left(1 - e^{-is_{k_1}}\right) \left(1 - e^{-is_{k_2}}\right)^{-1}$. The set $\{\mathbf{s} \in [-\pi, \pi]^d : s_{k_2} = 0\}$ has measure zero (in the measure $\prod_{j=1}^d \mathrm{d}s_j = \mathrm{d}^d\mathbf{s}$). We rewrite the equality as

$$\tilde{v}_{k_1}(\mathbf{s}) \mathrel{\overset{\circ}{=}} \tilde{v}_{k_2}(\mathbf{s}) \frac{1 - e^{-is_{k_1}}}{1 - e^{-is_{k_2}}} \ , \tag{10}$$

where $\overset{\circ}{=}$ means 'equal everywhere in $[-\pi, \pi]^d$ except for a set with measure zero'.

It is be convenient for us to introduce another pair of auxiliary quantities. Let us define $y : \mathbb{Z}^d \otimes [1, d]^2 \mapsto \mathbb{C}$ as

$$y_{\mathbf{n},kl} := b_{\mathbf{n},kl} v_{\mathbf{n},l} \ . \tag{11}$$

Due to the bounds (2) on $b_{\mathbf{n}}$ and the fact that it is a Hermitian matrix, we have $|b_{\mathbf{n},kl}| \le b_2$. Inserting this into $\sum_{\mathbf{n} \in \mathbb{Z}^d} |y_{\mathbf{n},kl}|^2$, we get

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} |y_{\mathbf{n},kl}|^2 \le b_2 \sum_{\mathbf{n} \in \mathbb{Z}^d} |v_{\mathbf{n},k}|^2 < \infty \tag{12}$$

for each $1 \le k, l \le d$. Because of (12), we can define another quantity, $\tilde{y} : [1, d]^2 \otimes [-\pi, \pi]^d \mapsto \mathbb{C}$, as the Fourier transform of $y_{\mathbf{n},kl}$,

$$\tilde{y}_{kl}(\mathbf{s}) := \frac{1}{(2\pi)^{d/2}} \sum_{\mathbf{n} \in \mathbb{Z}^d} y_{\mathbf{n},kl} e^{-i\mathbf{n}\cdot\mathbf{s}} \ , \tag{13}$$

$\tilde{y}_{kl}$ is a square-integrable function on the domain $[-\pi, \pi]^d$,

$$\int_{[-\pi,\pi]^d} |\tilde{y}_{kl}(\mathbf{s})|^2 \, \mathrm{d}^d\mathbf{s} < \infty \ .$$

With the help of (7) and (11), we write (5) as

$$\sum_{k=1}^d \sum_{l=1}^d (y_{\mathbf{n},kl} - y_{\mathbf{n}+\mathbf{e}_k,kl}) = 0 \ . \tag{14}$$

Using the definition of $\tilde{y}$, we can calculate the Fourier transform of (14),

$$\sum_{k=1}^d \sum_{l=1}^d \tilde{y}_{kl}(\mathbf{s}) \left(1 - e^{is_k}\right) = 0 \ .$$

Multiplying both sides by $\overline{\tilde{v}_1(\mathbf{s})}(1 - e^{is_1})^{-1}$ and using (10), we obtain

$$\sum_{k=1}^{d} \left(\sum_{l=1}^{d} \tilde{y}_{kl}(\mathbf{s}) \right) \overline{\tilde{v}_k(\mathbf{s})} \mathring{=} 0 \ .$$

Recalling the definition of $\mathring{=}$, we can integrate this formula over $\mathbf{s}$, obtaining

$$\sum_{k=1}^{d} \int_{[-\pi,\pi]^d} \overline{\tilde{v}_k(\mathbf{s})} \left(\sum_{l=1}^{d} \tilde{y}_{kl}(\mathbf{s}) \right) \mathrm{d}^d\mathbf{s} = 0 \ .$$

Since the Fourier transform preserves the L^2 scalar product, we have from (9), (11) and (13)

$$\sum_{k=1}^{d} \sum_{\mathbf{n}\in\mathbb{Z}^d} \overline{v_{\mathbf{n},k}} \left(\sum_{l=1}^{d} b_{\mathbf{n},kl} v_{\mathbf{n},l} \right) = 0 \ . \qquad (15)$$

By (2) and (8), we have for each $1 \le k \le d$

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \left| \sum_{l=1}^{d} b_{\mathbf{n},kl} v_{\mathbf{n},l} \right|^2 < \infty \ .$$

The last result, together with (8) and the Schwartz inequality, ensures the convergence of the series

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \overline{v_{\mathbf{n},k}} \left(\sum_{l=1}^{d} b_{\mathbf{n},kl} v_{\mathbf{n},l} \right)$$

for each $1 \le k \le d$. Thus, we may transform (15) into

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \sum_{k=1}^{d} \sum_{l=1}^{d} b_{\mathbf{n},kl} \overline{v_{\mathbf{n},k}} v_{\mathbf{n},l} = 0 \ . \qquad (16)$$

From (2) we know that for each $\mathbf{n} \in \mathbb{Z}^d$, we have

$$\sum_{k=1}^{d} \sum_{l=1}^{d} b_{\mathbf{n},kl} \overline{v_{\mathbf{n},k}} v_{\mathbf{n},l} \ge b_1 \sum_{k=1}^{d} |v_{\mathbf{n},k}|^2 \ .$$

Since $b_1 > 0$, equation (16) may be true if and only if $v_{\mathbf{n},k} = 0$ for all $\mathbf{n} \in \mathbb{Z}^d$ and all $1 \le k \le d$. From (7), we get $x_{\mathbf{n}} - x_{\mathbf{n}-\mathbf{e}_k} = 0$, which means that Theorem 1 is true,

$$x_{\mathbf{n}} = \text{const} \ .$$

$\square$

Using the above result, we prove the main theorem:

Theorem 2 *Let $f, f' : \mathbb{Z}^d \mapsto \mathbb{C}$ be solutions of equation* (4)*, which fulfill condition* (3)*. Then*

$$f_{\mathbf{n}} = f'_{\mathbf{n}} + \text{const} \ ,$$

which means that solutions of (4) *which fulfill condition* (3) *are unique up to a constant.*

Proof The difference $x := f - f'$ is a solution of equation (5) and fulfills condition (6). Therefore, Theorem 1 applies and we have

$$x_{\mathbf{n}} = \text{const} \ .$$

Thus,

$$f_{\mathbf{n}} = f'_{\mathbf{n}} + \text{const} \ .$$

This ends the proof.

$\square$

3 Necessary condition for the existence of solutions

Theorem 3 *If f, fulfilling condition* (3)*, is the solution of* (4)*, then its right-hand side g must be square-summable.*

Proof Indeed, from (4) we have

$$\begin{aligned}
\sum_{\mathbf{n}\in\mathbb{Z}^d} |g_{\mathbf{n}}|^2 = & \sum_{\mathbf{n}\in\mathbb{Z}^d} \sum_{k,l=1}^{d} \sum_{k',l'=1}^{d} \overline{b_{\mathbf{n},kl}} b_{\mathbf{n},k'l'} \overline{(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l})} (f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_{l'}}) \\
& - \sum_{\mathbf{n}\in\mathbb{Z}^d} \sum_{k,l=1}^{d} \sum_{k',l'=1}^{d} \overline{b_{\mathbf{n},kl}} b_{\mathbf{n}+\mathbf{e}_{k'},k'l'} \overline{(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l})} (f_{\mathbf{n}+\mathbf{e}_{k'}} - f_{\mathbf{n}+\mathbf{e}_{k'}-\mathbf{e}_{l'}}) \\
& + \sum_{\mathbf{n}\in\mathbb{Z}^d} \sum_{k,l=1}^{d} \sum_{k',l'=1}^{d} \overline{b_{\mathbf{n}+\mathbf{e}_k,kl}} b_{\mathbf{n}+\mathbf{e}_{k'},k'l'} \overline{(f_{\mathbf{n}+\mathbf{e}_k} - f_{\mathbf{n}+\mathbf{e}_k-\mathbf{e}_l})} \\
& \quad \times (f_{\mathbf{n}+\mathbf{e}_{k'}} - f_{\mathbf{n}+\mathbf{e}_{k'}-\mathbf{e}_{l'}}) \\
& - \sum_{\mathbf{n}\in\mathbb{Z}^d} \sum_{k,l=1}^{d} \sum_{k',l'=1}^{d} \overline{b_{\mathbf{n}+\mathbf{e}_k,kl}} b_{\mathbf{n},k'l'} \overline{(f_{\mathbf{n}+\mathbf{e}_k} - f_{\mathbf{n}+\mathbf{e}_k-\mathbf{e}_l})} (f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_{l'}})
\end{aligned} \tag{17}$$

Each of these terms is bounded, for example:

$$\left| \sum_{k,l=1}^{d} \sum_{k',l'=1}^{d} \overline{b_{\mathbf{n},kl}} b_{\mathbf{n},k'l'} \overline{(f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l})} (f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_{l'}}) \right| \leq$$

$$\leq \sum_{k,l=1}^{d} \sum_{k',l'=1}^{d} |b_{\mathbf{n},kl}| \, |b_{\mathbf{n},k'l'}| \, |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l}| \, \left| f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_{l'}} \right| \leq$$

$$\leq b_2^2 \sum_{k,l=1}^{d} \sum_{k',l'=1}^{d} |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l}| \, \left| f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_{l'}} \right| \ , \quad (18)$$

since for Hermitian $b_{\mathbf{n}}$, we have $|b_{\mathbf{n},kl}| \leq \|b_{\mathbf{n}}\|$. For each product of the type $|f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l}| \, \left| f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_{l'}} \right|$, we have

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_l}| \, \left| f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_{l'}} \right| \leq \sum_{k=1}^{d} \sum_{\mathbf{n}\in\mathbb{Z}^d} |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_k}|^2 < \infty \quad (19)$$

and analogously for other terms in (17). Using these results, we obtain

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} |g_{\mathbf{n}}|^2 \leq b_2^2 d^4 \sum_{k=1}^{d} \sum_{\mathbf{n}\in\mathbb{Z}^d} |f_{\mathbf{n}} - f_{\mathbf{n}-\mathbf{e}_k}|^2 < \infty \ . \quad (20)$$

Hence, g is square-summable.

□

4 Summary

We introduced the generalized discrete Poisson equation in d dimensions. With the use of the Fourier transform, we proved the uniqueness up to a constant of the solutions with square-summable discrete derivatives. We also provide the necessary condition for the existence of solution. Because of the ubiquity of discrete equations and the Poisson equation in particular, this result is important for many areas of physics and mathematics.

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