Analytic smoothing of geometric maps with applications to KAM theory

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Abstract

We prove that finitely differentiable diffeomorphisms preserving a geometric structure can be quantitatively approximated by analytic diffeomorphisms preserving the same geometric structure. More precisely, we show that finitely differentiable diffeomorphisms which are either symplectic, volume-preserving, or contact can be approximated with analytic diffeomorphisms that are, respectively, symplectic, volume-preserving or contact. We prove that the approximating functions are uniformly bounded on some complex domains and that the rate of convergence of the approximation can be estimated in terms of the size of such complex domains and the order of differentiability of the approximated function. As an application to this result, we give a proof of the existence, local uniqueness and bootstrap of regularity of KAM tori for finitely differentiable symplectic maps. The symplectic maps considered here are not assumed to be written either in action-angle variables or as perturbations of integrable ones.

Keywords: *smoothing, symplectic maps, volume-preserving maps, contact maps, KAM tori, uniqueness, bootstrap of regularity.*

1 Introduction

It is known that finitely differentiable functions can be approximated by C^∞ or analytic ones, in such a way that the quantitative properties of the approximation are related to the order of differentiability of the approximated function [Kra83, Mos66, Ste70, Zeh75]. In view of applications to KAM theory, it is natural to ask whether it is possible to approximate finitely differentiable diffeomorphisms preserving a symplectic or volume form with C^∞ or analytic diffeomorphisms preserving the same form. Here we show that finitely differentiable diffeomorphisms which are either symplectic, volume-preserving, or contact can be approximated with analytic diffeomorphisms that are, respectively, symplectic, volume-preserving or contact. We prove that the approximating functions are uniformly bounded on some complex domains and give quantitative relations between: the rate of convergence, the degree of regularity of the approximated function, and the size of the complex domains where the approximating functions are uniformly bounded. As an application we give a proof of the existence, local uniqueness and bootstrap of regularity of KAM tori for finitely differentiable symplectic diffeomorphism. The novelty of these KAM-results is that

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the symplectic diffeomorphisms considered here are not assumed to be written either in action-angle variables or as perturbations of integrable ones. Besides the just mentioned results, in this work we also develop several results which may be of independent interest. We present a detailed study of the relation between an analytic linear *smoothing operator* (c.f. [Zeh75], see also Definition 5) and the nonlinear operators: composition and pull-back.

The case of approximating finitely differentiable symplectic or volume-preserving diffeomorphisms on a compact manifold with symplectic, respectively volume-preserving, C^∞-diffeomorphisms has been considered in [Zeh77], where it was proved that: *i) symplectic C^k-diffeomorphisms, with $k \geq 1$, can be approximated in the C^k-norms with symplectic C^∞-diffeomorphisms* and *ii) volume-preserving $C^{k+\alpha}$-diffeomorphisms, with $k \geq 1$ an integer and $0 < \alpha < 1$, can be approximated by volume-preserving C^∞-diffeomorphisms in the C^k-norms.* The method used in the present work differs from that in [Zeh77], because we avoid the use of *generating functions*. As it is well known, generating functions may fail to be globally defined for some maps. One advantage of not using generating functions is that the result given here can be applied directly to *non-twist* maps. We are currently working on such application [GEHdlL].

Even though the proofs of our results involve many technicalities, the main ideas are rather simple. Let us explain briefly the methodology used in this work. First, we define an analytic linear smoothing operator S_t, taking differentiable functions into analytic ones. The definition of S_t depends on the domain of definition of the functions we wish to smooth. We consider three situations: *i) the d-dimensional torus $\mathbb{T}^d \stackrel{\text{def}}{=} \mathbb{R}^d/\mathbb{Z}^d$; ii) $U \subset \mathbb{R}^d$ satisfying certain conditions, specified in Section 2.1, that guarantee the existence of a bounded linear extension operator [Kra83] (see Definition 9) and the validity of the Mean Value Theorem*; and *iii) $\mathbb{T}^n \times U$ with $U \subset \mathbb{R}^{d-n}$ as in ii).* Following [Zeh75] we smooth functions defined on $\mathbb{R}^d$ by an operator S_t defined by the convolution operator with an analytic kernel (see Section 2.1). By defining S_t in this way, we obtain a linear operator which takes periodic functions into periodic functions. Hence, by considering *lifts* to $\mathbb{R}^d$, the universal covering of $\mathbb{T}^n$, S_t can be applied to differentiable functions defined on the torus $\mathbb{T}^d$: this is important in applications to KAM theory. It is known [Kra83, Ste70] that if $U \subset \mathbb{R}^d$ has smooth boundary then there exists a bounded linear extension operator taking differentiable functions defined on U into differentiable functions defined on $\mathbb{R}^d$. Hence, for functions defined on $U \subset \mathbb{R}^d$ with smooth boundary, we define an analytic linear smoothing operator by taking extensions and then applying the operator S_t described above. It is easy to check that if $U \subset \mathbb{R}^{d-n}$ has smooth boundary then $\mathbb{R}^n \times U \subset \mathbb{R}^d$ also has smooth boundary. Hence functions defined on $\mathbb{T}^n \times U$ are smoothed by considering the universal covering $\mathbb{R}^n \times U$ and using a linear extension operator (see Section 2.1).

Given a finite differentiable diffeomorphism f that preserves a form Ω, it is not necessarily true that $S_t[f]$ preserves Ω. More generally, the form $S_t[f]^*\Omega$ is not necessarily equal to $f^*\Omega$. So we use Moser's *deformation method* [Mos65] to prove that, for t sufficiently large, there is a diffeomorphism φ_t such that $\varphi_t^*(S_t[f]^*\Omega) = f^*\Omega$. Hence, given a finitely differentiable diffeomorphism f which is either symplectic, volume-preserving, or contact, for t sufficiently large, $T_t[f] = S_t[f] \circ \varphi_t$ gives a symplectic, respectively, volume-preserving, or contact diffeomorphism approximating f. Furthermore, using the calculus of deformations [dlLMM86], we prove that that if f is exact symplectic, then it is possible to construct analytic approximating functions $T_t[f]$ which are also exact. The method used in the present work produces quantitative properties of the nonlinear operators T_t in terms of the degree of differentiability of f. More precisely, for t sufficiently large, $T_t[f]$ is bounded uniformly, with respect to t, on some complex domains and the rate of convergence of $T_t[f]$ to f is given in terms of t and the degree of differentiability of f. Obtaining such quantitative properties involves estimates on complex domains of the difference between: *i) smoothing a composition of two functions and composing their smoothings*, and *ii) smoothing the pulled-back form $f^*\alpha$ and pulling-back the form α with the smoothed function $S_t[f]$, for a k-form α*. Estimating these differences on complex domains requires many technicalities but, once this is done, proving the quantitative properties of T_t is rather easy as we show in Section 2.4. An estimate, on complex domains, of the difference between smoothing a composition of two functions and composing their smoothings was previously obtained in [GEV].

We emphasize that the geometric form Ω is assumed to be analytic. This is important because in this case, if f is symplectic, respectively, volume-preserving or contact, we have that both $f^*\Omega$ and $S_t[f]^*\Omega$ are analytic so that Moser's deformation method produces, for t sufficiently large, an analytic diffeomorphism φ_t such that: $\varphi_t^* S_t[f]^*\Omega = f^*\Omega$. The analyticity assumption on Ω is of particular importance in the volume case because the existence of a diffeomorphism φ such that $\varphi^*\alpha = \beta$ for two arbitrary volume forms depends on the regularity of the forms and on their domain of definition. The existence of such diffeomorphism for volume forms has been studied under different hypotheses in [Ban74, DM90, GS79, Mos65, Zeh77]. Nevertheless, to the best knowledge of the authors the question proposed in [Zeh77] whether C^1-volume forms can be approximated in C^1-norm by C^∞-volume forms on d-dimensional manifolds, with $d \geq 3$, is still open.

As an application we prove existence, local uniqueness and bootstrap of regularity of KAM tori for finitely differentiable symplectic diffeomorphisms that are not necessarily written in either action-angle variables or as a perturbation of an integrable symplectic diffeomorphism. The existence, formulated in Theorem 5, is a finitely differentiable version of Theorem 1 in [dlLGJV05] (the latter is reported as Theorem 4 in the present work). Roughly, Theorem 4 establishes the existence of a maximal dimensional invariant torus K^* with Diophantine rotation vector ω for a given analytic exact symplectic map f. The main hypotheses of Theorem 4 are the existence of an analytic parameterization of an n-dimensional torus K such that *i) certain non-degeneracy conditions are satisfied* and *ii) K is approximately invariant, in the sense that the sup norm of the error function $f \circ K - K \circ R_\omega$ on a complex set $\{x \in C^n : |\mathrm{Im}\,(x)| < \rho\}$, for some $\rho > 0$, is 'sufficiently small', where R_ω represents the translation by ω.* Theorem 4 also gives an estimate of the distance between the initial, approximately invariant torus K and the invariant torus K^* in terms of the size of the initial error. Theorem 5 is a finitely differentiable version of Theorem 4: the analyticity hypotheses for f and K are replaced by 'sufficiently large' differentiability of both f and K and by asking the norm, in suitable spaces of differentiable functions, of $f \circ K - K \circ R_\omega$ to be 'sufficiently small'. In Theorem 6 we prove that finitely differentiable invariant tori for finitely differentiable symplectic diffeormophisms are locally unique. Theorem 6 is a finitely differentiable version of Theorem 2 in [dlLGJV05].

We emphasize that in the KAM-results of the present work, as well as in those given in [dlLGJV05], the symplectic diffeomorphisms are not assumed to be written either in action-angle variables or as a perturbation of an integrable map. One application of these results is the *validation* of numerical computations of invariant tori, because our results give an explicit condition on the size of the error $f \circ K - K \circ R_\omega$ in analytic (Theorem 4) or differentiable (Theorem 5) norms that guarantee the existence of a true invariant torus near a numerically computed approximately invariant torus. For such application, it is important that we do not assume that the system is close to integrable or written in action-angle variables, because in this way we do not have to compute local coordinates before the verification of the size of the error. Having a condition on the size of the error $f \circ K - K \circ R_\omega$ in finitely differentiable norms is also useful because for some numerical methods it is easier to estimate the finitely differentiable norms than the analytic ones, for example when using splines. Another application of this result is when studying invariant tori restricted to normally hyperbolic manifolds – which are only finitely differentiable. This analysis occurs in some mechanisms for the study of instability. In particular, in [DdlLS03, DLS06] it is shown that secondary tori close to resonances play an important role. The present result is particularly useful for this study since for these tori, the action-angle coordinates are singular and their construction and their estimates require extra work and extra assumptions, see [DLS06, 8.5.4]. The present work allows to simplify the proof of some of the results in [DLS06] and lowers the regularity assumptions of the main result of [DLS06]. This improvements are crucial in the higher dimensional extensions of the model.

Moser's smoothing technique [Jac72, Mos66, Zeh75] provides a method to obtain finitely differentiable versions of *Generalized Implicit Function* theorems from the corresponding analytic ones. Briefly, Moser's method goes as follows: Let F be defined on Banach spaces of analytic functions and assume that a Generalized Implicit Function Theorem holds in these Banach spaces. Assume that the functional equation $F(f, K) = 0$ has an analytic solution (f_0, K_0), and that there exists an analytic smoothing operator. Then one finds, using the analytic smoothing operator, a solution $(f, \Phi(f))$ for f in a small neighbourhood of f_0 in a space of finitely differentiable functions. One important hypothesis of Moser's technique is the existence of an approximate right inverse of the linear operator $D_2F(f, K)$. The approximate right-invertibility yields a loss of differentiability: in KAM theory this is related to the so called *'small denominators'*. At this point it becomes crucial to have quantitative properties of the smoothing in terms of the degree of differentiability of the smoothed functions. For a more detailed explanation of Moser's method see for example [Jac72, Mos66, Sal04, Zeh75].

To prove the existence of finitely differentiable solutions of the equation $f \circ K = K \circ R_\omega$ we use the following 'modified' smoothing technique: Rather than assuming the existence of an analytic initial solution of the functional equation we just assume the existence of a finitely differentiable approximate solution and find conditions under which there is an analytic solution nearby. The analytic Generalized Implicit Function Theorem for the functional $f \circ K - K \circ R_\omega$ is provided by Theorem 4, which only holds for exact symplectic maps. Hence, to apply the smoothing technique we use the nonlinear operator T_t, described above, to smooth the exact symplectic map f. Parameterizations of approximately invariant tori are smoothed using the operator S_t described at the beginning of this introduction. Then, given a finitely differentiable approximate solution (f, K) of $f \circ K = K \circ R_\omega$, the existence of an analytic solution close to (f, K) is guaranteed by: *i) a non-degeneracy condition on K* and *ii) a 'smallness' condition on the* sup *norm on complex domains of the difference $T_t[f] \circ S_t[K] - S_t[f \circ K]$ in terms of the the size of the initial error $f \circ K - K \circ R_\omega$ in a finite differentiable norm.*

In Section 7.1 of [Van02] a Generalised Implicit Function Theorem in spaces of finitely differentiable functions has been proved using the modified smoothing technique in which, rather than assuming the existence of a solution in analytic spaces, one assumes the existence of an approximate solution in finitely differentiable spaces of the equation $F(f, K) = 0$. The condition used in [Van02] to guarantee the existence of an analytic solution near a given finitely differentiable approximate solution is stated on page 71 of [Van02] and it requires that the norm – in suitable spaces – of the difference $F(S_t[f], S_t[K]) - S_t[F(f, K)]$ is 'sufficiently small'. In [Van02] the verification of this was left open for the composition operator, which is customary used in KAM theory. Reference [GEV] contains this verification.

As a consequence of the fact that, under certain general conditions, near a finitely differentiable solution (f, K) of the equation $f \circ K = K \circ R_\omega$ there is an analytic solution, we obtain the bootstrap of regularity of invariant tori with Diophantine rotation vector for exact symplectic maps that are either finitely differentiable or analytic. The bootstrap of regularity is stated in Theorem 7. To prove Theorem 7, first in Theorem 6 we prove a finitely differentiable version of the local uniqueness of invariant tori for symplectic maps. Theorem 6 and Theorem 7 are similar to Theorem 4 and Theorem 5 in [SZ89]. However, while the results in [SZ89] are stated and proved for Hamiltonian vector fields written in the Lagrangian formalism, Theorem 6 and Theorem 7 in the present work are stated and proved for exact symplectic maps that are not necessarily written either in action-angle variables or as perturbation of integrable ones.

This paper is divided into two parts. In Section 2 we show how to approximate finitely differentiable functions that preserve a geometric structure (exact symplectic, volume or contact) with analytic functions preserving the same geometric structure. In Section 3 we give an application of the symplectic smoothing result to KAM theory, proving of the existence, local uniqueness and bootstrap of regularity of Diophantine invariant tori for finitely differentiable symplectic maps. In Section 3 we also prove the bootstrap of regularity of KAM tori for analytic exact symplectic maps. That is, we prove that given an analytic exact symplectic map and an invariant torus with Diophantine frequency vector, if the invariant torus is sufficiently differentiable, then it is analytic.

2 Smoothing geometric diffeomorphisms

In this section we show that finitely differentiable diffeomorphisms which are either symplectic, volume-preserving or contact can be approximated by analytic diffeomorphisms having the same geometric property. We give quantitative properties of the approximation in terms of the degree of differentiability of the approximated functions.

Since obtaining such geometric approximating functions involves many technicalities, we have divided the present section as follows. In Section 2.1 we define the norms used and set the conditions on the domain of definition of the diffeomorphism to be smoothed. In Section 2.2 the geometric smoothing results are stated. The technical part of the proofs is given in Section 2.3 and the proofs are concluded in Sections 2.4 and 2.5.

2.1 Setting

Informally, the method we use to smooth symplectic, volume-preserving or contact diffeomorphism with analytic diffeomorphism having the same geometric property is the following. First, for $t \geq 1$, we define a linear operator S_t that takes finitely differentiable functions into analytic ones and such that $S_t[f]$ tends to f when t goes to infinity. Then, if f is a finitely differentiable symplectic, volume-preserving, or contact diffeomorphism we find, for t sufficiently large, a diffeomorphism φ_t such that $\varphi_t^* \left(S_t[f]^*\Omega\right) = f^*\Omega$. The analytic approximating functions satisfying the same geometric property of f are then defined by $S_t[f] \circ \varphi_t$. In view of the applications we are interested in symplectic, volume-preserving or contact diffeomorphisms defined on either $\mathbb{T}^d$, $\mathbb{U} \subset \mathbb{R}^d$ or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$. First, by using the convolution operator with an analytic kernel, we define S_t for continuous and bounded functions defined on $\mathbb{R}^d$. It turns out that, if f is a $\mathbb{Z}^d$-periodic (or partially periodic) continuous and bounded function defined on $\mathbb{R}^d$ then $S_t[f]$ is also $\mathbb{Z}^d$-periodic (respectively, partially periodic). Hence to extend the definition of S_t to torus maps we use lifts of torus maps to $\mathbb{R}^d$ (the universal covering of $\mathbb{T}^d$). To define S_t on functions with domain $U \subset \mathbb{R}^d$ we use a linear bounded extension operator. Then, by taking lifts, the definition of S_t is extended to functions defined on the annulus $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$. Before making these definitions explicit, let us introduce the Banach spaces of functions we work with.

Definition 1. *Let $\mathbb{Z}_+$ denote the set of positive integers. Given $U \subset \mathbb{C}^d$ an open set, $C^0(U)$ denotes the space of continuous functions $f : U \to \mathbb{R}$, such that*

$$|f|_{C^0(U)} \stackrel{\text{def}}{=} \sup_{x\in U} |f(x)| < \infty\,.$$

For $\ell \in \mathbb{N}$, $C^\ell(U)$ denotes the space of functions $f : U \to \mathbb{R}$ with continuous derivatives up to order ℓ such that

$$|f|_{C^\ell(U)} \stackrel{\text{def}}{=} \sup_{\substack{x\in U\\ |k|\leq \ell}} \left\{|D^k f(x)|\right\} < \infty\,.$$

Let $\ell = p + \alpha$, with $p \in \mathbb{Z}_+$ and $0 < \alpha < 1$. Define the Hölder space $C^\ell(U)$ to be the set of all functions $f : U \to \mathbb{R}$ with continuous derivatives up to order p for which

$$|f|_{C^\ell(U)} \stackrel{\text{def}}{=} |f|_{C^p} + \sup_{\substack{x,y\in U, x\neq y\\ |k|=p}} \left\{ \frac{|D^k f(x) - D^k f(y)|}{|x-y|^\alpha} \right\} < \infty\,.$$

For $\rho > 0$, and $U \subseteq \mathbb{R}^d$ let $U + \rho$ denote the complex strip:

$$U + \rho = \{x + iy \in \mathbb{C}^d : x \in U,\ |y| < \rho\}.$$

Definition 2. *Let $\ell \geq 0$. Given $U \subseteq \mathbb{R}^d$ open, define the Banach space $\mathcal{A}(U + \rho, C^\ell)$ to be the set of all holomorphic functions $f : U + \rho \to \mathbb{C}$ which are real valued on U (i.e. $\overline{f(x)} = f(\bar{x})$ for all $x \in U$) and such that $|f|_{C^\ell(U+\rho)} < \infty$.*

For a matrix or vector-valued function G with components $G_{i,j}$ in either $C^\ell(U)$ or in $\mathcal{A}(U + \rho, C^\ell)$ we use the norm, respectively,

$$|G|_{C^\ell(U)} \stackrel{\text{def}}{=} \max_{i,j} |G_{i,j}|_{C^\ell(U)} \qquad \text{or} \qquad |G|_{C^\ell(U+\rho)} \stackrel{\text{def}}{=} \max_{i,j} |G_{i,j}|_{C^\ell(U+\rho)}\,.$$

The space of all functions $g = (g_1, \ldots, g_d) : V \subseteq \mathbb{C}^n \to U \subseteq \mathbb{C}^d$ such that $g_i \in C^\ell(U)$, for $i = 1, \ldots, d$, is denoted by $C^\ell(U, V)$. Since it will not lead to confusion, $\mathcal{A}(U + \rho, C^\ell)$ will also denote the set of functions $g = (g_1, \ldots, g_d)$ with components in $\mathcal{A}(U + \rho, C^\ell)$.

Definition 3. *Let $U \subset \mathbb{R}^m$. A* lift *of a continuous map f, defined on the annulus $\mathbb{T}^n \times U$, to $\mathbb{R}^n \times U$ (the universal cover of $\mathbb{T}^n \times U$) is a continuous map $\hat{f}$ defined on $\mathbb{R}^n \times U$ such that:*

1. *$\hat{f}(x, y) = f(x \mod \mathbb{Z}^n, y)$, if f takes values in $\mathbb{R}$.*
2. *$\hat{f}(x, y) \mod \mathbb{Z} = f(x \mod \mathbb{Z}^n, y)$ for $(x, y) \in \mathbb{R}^n \times U$, if f takes values in $\mathbb{T}$.*

It is well known that given a continuous map f defined on $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^m$, any lift $\hat{f} : \mathbb{R}^n \times U$ has the following form

$$\hat{f}(x, y) = P\,x + u(x, y)\,, \qquad (x, y) \in \mathbb{R}^n \times U \tag{1}$$

where $u \in C^0(\mathbb{R}^n \times U, \mathbb{R}^s)$ is $\mathbb{Z}$-periodic in the first n-variables and P is an $(n \times 1)$-matrix with components in $\mathbb{Z}$. Furthermore, if f takes values in $\mathbb{R}$ then $P = 0$. Moreover, if f has additional regularity, the corresponding function u has the same regularity. Even though lifts of continuous annulus maps are not unique, they differ by a constant vector in $\mathbb{Z}$. This, together with the fact that any map of the form (1) defines an annulus map, enable us to work with lifts of torus and annulus maps (considering torus maps as particular cases of annulus maps). For notational reasons we use the same symbol to denote the annulus (torus) map and a lift of it.

Definition 4. *For $\ell \geq 0$, denote by $C^\ell(\mathbb{T}^n \times U, V)$, and $\mathcal{A}(\mathbb{T}^n \times U + \rho, C^\ell)$ the set of annulus maps with lift of the form (1) with $u \in C^\ell(\mathbb{R}^n \times U, V)$ (respectively in $\mathcal{A}(\mathbb{R}^n \times U + \rho, C^\ell)$) $\mathbb{Z}$-periodic in the first n-variables. The corresponding norms are defined as follows:*

$$|f|_{C^\ell(\mathbb{T}^d \times U)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d \times U)} \qquad \textit{and} \qquad |f|_{C^\ell(\mathbb{T}^d \times U+\rho)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d \times U+\rho)}\,.$$

In the case of torus maps, denote by $C^\ell(\mathbb{T}^d, V)$, and $\mathcal{A}(\mathbb{T}^d + \rho, C^\ell)$ the set of torus maps with lift of the following form:

$$f(x) = P\,x + u(x)\,, \tag{2}$$

where P is a matrix with components in $\mathbb{Z}$ and $u \in C^\ell(\mathbb{R}^d, V)$ (respectively $u \in \mathcal{A}\left(\mathbb{R}^d + \rho, C^\ell\right)$) is $\mathbb{Z}^d$-periodic. The corresponding norms are defined as follows:

$$|f|_{C^\ell(\mathbb{T}^d)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d)} \qquad \textit{and} \qquad |f|_{C^\ell(\mathbb{T}^d+\rho)} \stackrel{\text{def}}{=} |P| + |u|_{C^\ell(\mathbb{R}^d+\rho)}\,.$$

Moreover, for $r \geq 0$ denote by $\mathrm{Diff}^{\,r}(\mathbb{U})$ the set of C^r-diffeomorphisms of $\mathbb{U}$, where $\mathbb{U}$ is either $\mathbb{U} \subseteq \mathbb{R}^d$ open, $\mathbb{T}^d$, or $\mathbb{T}^n \times U$.

For $U \subseteq \mathbb{R}^d$ open, denote by $\Lambda^k(U)$ the space of real analytic k-forms in U. Let $\Omega \in \Lambda^k(U)$ have the following form:

$$\Omega(x) = \sum_{1 \le i_1 < \cdots < i_k \le d} \Omega_{\mathbf{i}}(x)\, dx_{\mathbf{i}}\,,$$

where $\mathbf{i}$ represents the multi-index $(i_1, \ldots, i_k)$ and $dx_{\mathbf{i}} \stackrel{\text{def}}{=} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. If $|\Omega_{\mathbf{i}}|_{C^\ell(U)} < \infty$ for all $1 \le i_1 < \cdots < i_k \le d$, define

$$|\,\Omega|_{C^\ell(U)} \stackrel{\text{def}}{=} \max_{1 \le i_1 < \cdots < i_k \le d} |\,\Omega_{\mathbf{i}}|_{C^\ell(U)}\ .$$

Definition 5. *Let $\mathbb{U}$ be either $\mathbb{U} \subseteq \mathbb{R}^d$ open, $\mathbb{T}^d$, $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$ an open set. We say that the linear operator $S_t : C^\ell(\mathbb{U}) \to \mathcal{A}(\mathbb{U} + t^{-1}, C^0)$ is an* analytic smoothing *operator if the following properties hold for any $f \in C^\ell(U)$:*

1. $|S_t[f]|_{C^0(\mathbb{U}+t^{-1})} \le c\, |f|_{C^\ell(\mathbb{U})}$ *for all $t \ge 1$.*
2. $\lim_{t\to\infty} |(S_t - \mathrm{Id})[\mathrm{f}]|_{C^0(\mathbb{U})} = 0$.
3. $|S_t - S_\tau[f]|_{C^0(\mathbb{U}+\tau^{-1})} \le c\, |f|_{C^\ell(\mathbb{U})}\, t^{-\ell}\,,$ *for $\tau \ge t \ge 1$.*

for some constant c depending on ℓ and $\mathbb{U}$, but independent of t.

Now we define the smoothing operator S_t we work with. First we define $S_t[f]$ for $f \in C^0(\mathbb{R}^d)$.

Definition 6. *Let $u : \mathbb{R}^d \to \mathbb{R}$ be C^∞, even, identically equal to 1 in a neighbourhood of the origin, and with support contained in the ball with center in the origin and radius 1. Let $\hat{u} : \mathbb{R}^d \to \mathbb{R}$ be the Fourier transform of u and denote by s the holomorphic continuation of $\hat{u}$. Define the linear operator S_t as*

$$S_t[f](z) \stackrel{\text{def}}{=} t^d \int_{\mathbb{R}^d} s(t(y-z)) f(y) dy\,, \qquad \textit{for} \quad f \in C^0(\mathbb{R}^d). \tag{3}$$

Applying obvious modifications, Definition 6 can be extended to functions in $C^0(\mathbb{R}^n, \mathbb{R}^d)$. In the sequel these latter operators are denoted by the same symbol S_t. We now summarize some elementary properties of S_t that follow from Definition 6.

Remark 7.

1. *S_t transforms functions in $C^0(\mathbb{R}^d)$ into entire functions on $\mathbb{C}^d$.*
2. *Using the change of variables $\xi = t\,\mathrm{Re}\,(y-z) = ty - t\,\mathrm{Re}\,(z)$, one has for $f \in C^0(\mathbb{R}^d)$*

$$S_t[f](z) = \int_{\mathbb{R}^d} s(\xi - it\,\mathrm{Im}\,(z)) f(\mathrm{Re}\,(z) + \xi/t) d\xi. \tag{4}$$

3. *S_t commutes with constant coefficient differential operators.*
4. *S_t acts as the identity on polynomials.*
5. *From (4) one has that S_t takes (partially) periodic functions into (partially) periodic functions.*
6. *From (4) we have that $S_t[f](x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d$.*

Remark 8. *In the applications of Moser's smoothing method to KAM theory it is of particular importance to know how to define* S_t *for functions defined on the d-dimensional torus* $\mathbb{T}^d$ *as well as functions defined on* $\mathbb{T}^n \times \mathbb{R}^m$*. Notice that since* S_t *in Definition 6 acts as the identity on polynomials and it takes partially periodic functions into partially periodic functions, we have that for any annulus map* f*, with lift of the form (1)* $S_t[\hat{f}]$ *is also of the form (1):*

$$S_t[\hat{f}](x, y) = P\,x + S_t[u](x, y)\,.$$

Hence to extend the definition of S_t *to torus maps as well as to maps defined on* $\mathbb{T}^n \times \mathbb{R}^m$*, we apply* S_t *in Definition 6 to any lift of it. This is well-defined because two lifts of the same torus map (respectively annulus map) differ by a constant vector with components in* $\mathbb{Z}$*.*

Definition 9. *Let* $\ell > 0$ *be not an integer. A* bounded linear extension operator *is a linear operator* $\mathscr{E}_U : C^\ell(U) \to C^\ell(\mathbb{R}^d)$ *such that* $\mathscr{E}_U(f)|_U = f$ *for all* $f \in C^\ell(U)$ *and* $|\mathscr{E}_U(f)|_{C^\ell(\mathbb{R}^d)} \leq c_U\, |f|_{C^\ell(U)}$ *.*

In order to extend the definition of the linear operator S_t to functions defined on $U \subset \mathbb{R}^d$ and to the annulus $\mathbb{T}^n \times U$, it suffices to have a linear bounded linear extension operator from $C^\ell(U)$ to $C^\ell(\mathbb{R}^d)$. The sufficient condition we adopt here to have such extension operator is that given in Theorem 14.9 in [Kra83]. It amounts to the regularity of the boundary of U.

Definition 10. *Let* $\varrho : \mathbb{R}^d \to \mathbb{R}$ *be a function with continuous derivatives up to order* m*, for some* $m \in \mathbb{N}$*, and assume that* $\operatorname{grad} \varrho(x) \neq 0$ *for all* $x \in \{x : \varrho(x) = 0\}$*. The set* $U = \{x \in \mathbb{R}^d : \varrho(x) \leq 0\}$ *is called a closed domain with* C^m*-boundary. An open domain is defined by* $\{x \in \mathbb{R}^d : \varrho(x) < 0\}$*.*

The following result guarantees the existence of a bounded extension operator for functions with domain of definition $U \subset \mathbb{R}^d$ provided that U has smooth boundary. For a proof we refer the reader to [Kra83, Ste70].

Theorem 1. *If* $0 < \ell < m \in \mathbb{N}$ *with* $\ell \notin \mathbb{N}$*, and* $U \subset \mathbb{R}^d$ *has* C^m*-boundary, then there is a linear extension operator* $\mathscr{E}_U^\ell : C^\ell(U) \to C^\ell(\mathbb{R}^d)$ *such that*

$$\left| \mathscr{E}_U^\ell(f) \right|_{C^\ell(\mathbb{R}^d)} \leq c_U\, |f|_{C^\ell(U)}\,. \tag{5}$$

for some constant c_U*, depending on* U*.*

Hence, for functions that are defined on a subset of $\mathbb{R}^d$ with regular boundary we have the following

Definition 11. *Let* $0 < \ell < m$*, with* $m \in \mathbb{N}$ *and* $\ell \notin \mathbb{N}$*. Let* $U \subset \mathbb{R}^d$ *be an open domain with* C^m*-boundary and* $\mathscr{E}_U^\ell$ *a linear extension operator as in Theorem 1. For* $f \in C^\ell(U)$ *and for any* $x \in \mathbb{C}^d$ *we define*

$$\hat{S}_t[f](x) \stackrel{\text{def}}{=} S_t[\,\mathscr{E}_U^\ell(f)\,](x)\,, \tag{6}$$

where S_t *is as in (3).*

The following remark is related to Remark 7

Remark 12. *Notice that the operator* $\hat{S}_t$*, defined in Definition 11 for functions in* $C^\ell(U)$*, satisfies the following properties:*

1. $\hat{S}_t$ *is linear.*
2. $\hat{S}_t$ *transforms functions in* $C^\ell(U)$ *into entire functions on* $\mathbb{C}^d$*.*

Remark 13. *Notice that if $U \subset \mathbb{R}^{d-n}$ is an open domain with C^m-boundary then $\mathbb{U} = \mathbb{R}^n \times U \subset \mathbb{R}^d$ also is an open domain with C^m-boundary. Moreover, it follows from Remark 8 and Definition 11 that, if $u(x,y)$ is defined for $(x,y) \in \mathbb{U} = \mathbb{R}^n \times U$ and $\mathbb{Z}^n$-periodic on the x-variable and $\hat{S}_t$ is as in Definition 11, then $\hat{S}_t[u]$ is also $\mathbb{Z}^n$-periodic on the x-variable. Indeed, since $\mathscr{E}^\ell_{\mathbb{U}}(u)(x,y) = u(x,y)$ for all $(x,y) \in \mathbb{R}^n \times U$, we have that $\mathscr{E}^\ell_{\mathbb{U}}(u)$ is $\mathbb{Z}^n$-periodic in the x-variable, where $\mathscr{E}^\ell_{\mathbb{U}}$ is as in Theorem 1. Therefore, any map defined on the annulus $\mathbb{T}^n \times U$ with U as in Theorem 1 and lift given by (1), is smoothed by*

$$\hat{S}_t[f](x,y) = P\,x + S_t[\mathscr{E}^\ell_{\mathbb{U}}(u)](x,y)\,,$$

where S_t is as Definition 6 and $\mathscr{E}^\ell_{\mathbb{U}}$ is as in Theorem 1.

Since it will not lead to confusion, the operator $\hat{S}_t$ defined in (6) will be denoted (dropping the hat) as the operator S_t in (3).

Remark 14. *Summarizing, a function f is smoothed, depending on its domain of definition as follows:*

1. *If $f \in C^0(\mathbb{R}^d)$, $S_t[f]$ is given by (3).*
2. *If $f \in C^\ell(\mathbb{U})$, with $\mathbb{U} \subset \mathbb{R}^d$ an open domain with C^m-boundary, we define*
$$S_t[f] = S_t[\mathscr{E}^\ell_{\mathbb{U}}(f)]\,,$$
where $\mathscr{E}^\ell_{\mathbb{U}}$ is as in Theorem 1 and S_t on the right hand side is defined by (3).
3. *If $f \in C^0(\mathbb{T}^d)$, with lift as in (2), where $u \in C^0(\mathbb{R}^d)$ is $\mathbb{Z}^d$-periodic, and P an $(d \times 1)$-matrix with components in $\mathbb{Z}$, then*
$$S_t[f](x) \stackrel{\text{def}}{=} P\,x + S_t[u](x)\,,$$
where S_t on the right hand side is defined by (3).
4. *For $U \subset \mathbb{R}^{d-n}$ an open domain with C^m-boundary and $f \in C^\ell(\mathbb{T}^n \times U)$, with lift given by (1) where $u \in C^\ell(\mathbb{R}^n \times U)$ $\mathbb{Z}$-periodic on the first n-variables, and P an $(n \times 1)$-matrix with components in $\mathbb{Z}$, we define*
$$S_t[f](x,y) \stackrel{\text{def}}{=} P\,x + S_t[\mathscr{E}^\ell_{\mathbb{U}}(u)](x,y)\,,$$
where $\mathbb{U} = \mathbb{R}^n \times U$, $\mathscr{E}^\ell_{\mathbb{U}}$ is as in Theorem 1, and S_t on the right hand side is defined by (3).

To define an analytic smoothing operator such that it takes finitely diffeomorphisms preserving either an exact symplectic, volume or contact form into analytic diffeomorphisms preserving the same structure, we set more conditions on the domain of definition of the maps. We assume the following.

Definition 15. *Given $U \subseteq \mathbb{C}^n$, for $x, y \in U$ denote by $d_U(x,y)$ the minimum length of arcs inside U joining x and y. We say that U is* compensated *if there exists a constant c_U such that $d_U(x,y) \le c_U\,|x-y|$, for all $x, y \in U$.*

It turns out that on compensated domains it is possible to apply the Mean Value Theorem to obtain estimates of the C^ℓ-norm of the composition of two functions in terms of the C^ℓ-norms of the composed functions (see [dlLO99] and Lemma 29 in the present paper). We finish this section recalling some geometric definitions.

Definition 16. 1. *Given a k-form Ω on a d-dimensional manifold, denote by $\mathscr{I}_\Omega$ the application $X \to i_X\,\Omega$, sending the vector field X into the inner product $i_X\Omega \stackrel{\text{def}}{=} \Omega(x)\,(X(x),\cdot)$. A k-form Ω is* non-degenerate *if $\mathscr{I}_\Omega$ is an isomorphism.*

2. A volume *element on a d-dimensional manifold is a d-form which is non-degenerate.*

3. A symplectic *form on a 2n-dimensional manifold is a non-degenerate closed 2-form.*

4. A contact *form on a $(2n+1)$-dimensional manifold is a 1-form Ω, such that $\Omega \wedge (d\Omega)^n$ is a volume element.*

5. *A diffeomorphism f of a contact manifold (M, Ω) is a contact diffeomorphism if there exists a nowhere zero function $\lambda : M \to \mathbb{R}$ such that $f^*\Omega = \lambda\,\Omega$.*

6. *Let $\Omega = d\alpha$ be an exact symplectic form on a symplectic manifold. The diffeomorphism f is exact symplectic if $f^*\alpha - \alpha$ is an exact 1-form.*

2.2 Statement of results

In this section we formulate the results guaranteeing the existence of an analytic smoothing operator that preserves the prescribed geometric structure. In Theorem 2 the symplectic and volume cases are considered; the contact case is considered in Theorem 3.

Theorem 2. *Let $2 < \ell < m$, with $m \in \mathbb{N}$ and $\ell \notin \mathbb{N}$, and let $C, \beta > 0$ and $1 < \mu < \ell - 1$ be given. Assume that the following hypotheses hold:*

H1. $\mathbb{U}$ is either: i) $\mathbb{T}^d$, ii) a compensated bounded open domain in $\mathbb{R}^d$ with C^m-boundary (see definitions 10 and 15), or iii) $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$ a compensated bounded open domain with C^m-boundary and $n < d$.

H2. $\mathbb{V}$ is C^m-diffeomorphic to $\mathbb{U}$ and such that $\mathbb{U} \subseteq \mathbb{V}$. $\Omega = d\alpha$ is either a real analytic symplectic form (with $d = 2n$) or a real analytic volume element on $\mathbb{V}$ such that $|\Omega|_{C^\ell(\mathbb{V}+\rho)} < \infty$ for some $\rho > 0$.

H3. Let $\mathscr{I}_\Omega$ be as in Definition 16 and let $\mathscr{I}_\Omega^{-1}$ denote the inverse of $\mathscr{I}_\Omega$. Let $k = 2$ if Ω is a symplectic form, and $k = d$ if Ω is a volume form and assume that for any $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho')} < \infty$, with $\rho' \geq 0$, the following holds

$$\left|\mathscr{I}_\Omega^{-1}\theta\right|_{C^0(\mathbb{U}+\rho')} \leq M_\Omega \, |\theta|_{C^0(\mathbb{U}+\rho')} \, .$$

Then, there exists two constants $t^* = t^*(d, \ell, \mathbb{V}, C, \mu, \beta, M_\Omega\,, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, and $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ and a family of nonlinear operators $\{T_t\}_{t \geq t^*}$, taking functions belonging to $\{f \in \mathrm{Diff}^\ell(\mathbb{U}) : |f|_{C^\ell(\mathbb{U})} \leq \beta,\ f^*\Omega = \Omega\,,\mathrm{closure\ of}\ f(\mathbb{U}) \subseteq \mathbb{V}\}$ into real analytic functions. Moreover, if $\mathbb{U}_t$ is defined as follows:

$$\mathbb{U}_t \stackrel{\text{def}}{=} \begin{cases} \mathbb{T}^d\,, & \mathbb{U} = \mathbb{T}^d \\ \{x \in \mathbb{U} : \bar{B}(x, t^{-1}) \subset \mathbb{U}\}\,, & \mathbb{U} \subset \mathbb{R}^d \\ \mathbb{T}^n \times \{x \in U : \bar{B}(x, t^{-1}) \subset U\}\,, & \mathbb{U} = \mathbb{T}^n \times U \end{cases} \tag{7}$$

where $\bar{B}(x, t^{-1})$ represents the closed ball with center at x and radius t^{-1}, then the following properties hold:

T0. $T_t[f]$ is a diffeomorphism on $\mathbb{U}_t$.

T1. $T_t[f]^*\Omega = f^*\Omega$.

T2. $|\,T_t[f]\,|_{C^1(\mathbb{U}_t + C\,t^{-1})} \leq \kappa\, M_f$.

T3. $|\,T_t[f] - S_t[f]\,|_{C^0(\mathbb{U}_t + C\,t^{-1})} \leq \kappa\, M_f\, t^{-\mu+1}$,

T4. If $2 < \mu < \ell - 1$, then $|\,T_t[f]\,|_{C^2(\mathbb{U}_t + C\,t^{-1})} \leq \kappa\, M_f$.

T5. $|(T_t - \mathrm{Id})[\mathrm{f}]|_{C^r(\mathbb{U}_t)} \leq \kappa\, M_f\, t^{-(\mu-r-1)}$ *, for all* $0 \leq r \leq \mu - 1$.

T6. $|(T_\tau - T_t)\,[f]\,|_{C^0(\mathbb{U}_t + C\,t^{-1})} \leq \kappa\, M_f\, t^{-\mu+1}$ *, for all* $\tau \geq t \geq t^*$.

T7. If f is exact symplectic so is $T_t[f]$.

where M_f depends on ℓ, k, $|\Omega|_{C^\ell(\mathbb{U}+\rho)}$, and β, but it is independent of t.

Remark 17. *We remark that, in hypothesis H2 of Theorem 2, if $\mathbb{U} = \mathbb{T}^d$ then $\mathbb{V}$ can be chosen to be also $\mathbb{T}^d$. Actually, we asked Ω to be defined on a neighbourhood of $\mathbb{U}$ that contains the closure of $f(\mathbb{U})$, to guarantee that $S_t[f](\mathbb{U})$ is contained in the domain of definition of Ω, for t sufficiently large. And so $S_t[f]^*\Omega$ is defined on $\mathbb{U}$. If Ω is defined only on $\mathbb{U}$ and it cannot be extended to a neighbourhood of $\mathbb{U}$, then $S_t[f]^*\Omega$ is defined on $S_t[f]^{-1}(\mathbb{U})$. By modifying the definition of $\mathbb{U}_t$ in (7), the proof of Theorem 2 given in Section 2.4 also works in this latter case. However this just yields a more complicated notation and does not change the proof of Theorem 2. To avoid this notational complication we assume that Ω is defined on a neighbourhood of $\mathbb{U}$ as in H2 in Theorem 2.*

Theorem 3. *Let m, ℓ, $\mathbb{V}$, and $\mathbb{U}$ be as in Theorem 2. Let Ω be a contact form on $\mathbb{V}$ such that $|\Omega|_{C^\ell(\mathbb{V}+\rho)}\,, |d\,\Omega|_{C^\ell(\mathbb{V}+\rho)} < \infty$, for some $\rho > 0$. Assume that for any $\theta \in \mathscr{I}_{d\,\Omega}(Ker(\Omega))$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho')} < \infty$, the following holds:*

$$\left| \left(\mathscr{I}_{d\,\Omega}|_{Ker(\Omega)} \right)^{-1} \theta \right|_{C^0(\mathbb{U}+\rho')} \leq M_\Omega\ |\theta|_{C^0(\mathbb{U}+\rho')}\ .$$

Then, given $N, C, \beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa' = \kappa'(d, \ell, C, \beta, \mu, M_\Omega)$, and $t^{**} = t^{**}(d, \ell, \mathbb{V}, C, \beta, \mu, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)}\,, |d\,\Omega|_{C^\ell(\mathbb{U}+\rho)})$, and a family of – nonlinear – operators $\{T_t\}_{t \geq t^{**}}$, taking contact diffeomorphisms belonging to the set of diffeomorphisms $f \in \mathrm{Diff}^\ell(\mathbb{U})$ such that: i) $|f|_{C^\ell(\mathbb{U})} \leq \beta$ and ii) $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, into real analytic functions such that properties T0-T6 in Theorem 2 hold.

2.3 Analytic smoothing

This section contains the technical part of our proof of Theorem 2 and Theorem 3. We begin by collecting the properties of the operator S_t defined in Section 2.1 (see Remark 14). First we prove that S_t is a linear smoothing operator (see Definition 5) and then, using the fact that S_t is a linear smoothing operator, we show that given a k-form Ω, the C^0-norm of the k-form given by

$$S_t[f]^*\Omega - f^*\Omega \tag{8}$$

goes to zero as t goes to infinity. However, to prove Theorem 2 we need more accurate estimates. Actually, as we will see in Section 2.4, we need an estimate for the C^0-norm of (8) on complex strips, which is given in Proposition 26. To obtain such an estimate we extend the definition of S_t to k-forms and prove several analytic estimates which are given in Section 2.3.1. Estimates of particular importance are those given in Proposition 28, and in Proposition 34. Proposition 28 contains an estimate of the norm of $S_t[f]^*\Omega - S_t[f^*\Omega]$ on complex strips of width $C\,t^{-1}$ for arbitrary $C \geq 0$. In Proposition 34 we give an estimate of the difference between smoothing a composition of two functions and composing their smoothings.

To describe the behaviour of S_t we found very useful to write $S_t[f]$ in terms of the Taylor expansion of f, for $f \in C^\ell(\mathbb{R}^d)$. This is done in the following:

Lemma 18. *For any $f \in C^\ell(\mathbb{R}^d)$, with ℓ not an integer, we have*

$$S_t[f](z) = P_{f,\ell}(\mathrm{Re}\,(z)\,, i\ \mathrm{Im}\,(z)) + \hat{R}_{f,\ell}(z,t), \tag{9}$$

where

$$P_{f,\ell}(x,y) \stackrel{\text{def}}{=} \sum_{|k|_1<\ell} \frac{1}{k\,!} D^k f(x)\, y^k \,,$$

and

$$\left|\hat{R}_{f,\ell}(z,t)\right| \le \tilde{c}\, |f|_{C^\ell(\mathbb{R}^d)}\; t^{-\ell}\, e^{t\,|\mathrm{Im}(z)|}, \tag{10}$$

where $\tilde{c} = \tilde{c}(\ell, d)$.

Proof. Following [Sal04, Zeh75], we apply Taylor's theorem to f:

$$f(x+y) = P_{f,\ell}(x,y) + R_{f,\ell}(x;y)$$

where $R_{f,\ell}$ is the remainder. Then, using (4) and since S_t acts as the identity on polynomials, we have (9) with

$$\hat{R}_{f,\ell}(z,t) \stackrel{\text{def}}{=} \int s\left(\xi - i\,t\,\mathrm{Im}\,(z)\right)\; R_{f,\ell}(\mathrm{Re}\,(z)\,;\,\xi/t)\, d\,\xi.$$

We note that from the properties of s in Definition 6, for any $r, N > 0$ there exists a constant $c = c(r,N) > 0$ such that for all $k \in \mathbb{N}^d$ with $|k|_1 \le r$ then

$$|D^k s(z)| \le c\, (1 + |\,\mathrm{Re}\,(z)\,|)^{-N}\, e^{|\,\mathrm{Im}(z)|}\,.$$

Then, from Taylor's Theorem we have

$$\begin{aligned}
\left|\hat{R}_{f,\ell}(z,t)\right| &\le\; c\,t^{-\ell}\; |f|_{C^\ell(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\, s(\xi - i\,t\;\mathrm{Im}\,(z))|\; |\xi|^{\,\ell}\; d\xi \\
&\le t^{-\ell}\; |f|_{C^\ell(\mathbb{R}^d)}\; c\, e^{|t\;\mathrm{Im}(z)|} \int_{\mathbb{R}^d} \frac{|\xi|^{\,\ell}}{(1+|\xi|)^N} d\xi \\
&\le \tilde{c}\; |f|_{C^\ell(\mathbb{R}^d)}\; t^{-\ell}\, e^{|t\;\mathrm{Im}(z)|},
\end{aligned}$$

where we have fixed $N > \ell + d$. $\square$

Remark 19. *The constants appearing in our estimates depend on certain quantities. In particular if $f \in C^\ell(\mathbb{U})$, with $\mathbb{U}$ an open domain with smooth boundary, these constants also depend on $\mathbb{U}$. In what follows we do not write explicitly this dependence and represent a generic constant by κ.*

The following result ensures that S_t is an analytic linear smoothing operator in the sense of [Zeh75]. The case in which S_t is applied to functions in $C^0(\mathbb{R}^d)$ is proved in [Zeh75].

Proposition 20. *Let $1 < \ell < m$ with $\ell \notin \mathbb{N}$, $m \in \mathbb{N}$ and let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 2. Assume that S_t is as in Remark 14. Then, for any $C \ge 0$, there exists a constant $\kappa = \kappa(d,\ell,C)$ such that for all $t \ge 1$ and $f \in C^\ell(\mathbb{U})$ the following holds:*

1. $|(S_t - \mathrm{Id})\,[\mathrm{f}]\,|_{C^r(\mathbb{U})} \le \kappa\; |f|_{C^\ell(\mathbb{U})}\;\; t^{-\ell+r}\,, \qquad 0 \le r < \ell.$
2. $|S_t[f]\,|_{C^0(\mathbb{U}+C\,t^{-1})} \le\; \kappa\,|f|_{C^0(\mathbb{U})}.$
3. $|(S_\tau - S_t)[f]\,|_{\,C^0(\mathbb{U}+\tau^{-1})} \le \kappa\; |f|_{C^\ell(\mathbb{U})}\;\; t^{-\ell}$*, for all* $\tau \ge t$.
4. $|\,\mathrm{Im}\,(\,S_t[f]\,)|_{C^0(\mathbb{U}+C\,t^{-1})} \le \kappa\, C\, t^{-1}\; |f|_{C^\ell(\mathbb{U})}\,.$

Proof. We first prove Proposition 20 for functions in $C^\ell(\mathbb{U})$, with $\mathbb{U} \subset \mathbb{R}^d$ a compensated open domain with C^m-boundary. In this case S_t is defined by equation (6). The linearity of S_t follows from the linearity of the extension operator $\mathscr{E}^\ell_U$ in Theorem 1 and from the linearity of the convolution operator. To prove part 1, first notice that if $f \in C^\ell(\mathbb{U})$ and $x \in \mathbb{U}$ then

$\mathscr{E}^{\ell}_{\mathbb{U}}(f)(x) = f(x)$, then using the fact that part 1 holds for functions in $C^0(\mathbb{R}^d)$ and estimate (5) we have:

$$|\,(S_t - \mathrm{Id})[\mathrm{f}]\,|_{C^r(\mathbb{U})} = \left|\,(S_t - \mathrm{Id})[\mathscr{E}^{\ell}_{\mathbb{U}}(\mathrm{f})]\right|_{C^r(\mathbb{U})} \le \kappa' \,\left|\mathscr{E}^{\ell}_{\mathbb{U}}(f)\right|_{C^{\ell}(\mathbb{R}^d)} \; t^{-\ell+r} \le \kappa \; |\, f|_{C^{\ell}(\mathbb{U})} \; t^{-\ell+r}\,.$$

To prove part 2 we use (4) and Theorem 1 to obtain

$$|S_t[f]|_{C^0(\mathbb{R}^d + C\,t^{-1})} \le \left(\sup_{0\le\eta<C} \int_{\mathbb{R}^d} |s(\xi - i\,\eta)| \; d\xi\right) \left|\mathscr{E}^{\ell}_{U}(f)\right|_{C^0(\mathbb{R}^d)} \le \kappa \; |f|_{C^0(U)}\,.$$

Part 3 is a consequence of Lemma 18 and Theorem 1. To prove part 4 we use the Mean Value Theorem and the fact that the convolution commutes with the derivative to obtain

$$\begin{aligned}
|\,\mathrm{Im}\,(\,S_t[f]\,)|_{C^0(\mathbb{R}^d + C\,t^{-1})} &\le C\,t^{-1}\; |DS_t[f]|_{C^0(\mathbb{R}^d + C\,t^{-1})} \\
&\le C\,t^{-1} \left|D\mathscr{E}^{\ell}_{\mathbb{U}}(f)\right|_{C^0(\mathbb{R}^d)} \\
&\le C\,t^{-1} \left|\mathscr{E}^{\ell}_{\mathbb{U}}(f)\right|_{C^{\ell}(\mathbb{R}^d)}\,.
\end{aligned}$$

To prove Proposition 20 for $f \in C^{\ell}(\mathbb{T}^d)$ we use a lift of f. Let $f \in C^{\ell}(\mathbb{T}^d)$ with lift (see Definition 3) given by $P\,x + u(x)$, where P is a $d\times 1$ matrix with components in $\mathbb{Z}$ and $u \in C^{\ell}(\mathbb{R}^d)$ is a $\mathbb{Z}^d$-periodic function. Then (see part 3 in Remark 14):

$$(S_t - \mathrm{Id})\,[f](x) = P\,x + S_t[u](x) - (P\,x + u(x)) = (S_t - \mathrm{Id})\,[u]\,. \tag{11}$$

Moreover from Definition 4 one has

$$\begin{aligned}
|S_t[f]\,|_{C^0\left(\mathbb{T}^d + Ct^{-1}\right)} &\le |P| + |S_t[u]|_{C^0\left(\mathbb{R}^d + C\,t^{-1}\right)}\,, \\
(S_t - S_\tau)\,[f] &= (S_t - S_\tau)\,[u]\,, \\
|\mathrm{Im}\,(S_t[f])|_{C^0\left(\mathbb{T}^d + Ct^{-1}\right)} &\le |P|\; C\,t^{-1} + |\mathrm{Im}\,(S_t[u])|_{C^0\left(\mathbb{R}^d + C\,t^{-1}\right)}\,,
\end{aligned}$$

where S_t on the right hand side is given by (3). Hence properties 1-4 of Proposition 20 follow from the same properties for $u \in C^{\ell}(\mathbb{R}^d)$ and the fact that $S_t[u]$ is $\mathbb{Z}^d$-periodic. The annulus case $\mathbb{U} = \mathbb{T}^n \times U$, with $U \subset \mathbb{R}^{d-n}$ a bounded compensated open domain with C^m-boundary, is proved in a similar way. $\square$

Remark 21. *From (11) we have that in the case that $\mathbb{U} = \mathbb{T}^d$, a better estimate holds than that given in part 1 of Proposition 20:*

$$|(S_t - \mathrm{Id})\,[f]|_{C^0(\mathbb{U})} \le \kappa \; |u|_{C^0(\mathbb{R}^d)} \; t^{-\ell}\,,$$

where u is the periodic part of a lift of f. A similar result holds for $f \in C^{\ell}(\mathbb{T}^n \times U)$, with U a compensated open domain in $\mathbb{R}^{d-n}$ with C^m-boundary for some $\ell < m \in \mathbb{N}$.

Remark 22. *From the proof of Proposition 20 one notices that, if $\mathbb{U} \subset \mathbb{R}^d$ is an open domain with C^m-boundary, then the estimates in parts 2, 3, and 4 in Proposition 20 also hold if one replaces $\mathbb{U} + C\,t^{-1}$ with $\mathbb{R}^d + C\,t^{-1}$.*

Remark 23. *Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 2 and let $\mathbb{V} \subseteq \mathbb{R}^p$ be open, and assume that $f \in C^{\ell}(\mathbb{U}, \mathbb{V})$. Then for any $\Omega \in \Lambda^k(\mathbb{V})$ one has $f^*\Omega \in \Lambda^k(\mathbb{U})$. Notice that since the domain of definition of $\Omega \circ S_t[f]$ is $S_t[f]^{-1}(\mathbb{V})$, and since we know an estimate of $|S_t[f](x) - f(x)|$ only when $x \in \mathbb{U}$, to estimate the norm of the difference between $S_t[f]^*\Omega(x)$ and $f^*\Omega(x)$ we have to restrict x to be in $S_t[f]^{-1}(\mathbb{V}) \cap \mathbb{U} \subseteq \mathbb{U}$. It is not difficult to see that*

$$\mathbb{U} = \bigcup_{t\ge 1} \left(\,S_t[f]^{-1}(\mathbb{V}) \cap \mathbb{U}\,\right).$$

Furthermore,

$$S_t[f](\mathbb{U}) \subseteq \mathbb{V} \qquad \Longleftrightarrow \qquad S_t[f]^{-1}(\mathbb{V}) \cap \mathbb{U} = \mathbb{U}.$$

We consider functions $f : \mathbb{U} \to \mathbb{V}$, with $\mathbb{U}$ either $\mathbb{R}^d$ or as in H1 in Theorem 2, and $\mathbb{V}$ either $\mathbb{R}^p$ or H2 in Theorem 2. In Lemma 24 we prove that if moreover $\mathbb{V}$ contains the closure of $f(\mathbb{U})$ then, for t sufficiently large, $S_t[f]^*\Omega \in \Lambda^k(\mathbb{U})$.

Lemma 24. *Let $1 < \ell < m$ with $\ell \notin \mathbb{N}$ and $m \in \mathbb{N}$. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in Theorem 2, let $\mathbb{V}$ be either $\mathbb{V} \subseteq \mathbb{R}^p$ an open subset, or $\mathbb{T}^p$, or $\mathbb{T}^s \times V$, with $V \subset \mathbb{R}^{p-s}$ an open subset. Let $f \in C^\ell(\mathbb{U}, \mathbb{V})$, and assume that $\mathbb{V}$ contains the closure of $f(\mathbb{U})$. Then there exists $\bar{t} = \bar{t}(\ell, d, |f|_{C^\ell(\mathbb{U})}, \mathbb{V})$ such that for all $t \geq \bar{t}$ the following holds:*

1. *$S_t[f](\mathbb{U}) \subseteq \mathbb{V}$.*
2. *$S_t[f]\left(\mathbb{U} + C\,t^{-1}\right) \subseteq \mathbb{V} + (C\,\beta\,\kappa)\,t^{-1}$.*

Proof. To prove part 1 of Lemma 24 first, notice that from Remark 14 and part 6 of Remark 7, one has that $S_t[f](x)$ is real if x is real. Hence, if $\mathbb{V} = \mathbb{R}^p$, or $\mathbb{V} = \mathbb{T}^p$, we have $S_t[f](\mathbb{U}) \subseteq \mathbb{V}$. Now, assume that $\mathbb{V} \subsetneq \mathbb{R}^d$ is open. Then from part 1 in Proposition 20 we have that for $t \geq 1$, the following holds:

$$S_t[f](\mathbb{U}) \subset \left\{ y \in \mathbb{R}^d : \sup_{x \in \mathbb{U}} |y - f(x)| \leq \kappa\, |f|_{C^\ell(U)}\, t^{-\ell} \right\} . \tag{12}$$

Hence, if $\mathbb{V} \subsetneq \mathbb{R}^d$ is open and the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$, one has from (12) that for t sufficiently large $S_t[f](\mathbb{U}) \subset \mathbb{V}$. By taking coordinate functions, the case $\mathbb{V} = \mathbb{T}^s \times V$, with $V \subset \mathbb{R}^{p-s}$ an open subset, follows from the previous two cases.

Part 2 of Lemma 24 follows from part 1 of Lemma 24 and part 4 of Proposition 20. □

A consequence of Proposition 20 is the following.

Proposition 25. *Let $1 < \ell \notin \mathbb{N}$ and let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 2. Let $\mathbb{V}$ be as in Lemma 24, and let $\Omega \in \Lambda^k(\mathbb{V})$ be such that $|\,\Omega|_{C^1(\mathbb{V}+\rho)} < \infty$, for some $\rho > 0$. Let $f \in C^\ell(\mathbb{U}, \mathbb{V})$, and assume that $\mathbb{V}$ contains the closure of $f(\mathbb{U})$. Then there exist two positive constants $\kappa = \kappa(d, \ell, k, |\,\Omega|_{C^1(\mathbb{V}+\rho)})$ and $\bar{\bar{t}} = \bar{\bar{t}}(d, \ell, \rho, |f|_{C^\ell(\mathbb{U})}, \mathbb{V})$ such that for all $t \geq \bar{\bar{t}}$ the following holds:*

1. *$|\,S_t[f]^*\Omega - f^*\Omega\,|_{C^0(\mathbb{U})} \leq \kappa\,\left(t^{-k(\ell-1)}\,|f|^k_{C^\ell(\mathbb{U})} + t^{-\ell}\,|f|_{C^\ell(\mathbb{U})}\right)$.*
2. *$|\,S_t[f]^*\Omega\,|_{C^0(\mathbb{U}+t^{-1})} \leq \kappa\,|f|^k_{C^\ell(\mathbb{U})}$.*

Proof. Assume that Ω has the following form:

$$\Omega(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq p} \Omega_{\mathbf{i}}(x)\, dx_{\mathbf{i}} .$$

Since $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, we have that part 1 of Lemma 24 implies that, for index $\mathbf{i} = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq p$, $\Omega_{\mathbf{i}} \circ S_t[f]$ is defined on $\mathbb{U}$ for all $t \geq \bar{t}$, where $\bar{t}$ is as in Lemma 24. Hence for $t \geq \bar{t}$ the following holds:

$$(S_t[f]^*\Omega)\,(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq p} \Omega_{\mathbf{i}}(S_t[f](x))\; S_t[f]^*\, dx_{\mathbf{i}}\,, \qquad \forall x \in \mathbb{U}. \tag{13}$$

Then part 1 follows from Proposition 20 and the following equality

$$\begin{aligned}(S_t[f]^*\Omega \,-\, f^*\Omega)\,(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq p} &\big[\,\Omega_{\mathbf{i}}(f(x))\,\{\,S_t[f] - f\}^*\, dx_{\mathbf{i}} \\ &+ \{\Omega_{\mathbf{i}} \circ S_t[f] - \,\Omega_{\mathbf{i}} \circ f\}\,(x)\, S_t[f]^*\, dx_{\mathbf{i}}\,\big]\,,\end{aligned}$$

for $x \in \mathbb{U}$, where we have used the equality $\{\,f^* - g^*\,\}\; dx_{\mathbf{i}} = \{\,f - g\,\}^*\; dx_{\mathbf{i}}$, which is true because the k-form $dx_{\mathbf{i}}$ does not depend on the base point.

Let us prove part 2 of Proposition 25. From part 4 of Proposition 20 and Lemma 24 we have that

$$S_t[f]\left(\mathbb{U}+t^{-1}\right) \subseteq S_t[f](\mathbb{U})+\kappa\, t^{-1}\, |f|_{C^\ell(\mathbb{U})} \subseteq \mathbb{V}+\kappa\, t^{-1}\, |f|_{C^\ell(\mathbb{U})}\,.$$

Hence, if $t \geq \bar{t}$ is sufficiently large so that $t^{-1}\kappa\, |f|_{C^\ell(\mathbb{U})} < \rho$, then for any multi-index $\mathbf{i} = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq p$, the following holds:

$$|\,\Omega_{\mathbf{i}} \circ S_t[f]\,|_{C^0(\mathbb{U}+t^{-1})} \leq |\,\Omega_{\mathbf{i}}\,|_{C^0(S_t[f](\mathbb{U}+t^{-1}))} \leq |\,\Omega_{\mathbf{i}}\,|_{C^0(\mathbb{V}+\rho)}\,, \tag{14}$$

Hence, part 2 of Proposition 25 follows from (13) and (14). □

To prove Theorem 2 and Theorem 3 we need more accurate estimates than those given in Proposition 25. Actually (see Section 2.4 and Section 2.5) we need an estimate for

$$|S_t[f]^*\Omega \;-\; f^*\Omega|_{C^0(\mathbb{U}+C\,t^{-1})}\,,$$

with $C \geq 0$, in the case that both Ω and $f^*\Omega$ are real analytic k-forms. This is given in the following.

Proposition 26. *Let $1 < \ell < m$, with $\ell \notin \mathbb{N}$ and $m \in \mathbb{N}$. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 2 and let $\mathbb{V}$ be either $\mathbb{R}^p$, $\mathbb{T}^p$, or $\mathbb{T}^s \times V$, $V \subset \mathbb{R}^{p-s}$ a compensated open domain with C^m-boundary, or $\mathbb{V} \subset \mathbb{R}^p$ a compensated open domain with C^m-boundary.*

Assume that $\Omega \in \Lambda^k(\mathbb{V})$ and $\tilde{\Omega} \in \Lambda^k(\mathbb{U})$ are two real analytic k-forms such that $|\Omega|_{C^\ell(\mathbb{V}+\rho)}$, $\left|\tilde{\Omega}\right|_{C^\ell(\mathbb{U}+\rho)} < \infty$, for some $\rho > 0$. Then, for each $C \geq 0, \beta > 0$, and $0 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, p, \ell, C, \beta, \mu, k)$ and $\hat{t} = \hat{t}(d, p, \ell, \mathbb{V}, C, \beta, \mu)$ such that for all $f \in C^\ell(\mathbb{U}, \mathbb{V})$ satisfying: i) the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$, ii) $|f|_{C^\ell(\mathbb{U})} \leq \beta$, and iii) $f^*\Omega = \tilde{\Omega}$, the following holds:

$$\left|\, S_t[f]^*\Omega \;-\; \tilde{\Omega}\,\right|_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\, \hat{M}_f\, t^{-\mu}\,, \qquad \forall \quad t \geq \hat{t}\,,$$

where $\hat{M}_f$ depends on k, $|\Omega|_{C^\ell(\mathbb{U}+\rho)}$, $\left|\tilde{\Omega}\right|_{C^\ell(\mathbb{U}+\rho)}$, and β, but is independent of t.

To prove Proposition 26 we extend the definition of the analytic smoothing operator S_t to k-forms in the following way. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 2. Let $\tilde{\Omega} \in \Lambda^k(\mathbb{U})$ be of the form:

$$\tilde{\Omega}(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq d} \tilde{\Omega}_{\mathbf{i}}(x)\, dx_{\mathbf{i}}\,,$$

with $\tilde{\Omega}_{\mathbf{i}} \in C^\ell(\mathbb{U})$ for all $1 \leq i_1 < \cdots < i_k \leq d$, define the k-form $S_t[\tilde{\Omega}] \in \Lambda^k(\mathbb{U})$ by

$$S_t[\,\tilde{\Omega}] \stackrel{\text{def}}{=} \sum_{1 \leq i_1 < \cdots < i_k \leq d} S_t[\,\Omega_{\mathbf{i}}]\, dx_{\mathbf{i}}. \tag{15}$$

Notice that

$$S_t[f]^*\Omega \;-\; \tilde{\Omega} = \{S_t[f]^*\Omega \;-\; S_t[f^*\Omega]\,\} \;+\; (\,S_t - \mathrm{Id}\,)\,[\,\tilde{\Omega}\,]\,, \tag{16}$$

so to prove Proposition 26 it suffices to estimate the norm of the differences on the right hand side of (16). An estimate of the norm of the second difference on the right hand side of (16) follows from the following lemma and (15).

Lemma 27. *Let $1 < \ell \notin \mathbb{N}$ and let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in H1 in Theorem 2. Then, there exists a constant $\kappa = \kappa(d, \ell, C)$ such that if $g \in \mathcal{A}(\mathbb{U} + C\,t^{-1}, C^\ell)$ then the following holds:*

$$|(S_t - \mathrm{Id})[\mathrm{g}]\,|_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\; |g|_{C^\ell(\mathbb{U}+C\,t^{-1})}\;\, t^{-\ell}. \tag{17}$$

Proof. First one proves Lemma 27 in the case that $\mathbb{U} \subseteq \mathbb{R}^d$ is either $\mathbb{R}^d$ or a compensated open domain with C^m-boundary. Then the cases $g \in \mathcal{A}(\mathbb{T}^d + C\,t^{-1}, C^\ell)$ and $g \in \mathcal{A}(\mathbb{T}^n \times U + C\,t^{-1}, C^\ell)$, with U a compensated open domain in $\mathbb{R}^{d-n}$ with C^m-boundary, follow by taking a lift of g and using that (17) holds for the periodic (respectively partially periodic) part of the lift of g (see Remarks 8, 13, and 14).

We prove Lemma 27 in the case that $\mathbb{U} \subset \mathbb{R}^d$ is a compensated open domain with C^m-boundary. The case $\mathbb{U} = \mathbb{R}^d$ is proved in the same way. From Lemma 18 and Theorem 1 we have that if $z \in \mathbb{U} + C\,t^{-1}$ then

$$|\, S_t[g](z) - P_{g,\ell}(\mathrm{Re}\,(z)\,, i\;\mathrm{Im}\,(z))| \,\leq e^C\,\tilde{c}\,c_U\;|g|_{C^\ell(\mathbb{U})}\;t^{-\ell}\,. \tag{18}$$

Moreover, from the Taylor Theorem we have for all $z \in \mathbb{U} + C\,t^{-1}$

$$|\,g(z) \;-\; P_{g,\ell}(\mathrm{Re}\,(z)\,, i\,\mathrm{Im}\,(z))\,|\;\leq \hat{c}\;|g|_{C^\ell(\mathbb{U}+C\,t^{-1})}\;|\,\mathrm{Im}\,(z)\,|^\ell \tag{19}$$

for some constant $\hat{c}$. Therefore Lemma 27 follows from (18) and (19). □

Giving an estimate for the norm of the first difference on the right hand side of (16) is more intricate. In Section 2.3.1 we give several results from which the following proposition follows easily (see Section 2.3.2).

Proposition 28. *Assume that ℓ, m, $\mathbb{U}$, $\mathbb{V}$ and $\Omega \in \Lambda^k(\mathbb{V})$ are as in Proposition 26. Then, for each each $C \geq 0, \beta > 0$, and $0 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, p, \ell, C, \beta, \mu, k)$ and $\tilde{t} = \tilde{t}(d, p, \ell, \mathbb{V}, C, \beta, \mu)$ such that for all $f \in C^\ell(\mathbb{U}, \mathbb{V})$ satisfying i) and ii) in Proposition 26, and for all $t \geq \tilde{t}$, the following holds:*

$$|S_t\,[f^*\Omega] \;-\; S_t\,[f]^*\,\Omega\,|_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\,\tilde{M}_f\;t^{-\mu}\,,$$

where $\tilde{M}_f$ depends on $|\Omega|_{C^\ell(\mathbb{V}+\rho)}$, and $|f|_{C^\ell(\mathbb{U})}$, but is independent of t.

2.3.1 Analytic estimates

In this section we give analytic estimates of certain quantities that enable us to estimate the norm on complex strips of the difference between $S_t[f]^*\Omega$ and $S_t[f^*\Omega]$: Since the pull-back involves the composition and the multiplication of functions, the quantities to be estimated depend on the norm of the difference between:

i) Smoothing a multiplication of two functions and multiplying their smoothings (Lemma 30),

ii) Smoothing a composition of two functions and composing their smoothings (Proposition 34).

Let us start by estimating the C^ℓ-norms on complex strips of the product and composition of functions in terms of the C^ℓ-norms of the original functions.

Lemma 29. *1. Let $\mathbb{U}$ be either $\mathbb{R}^d$ or as in Theorem 2. Assume that $g_1, \ldots, g_k \in C^r(\mathbb{U} + \rho)$, for some $\rho \geq 0$. Then, the following holds:*

$$|\,g_1\,g_2\,|_{C^r(\mathbb{U}+\rho)} \leq \kappa\,\left(|g_1|_{C^r(\mathbb{U}+\rho)}\;|g_2|_{C^0(\mathbb{U}+\rho)} \,+\, |g_1|_{C^0(\mathbb{U}+\rho)}\;|g_2|_{C^r(\mathbb{U}+\rho)}\right) \tag{20}$$

and

$$|\,g_1\,g_2\,\ldots g_k|_{C^r(\mathbb{U}+\rho)} \leq \kappa\,\sum_{i=1}^{k}\left(|g_i|_{C^r(\mathbb{U}+\rho)}\prod_{\substack{j\in\{1,\ldots,k\,\}\\ j\neq i}}|g_j|_{C^0(\mathbb{U}+\rho)}\right). \tag{21}$$

2. Let $W \subset \mathbb{C}^n$ and $Z \subset \mathbb{C}^p$ be compensated domains (Definition 15), $s, \sigma \geq 0$, and $h \in \mathcal{A}(Z, C^s)$. Assume that, $f \in \mathcal{A}(W, C^\sigma)$ is such that $f(W) \subset Z$, then:

(a) *If* $\max(s,\sigma)<1$, *then* $h\circ f\in\mathcal{A}(W,C^{s\,\sigma})$ *and*

$$|\,h\circ f|_{C^{s\,\sigma}(W)}\leq |h|_{C^s(Z)}\;|f|^s_{C^\sigma(W)}\;+|h|_{C^0(Z)}\;.$$

(b) *If* $\max(s,\sigma)\geq 1$, *then* $h\circ f\in\mathcal{A}(W,C^\ell)$, *with* $\ell=\min(s,\sigma)$. *Moreover*

i. *If* $0\leq s<1\leq\sigma$, *then*

$$|\,h\circ f|_{C^{\,s}(W)}\leq\kappa\;|h|_{C^s(Z)}\;|f|^s_{C^1(W)}\;+|h|_{C^0(Z)}\;.$$

ii. *If* $0\leq\sigma\leq 1\leq s$, *then*

$$|h\circ f|_{C^\sigma(W)}\leq\kappa\;|h|_{C^1(Z)}\;|f|_{C^\sigma(W)}\;+|h|_{C^0(Z)}\;.$$

iii. *If* $\ell=\min(s,\sigma)\geq 1$, *then*

$$|h\circ f|_{C^\ell(W)}\leq\kappa\;|h|_{C^\ell(Z)}\;\left(1+|f|^\ell_{C^\ell(W)}\right)\;.$$

Proof. To prove estimate (20) use the Leibniz's rule to write the derivative of the product function $h=g_1\,g_2$ in terms of the derivatives of g_1 and g_2 and use the interpolation estimates [dlLO99, Zeh75]. Estimate (21) follows from (20). Part 2 follows from Theorem 4.3 in [dlLO99]. □

In the following lemma we give an estimate for the norm of the difference between smoothing a multiplication of two functions and multiplying their smoothings.

Lemma 30. *Let* $1<\ell<m$, *with* $m\in\mathbb{N}$ *and* $\ell\notin\mathbb{N}$, *and let* $\mathbb{U}$ *be either* $\mathbb{R}^d$ *or as in H1 in Theorem 2. Then for each* $C\geq 0$, $0\leq\mu<\ell$ *and* $r\in(0,1)$, *with* $0<r+\mu<\ell$, *there exists a constant* $\kappa=\kappa(d,\ell,C\,,\mu,r)$, *such that for all* $t\geq e^{1/r}$ *satisfying*

$$t^{-1}(C+r\log(t))\leq 1\,, \tag{22}$$

the following holds:

1. $|\,S_t[g]\,|_{C^{\,\mu}(\mathbb{U}+C\,t^{-1})}\leq\kappa\;|g|_{C^\ell(\mathbb{U})}\;,\quad$ *for* $g\in C^\ell(\mathbb{U})$.

2. *For* $g_1,g_2\in C^\ell(\mathbb{U})$

$$|\,S_t[g_1]\,S_t[g_2]\,|_{C^{\,\mu}(\mathbb{U}+C\,t^{-1})}\leq\kappa\;\left(|g_1|_{C^0(\mathbb{U})}\;|g_2|_{C^\ell(\mathbb{U})}\;+|g_1|_{C^\ell(\mathbb{U})}\;|g_2|_{C^0(\mathbb{U})}\right).$$

3. *For* $g_1,g_2\in C^\ell(\mathbb{U})$

$$|\,S_t[\,g_1\,g_2]-S_t[g_1]\,S_t[g_2]\,|_{C^0(\mathbb{U}+C\,t^{-1})}\leq\;\kappa\;\left(|g_1|_{C^0(\mathbb{U})}\;|g_2|_{C^\ell(\mathbb{U})}\;+|g_1|_{C^\ell(\mathbb{U})}\;|g_2|_{C^0(\mathbb{U})}\right)\;t^{-\mu}\,.$$

Proof. To prove part 1 of Lemma 30, one first proves that it holds for $g\in C^\ell(\mathbb{U})$, when $\mathbb{U}$ is either $\mathbb{R}^d$ or an open domain with C^m-boundary. That part 1 of Lemma 30 holds for $g\in C^\ell(\mathbb{U})$, when $\mathbb{U}$ is either $\mathbb{T}^d$ or $\mathbb{T}^n\times U$, with $U\subset\mathbb{R}^{d-n}$ an open domain with C^m-boundary, follows by taking a lift of g, applying part 1 of Lemma 30 to the periodic (respectively, partially periodic) part of the lift of g, and using the norms introduced in Definition 4 (see Remarks 8, 13, and 14).

We only prove part 1 of Lemma 30 in the case that $\mathbb{U}$ is an open domain with C^m-boundary. The case $\mathbb{U}=\mathbb{R}^d$ is proved in the same way. For $t\geq 1$, define $\rho(t)=t^{-1}\,(C+r\,\log(t))$ and let k be such that $2^k\leq t<2^{k+1}$. Using Lemma 18 and Theorem 1 one proves that if $g\in C^\ell(\mathbb{U})$ then the following estimates hold:

$$\begin{aligned}|(S_{2t}-S_t)\,[g]\,|_{C^0(\mathbb{R}^d+\rho(2\,t))}&\leq\;\kappa\;|g|_{C^\ell(\mathbb{U})}\;t^{-\ell+r}\;,\\ |(S_t-S_{2^k})\,[g]\,|_{C^0(\mathbb{R}^d+\rho(t))}&\leq\kappa\;|g|_{C^\ell(\mathbb{U})}\;2^{-k\ell}\,t^r\,.\end{aligned}$$

Then part 1 of Lemma 30 in the case that $\mathbb{U}$ is an open subset of $\mathbb{R}^d$ with C^m-boundary, follows using Cauchy's estimates and the following inequality:

$$\begin{aligned}|S_t[g]|_{C^\mu(\mathbb{R}^d+t^{-1}C)} \leq |(S_t - S_{2^k})\,[g]|_{C^\mu(\mathbb{R}^d+t^{-1}C)} &+\\ + \sum_{j=1}^{k} |(S_{2^j} - S_{2^{j-1}})\,[g]|_{C^\mu(\mathbb{R}^d+t^{-1}C)} &+\\ + |S_1[g]\,|_{C^\mu(\mathbb{R}^d+t^{-1}C)}\,.\end{aligned}$$

Part 2 of Lemma 30 follows from estimate (20), and part 1 of Lemma 30. To prove part 3 of Lemma 30 write

$$\begin{aligned}S_t\,[g_1g_2] - S_t[g_1]\,S_t[g_2] = S_t\,[\,(\mathrm{Id} - S_t)\,[g_1]\,g_2\,] &+\\ + S_t\,[\,S_t[g_1]\,\,(Id -\,S_t)\,[g_2]] &+\\ + (S_t - 1)\,[\,S_t[g_1]\,S_t\,[g_2]\,]\,.&\end{aligned} \tag{23}$$

Part 2 and part 1 of Proposition 20 imply

$$\begin{aligned}|\,S_t\,[\,(1 - S_t)\,[g_1]\,g_2\,]\,|_{C^0(\mathbb{U}+C\,t^{-1})} &\leq \kappa\,\,|(1 - S_t)\,[g_1]\,g_2\,|_{C^0(\mathbb{U})}\\ &\leq \kappa\,\,|g_2\,|_{C^0(\mathbb{U})}\,\,|\,(1 - S_t)\,[g_1]|_{C^0(\mathbb{U})}\\ &\leq \kappa\,\,|g_2\,|_{C^0(\mathbb{U})}\,\,|\,g_1|_{C^\ell(\mathbb{U})}\,t^{-\ell}.\end{aligned} \tag{24}$$

and

$$\begin{aligned}|\,S_t\,[\,S_t[g_1]\,\,(1 -\,S_t)\,[g_2]]\,|_{C^0(\mathbb{U}+C\,t^{-1})} &\leq \kappa\,\,|\,S_t[g_1]\,\,(1 -\,S_t)\,[g_2]\,|_{C^0(\mathbb{U})}\\ &\leq \kappa\,\,|\,S_t[g_1]\,|_{C^0(\mathbb{U})}\,\,|(1 -\,S_t)\,[g_2]\,|_{C^0(\mathbb{U})}\\ &\leq \kappa\,|\,g_1|_{C^0(\mathbb{U})}\,\,|g_2\,|_{C^\ell(\mathbb{U})}\,\,t^{-\ell}.\end{aligned} \tag{25}$$

Moreover, because of part 2 of Lemma 30 we have $S_t[g_1]\,S_t[g_1] \in \mathcal{A}(\mathbb{R}^d + C\,t^{-1}\,, C^\mu)$, then Lemma 27 and part 2 of Lemma 30 imply

$$\begin{aligned}|\,(S_t - 1)\,[\,S_t[g_1]\,S_t[g_2]]\,|_{C^0(\mathbb{U}+C\,t^{-1})} &\leq \kappa\,|\,S_t[g_1]\,S_t[g_2]\,|_{C^{\,\mu}(\mathbb{R}^d+C\,t^{-1})}\,\,t^{-\mu}\\ &\leq \kappa\,\left(|g_1|_{C^0(\mathbb{U})}\,\,|g_2|_{C^\ell(\mathbb{U})}\,+|g_1|_{C^\ell(\mathbb{U})}\,\,|g_2|_{C^0(\mathbb{U})}\right)\,\,t^{-\mu}\end{aligned} \tag{26}$$

Hence part 3 of Lemma 30 follows from equality (23) and estimates (24), (25), and (26). □

We emphasize that the proof of Lemma 30 is based on the linearity of S_t. As a consequence of Lemma 30 we have the following.

Lemma 31. *Let ℓ and $\mathbb{U}$ be as in Lemma 30. Let k, n be two non-negative integers such that $0 \leq n + k \leq d$. For each $0 \leq \mu < \ell$, $C \geq 0$ and $r \in (0,1)$, with $0 < r + \mu < \ell$, there exists a constant $\kappa = \kappa(d, \ell, C, \mu, r, k, n)$, such that for all $\vartheta \in \Lambda^n(\mathbb{U})$ and $\alpha \in \Lambda^k(\mathbb{U})$ with*

$$|\vartheta|_{C^\ell(\mathbb{U})} < \infty\,, \qquad \textit{and} \qquad |\alpha|_{C^\ell(\mathbb{U})} < \infty\,,$$

and for all $t \geq e^{1/r}$ satisfying (22) the following holds:

$$|\,S_t[\vartheta]\,\wedge S_t[\alpha]\,-\,S_t[\vartheta\wedge\alpha\,]\,|_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\,\left(|\vartheta|_{C^0(\mathbb{U})}\,\,|\alpha|_{C^\ell(\mathbb{U})} + |\vartheta|_{C^\ell(\mathbb{U})}\,\,|\alpha|_{C^0(\mathbb{U})}\right)t^{-\mu}\,.$$

Proof. Let $\vartheta \in \Lambda^n(\mathbb{U})$ and $\alpha \in \Lambda^k(\mathbb{U})$ be given by

$$\vartheta(x) = \sum_{1\leq i_1<\cdots<\,i_n\leq d} \vartheta_{\mathbf{i}}(x)\,dx_{\mathbf{i}}\,, \qquad \alpha(x) = \sum_{1\leq j_1<\cdots<\,j_k\leq d} \alpha_{\mathbf{j}}(x)\,dx_{\mathbf{j}}\,,$$

with $\vartheta_{\mathbf{i}}, \alpha_{\mathbf{j}} \in C^\ell(\mathbb{U})$ for all $\mathbf{i} = (i_1, \dots, i_n)$, with $1 \le i_1 < i_2 < \cdots < i_n \le d$, and $\mathbf{j} = (j_1, \dots, j_k)$, with $1 \le j_1 < \cdots < j_k \le d$. Then, performing some simple computations one obtains

$$S_t[\vartheta] \wedge S_t[\alpha] - S_t[\vartheta \wedge \alpha](x)(\xi_1, \dots, \xi_{m+k}) = \\ = \sum_{\sigma \in \mathcal{S}(n,k)} \sum_{\substack{1 \le i_1 < \cdots < i_n \le d \\ 1 \le j_1 < \cdots < j_k \le d}} c_{\mathbf{i},\mathbf{j}}(x)\, dx_{\mathbf{i}} \left(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)}\right) dx_{\mathbf{j}}(\xi_{\sigma(n+1)}, \dots \xi_{\sigma(n+k)}),$$

where $\mathcal{S}(n,k)$ represents the set of all permutations σ of $\{1, 2, \dots, n+k\}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\sigma(n+1) < \cdots < \sigma(n+k)$, and

$$c_{\mathbf{i},\mathbf{j}}(x) \stackrel{\text{def}}{=} (\, S_t[\vartheta_{\mathbf{i}}]\, S_t[\alpha_{\mathbf{j}}] - S_t[\,\vartheta_{\mathbf{i}}\,\alpha_{\mathbf{j}}]\,)\,(x)\,.$$

Hence the proof is finished applying part 3 of Lemma 30. □

The following lemma is the same result as Proposition 28, applied to the components of $\Lambda^k(\mathbb{V})$ with respect to the basis: $dx_{\mathbf{i}} = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, with $1 \le i_1 < \cdots < i_k \le p$.

Lemma 32. . *Let ℓ, $\mathbb{U}$ and $\mathbb{V}$ be as in Lemma 24. For each natural number $1 \le k \le p$, and all real numbers $0 \le \mu < \ell - 1$, $C \ge 0$, and $r \in (0,1)$, satisfying: $0 < r + \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \mu, r, k)$ and $t_0 = t_0(d, \ell, \beta, \mathbb{V}, r)$ such that for any $f \in C^\ell(\mathbb{U}, \mathbb{V})$ satisfying i) the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$, ii) $|f|_{C^\ell(\mathbb{U})} \le \beta$, and any multi-index $\mathbf{i} = (i_1, \dots, i_k)$, with $1 \le i_1 < \cdots < i_k \le p$, the following holds for all $t \ge t_0$ satisfying (22)*

$$|S_t\,[f^* dx_{\mathbf{i}}] \,-\, S_t\,[f]^*\, dx_{\mathbf{i}}\,|_{C^0(\mathbb{U}+C\,t^{-1})} \le \kappa\, |f|^k_{C^\ell(\mathbb{U})}\;\; t^{-\mu}\,.$$

Proof. Let $\mathbf{i} = (i_1, \dots, i_k)$ be a multi-index with $1 \le i_1 < \cdots < i_k \le p$, and let $f \in C^\ell(\mathbb{U}, \mathbb{V})$ be such that the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$ and $|f|_{C^\ell(\mathbb{U})} \le \beta$. Performing some computations and using the linearity of S_t one obtains

$$S_t\,[f^* dx_{\mathbf{i}}] \,-\, S_t\,[f]^*\, dx_{\mathbf{i}} \,= \sum_{n=1}^{k-1} \alpha_n \,\wedge \varphi_n\,, \tag{27}$$

where, for $n \in \{1, \dots, k-1\}$,

$$\varphi_n \stackrel{\text{def}}{=} \begin{cases} S_t[f^* dx_{i_{n+2}}] \wedge \cdots \wedge\, S_t[f^* dx_{i_k}]\,, & n \in \{1, \dots, k-2\} \\ 1\,, & n = k-1 \end{cases}\,,$$

and

$$\alpha_n \stackrel{\text{def}}{=} S_t\left[\,\vartheta_n \wedge\, f^* dx_{i_{n+1}}\,\right] \,-\, S_t[\,\vartheta_n] \,\wedge\, S_t[f^* dx_{i_{n+1}}]\,, \tag{28}$$

where

$$\vartheta_n \stackrel{\text{def}}{=} f^*\,(dx_{i_1} \wedge\, \cdots \wedge\, dx_{i_n})\,. \tag{29}$$

Notice that, because of part 2 of Proposition 20, for all $n \in \{1, \dots, k-2\}$ the following estimate holds:

$$|\,\varphi_n\,|_{C^0(\mathbb{U}+C\,t^{-1})} \le\, \kappa \left(|\,S_t[Df]\,|_{C^0(\mathbb{U}+C\,t^{-1})}\right)^{k-(n+1)} \le \kappa\, \left(|Df|_{C^0(\mathbb{U})}\right)^{k-(n+1)}\,.$$

Hence using (27) one has

$$|S_t\,[f^* dx_{\mathbf{i}}] \,-\, S_t\,[f]^*\, dx_{\mathbf{i}}\,|_{C^0(\mathbb{U}+C\,t^{-1})} \le |\,\alpha_{k-1}|_{C^0(\mathbb{U}+C\,t^{-1})} + \kappa \sum_{n=1}^{k-2} |\,\alpha_n\,|_{C^0(\mathbb{U}+C\,t^{-1})}\, |Df|^{k-(n+1)}_{C^0(\mathbb{U})}\,, \tag{30}$$

where we assumed, without loss of generality, that $\kappa \geq 1$. Moreover, from (21) and (29) we have for all $n \in \{1, \dots, k-1\}$

$$|\vartheta_n|_{C^{\ell-1}(\mathbb{U})} \leq \kappa \; n \; |Df|^{n-1}_{C^0(\mathbb{U})} \; |Df|_{C^{\ell-1}(\mathbb{U})} < \infty,$$

and

$$\left| \, f^* dx_{i_{n+1}} \right|_{C^{\ell-1}(\mathbb{U})} \leq \kappa \; |Df|_{C^{\ell-1}(\mathbb{U})} < \infty \, ,$$

for some constant κ. Hence, using (28) and Lemma 31, we have that given $0 \leq \mu < \ell - 1$, $C \geq 0$, and $r \in (0,1)$, with $0 < r < \ell - 1 - \mu$, there exists a constant $\kappa = \kappa(d, \ell, \mu, r, C)$, such that

$$\begin{aligned} |\, \alpha_n \,|_{C^0(\mathbb{U} + C\, t^{-1})} &\leq \kappa \, \left(|\, \vartheta_n \,|_{C^0(\mathbb{U})} \; \left| f^* dx_{i_{n+1}} \right|_{C^{\ell-1}(\mathbb{U})} + |\, \vartheta_n \,|_{C^{\ell-1}(\mathbb{U})} \; \left| \, f^* dx_{i_{n+1}} \, \right|_{C^0(\mathbb{U})} \right) \, t^{-\mu} \\ &\leq \kappa \, (n+1) \; |Df|^n_{C^0(\mathbb{U})} \; |Df|_{C^{\ell-1}(\mathbb{U})} \; t^{-\mu} \end{aligned} \tag{31}$$

Hence Lemma 32 follows from (30) and (31). □

In order to prove Proposition 28 for an arbitrary k-form we need an estimate for the norm of the difference between the composition of the smoothing and the smoothing of the composition. This was considered in [GEV] for functions in $C^\ell(\mathbb{R}^d)$. We use the following.

Lemma 33. *Let ℓ, m, $\mathbb{U}$ and $\mathbb{V}$ be as in Proposition 26. Given $0 < \mu < \ell$ and $C \geq 0$, $\beta > 0$ there exist two constants $\kappa = \kappa(d, p, \ell, C, \mu, \beta)$ and $t_1 = t_1(p, \ell, \mathbb{V}, C, \beta, \mu)$ such that for each $h \in C^\ell(\mathbb{V})$ and $f \in C^\ell(\mathbb{U}, \mathbb{V})$, satisfying i) the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$ and ii) $|f|_{C^\ell(\mathbb{U})} \leq \beta$, the following holds for all $t \geq t_1$:*

$$|\, S_t[h] \circ S_t[f]|_{C^\mu(\mathbb{U} + C\, t^{-1})} \leq \kappa \; |h|_{C^\ell(\mathbb{V})} \left(1 + |f|^\tau_{C^\ell(\mathbb{U})} \right) \, , \tag{32}$$

where

$$\begin{array}{llll} \tau & \textit{is any number in} \;\; (\mu, 1) \, , & \textit{if} & 0 < \mu < 1 < \ell \, , \\ \tau = \mu \, , & & \textit{if} & 1 \leq \mu < \ell \, . \end{array}$$

Proof. Let $0 \leq s, \sigma < \ell$, fix $r_1, r_2 \in (0,1)$ in such a way that $0 \leq s + r_1 < \ell$, and $0 \leq \sigma + r_2 < \ell$ (e.g. $r_1 = \min(1/2, (\ell - s)/2)$, $r_2 = \min(1/2, (\ell - \sigma)/2)$). Let κ be as in Proposition 20 and assume that $t \geq \max\left(e^{1/r_1}, e^{1/r_2}\right)$ is sufficiently large such that

$$t^{-1} \, \max\left(C + r_2 \, \log(t) \, , C\, \kappa\beta + r_1 \, \log(t) \,\right) \leq 1.$$

Then Lemma 30 implies for $h \in C^\ell(\mathbb{V})$ and $f \in C^\ell(\mathbb{U}, \mathbb{V})$,

$$\begin{aligned} |\, S_t[f] \,|_{C^\sigma(\mathbb{U} + C\, t^{-1})} &\leq \kappa(d, \ell, C, \sigma, r_2) \; |f|_{C^\ell(\mathbb{U})} \, , \qquad 0 \leq \sigma < \ell \\ |\, S_t[h]| \;_{C^s(\mathbb{V} + (C\, \beta\, \kappa)\, t^{-1})} &\leq \kappa(p, \ell, C, s, r_1) \; |h|_{C^\ell(\mathbb{V})} \, , \qquad 0 \leq s < \ell, \end{aligned} \tag{33}$$

Hence for all $0 \leq s, \sigma < \ell$ there exists $\tilde{t}_1 = \tilde{t}_1(p, \ell, C, \beta, s, \sigma)$ such that, for all $t \geq \tilde{t}_1$,

$$\begin{aligned} S_t[h] &\in \mathcal{A}(\mathbb{V} + (C\, \beta\, \kappa)\; t^{-1}, C^s) \, , \\ S_t[f] &\in \mathcal{A}(\mathbb{U} + C\, t^{-1}, C^\sigma). \end{aligned} \tag{34}$$

Inclusions in (34) and part 2 of Lemma 24 enable us to apply part 2 of Lemma 29 to the composition $S_t[h] \circ S_t[f]$ as follows. If $1 \leq \mu < \ell$, estimate (32) follows from estimates (33), and part 2(b)iii of Lemma 29. Finally, if $0 < \mu < 1 < \ell$, write $\mu = \sigma\, s$ with $s \in (\mu, 1) \subset [0, \ell)$ and $\sigma = \mu/s \in (0\,,1) \subset [0, \ell)$, then estimate (32) follows from estimates (33), and part 2a of Lemma 29 with

$$t_1(p, \ell, C, \beta, \mu) \stackrel{\text{def}}{=} \tilde{t}_1(p, \ell, C, \beta, s(\mu), \sigma(\mu)) \, .$$

□

Proposition 34. *Let ℓ, m, $\mathbb{U}$ and $\mathbb{V}$ be as in Proposition 26. Given the real numbers $C \geq 0$, $\beta > 0$, and $0 < \mu < \ell$, there exist two positive constants $\kappa = \kappa(p, d, \ell, C, \mu, \beta)$ and $t_2 = t_2(p, \ell, \mathbb{V}, C, \mu, \beta)$ such that for every $h \in C^\ell(\mathbb{V})$ and $f \in C^\ell(\mathbb{U}, \mathbb{V})$, satisfying i) the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$ and ii) $|f|_{C^\ell(\mathbb{U})} < \beta$, the following holds for all $t \geq t_2$:*

$$| S_t[h] \circ S_t[f] - S_t[h \circ f] |_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\, M_1\, t^{-\mu}\,, \tag{35}$$

where and

$$M_1 \stackrel{\text{def}}{=} |h|_{C^\ell(\mathbb{V})} \left(1 + |f|^\tau_{C^\ell(\mathbb{U})}\right) + |h|_{C^\ell(\mathbb{V})}\, |f|_{C^\ell(\mathbb{U})}\,.$$

and

$$\begin{array}{lll} \tau \quad \textit{is any number in} \quad (\mu, 1)\,, & \textit{if} & 0 < \mu < 1 < \ell\,, \\ \tau = \mu\,, & \textit{if} & 1 \leq \mu < \ell\,. \end{array}$$

Proof. That the composition $h \circ f$ belongs to $C^\ell(\mathbb{U})$ follows from part 2 of Lemma 29 (the torus and annulus cases this is obtained by using lifts). To prove estimate (35), first write

$$\begin{aligned} S_t[h] \circ S_t[f] - S_t[h \circ f] = (1 - S_t)\, [\, S_t[h] \circ S_t[f]\,] \,+ & \\ + S_t\, [\, S_t[h] \circ S_t[f]] - S_t\, [S_t[h] \circ f\,] \,+ & \\ + S_t\, [\; S_t[h] \circ f - h \circ f\;]\,. & \end{aligned} \tag{36}$$

Let us estimate the first term on the right hand side of (36). Let $C \geq 0$, $\beta > 0$, and $0 < \mu < \ell$ be given and let κ and t_1 be as in Lemma 33. Then from Lemma 33 and Lemma 27 one obtains for all $t \geq t_1$

$$\begin{aligned} |(\mathrm{Id} - \mathrm{S_t})\, [\, S_t[h] \circ S_t[f]\,]\,|_{C^0(\mathbb{U}+C\,t^{-1})} &\leq \kappa\; |S_t[h] \circ S_t[f]|_{C^\mu(\mathbb{U}+C\,t^{-1})}\;\; t^{-\mu} \\ &\leq \kappa\; |\,h|_{C^\ell(\mathbb{V})} \left(1 + |f|^\tau_{C^\ell(\mathbb{U})}\right)\; t^{-\mu}\,, \end{aligned} \tag{37}$$

where τ is as in Lemma 33. Now we consider the third term on the right hand side of (36). Using again part 2 of Proposition 20 we have

$$\begin{aligned} |\, S_t\, [\; S_t[h] \circ f - h \circ f\;]\,|_{C^0(\mathbb{U}+C\,t^{-1})} &\leq \kappa\; |\, S_t[h] \circ f - h \circ f\;\; |_{C^0(\mathbb{U}))} \\ &\leq \kappa\; |\, (S_t - 1)\, [h]\, |_{C^0(\mathbb{V}))} \\ &\leq \kappa\; |h|_{C^\ell(\mathbb{V})}\;\; t^{-\ell}\,, \end{aligned} \tag{38}$$

where in the last inequality we have used part 1 of Proposition 20. To estimate the second term on the right hand side of (36), we first consider the case $\mathbb{U} \subset \mathbb{R}^d$ is a compensated open domain with C^m-boundary. Notice that from Remark 14 one has

$$S_t\, [\, S_t[h] \circ S_t[f]\,] \;-\; S_t\, [S_t[h] \circ f\,] = S_t \left[\, S_t[h] \circ S_t[f] - \mathscr{E}^\ell_{\mathbb{U}}\, (S_t[h] \circ f)\,\right] \tag{39}$$

Moreover, if $x \in \mathbb{U}$, then

$$\mathscr{E}^\ell_{\mathbb{U}}(S_t[h] \circ f)(x) = (S_t[h] \circ f)(x)\,. \tag{40}$$

Then, from Proposition 20, and equalities (39) and (40) we have

$$|S_t\, [S_t[h] \circ S_t[f]] - S_t\, [S_t[h] \circ f]\, |_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\; |h|_{C^\ell(\mathbb{U})}\, |f|_{C^\ell(\mathbb{U})}\,. \tag{41}$$

In the same way, one proves that estimate (41) also holds in the case $\mathbb{U} = \mathbb{R}^d$. Indeed, if $\mathbb{U} = \mathbb{R}^d$ then (compare with (39))

$$S_t\, [S_t[h] \circ S_t[f]] - S_t\, [S_t[h] \circ f] = S_t\, [\, S_t[h] \circ S_t[f] - S_t[h] \circ f\,]\,.$$

Furthermore, taking lifts, using the norms introduced in Definition 4, and using that (41) holds when $\mathbb{U}$ is either $\mathbb{R}^d$ or a compensated open domain in $\mathbb{R}^d$ with C^m-boundary, one proves that estimate (41) also holds in the following cases: *i)* $\mathbb{U} = \mathbb{T}^d$, $\mathbb{V} = \mathbb{T}^p$, *ii)* $\mathbb{U} = \mathbb{T}^d$ and $\mathbb{V} \subset \mathbb{R}^p$ is a compensated open domain with C^m-boundary, *iii)* $\mathbb{U} = \mathbb{T}^n \times U$, with $U \subset \mathbb{R}^d$ a compensated open domain with C^m-boundary, and $\mathbb{V} = \mathbb{T}^p$, *iv)* $\mathbb{U} = \mathbb{T}^n \times U$, with $U \subset \mathbb{R}^d$ a compensated open domain with C^m-boundary, and $\mathbb{V}$ is a compensated open domain with C^m-boundary. Hence, estimate (41) holds for $\mathbb{U}$ and $\mathbb{V}$ as in the hypotheses of Proposition 34.

Proposition 34 follows from equality (36) taking t_2 sufficiently large such that estimates (37), (38), and (41), holds for all $t \geq t_2$. □

2.3.2 Smoothing and pull-back (Proof of Proposition 28)

We now have all the ingredients to prove Proposition 28. Let $\mathbb{U}$, $\mathbb{V}$, and $\Omega \in \Lambda^k(\mathbb{V})$ be as in Proposition 26. Throughout this section we assume that $C \geq 0$, $\beta > 0$, and $0 < \mu < \ell - 1$ are given. Fix $r \in (0,1)$ in terms of ℓ and μ in such a way that $0 < \mu + r < \ell - 1$ (e.g. $r = \min(1/2, (\ell - 1 - \mu)/2)$) so that the constants depending on r will actually depend on μ and ℓ. Let $f \in C^\ell(\mathbb{U}, \mathbb{V})$ be such that the closure of $f(\mathbb{U})$ is contained in $\mathbb{V}$, then Lemma 24 implies $S_t[f]^*\Omega \in \Lambda^k(\mathbb{U})$ for $t \geq \bar{t}$. Hence, to have $S_t[f]^*\Omega$ defined on $\mathbb{U}$ we assume from now on that $t \geq \bar{t}$. To prove Proposition 28 we first write

$$S_t[f]^*\Omega - S_t\,[f^*\Omega] = \{\, S_t[f]^*\Omega - S_t[f]^*\,(S_t[\Omega])\,\} + \{\, S_t[f]^*\,(S_t[\Omega]) - S_t\,[f^*\Omega]\,\} \tag{42}$$

Let us estimate the first term in brackets on the right hand side:

$$(S_t[f]^*\Omega - S_t[f]^*\,(S_t[\Omega]))\,(x) = \sum_{1 \leq i_i < \cdots < i_k \leq d} (1 - S_t)\,[\Omega_{\mathbf{i}}]\,(S_t[f](x))\; S_t[f]^* dx_{\mathbf{i}}\,. \tag{43}$$

From Lemma 24 we have

$$S_t[f]\,\left(\mathbb{U} + C\,t^{-1}\right) \subseteq \mathbb{V} + (C\,\beta\kappa)\;t^{-1} \subseteq \mathbb{V} + \rho, \qquad \forall t \geq \max\left(\rho^{-1}\beta\,C\,\kappa\,\bar{t}\right).$$

Assume that $\Omega \in \mathcal{A}(\mathbb{V}+\rho, C^\ell)$, then for all $\mathbf{i} = (i_1, \ldots, i_k)$, with $1 \leq i_i < \cdots < i_k \leq p$ and $t \geq \max\left(\rho^{-1}\beta\,C\,\kappa\,,\bar{t}\right)$, Lemma 27 implies

$$\begin{aligned} |(\mathrm{Id} - \mathrm{S_t})\,[\Omega_{\mathbf{i}}] \circ S_t[f]|_{C^0(\mathbb{U}+C\,t^{-1})} &\leq |\,(\mathrm{Id} - \mathrm{S_t})\,[\Omega_{\mathbf{i}}]\,|_{C^0(\mathbb{V}+\rho)} \\ &\leq \left(\kappa\;|\Omega_{\mathbf{i}}|_{C^\sigma(\mathbb{V}+\rho)}\right)\;t^{-\sigma}\,, \end{aligned} \tag{44}$$

for all $0 \leq \sigma \leq \ell$. Hence, part 2 of Proposition 20, estimate (44), and equality (43), yield for all $t \geq \max\left(\rho^{-1}\beta\,C\,\kappa\,,\bar{t}\right)$,

$$|S_t[f]^*\Omega - S_t[f]^*\;(S_t[\Omega])\,|_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\;|\Omega|_{C^\mu(\mathbb{V}+\rho)}\;|f|^k_{C^\ell(\mathbb{U})}\;t^{-\mu}\,, \tag{45}$$

where $\kappa = \kappa(p, d, \ell, C, \mu, \beta, k)$.

Now write the second term on the right hand side of (42) in the following way:

$$\begin{aligned} S_t[f]^*\,(S_t[\Omega]) - S_t\,[f^*\Omega] = \sum_{1 \leq i_i < \cdots < i_k \leq d} \Big\{ &(S_t[\Omega_{\mathbf{i}}] \circ S_t[f] - S_t[\Omega_{\mathbf{i}} \circ f])\; S_t[f]^* dx_{\mathbf{i}} + \\ &S_t[\Omega_{\mathbf{i}} \circ f\,]\;(\,S_t[f]^* dx_{\mathbf{i}} - S_t\,[f^* dx_{\mathbf{i}}]\,) + \\ &S_t[\Omega_{\mathbf{i}} \circ f\,]\,S_t[\,f^* dx_{\mathbf{i}}] - S_t[\,(\Omega_{\mathbf{i}} \circ f)\;f^* dx_{\mathbf{i}}]\,\Big\}\,. \end{aligned} \tag{46}$$

In what follows we give estimates for the three terms on the right hand side of (46). The first term is estimated as follows: Let t_2 be as in Proposition 34, then Proposition 34 and part 2 of Proposition 20 yield for all $t \geq t_2$:

$$\begin{aligned} &|\,(S_t[\Omega_{\mathbf{i}}] \circ S_t[f] - S_t[\Omega_{\mathbf{i}} \circ f])\;S_t[f]^* dx_{\mathbf{i}}\,|_{C^0(\mathbb{R}^d+C\,t^{-1})} \leq \\ &\qquad \kappa\;|f|^k_{C^\ell(\mathbb{U})}\;\left\{|\Omega_{\mathbf{i}}|_{C^\ell(\mathbb{U})}\;\left(1 + |f|^\tau_{C^\ell(\mathbb{U})}\right) + |\Omega_{\mathbf{i}}|_{C^\ell(\mathbb{U})}\;|f|_{C^\ell(\mathbb{U})}\right\}\;t^{-\mu}\,, \end{aligned} \tag{47}$$

where $\kappa = \kappa(p, d, \ell, C, \mu, \beta, k)$ and τ is as in Proposition 34.

An estimate for the second term on the right hand side of (46) follows from part 2 of Proposition 20 and Lemma 32:

$$| S_t[\Omega_{\mathbf{i}} \circ f] \; (S_t[f]^* dx_{\mathbf{i}} - S_t \, (f^* dx_{\mathbf{i}}) \,)|_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa \; |\Omega_{\mathbf{i}}|_{C^0(\mathbb{V})} \; |f|^k_{C^\ell(\mathbb{U})} \; t^{-\mu} \tag{48}$$

where $t \geq t_0$, with t_0 as in Lemma 32, and $\kappa = \kappa(d, \ell, C, \mu, k)$.

Finally, applying Lemma 31 to the 0-form $\Omega_{\mathbf{i}} \circ f$ and the k-form $f^* dx_{\mathbf{i}}$ and using Lemma 29, one has that there exists a constant $\kappa = \kappa(d, \ell, C, \mu, k)$ such that, for all $t \geq e^{1/r}$ satisfying (22), the following holds:

$$\begin{aligned} &|S_t[\Omega_{\mathbf{i}} \circ f] \; S_t[f^* dx_{\mathbf{i}}] - S_t[\, (\Omega_{\mathbf{i}} \circ f) \; f^* dx_{\mathbf{i}}] \; |_{C^0(\mathbb{U}+C\,t^{-1})} \leq \\ &\qquad \leq \kappa \, \left(|\Omega_{\mathbf{i}} \circ f|_{C^0(\mathbb{U})} \, |f^* dx_{\mathbf{i}}|_{C^{\ell-1}(\mathbb{U})} + |\Omega_{\mathbf{i}} \circ f|_{C^{\ell-1}(\mathbb{U})} \, |f^* dx_{\mathbf{i}}|_{C^0(\mathbb{U})} \right) \, t^{-\mu} \\ &\qquad \leq \kappa \, \left\{ |\Omega_{\mathbf{i}}|_{C^0(\mathbb{V})} \, |Df|^{k-1}_{C^0(\mathbb{U})} \; |f|_{C^\ell(\mathbb{U})} + |\Omega_{\mathbf{i}}|_{C^\ell(\mathbb{V})} \left(1 + |f|^\ell_{C^\ell(\mathbb{U})} \right) \, |Df|^k_{C^0(\mathbb{U})} \right\} \, t^{-\mu} \, , \end{aligned} \tag{49}$$

where we have used the inequality $|\Omega_{\mathbf{i}} \circ f|_{C^{\ell-1}(\mathbb{U})} \leq |\Omega_{\mathbf{i}} \circ f|_{C^\ell(\mathbb{U})}$ and part 2(b)iii of Lemma 29.

Define

$$\tilde{t} \stackrel{\text{def}}{=} \max \left(C \, \beta \, \kappa \, \rho^{-1}, e^{1/r} \, , t_2, \bar{t} \right) \, ,$$

where $\kappa = \kappa(d, C)$ is as in Proposition 20, $\bar{t}$ is as in Lemma 24, and t_2 is as in Proposition 34. Let $t \geq \tilde{t}$ satisfy (22), then equality (46) and estimates (47), (48) and (49) imply

$$|S_t[f]^* \, (S_t[\Omega]) - S_t \, [f^* \Omega]|_{C^0(\mathbb{U}+C\; t^{-1})} \leq \kappa \, M_2 \; t^{-\mu} \, , \tag{50}$$

where κ is a constant depending on d, ℓ, k, r, μ, and C, and M_2 is defined by

$$M_2 \stackrel{\text{def}}{=} |f|^k_{C^\ell(\mathbb{U})} \; |\Omega|_{C^\ell(\mathbb{U})} \; \left\{ 1 + |f|^\tau_{C^\ell(\mathbb{U})} + |f|^\ell_{C^\ell(\mathbb{U})} + |f|_{C^\ell(\mathbb{U})} \right\} \, .$$

Hence Proposition 28 follows from estimates (45) and (50).

2.4 The symplectic and volume cases (Proof of Theorem 2)

Let ℓ and $\mathbb{U}$ be as in Theorem 2 and let $f \in \mathrm{Diff}^\ell(\mathbb{U})$. We prove Theorem 2 in several lemmas. First in Lemma 36 we prove that if Ω is a non-degenerate form, then for sufficiently large t, the form defined by

$$\Omega_t^\varepsilon \stackrel{\text{def}}{=} \Omega + \varepsilon \, (S_t[f]^* \Omega - \Omega) \, , \tag{51}$$

is also non-degenerate for all $\varepsilon \in [0, 1]$. We also give explicit estimates for the norm of $\mathscr{I}^{-1}_{\Omega_t^\varepsilon} \theta$ on complex strips in terms of the corresponding norm of θ. In Lemma 38 we use the deformation method [Mos65] to prove that, for t sufficiently large, there exists a diffeomorphism such that $(\phi_t^\varepsilon)^* \, \Omega_t^\varepsilon = \Omega$. Moreover, in Lemma 38 we also give quantitative properties of ϕ_t^ε. More precisely, using Lemma 36 we prove that the diffeomorphism ϕ_t^ε is real analytic, close to the identity and with first and second derivatives bounded on the complex strips $\mathbb{U}_t + C\, t^{-1}$, with $\mathbb{U}_t$ defined in (7). In Lemma 39 we prove that if $\varphi_t \stackrel{\text{def}}{=} \phi_t^1$, then $T_t[f] \stackrel{\text{def}}{=} S_t[f] \circ \varphi_t$ satisfies properties T1-T6 of Theorem 2. Property T7 is proved in Section 2.4.1.

Remark 35. *Notice that if $f \in \mathrm{Diff}^\ell(\mathbb{U})$ then from part 1 of Proposition 20 we have that, for t sufficiently large, $S_t[f]$ is a diffeomorphism on $\mathbb{U}$.*

Lemma 36. *Let ℓ, $\mathbb{U}$, $\mathbb{V}$, Ω and $\mathscr{I}_\Omega$ satisfy the hypotheses of Theorem 2. Then, given $C \geq 0$ and $\beta > 0$ there exists a constant $t_3 = t_3(d, \ell, \mathbb{V}, C, \beta, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, such that for all $t \geq t_3$ and for all $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying i) $|f|_{C^\ell(\mathbb{U})} < \beta$, ii) $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, and iii) $f^* \Omega = \Omega$, the k-from defined by (51) is non-degenerate for all $\varepsilon \in [0, 1]$. Furthermore, for any real analytic $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho)} < \infty$, and any $t \geq t_3$, the application taking (ε, x) into $\mathscr{I}^{-1}_{\Omega_t^\varepsilon}(\theta)(x)$ is continuous on $(\varepsilon, x) \in [0, 1] \times \mathbb{U} + C\, t^{-1}$ and real analytic with respect to x. Moreover*

$$\left| \mathscr{I}^{-1}_{\Omega_t^\varepsilon}(\theta) \right|_{C^0(\mathbb{U}+C\,t^{-1})} \leq 2 \, M_\Omega \; |\theta|_{C^0(\mathbb{U}+C\,t^{-1})} \; , \qquad \forall \, \varepsilon \in [0, 1] \, .$$

Proof. This follows from Proposition 26. Indeed, first notice that for all $t \geq 1$ and $\varepsilon \in [0,1]$ the following equality holds:

$$\mathscr{I}_{\Omega_t^\varepsilon} = \mathscr{I}_\Omega + \varepsilon\, \mathscr{I}_{(S_t[f]^*\Omega - \Omega)} = \left(\mathrm{Id} + \varepsilon\, \mathscr{I}_{(\mathrm{S_t[f]^*}\Omega-\Omega)} \circ \mathscr{I}_\Omega^{-1}\right) \circ \mathscr{I}_\Omega\,, \tag{52}$$

where Id represents the identity map on $\Lambda^{k-1}(\mathbb{U})$. Let $\kappa = \kappa(d, d, \ell, C, \beta, (\ell-1)/2, k)$, $\hat{t} = \hat{t}(d, d, \ell, \mathbb{V}, C, \beta, (\ell-1)/2)$, and $\hat{M}_f$ be as in Proposition 26, then for all $t \geq \max(\hat{t}, C)$ and for any $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho)} < \infty$, the following estimate holds:

$$\left|\mathscr{I}_{(S_t[f]^*\Omega-\Omega)}(\mathscr{I}_\Omega^{-1}\theta)\right|_{C^0(\mathbb{U}+C\,t^{-1})} \leq \kappa\, \hat{M}_f\, t^{-(\ell-1)/2}\, M_\Omega\, |\theta|_{C^0(\mathbb{U}+C\,t^{-1})}\,. \tag{53}$$

Assume that t_3 is sufficiently large so that for all $t \geq t_3$ estimate (53) holds and moreover

$$t^{-(\ell-1)/2}\, \kappa\, \hat{M}_f\, M_\Omega \leq 1/2\,.$$

Then for all $t \geq t_3$, $\varepsilon \in [0,1]$, the application $\left(\mathrm{Id} - \varepsilon\, \mathscr{I}_{\mathrm{S_t[f]^*}\Omega-\Omega} \circ \mathscr{I}_\Omega^{-1}\right)$ is an isomorphism on $\Lambda^{k-1}(\mathbb{U})$, and moreover the following holds for any $\theta \in \Lambda^{k-1}(\mathbb{U})$, satisfying $|\theta|_{C^0(\mathbb{U}+\rho)} < \infty$:

$$\left|\left(\mathrm{Id} - \varepsilon\, \mathscr{I}_{\mathrm{S_t[f]^*}\Omega-\Omega} \circ \mathscr{I}_\Omega^{-1}\right)^{-1} \theta\right|_{C^0(\mathbb{U}+C\,t^{-1})} \leq 2\, |\theta|_{C^0(\mathbb{U}+C\,t^{-1})}\,.$$

Hence from (52) we have that for all $t \geq t_3$, and $\varepsilon \in [0,1]$, the application $\mathscr{I}_{\Omega_t^\varepsilon}$ is invertible with inverse given by

$$\mathscr{I}_{\Omega_t^\varepsilon}^{-1} = \mathscr{I}_\Omega^{-1} \circ \left(\mathrm{Id} + \varepsilon\, \mathscr{I}_{\mathrm{S_t[f]^*}\Omega-\Omega} \circ \mathscr{I}_\Omega^{-1}\right)^{-1}\,,$$

from which Lemma 36 follows. $\square$

Lemma 37. *Let ℓ, $\mathbb{U}$, $\mathbb{V}$, Ω and $\mathscr{I}_\Omega$ satisfy the hypotheses of Theorem 2. Then, given $C \geq 0$, $\beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ and $t_4 = t_4(d, \ell, \mathbb{V}, C, \beta, \mu, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, such that for all $t \geq t_4$ and for all $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying i) $|f|_{C^\ell(\mathbb{U})} < \beta$, ii) $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, and iii) $f^*\Omega = \Omega$, there exists a vector field u_t^ε satisfying*

$$d\, i_{u_t^\varepsilon}\,(\,\Omega_t^\varepsilon\,) = -\,(S_t[f]^*\Omega - f^*\Omega)\,, \tag{54}$$

where d represents the exterior derivative and Ω_t^ε is defined in (51). Furthermore, the vector field vector field u_t^ε is continuous on $(\varepsilon, x) \in [0,1] \times \mathbb{U} + 2\,C\,t^{-1}$, real analytic with respect to x on $\mathbb{U} + 2\,C\,t^{-1}$, and it satisfies the following estimates:

$$|u_t^\varepsilon|_{C^0(\mathbb{U}+2\,C\,t^{-1})} \leq \kappa\, \hat{M}_f\, t^{-\mu}, \qquad \forall\, \varepsilon \in [0,1]\,, \tag{55}$$

where $\hat{M}_f$ is as in Proposition 26.

Proof. First notice that since $\Omega = d\alpha$ is exact and analytic, then the right hand side of (54) is also exact and analytic. Then, the Poincaré's formula implies the existence of an analytic 1-form γ_t such that: $d\gamma_t = S_t[f]^*\Omega - \Omega$ and

$$|\gamma_t|_{\mathbb{U}+2\,C\,t^{-1}} \leq \hat{\kappa}\, |S_t[f]^*\Omega - \Omega|_{\mathbb{U}+C\,t^{-1}} \leq \kappa\, \hat{M}_f\, t^{-\mu}\,,$$

where we have used Proposition 26 and the fact that $\mathbb{U}$ is bounded. Lemma 37 follows from Lemma 36 by solving the following equation:

$$i_{u_t^\varepsilon}\, \Omega_t^\varepsilon = -\,\gamma_t\,.$$

$\square$

Lemma 38. *Let ℓ, $\mathbb{U}$, $\mathbb{V}$, Ω and $\mathscr{I}_\Omega$ satisfy the hypotheses of Theorem 2. Then, given $C \geq 0$, $\beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ and $t_5 = t_5(d, \ell, \mathbb{V}, C, \beta, \mu, M_\Omega, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, such that for any $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying i) $|f|_{C^\ell(\mathbb{U})} < \beta$, ii) $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, and iii) $f^*\Omega = \Omega$, any $t \geq t_5$, and any $\varepsilon \in [0,1]$, there exists an analytic diffeomorphism ϕ_t^ε on $\mathbb{U}_t$, with $\mathbb{U}_t$ defined in (7), such that the following hold:*

1. $(\phi_t^\varepsilon)^* \, \Omega_t^\varepsilon = \Omega$.

2. $\phi_t^0 = \mathrm{id}$.

3. $\left|\phi_t^1 - \mathrm{id}\right|_{C^0(\mathbb{U}_t + C\,t^{-1})} \leq \kappa \, \hat{M}_f \, t^{-\mu}$ *, where* id *represents the identity map.*

4. $\left|D\phi_t^1\right|_{C^0(\mathbb{U}_t + C\,t^{-1})} \leq \exp\left(C^{-1} \, \kappa \, \hat{M}_f \, t^{-\mu+1}\right)$.

5. $\left|D^2\phi_t^1\right|_{C^0(\mathbb{U}_t + C\,t^{-1})} \leq C^{-2} \, \kappa \, \hat{M}_f \, t^{-\mu+2} \exp\left(3\, C^{-1} \, \kappa \, \hat{M}_f t^{-\mu+1}\right)$

Proof. Following the proof of Theorem 2 in [Mos65], we determine ϕ_t^ε by solving the differential equation

$$\frac{d}{d\varepsilon} \, \phi_t^\varepsilon = u_t^\varepsilon \circ \phi_t^\varepsilon \, , \qquad\qquad 0 \leq \varepsilon \leq 1 \, , \tag{56}$$

with ϕ_t^0 the identity mapping, where the vector field u_t^ε is as in Lemma 37. Notice that, in the case $\mathbb{U} = \mathbb{T}^d$, the properties of u_t^ε given in Lemma 37 imply the existence of a unique solution ϕ_t^ε of (56) for all $\varepsilon \in [0,1]$ and all x in the closure of $\mathbb{T}^d + C\,t^{-1}$. To guarantee a solution of (56) for all $\varepsilon \in [0,1]$ in the non-compact cases: *i)* $\mathbb{U} \subset \mathbb{R}^d$ a compensated bounded open domain with C^m-boundary, and *ii)* $\mathbb{U} = \mathbb{T}^n \times U$ with $U \subset \mathbb{R}^{d-n}$ a compensated bounded open domain with C^m-boundary, we solve (56) for initial conditions in the closure of $\mathbb{U}_t + C\,t^{-1}$, with $\mathbb{U}_t \subset \mathbb{U}$ defined in (7). Notice that if t_4 as in Lemma 37 and $t \geq t_4$ is sufficiently large so that

$$\kappa \, \hat{M}_f \, t^{-\mu+1} < 1 \, , \tag{57}$$

which is possible because $1 < \mu < \ell - 1$, then (55) and (57) imply the existence of a unique solution ϕ_t^ε of (56) for all $\varepsilon \in [0,1]$ and all x in the closure of $\mathbb{U}_t + C\,t^{-1}$, with $\mathbb{U}_t \subset \mathbb{U}$ defined in (7).

Hence, if $\mathbb{U}$ and $\mathbb{U}_t$ are as in Theorem 2 and $t \geq t_4$ satisfies (57), then equation (56) has a unique solution $\phi_t^\varepsilon(x)$, defined for $\varepsilon \in [0,1]$, and x in the closure of $\mathbb{U}_t + C\,t^{-1}$. Moreover, the following holds:

$$\begin{aligned}
\left|\phi_t^1 - \mathrm{id}\,\right|_{C^0(\mathbb{U}_t + C\,t^{-1})} &= \sup_{x \in \, \mathbb{U}_t + C\,t^{-1}} \left| \int_0^1 u_t^s(\phi_t^s(x)) ds \right| \\
&\leq \sup_{s \in [0,1]} |\, u_t^s|_{C^0(\mathbb{U} + C\,t^{-1})} \\
&\leq \kappa \, \hat{M}_f \, t^{-\mu} \, ,
\end{aligned}$$

from which part 3 of Lemma 38 follows.

Part 1 follows from (54), (56) and the E. Cartan's formula for the Lie derivatives (c.f. [Ste64]):

$$\frac{d}{d\varepsilon} \left((\phi_t^\varepsilon)^* \; \Omega_t^\varepsilon\right) = (\phi_t^\varepsilon)^* \; \left\{ d\left(i_{u_t^\varepsilon} \, \Omega_t^\varepsilon\right) + i_{u_t^\varepsilon} \left(d\Omega_t^\varepsilon\;\right) + \frac{d}{d\varepsilon} \, \Omega_t^\varepsilon \right\} = 0$$

Parts 4 and 5 follow from the Gronwall's and Cauchy's estimates, and and (55) as follows: From (56) we have for $t \geq t_4$ satisfying (57) and x in the closure of $\mathbb{U}_t + C\,t^{-1}$

$$|D\phi_t^\varepsilon(x)| \, \leq 1 + \int_0^\varepsilon |Du_t^s|_{C^0(\mathbb{U} + C\,t^{-1})} \; |D\phi_t^s(x)| \; ds, \qquad \varepsilon \in [0,1] \, ,$$

then the Gronwall's and Cauchy's estimates and (55) imply

$$\begin{aligned}
|D\phi_t^\varepsilon|_{C^0(\mathbb{U}_t + C\,t^{-1})} &\leq \exp\left(\sup_{s \in [0,1]} |Du_t^s|_{C^0(\mathbb{U} + C\,t^{-1})} \right) \\
&\leq \exp\left(C^{-1} \, \kappa \, \hat{M}_f \, t^{-\mu+1}\right) .
\end{aligned}$$

Similarly

$$\begin{aligned}
|D^2\phi_t^1|_{C^0(\mathbb{U}_t+C\,t^{-1})} \le & \int_0^1 |D^2u_t^s|_{C^0(\mathbb{U}+C\,t^{-1})}\,|D\phi_t^s|^2_{C^0(\mathbb{U}_t+C\,t^{-1})}\,ds + \\
& + \int_0^1 |Du_t^s|_{C^0(\mathbb{U}+C\,t^{-1})}\,|D^2\phi_t^s|_{C^0(\mathbb{U}_t+C\,t^{-1})}\,ds \\
\le & \sup_{\varepsilon\in[0,1]} |u_t^\varepsilon|_{C^2(\mathbb{U}+C\,t^{-1})} \exp\Big(2\,C^{-1}\,\kappa\,\hat{M}_f\,t^{-\mu+1}\Big) \\
& + |u_t^\varepsilon|_{C^1(\mathbb{U}+C\,t^{-1})} \int_0^1 |D^2\phi_t^s|_{C^0(\mathbb{U}_t+C\,t^{-1})}\,ds \\
\le & \;\kappa\,C^{-2}\,\hat{M}_f\,t^{-\mu+2}\,\exp\Big(2\,C^{-1}\,\kappa\,\hat{M}_f t^{-\mu+1}\Big) \\
& + \kappa\,C^{-1}\,\hat{M}_f\,t^{-\mu+1} \int_0^1 |D^2\phi_t^s|_{C^0(\mathbb{U}_t+C\,t^{-1})}\,ds
\end{aligned}$$

from which part 5 of Lemma 38 follows. □

Lemma 39. *Assume that the hypotheses of Theorem 2 hold. Let t_5 and ϕ_t^1 be as in Lemma 38, define for $t \ge t_5$*

$$\varphi_t \stackrel{\text{def}}{=} \phi_t^1 \,.$$

Then, given given $C \ge 0$, $\beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \beta, \mu, k, M_\Omega)$ and $t^* = t^*(d, \ell, \mathbb{V}, C, \mu, \beta, M_\Omega\,, |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, such that if the elements of the family of – non-linear – operators $\{T_t\}_{t\ge t^*}$ are defined for $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying i) $|f|_{C^\ell(\mathbb{U})} < \beta$, ii) $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, and iii) $f^*\Omega = \Omega$, by:

$$T_t[f](x) \stackrel{\text{def}}{=} S_t[f](\varphi_t(x))\,, \qquad x \in \mathbb{U}_t\,,$$

where $\mathbb{U}_t$ is as in Theorem 2, then $T_t[f]$ satisfies T0-T1-T2-T4 of Theorem 2 and the following properties:

T3'. $|\,T_t[f] - S_t[f]\,|_{C^0(\mathbb{U}_t+C\,t^{-1})} \le \kappa\,M_f\,t^{-\mu}\,,$

T5'. $|\,(T_t - \mathrm{Id})[\mathrm{f}]\,|_{C^r(\mathbb{U}_t)} \le \kappa\,M_f\,t^{-(\mu-r)}$*, for all* $0 \le r \le \mu$.

T6'. $|\,(T_\tau - T_t)\,[f]\,|_{C^0(\mathbb{U}_t+C\,t^{-1})} \le \kappa\,M_f\,t^{-\mu}$*, for all* $\tau \ge t \ge t^*$.

Proof. That $T_t[f]$ is a diffeomorphism on $\mathbb{U}_t$ follows from Remark 35 and Lemma 38. Notice that property T1 of Theorem 2 follows from part 1 of Lemma 38. Now, assume that $t^* \ge t_5$ is sufficiently large so that for all $t \ge t^*$ the following holds:

$$\kappa\,\hat{M}_f\,t^{-(\mu-1)} < C\,\log(2)\,, \tag{58}$$

which is possible because $1 < \mu < \ell - 1$. Then using (58) and parts 3 and 4 of Lemma 38 one has for all $t \ge t_5$

$$|\,\varphi_t - \mathrm{id}\,|_{C^0(\mathbb{U}_t+C\,t^{-1})} \le \kappa\,\hat{M}_f\,t^{-\mu} < C\,t^{-1}\,, \tag{59}$$

and

$$|\,D\varphi_t\,|_{C^0(\mathbb{U}_t+C\,t^{-1})} \le \exp(C^{-1}\,\kappa\,\hat{M}_f\,t^{-(\mu-1)}) < 2\,. \tag{60}$$

Notice that if $2 < \mu < \ell - 1$ then it is possible to choose t^* sufficiently large such that (compare with (58))

$$\kappa\,\hat{M}_f\,t^{-(\mu-2)} < \min(C^2, C\,\log(2))\,, \qquad \forall \quad t \ge t^*\,.$$

Then part 5 of Lemma 38 implies for such t that

$$|\,D^2\varphi_t\,|_{C^0(\mathbb{U}_t+C\,t^{-1})} < 2^3\,. \tag{61}$$

A consequence of (60) is that we can control the domain of the composition $S_t[f] \circ \varphi_t$ on complex strips because of the following estimate:

$$\begin{aligned} |\mathrm{Im}\,(\varphi_t)|_{C^0(\mathbb{U}_t + C\,t^{-1})} &\le C\,t^{-1}\,|D\varphi_t|_{C^0(\mathbb{U}_t + C\,t^{-1})} \\ &< 2\,C\,t^{-1}\,. \end{aligned}$$

From which we have

$$\varphi_t\left(\mathbb{U}_t + C\,t^{-1}\right) \subset \mathbb{U} + 2\,C\,t^{-1} \tag{62}$$

Now property T2 of Theorem 2 follow easily. First, using (62) and part 2 of Proposition 20 one has:

$$\begin{aligned} |\,T_t[f]\,|_{C^0(\mathbb{U}_t + C\,t^{-1})} &= |S_t[f] \circ \varphi_t|_{C^0(\mathbb{U}_t + C\,t^{-1})} \\ &\le |S_t[f]|_{C^0(\mathbb{U} + (2\,C\,t^{-1}))} \\ &\le \kappa\,|f|_{C^0(\mathbb{U})}\;. \end{aligned}$$

Now, using (62), part 1 of Lemma 30, and estimate (60) one has:

$$\begin{aligned} |\,DT_t[f]\,|_{C^0(\mathbb{U}_t + C\,t^{-1})} &\le |S_t[f]|_{C^1(\mathbb{U} + 2\,C\,t^{-1})}\;|D\varphi_t(x)|_{C^0(\mathbb{U}_t + C\,t^{-1})} \\ &\le \kappa\,|f|_{C^\ell(\mathbb{U})}\;. \end{aligned}$$

To prove T3' use (62), (59) and part 1 of Lemma 30 to obtain:

$$\begin{aligned} |\,(T_t - S_t)\,[f]\,|_{C^0(\mathbb{U}_t + C\,t^{-1})} &\le |\,S_t[f]\,|_{C^1(\mathbb{U} + 2\,C\,t^{-1})}\;|\,\varphi_t - \mathrm{id}\,|_{C^0(\mathbb{U}_t + C\,t^{-1})} \\ &\le \kappa\,\hat{M}_f\;|f|_{C^\ell(\mathbb{U})}\;t^{-\mu}\,. \end{aligned}$$

Furthermore, if $2 < \mu < \ell - 1$, then the chain rule, (62), part 1 of Lemma 30, and estimates (60) and (61) imply

$$\begin{aligned} \left|\,D^2\,T_t[f]\,\right|_{C^0(\mathbb{U}_t + C\,t^{-1})} &\le 2^2\;|\,S_t[f]\,|_{C^2(\mathbb{U} + 2\,C\,t^{-1})} + 2^3\;|\,S_t[f]\,|_{C^1(\mathbb{U} + 2\,C\,t^{-1})} \\ &\le \kappa\;|f|_{C^\ell(\mathbb{U})}\;. \end{aligned}$$

This proves property T4 of Theorem 2.

Finally, properties T5' and T6' of Lemma 39 follow from Proposition 20, property T3' of Lemma 39, and the following inequalities

$$|\,(T_t - 1)\,[f]\,|_{C^r(\mathbb{U}_t)} \le |\,(T_t - S_t)\,[f]\,|_{C^r(\mathbb{U}_t)} + |\,(S_t - 1)[f]\,|_{C^r(\mathbb{U})}\;,$$

and for $\tau \ge t$

$$\begin{aligned} |\,(T_\tau - T_t)\,[f]\,|_{C^0(\mathbb{U}_t + C\,\tau^{-1})} \le\;& |\,(T_\tau - S_\tau)\,[f]\,|_{C^0(\mathbb{U}_t + C\,\tau^{-1})} + \\ &+ |\,(S_t - T_t)\,[f]\,|_{C^0(\mathbb{U}_t + C\,t^{-1})} + \\ &+ |\,(S_\tau - S_t)\,[f]\,|_{C^0(\mathbb{U} + C\,\tau^{-1})}\;. \end{aligned}$$

□

2.4.1 Exactness considerations

In this section we show that in the case that the diffeomorphism f is exact symplectic, then it is possible to construct analytic approximating functions $T_t[f]$ which are also exact symplectic, as claimed in part T7 of Theorem 2. Here use the calculus of deformations, similar constructions are obtained in [dlLMM86].

Let $\mathbb{U}$ be as in Theorem 2. Of course, exactness is a problem only in the case that $\mathbb{U} = \mathbb{T}^n \times U$. In the other cases, Poincaré's Lemma shows that all symplectic maps are exact. Hence, throughout this section we assume that $\mathbb{U} = \mathbb{T}^n \times U$.

Let $T_t[f]$ be as in Lemma 39, we show that for t sufficiently large, there exists a diffeomorphism h_t such that $h_t \circ T_t[f]$ is exact and satisfies properties T1-T6 of Theorem 2. Notice that since $T_t[f]^*\Omega = \Omega$ we have that the form $(T_t[f]^*\alpha - \alpha)$ is closed. Recall that if $\Omega = d\alpha$, then $h_t \circ T_t[f]$ is exact if and only if the form $(h_t \circ T_t[f])^* \alpha - \alpha$ is exact. Equivalently

$$[\, T_t[f]^* \, (h_t^* \alpha - \alpha) \,] = - \, [\, T_t[f]^* \alpha - \alpha \,] \, , \tag{63}$$

where $[\, \beta \,]$ represents the de Rham cohomology class of the closed form β. The existence of a diffeomorphism h_t satisfying (63) is proved in the following lemma where, moreover, we estimate the distance between h_t and the identity.

Lemma 40. *Let $\Omega = d\alpha$ be an exact symplectic form and $2 < \ell \notin \mathbb{Z}$. Assume that the hypotheses of Theorem 2 hold. Let $\mathbb{U}_t$ be defined by (7). Then, given $C \geq 0$, $\beta > 0$ and $1 < \mu < \ell - 1$, there exist two constants $\kappa = \kappa(d, \ell, C, \beta, \mu, M_\Omega)$ and $t^* = t^*(d, \ell, \mathbb{V}, C, \mu, \beta, M_\Omega \, , |\Omega|_{C^\ell(\mathbb{U}+\rho)})$, such that for any $t \geq t^*$ and any $f \in \mathrm{Diff}^\ell(\mathbb{U})$ satisfying i) $|f|_{C^\ell(\mathbb{U})} < \beta$, ii) $\mathbb{V}$ contains the closure of $f(\mathbb{U})$, and iii) f is exact, there exists a diffeomorphism h_t satisfying equality (63) and such that the following holds:*

$$|\, h_t - \mathrm{id} \,|_{C^0(\mathbb{U}+\rho)} \leq \kappa \, \hat{M}_f \, t^{-\mu+1} \, , \tag{64}$$

$$|\, h_t|_{C^1(\mathbb{U}+\rho)} \leq \, \kappa \tag{65}$$

Proof. Let $H^1(M, \mathbb{R})$ denote *the first de Rham cohomology group* of the manifold M. Let $\mathbb{V}$ be as in Theorem 2 and let $\mathbb{U}_t$ be as in (7). We note that if $\mathbb{U} = \mathbb{T}^n \times U$, and if $\mathbb{V}$ diffeomorphic to $\mathbb{U}$, then $H^1(\mathbb{U}_t, \mathbb{R}) = \mathbb{R}^n$ and $H^1(\mathbb{V}, \mathbb{R}) = \mathbb{R}^n$. For t sufficiently large, let $T_t[f]$ be as in Lemma 39. Consider

$$\begin{array}{rccc} T_t[f]_\# : & H^1(\mathbb{V}, \mathbb{R}) & \to & H^1(\mathbb{U}_t, \mathbb{R}) \\ & [\, \gamma \,] & \to & [\, T_t[f]^* \gamma \,] \end{array}$$

Notice that since $T_t[f]$ is a diffeomorphism on $\mathbb{U}_t$ and since the pull-back commutes with the exterior derivative one has: *a) $T_t[f]_\#$ is well defined, b) $T_t[f]_\#$ at zero is equal to zero,* and *c) $T_t[f]_\#$ is differentiable with invertible derivative at zero.* If moreover, f is exact we have

$$\begin{aligned} |\, [\, T_t[f]^* \alpha - \alpha \,] \,| &= |\, [\, T_t[f]^* \alpha - f^* \alpha \,] \,| \\ &\leq \hat{\kappa} \, |(T_t - \mathrm{Id}) \, [f] \,|_{C^1(\mathbb{U}_t)} \\ &\leq \kappa M_f \, t^{-\mu+1} \, , \end{aligned}$$

where we have used the fact that $T_t[f]$ satisfies property T5' of Lemma 39. Hence a finite dimensional version of the Implicit Function Theorem implies that, for t sufficiently large, there exists $[\, \gamma_t \,] \in H^1(\mathbb{V}, \mathbb{R})$ such that

$$T_t[f]_\# \, ([\, \gamma_t \,]) = - \, [\, T_t[f]^* \alpha - \alpha \,] \, , \tag{66}$$

and

$$|\, [\, \gamma_t \,] \,| \leq \kappa \, M_f \, t^{-\mu+1} \, . \tag{67}$$

Let $\gamma_1, \ldots, \gamma_n$ be closed forms, analytic on $\mathbb{V} + \rho$, and such that $\{\, [\, \gamma_j \,] \,\}_{j=1}^n$ is a basis of $H^1(\mathbb{V}, \mathbb{R})$. For t sufficiently large, let $\eta_t = (\eta_t^1, \ldots, \eta_t^n) \in \mathbb{R}^n$ be such that

$$[\, \gamma_t \,] = \sum_{j=1}^n \eta_t^j \; [\, \gamma_i \,] \, .$$

Then estimate (67) implies

$$|\, \eta_t| \leq \tilde{\kappa} \, M_f \, t^{-\mu+1} \, . \tag{68}$$

Following [Ban78], we construct, for t sufficiently large, a diffeomorphism h_t satisfying

$$[\, h_t^* \alpha - \alpha \,] = \sum_{j=1}^{n} \eta_t^j \; [\, \gamma_i \,] \; . \tag{69}$$

The non-degeneracy of Ω implies the existence of a vector field X_t such that

$$i_{X_t}(\Omega) = \sum_{j=1}^{n} \eta_t^j \, \gamma_j \; . \tag{70}$$

From (68) and (70) we have

$$|X_t|_{C^0(\mathbb{V}+\rho)} \leq \kappa' \; |\eta| \leq \kappa \, M_f \, t^{-\mu+1} \, . \tag{71}$$

Let h_t^ε be the flow generated by X_t:

$$\frac{d}{d\varepsilon} h_t^\varepsilon = X_t \circ h_t^\varepsilon \qquad h_t^0 = \mathrm{id} \, . \tag{72}$$

The existence of h_t^ε for all $\varepsilon \in [0,1]$ is obtained by assuming that t is sufficiently large and using (70). Using Proposition I.1.3. in [Ban78] we have

$$(h_t^\varepsilon)^* \, \alpha - \alpha = \int_0^\varepsilon \frac{d}{ds} \, (h_t^s)^* \, \alpha \, ds \; = \varepsilon \sum_{j=1}^{n} \eta_j \gamma_j + d\beta_t^\varepsilon \, , \tag{73}$$

with

$$\beta_t^\varepsilon = \int_0^\varepsilon \left(\int_0^s (h_t^r)^* \, i_{X_t} \left(\sum_{j=1}^{n} \eta_j \gamma_j \right) dr \right) ds + \int_0^\varepsilon \, (h_t^s)^* \; i_{X_t}(\alpha) \, ds \, ,$$

where we have used the Cartan's formula and the fact that the right hand side of (70) is closed. From (73) one has that, for all for $\varepsilon \in [0,1]$, h_t^ε preserves the exact symplectic form $\Omega = d\alpha$. Define $h_t \stackrel{\text{def}}{=} h_t^1$, then considering the first cohomology class in (73) we have that h_t satisfies (69). Finally notice that (69) and (66) imply (63)

Estimate (64) follows from (71) and (72). Now taking t sufficiently large, using (71), (72), and Gronwall's inequality we obtain, for t sufficiently large, the following estimate:

$$|Dh_t|_{C^0(\mathbb{U}+\rho)} < 2 \, ,$$

from which and (64) estimate (65) follows, for t sufficiently large. □

It is clear from Lemma 40 that the composition $\tilde{T}_t[f] \stackrel{\text{def}}{=} h_t \circ T_t[f]$ is exact symplectic on $\mathbb{U}_t$ The verification of properties T1-T6 of Theorem 2 for the diffeomorphism $\tilde{T}_t[f]$ is performed by using Lemma 40, Lemma 39, and the following estimates

$$| \, h_t \circ T_t[f] \, |_{C^1(\mathbb{U}_t + C\, t^{-1})} \leq \kappa \; |h_t|_{C^1(\mathbb{U}+\rho)} \, \left(1 + |T_t[f]|_{C^1(\mathbb{U}_t + C\, t^{-1})} \right) \, ,$$

$$|h_t \circ T_t[f] - S_t[f]|_{C^0(\mathbb{U}_t + C\, t^{-1})} \leq \kappa \; |h_t|_{C^1(\mathbb{U}+\rho)} \; |T_t[f] - S_t[f]|_{C^0(\mathbb{U}_t + C\, t^{-1})} + \kappa \; |h_t - \mathrm{id}|_{C^0(\mathbb{U}+\rho)}$$

$$| \, h_t \circ T_t[f] \, |_{C^2(\mathbb{U}_t + C\, t^{-1})} \leq \kappa \; |h_t|_{C^2(\mathbb{U}+\rho)} \, \left(1 + |T_t[f]|_{C^2(\mathbb{U}_t + C\, t^{-1})} \right) \, ,$$

$$|h_t \circ T_t[f] - f|_{C^0(\mathbb{U}_t)} \leq \kappa \; |h_t - \mathrm{id}|_{C^0(\mathbb{U})} + \kappa \; |T_t[f] - f|_{C^0(\mathbb{U}_t)} \; .$$

2.5 The contact case (Proof of Theorem 3)

Theorem 3 is proved following the same steps of the proof of Theorem 2 given in Section 2.4. Here we just mention the necessary modifications. Let ℓ, $\mathbb{U}$, $\mathbb{V}$, and $\mathbb{U}_t$ be as in Theorem 2. Let Ω be a contact form on $\mathbb{V}$, $f \in \mathrm{Diff}^\ell(\mathbb{U})$ a contact diffeomorphism, and let λ be a nowhere zero function such that $f^*\Omega = \lambda\,\Omega$. Define for $t \geq 1$ and $\varepsilon \in [0,1]$

$$\Omega_t^\varepsilon \stackrel{\text{def}}{=} \lambda\,\Omega + \varepsilon\,(S_t[f]^*\Omega - \lambda\,\Omega)\ .$$

Notice that, since the 2-form $d\,\Omega$ is a symplectic form on the fibres of the $2n$-dimensional subbundle $Ker(\Omega) \subset T(\mathbb{U})$, with the obvious modifications, Lemma 36 holds for the isomorphism $\mathscr{I}_{d\,\Omega}|_{Ker(\Omega)}$. Roughly speaking, this means that, for t sufficiently large, $\mathscr{I}_{d\,\Omega_t^\varepsilon}|_{Ker(\Omega_t^\varepsilon))}$ is also an isomorphism. Hence, for t sufficiently large and $\varepsilon \in [0,1]$, there exists a vector field u_t^ε satisfying

$$u_t^\varepsilon = -\left(\mathscr{I}_{d\Omega_t^\varepsilon}|_{Ker(\Omega_t^\varepsilon)}\right)^{-1}\left(\frac{\partial}{\partial\varepsilon}\,\Omega_t^\varepsilon\right)\,, \qquad u_t^\varepsilon \in Ker\,(\Omega_t^\varepsilon)\ ,$$

equivalently,

$$\begin{aligned} i_{u_t^\varepsilon}\,(d\,\Omega_t^\varepsilon) \quad &= -\,(S_t[f]^*\Omega - \lambda\,\Omega) \\ i_{u_t^\varepsilon}\,(\Omega_t^\varepsilon) \quad &= 0\,. \end{aligned} \tag{74}$$

Applying Proposition 26 to the 1-forms Ω and $\lambda\,\Omega$ we obtain, for t sufficiently large,

$$|\,S_t[f]^*\Omega - \lambda\,\Omega|_{C^0(\mathbb{U}+(2\,C\,t^{-1}))} \leq \kappa\,\hat{M}_f\,t^{-\mu}\,.$$

Then, following the same steps of the proof of Lemma 36 we obtain that the solution $u_t^\varepsilon(x)$ of (74), is continuous on $(\varepsilon, x) \in [0,1] \times \mathbb{U} + 2\,C\,t^{-1}$, real analytic with respect to $x \in \mathbb{U} + 2\,C\,t^{-1}$, and moreover

$$\begin{aligned} |u_t^\varepsilon|_{C^0(\mathbb{U}+2\,C\,t^{-1})} &\leq 2\,M_\Omega\ |\,S_t[f]^*\Omega - \lambda\,\Omega|_{C^0(\mathbb{U}+2\,C\,t^{-1})} \\ &\leq \kappa\,\hat{M}_f\,t^{-\mu}\,. \end{aligned} \tag{75}$$

Now, for t sufficiently large, let ϕ_t^ε be the solution of the following differential equation

$$\frac{d}{d\,\varepsilon}\,\phi_t^\varepsilon = u_t^\varepsilon\,, \qquad \phi_t^0 = \mathrm{id}\,,$$

then (74) and the Cartan's formula for the Lie derivative imply

$$\frac{d}{d\varepsilon}\,((\phi_t^\varepsilon)^*\ \Omega_t^\varepsilon) = (\phi_t^\varepsilon)^*\,\left[d\left(i_{u_t^\varepsilon}\,\Omega_t^\varepsilon\right) + i_{u_t^\varepsilon}\,(d\Omega_t^\varepsilon) + \frac{d}{d\varepsilon}\Omega_t^\varepsilon\right] = 0.$$

Hence

$$(\phi_t^\varepsilon)^*\ \Omega_t^\varepsilon = \lambda\,\Omega\,, \qquad \forall\,\varepsilon \in [0,1]\,.$$

Moreover, following the proof of Lemma 38 and using (75) one obtains that ϕ_t^1 satisfies estimates 3, 4, and 5 of Lemma 38. The proof of Theorem 3 is finished by following the same steps in the proof of Lemma 39.

3 An application: KAM theory without action-angle variables for finitely differentiable symplectic maps

Let $(\mathbb{U},\,\Omega = d\alpha)$ be a $2n$-dimensional analytic exact symplectic manifold and let $f \in \mathrm{Diff}^\ell(\mathbb{U})$ be an exact symplectic map. The study of the existence of n-dimensional invariant tori with quasi-periodic motion is based on the study of the equation

$$F(f, K) = 0\,, \tag{76}$$

where

$$F(f,K)(\theta) \stackrel{\text{def}}{=} (f \circ K)(\theta) - K(\theta + \omega), \tag{77}$$

$K : \mathbb{T}^n \to \mathbb{U}$ is the function to be determined, and $\omega \in \mathbb{T}^n$ satisfies a Diophantine condition. In [dlLGJV05] it is proved that if f is a real analytic diffeomorphism and if there exists a real analytic parameterization of an n-dimensional torus K, satisfying a non-degeneracy condition, such that (f, K) is an approximate solution of (76) in the sense that $|F(f,K)|_{C^0(\mathbb{T}^n+\rho)}$ is 'sufficiently small', with respect to the non-degeneracy conditions, then there exists a true real analytic solution of (76), which moreover is close to the approximate solution. In Theorem 4 we give the rigorous formulation of this result. We emphasize that in Theorem 4 we do not assume the symplectic map is written either in action-angle variables or as perturbation of an integrable one. Moreover, the proof of Theorem 4 produces a algorithm to compute invariant tori for exact symplectic maps.

In this section we show that a finitely differentiable version of Theorem 4 also holds, see Theorem 5 for the formulation. The proof of Theorem 5 we present here is a slightly modified Moser's analytic smoothing method. We remark that, since Theorem 4 holds for exact symplectic maps, then the symplectic map f is smoothed using the operator T_t of Theorem 2. Moreover, rather than assuming the existence of an analytic solution of (76) we assume the existence of a finitely differentiable initial approximate solution of (76) and give conditions guaranteeing the existence of an analytic solution, which is close to the approximated one in finitely differentiable norms. This is achieved by using the estimates given in Theorem 2 and Proposition 34.

In this section we also prove the bootstrap of regularity of invariant tori with Diophantine rotation vector for exact symplectic diffeomorphisms. To obtain the bootstrap of regularity first, we prove the local uniqueness of finitely differentiable invariant tori for finitely differentiable symplectic maps. We remark that the uniqueness result stated in Theorem 6 is the finitely differentiable version of Theorem 2 in [dlLGJV05].

The local uniqueness and the bootstrap of regularity are stated in Theorem 6 and Theorem 7, respectively. Theorem 6 and Theorem 7 are similar to Theorem 4 and Theorem 5 in [SZ89], respectively. However, while the latter are stated and proved for Hamiltonian vector fields written in a Lagrangian formalism, Theorem 6 and Theorem 7 are stated for symplectic maps that are not neither written in action-angle variables nor perturbation of integrable ones, and proved using the symplectic formalism rather than the Lagrangian one.

The existence of the operator T_t in Theorem 2, in the exact symplectic case, enables us to obtain analytic approximate solutions of equation (76) close to a given finitely differentiable one. This together with the uniqueness argument yield the bootstrap of regularity for solutions of (76).

Let $\mathbb{U}$ be either an open subset of $\mathbb{R}^{2n}$ or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^n$. In addition to the notation introduced in Section 2.1 we use the following notation. For each $x \in \mathbb{U}$ let $J(x) : T_x\,\mathbb{U} \to T_x\,\mathbb{U}$ be linear isomorphism satisfying

$$\Omega(x)\,(\xi,\eta) = \xi^\top \cdot\, J(x)\,\eta\,, \tag{78}$$

where $\cdot$ is the Euclidean scalar product on $\mathbb{R}^{2n}$. The average of a mapping $K \in C^0(\mathbb{T}^n, \mathbb{U})$ is defined by

$$\operatorname{avg}\{K\}_\theta \stackrel{\text{def}}{=} \int_{\mathbb{T}^n} K(\theta)\, d\theta.$$

Definition 41. *Given $\gamma > 0$ and $\sigma \geq n$, we define $D(\gamma,\sigma)$ as the set of frequency vectors $\omega \in \mathbb{T}^n$ satisfying the Diophantine condition:*

$$|\ell \cdot \omega - m| \geq \gamma\, |\ell|_1^{-\sigma} \qquad \forall \ell \in \mathbb{Z}^n \setminus \{0\},\, m \in \mathbb{Z},$$

where $|\ell|_1 = |\ell_1| + \cdots + |\ell_n|$.

Definition 42. *Given a symplectic diffeomorphism $f \in \operatorname{Diff}^1(\mathbb{U})$ and $\omega \in D(\gamma,\sigma)$, let $\mathcal{N}$ denote the set of functions in $K \in C^1(\mathbb{T}^n, \mathbb{U})$ satisfying the following conditions:*

N1 *There exists an $n \times n$ matrix-valued function $N(\theta)$ such that*

$$N(\theta) \left(DK(\theta)^\top DK(\theta) \right) = I_n,$$

where I_n is the n-dimensional identity matrix.

N2 *The average of the matrix-valued function*

$$A(\theta) \stackrel{\text{def}}{=} P(\theta+\omega)^\top \left[Df(K(\theta))J(K(\theta))^{-1}P(\theta) - J(K(\theta+\omega))^{-1}P(\theta+\omega) \right], \tag{79}$$

with J defined in (78) and

$$P(\theta) \stackrel{\text{def}}{=} DK(\theta)\, N(\theta),$$

is non-singular.

By the Rank Theorem, Condition **N1** guarantees that $\dim K(\mathbb{T}^n) = n$. For the KAM theorems 4 and 5, the main non-degeneracy condition is **N2**, which is a twist condition. Note that **N1** only depends on K whereas **N2** depends on both K and f.

From now on we assume that $\Omega = d\alpha$ is analytic exact symplectic form as in Theorem 2. Let J be the isomorphism defined by (78), and let J^{-1} denote its inverse. The following Theorem 4 is the main theorem from [dlLGJV05].

Theorem 4. *Let $\mathbb{U}$ be either a compensated open domain in $\mathbb{R}^{2n}$ or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^n$ a compensated open domain. Let f be an exact symplectic diffeomorphism on $\mathbb{U}$ and $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma > n$. Assume that the following hypotheses hold:*

A1. $K \in \mathcal{N} \cap \mathcal{A}\left(\mathbb{T}^n + \rho, C^1\right)$ *(see Definition 42 and Definition 4).*

A2. The map f is real analytic and it can be holomorphically extended to $\mathcal{B}$, a complex neighbourhood of $K\left(\mathbb{T}^n + \rho\right)$, such that $\operatorname{dist}\left(K\left(\mathbb{T}^n + \rho\right), \partial\mathcal{B}\right) > \eta > 0$. Furthermore, $|f|_{C^2(\mathcal{B})} < \infty$.

A3. $|J|_{C^1(\mathcal{B})}, \left| J^{-1} \right|_{C^1(\mathcal{B})}, |\alpha|_{C^2(\mathcal{B})} < \infty$

Then, there exists a constant $c > 0$ depending on σ, n, $|f|_{C^2(\mathcal{B})}$, $|\alpha|_{C^2(\mathcal{B})}$, $|J|_{C^1(\mathcal{B})}$, $\left|J^{-1}\right|_{C^1(\mathcal{B})}$, $|DK|_{C^0(\mathbb{T}^n+\rho)}$, $|N|_{C^0(\mathbb{T}^n+\rho)}$, $\left|(\operatorname{avg}\{A\}_\theta)^{-1}\right|$ (where N and A are as in Definition 42) such that, if

$$c\,\gamma^{-4}\,\rho^{-4\sigma}\,|F(f,K)|_{C^0(\mathbb{T}^n+\rho)} < \min(1\,,\eta),$$

then there exists $K^* \in \mathcal{N} \cap \mathcal{A}\left(\mathbb{T}^n + \rho/2, C^1\right)$ such that $F(f, K^*) = 0$. Moreover,

$$|K^* - K|_{C^0(\mathbb{T}^n+\rho/2)} \le c\,\gamma^{-2}\,\rho^{-2\sigma}\,|\,F(f,K)\,|_{C^0(\mathbb{T}^n+\rho)}, \tag{80}$$

and

$$|DK^* - DK|_{C^0(\mathbb{T}^n+\rho/2)} \le c\,\gamma^{-2}\,\rho^{-(2\sigma+1)}\,|\,F(f,K)\,|_{C^0(\mathbb{T}^n+\rho)}.$$

The finitely differentiable version of Theorem 4 we present here is the following.

Theorem 5. *Let $\omega \in D(\gamma, \sigma)$, for some $\gamma > 0$ and $\sigma > n$. Let $m \in \mathbb{N}$, $\ell \notin \mathbb{N}$ be such that $4\sigma + 3 < \ell < m$. Let $\mathbb{U}$ be either a compensated open domain in $\mathbb{R}^{2n}$ with C^m-boundary, or $\mathbb{T}^n \times U$, with $U \subset \mathbb{R}^n$ a compensated open domain with C^m-boundary. Let $f \in \operatorname{Diff}^\ell(\mathbb{U})$ be an exact symplectic diffeomorphism and let $K \in C^\ell(\mathbb{T}^n, \mathbb{U})$ be a parameterization of an n-dimensional torus. Assume that the following hypotheses hold:*

S1. $|DK|_{C^0(\mathbb{T}^n)} < \beta$ *and* $K(\mathbb{T}^n) \subset \mathbb{U}$, *with* $\eta \stackrel{\text{def}}{=} 2^{-1}\, dist\,(K(\mathbb{T}^n), \partial\mathbb{U}) > 0$.

S2. $K \in \mathcal{N}$ *(see Definition 42).*

S3. $\Omega = d\alpha$ *is real analytic on* $\mathbb{U}+\rho$*, and* $|J|_{C^{\ell}(\mathbb{U}+\rho)}, |\alpha|_{C^{\ell}(\mathbb{U}+\rho)} < \zeta$*, and* $\left|J^{-1}\right|_{C^{1}(\mathbb{U}+\rho)} < M_{\Omega}$*. for some* $\rho, \zeta > 0$*.*

Then, given $4\sigma+2 < \mu < \ell-1$ *there exists two positive constants* c *and* $\rho^* < 1$*, depending on* μ*,* n*,* ℓ*,* σ*,* ζ*,* β*,* M_Ω*,* $|f|_{C^{\ell}(\mathbb{U})}$*,* $|K|_{C^{\ell}(\mathbb{T}^n)}$*,* $|N|_{C^0(\mathbb{T}^n)}$*, and* $\left|(\mathrm{avg}\,\{A\}_\theta)^{-1}\right|$*, such that: given* $0 < \rho_1 \le \rho^*$*, if* $\mu - 2\sigma \notin \mathbb{Z}$ *and*

$$c\,\gamma^{-4}\,\rho_1^{-(4\sigma+1)}\,|F(f,K)|_{C^0(\mathbb{T}^n)} \le \min(1, \kappa\beta, \eta)\,, \tag{81}$$

where $\kappa = \kappa(n,\ell,1)$ *is as in Proposition 20, then there exists a parameterization of an* n*-dimensional torus,* $K^* \in C^{\mu-(2\sigma+1)}(\mathbb{T}^n, \mathbb{U})$ *such that* $F(f, K^*) = 0$ *and*

$$|K - K^*|_{C^{\nu}(\mathbb{T}^n)} \le \tilde{c}\,\gamma^{-2}\,\rho_1^{-(2\sigma+\nu)}\left(\rho_1^{\mu-1} + |F(f,K)|_{C^0(\mathbb{T}^n)}\right), \qquad \forall\, 0 \le \nu < \mu - (2\sigma+1)\,,$$

where F *is as in (77) and* $\tilde{c}$ *is a constant depending on the same quantities as* c*.*

Remark 43. *Let* f *and* K *be as in Theorem 5. Notice that since* $|f|_{C^{\ell}(U)}$ *and* $|K|_{C^{\ell}(\mathbb{T}^n)}$ *are bounded we have that* $|F(f,K)|_{C^{\ell}(\mathbb{T}^n)}$ *is also bounded. If moreover assumption (81) holds then there is a constant* κ *such that*

$$|F(f,K)|^{\ell}_{C^{\ell}(\mathbb{T}^n)} \le \kappa\,, \qquad \text{and} \qquad |F(f,K)|_{C^0(\mathbb{T}^n)} \le \kappa\,\rho_1^{4\sigma+1}\,.$$

Thus, by using the interpolation estimates [dlLO99, Zeh75], we have for any $0 \le s \le \ell$

$$|F(f,K)|_{C^s(\mathbb{T}^n)} \le \kappa\,|F(f,K)|^{s}_{C^{\ell}(\mathbb{T}^n)}\,|F(f,K)|^{1-s/\ell}_{C^0(\mathbb{T}^n)} \le \hat{\kappa}\,\rho_1^{(4\sigma+1)(1-s/\ell)}\,.$$

Hence assumption (81) implies that all the intermediate norms $|F(f,K)|_{C^s(\mathbb{T}^n)}$ *with* $0 \le s < \ell$ *are also small. We therefore have that hypothesis (81) is equivalent to assume the* C^s*-norms of the function error to be small, for* $0 \le s < \ell$*.*

The local uniqueness is stated in the following

Theorem 6. *Let* $\omega \in D(\gamma, \sigma)$ *for some* $\gamma > 0$ *and* $\sigma > n$*. Let* $\ell > 2\sigma$ *be such that* $\ell, \ell - 2\sigma \notin \mathbb{Z}$*. Let* $f \in \mathrm{Diff}^{\ell+2}(\mathbb{U})$ *be a symplectic diffeomorphism. Assume that* (f, K_1) *and* (f, K_2) *satisfy (76), with* $K_1, K_2 \in C^{\ell+1}(\mathbb{T}^n, \mathbb{U})$ *satisfying N1 and N2 in Definition 42. Then, there exists a constant* κ*, depending on* n*,* $\left|J^{-1}\right|_{C^0(\mathbb{U})}$*,* $|f|_{C^{\ell+2}(\mathbb{U})}$ $|K_2|_{C^{\ell+1}(\mathbb{T}^n)}$*,* $|K_1|_{C^{\ell+1}(\mathbb{T}^n)}$*, and* $|N_2|_{C^0}$*, with* N_2 *defined as in N1 in Definition 42 by replacing* K *with* K_2*, such that if*

$$\kappa\,\gamma^{-2}\,|K_1 - K_2|_{C^{\ell}(\mathbb{T}^n)} < 1$$

then $K_1 \circ R_{\hat{\theta}} = K_2$ *on* $\mathbb{T}^n$*, for some constant* $\hat{\theta} \in \mathbb{R}^n$*.*

The bootstrap of regularity is stated in the following

Theorem 7. *Let* $\omega \in D(\gamma, \sigma)$*, for some* $\gamma > 0$ *and* $\sigma > n$ *and let* $\varrho > 0$*. Let* $4\sigma + 3 < \ell_1 < m$*, with* $m \in \mathbb{N}$ *and* $\ell_1 \notin \mathbb{N}$*. Let* $\mathbb{U}$ *be as in Theorem 5. Let* (K, f) *be a solution of (76), with* $f \in \mathrm{Diff}^{\ell_1}(\mathbb{U})$ *an exact symplectic diffeomorphism and* $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$ *satisfying N1 and N2 in Definition 42. Let* $\ell \in [\ell_1, m)$ *be not an integer and assume that* $f \in \mathrm{Diff}^{\ell}(\mathbb{U})$*, and that hypotheses S1-S3 in Theorem 5 hold (replacing* ρ *with* ϱ *in S3). Then, for any* $4\sigma + 2 < \mu < \ell - 1$ *satisfying* $\mu - (2\sigma+1) \notin \mathbb{Z}$ *we have that* $K \in C^{\mu-(2\sigma+1)}(\mathbb{T}^n)$*. Moreover if* $m = \infty$ *and* $f \in \mathrm{Diff}^{\infty}(\mathbb{U})$ *then* $K \in C^{\infty}(\mathbb{T}^n, \mathbb{U})$*. Furthermore, if* $f \in \mathcal{A}(\mathbb{U} + \varrho, C^{\ell_1})$*, then* $K \in \mathcal{A}(\mathbb{T}^n, C^1)$*.*

3.1 Existence (Proof of Theorem 5)

Throughout this section we assume that the hypotheses of Theorem 5 hold. As we already mentioned, the proof of Theorem 5 given here is based on Moser's technique of analytic smoothing [Mos66, Zeh75]. What we do is the following:

Step 1: Obtain an analytic approximate solution (f_1, K_1) of (76), with f_1 exact symplectic map and K_1 satisfying properties N1-N2 of Definition 42.

Step 2: Apply Moser's smoothing technique and Theorem 4 to construct a sequence of analytic solutions of (76) converging to a finitely differentiable solution (f, K^*). More concretely, starting with (f_1, K_1), we assume that we have computed (f_m, K_m) an analytic solution of (76) and verify that, if T_t is as in Theorem 2 then, for a suitable t_m, $(T_{t_m}[f], K_m)$ is an the approximate solution of (76) that satisfies the hypotheses of Theorem 4, so that one obtains a new analytic solution (f_{m+1}, K_{m+1}) of (76), with $f_{m+1} = T_{t_m}[f]$. The convergence of the method is obtain by using (80) in Theorem 4.

In order perform Step 1 we use pairs of functions of the form $(T_t[f], S_t[K])$, where T_t is as in Theorem 2, and S_t is as in Definition 6. Notice that since S_t takes periodic functions into periodic functions then $S_t[K] \in \mathcal{A}\left(\mathbb{T}^n + t^{-1}, C^1\right)$ is an analytic parameterization of an n-dimensional torus that is close to K (see Remark 8).

To prove that $S_t[K]$ satisfies properties N1-N2 of Definition 42, since condition N2 in Definition 42 depends on both the parameterization and the map, it is necessary to fix the constants appearing in Theorem 2 and verify that $S_t[K]\left(\mathbb{T}^n + t^{-1}\right)$ belongs to the domain of $T_t[f]$. This is done in the following

Lemma 44. *Let $K \in C^\ell(\mathbb{T}^n, \mathbb{U})$ satisfy hypothesis S1 of Theorem 5. Let κ be as in Proposition 20, define $r \stackrel{\text{def}}{=} \kappa\, \beta$, then*

$$|DS_t[K]|_{C^0(\mathbb{T}^n + t^{-1})} < r\,, \qquad \text{for all} \quad t \geq 1. \tag{82}$$

Moreover, if $\mathbb{U}_t$ is defined by (7), then there exists $t_6 \geq 1$, depending on n, ℓ, $|K|_\ell$, and η such that for all $t > t_6$, $S_t[K](\mathbb{T}^n) \subset \mathbb{U}_t$, and

$$|\, S_t[K] - K\,|_{C^0(\mathbb{T}^n)} \, < \frac{1}{2}\,\eta. \tag{83}$$

Furthermore, if $2 < \mu < \ell - 1$ is given, let $t^* = t^*(d, \ell, \mathbb{V}, 2\,r, \mu, |f|_{C^\ell(\mathbb{U})}), M_\Omega\,, \zeta)$, be as in Theorem 2, then for all $t \geq \max(t^*, t_6)$, the components of the symplectic map $T_t[f]$ belong to $\mathcal{A}(\mathbb{U}_t + 2\,r\,t^{-1}, C^{\,2})$ and properties T0-T7 of Theorem 2 hold.

Proof. Part 2 of Proposition 20 implies (82), from which we have

$$|\mathrm{Im}\,(S_t[K])|_{C^0(\mathbb{T}^n + t^{-1})} \leq t^{-1}\; |DS_t[K]|_{C^0(\mathbb{T}^n + t^{-1})} \leq r\,t^{-1}.$$

Let $t_6 > 1$ be sufficiently large so that for any $t \geq t_6$ the following holds:

$$\max\left(\kappa\,|K|_{C^\ell(\mathbb{T}^n)}\; t^{-\ell},\, t^{-1}\right) < 2^{-1}\,\eta\,,$$

then part 1 of Proposition 20 implies (83). So we have $S_t[K]\left(\mathbb{T}^n + t^{-1}\right) \subset \mathbb{U}_t + r\,t^{-1}$. Now apply Theorem 2 to the constants $C = 2r$, and $\beta = |f|_{C^\ell(\mathbb{U})}$. □

Now we prove that, for t sufficiently large, $S_t[K]$ satisfies N1-N2 of Definition 42.

Lemma 45. *Let r and t_6 be as in Lemma 44, and let $2 < \mu < \ell - 1$ be fixed. Assume that $K \in C^\ell(\mathbb{T}^n, \mathbb{U})$ satisfies the hypothesis of Theorem 5 and let N and A be as in Definition 42. Then, there exists $t_7 \geq t_6$, depending on n, ℓ, $2\,r$, μ, η, M_Ω, $|K|_{C^\ell(\mathbb{T}^d)}$, $|N|_{C^0(\mathbb{T}^d)}$, $\left|\mathrm{avg}\,\{A\}_\theta^{-1}\right|$,*

$|f|_{C^\ell(\mathcal{B})}$ and M_f, with M_f as in Theorem 2, such that $S_t[K] \in \mathcal{A}\left(\mathbb{T}^n + t^{-1}, C^1\right) \cap \mathcal{N}$, for all $t \geq t_7$. Moreover, if

$$N_t(\theta) \stackrel{\text{def}}{=} \left(DS_t[K](\theta)^\top DS_t[K](\theta)\right)^{-1}$$

and

$$A_t(\theta) \stackrel{\text{def}}{=} P_t(\theta+\omega)^\top \left[\, DT_t[f](S_t[K](\theta))J(S_t[K](\theta))^{-1}P_t(\theta) \;-\; J(S_t[K](\theta+\omega))^{-1}P_t(\theta+\omega)\,\right]$$

where $P_t(\theta) \stackrel{\text{def}}{=} S_t[DK](\theta)\, N_t(\theta)$, then the following hold

$$|N_t|_{C^0(\mathbb{T}^n+t^{-1})} \leq 2\; |N|_{C^0(\mathbb{T}^n)} \left(1 + \kappa\, \hat{M}_{K,f}\; |N|_{C^0(\mathbb{T}^n)}\, t^{-1}\right)\,, \tag{84}$$

and

$$\left|\operatorname{avg}\{A_t\}_\theta^{-1}\right| \;\leq \left|\operatorname{avg}\{A\}_\theta^{-1}\right| \left(1 + \kappa\, \hat{M}_{K,f}\; \left|\operatorname{avg}\{A\}_\theta^{-1}\right| t^{-\mu+2}\right)\,,$$

where κ is a constant depending on n and ℓ and $\hat{M}_{K,f}$ depends on $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, $|f|_{C^\ell(\mathcal{B})}$ and M_f.

Proof. Notice that the conditions N1-N2 in Definition 42 deal with invertibility of matrices, hence Lemma 45 is a consequence of the openness of the invertibility of matrices. In what follows we obtain explicit estimates for the size of t. Performing some simple computations and using Proposition 20 one has

$$\left|DS_t[K](\theta)^\top DS_t[K](\theta) - N(\theta)^{-1}\right|_{C^0(\mathbb{T}^n)} \leq \kappa\; |K|^2_{C^\ell(\mathbb{T}^n)}\, t^{-\ell+1}\,.$$

Hence if t is sufficiently large such that

$$\kappa\; |K|^2_{C^\ell(\mathbb{T}^n)}\; |N|_{C^0(\mathbb{T}^n)}\, t^{-\ell+1} \;\leq 1/2\,,$$

the Neuman's series theorem implies that, for all $\theta \in \mathbb{R}^d$, the matrix $DS_t[K](\theta)^\top DS_t[K](\theta)$ is invertible and its inverse, denoted by N_t, satisfies

$$|N_t - N|_{C^0(\mathbb{T}^n)} \leq 2\,\kappa\; |K|^2_{C^\ell(\mathbb{T}^n)}\; |N|^2_{C^0(\mathbb{T}^n)}\, t^{-\ell+1} \;\leq\; |N|_{C^0(\mathbb{T}^n)}\,. \tag{85}$$

Now, let $\theta \in \mathbb{R}^n + t^{-1}$, then part 1 of Lemma 30 implies, for t sufficiently large,

$$|DS_t[K](\theta) - DS_t[K](\operatorname{Re}(\theta))| \leq \left|D^2 S_t[K]\right|_{C^0(\mathbb{T}^n+t^{-1})} |\operatorname{Im}(\theta)| \leq \kappa\; |K|_{C^\ell(\mathbb{T}^n)}\; t^{-1}\,,$$

κ is a constant depending on n, and ℓ. So one obtains

$$\left|DS_t[K]^\top(\theta) DS_t[K](\theta) - N_t^{-1}(Re(\theta))\right| \leq \kappa\; |K|^2_{C^\ell(\mathbb{T}^n)}\; t^{-1}\,.$$

Then, if t is sufficiently large so that

$$2\,\kappa\; |K|^2_{C^\ell(\mathbb{T}^n)}\; |N|_{C^0(\mathbb{T}^n)}\, t^{-1} \;\leq 1/2$$

we have from (85)

$$\kappa\; |K|^2_{C^\ell(\mathbb{T}^n)}\; |N_t|_{C^0(\mathbb{T}^n)}\, t^{-1} \;\leq 1/2\,.$$

Hence, Neuman's series theorem implies that the matrix $DS_t[K](\theta)^\top DS_t[K](\theta)$ is invertible for all $\theta \in \mathbb{R}^n + t^{-1}$ and

$$|N_t|_{C^0(\mathbb{T}^n+t^{-1})} \leq |N_t|_{C^0(\mathbb{T}^n)} + 2\; |N_t|^2_{C^0(\mathbb{T}^n)}\; \kappa\; |K|^2_{C^\ell(\mathbb{T}^n)}\; t^{-1},$$

from which and (85) estimate (84) follows.

It is clear that, for t sufficiently large, A_t is a perturbation of A defined in (79). In what follows we give an estimation of the size of $|A_t - A|_{C^0(\mathbb{T}^n)}$. Let $P(\theta) = DK(\theta)N(\theta)$, then using (85) and Proposition 20 we have

$$|P_t - P|_{C^0(\mathbb{T}^n)} \leq \kappa\, M_K\, t^{-\ell+1}\,,$$

where κ depends on n and ℓ, and M_K is a constant depending on $|K|_{C^\ell(\mathbb{T}^n)}$ and $|N|_{C^0(\mathbb{T}^n)}$. Moreover, performing some simple computations and using Theorem 2, Proposition 20, and the Cauchy's estimates we have

$$|DT_t[f](S_t[K](\theta)) - Df(K(\theta))\,|_{C^0(\mathbb{T}^n)} \leq \kappa\, t^{-\mu+2}\,, \tag{86}$$

where κ is a generic constant independent of t. Finally, using again Proposition 20 we have

$$|J \circ S_t[K] - J \circ K|_{C^0(\mathbb{T}^n)} \leq |J|_{C^1(\mathbb{U})}\ \kappa\, |K|_{C^\ell(\mathbb{T}^n)}\, t^{-\ell+2}\,, \tag{87}$$

Performing some computations and using (3.1), (86), and (87) one gets

$$|A_t - A|_{C^0(\mathbb{T}^n)} \leq \kappa\, M_{f,K}\, t^{-\mu+2}\,,$$

where κ is a constant depending on n, ℓ, C^{-1}, κ, κ, $|J|_{C^1(\mathbb{U})}$, and $M_{f,K}$ is a constant depending on $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, $|f|_{C^\ell(\mathcal{B})}$ and M_f. Hence the proof of Lemma 45 is finished by applying Neuman's series theorem and taking t is sufficiently large so that

$$\left|\operatorname{avg}\{A\}_\theta^{-1}\right|\ \kappa\, M_{f,K}\, t^{-\mu+2}\ \leq 1/2\,.$$

$\square$

From lemmas 44 and 45 we have that, for t sufficiently large, $(T_t[f], S_t[K])$ is a candidate for an analytic approximate solution of equation (76). In the following lemma we summarize the results of lemmas 44 and 45 and give an estimate of $|\,F(T_t[f], S_t[K])\,|_{C^0(\mathbb{T}^n + t^{-1})}$.

Lemma 46. *Let t_7 be as in Lemma 45, and let $2 < \mu < \ell - 1$. Assume that $K \in C^\ell(\mathbb{T}^n)$, $f \in \operatorname{Diff}^\ell(\mathbb{U})$ and that hypothesis of Theorem 5 hold. Then there exists $t_8 \geq t_7$, depending on n, ℓ, β, μ, ζ, M_Ω, η, $|K|_{C^\ell(\mathbb{T}^d)}$, $\left|\operatorname{avg}\{A\}_\theta^{-1}\right|$, $|N|_{C^0(\mathbb{T}^n)}$, $|f|_{C^\ell(\mathbb{U})}$, and M_f, with M_f as in Theorem 2, such that for all $t \geq t_8$ the following hold*

1. *$S_t[K] \in \mathcal{A}\left(\mathbb{T}^n + t^{-1}, C^1\right)$, and $|DS_t[K]|_{C^0(\mathbb{T}^n + t^{-1})} \leq r$, $r = r(n, \beta)$ as in Lemma 44.*
2. *$S_t[K](\mathbb{T}^n) \subset \mathbb{U}_t$, with $|S_t[K] - K|_{C^0(\mathbb{T}^n)} < \eta/2$.*
3. *$T_t[f] \in \mathcal{A}(\mathbb{U}_t + 2\,r\,t^{-1}, C^2)$ with $|T_t[f]|_{C^2(\mathbb{U}_t + (2\,r\,t^{-1}))} \leq \kappa\, M_f$.*
4. *$S_t[K] \in \mathcal{N}$ and if N_t and A_t are as in Lemma 45, then*

$$|N_t|_{C^0(\mathbb{T}^n + t^{-1})} \leq 3\ |N|_{C^0}\ , \qquad \left|\operatorname{avg}\{A_t\}_\theta^{-1}\right| \leq \frac{3}{2}\ \left|\operatorname{avg}\{A\}_\theta^{-1}\right|\,.$$

5. *There is a constant κ, depending on n, ℓ, C , μ, M_Ω, and $|Df|_{C^0(\mathbb{U})}$ such that*

$$|F(T_t[f], S_t[K])|_{C^0(\mathbb{T}^n + t^{-1})}\ \leq\ \kappa\, \hat{M}_{K,f}\, t^{-\mu+1}\ +\ \kappa\ |F(f, K)|_{C^0(\mathbb{T}^n)}\ ,$$

where

$$\hat{M}_{K,f} = \max\Big(M_f, |f|_{C^\ell(\mathbb{U})}\,\Big(1 + |K|^{\mu}_{C^\ell(\mathbb{T}^n)} + |K|_{C^\ell(\mathbb{T}^n)}\Big)\Big)\,,$$

with M_f as in Theorem 2.

Proof. Parts 1, 2 and 3 are stated in Lemma 44. Part 4 is consequence of Lemma 45. Part 5 is a consequence of property T3 in Theorem 2, Proposition 34, and part 2 in Proposition 20. Indeed, let t_2 be as in Proposition 34, and let t^* and $\{T_t\}_{t\geq t^*}$ be as in Lemma 44. Then part 5 follows by taking $t_8 = \max(t_2, t^*, t_7)$ and using the following equality

$$F(T_t[f], S_t[K]) = \{(T_t[f] - S_t[f]) \circ S_t[K]\} + \{S_t[f] \circ S_t[K] - S_t[f \circ K]\} + S_t[F(f, K)]\,.$$

$\square$

Now we give the sufficient conditions to have an iterative scheme to construct a sequence of analytic solutions (f_j, K_j^*) of equation (76)

Lemma 47. *Let $r = r(n, \beta)$ be as in Lemma 44. Let $2 < \mu < \ell - 1$ be given and assume that for fixed $m \geq 1$ there is a $\tau_m \geq 1$ such that $(f_m, K_m) = (T_{\tau_m}[f], S_{\tau_m}[K])$, satisfies the following conditions:*

A1(m) $K_m \in \mathcal{A}\left(\mathbb{T}^n + \rho_m, C^1\right)$, *and* $|DK_m|_{C^0(\mathbb{T}^n+\rho_m)} \leq r_m$, *with* $\rho_m \stackrel{\text{def}}{=} \tau_m^{-1}$ *and* $r_m = r \sum_{j=0}^{m-1} 2^{-j}$.

A2(m) $K_m(\mathbb{T}^n) \subset \mathbb{U}_{\tau_m}$, *with* $|K_m - K|_{C^0(\mathbb{T}^n)} < \eta_m$ *where* $\eta_m \stackrel{\text{def}}{=} \eta \sum_{j=1}^{m} 2^{-j}$

A3(m) $f_m \in A\left(\mathbb{U}_{\tau_m} + (2\, r_m\, \rho_m), C^2\right)$ *with* $|\, f_m\,|_{C^2(\mathbb{U}_{\tau_m}+2\, r_m\, \rho_m)} \leq \kappa\, M_f$.

A4(m) If N_m and A_m are defined as in Lemma 45, by replacing $S_t[K]$ with K_m, then

$$|N_m|_{C^0(\mathbb{T}^n+\rho_m)} \leq 2\, |N|_{C^0(\mathbb{T}^n)} \prod_{j=1}^{m} \left(1 + 2^{-j}\right)\,,$$

$$\left|\operatorname{avg}\{A_m\}_\theta^{-1}\right| \leq \left|\operatorname{avg}\{A\}_\theta^{-1}\right| \prod_{j=1}^{m} \left(1 + 2^{-j}\right)\,,$$

where N and A are as in Definition 42.

Then there exists two constants $\tilde{\lambda}$ and λ, depending on σ, n, η, M_Ω, $|f|_{C^\ell(\mathbb{U})}$, $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, $\left|(\operatorname{avg}\{A\}_\theta)^{-1}\right|$, such that, if

$$\gamma^{-4}\, \tilde{\lambda}\, \rho_m^{-(4\sigma+1)}\, |F(f_m, K_m)|_{C^0(\mathbb{T}^n+\rho_m)} < \min(1\,, r,\, \eta), \tag{88}$$

then there exists a parameterization $K_{m+1} \in \mathcal{A}\left(\mathbb{T}^n + \rho_{m+1}, C^1\right) \cap \mathcal{N}$, with $\rho_{m+1} = \rho_m/2$, such that $F(f_m, K_{m+1}) = 0$,

$$|K_{m+1} - K_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leq \gamma^{-2}\, \tilde{\lambda}\, \rho_m^{-2\,\sigma}\, |F(f_m, K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\,. \tag{89}$$

and

$$|DK_{m+1} - DK_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leq \tilde{\lambda}\, \gamma^{-2}\, \rho_m^{-(2\,\sigma+1)}\, |F(f_m, K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\,. \tag{90}$$

Furthermore, if $f_{m+1} \stackrel{\text{def}}{=} T_{2\,\tau_m}[f]$ and

$$2^{m+1}\lambda\, \left(\rho_m^{\mu-1} + \gamma^{-2}\, \rho_m^{-(2\sigma+1)}\, |F(f_m, K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\right) < \min(1, r, \eta), \tag{91}$$

then (f_{m+1}, K_{m+1}) satisfies properties A1(m+1)-A4(m+1) and the following estimate holds:

$$|F(f_{m+1}, K_{m+1})|_{C^0(\mathbb{T}^n+\rho_{m+1})} \leq \kappa\, M_f\, \rho_m^{\mu-1}\,, \tag{92}$$

where κ and M_f are as in Theorem 2.

Proof. Properties A1(m)-A4(m) and Theorem 4 imply the existence of a constant λ_m depending on σ, n, γ^{-4}, ζ, M_Ω, $|f_m|_{C^2(\mathbb{U}_{\tau_m}+(2\,r_m\,\rho_m))}$, $|DK_m|_{C^0(\mathbb{T}^n+\rho_m)}$, $|N_m|_{C^0(\mathbb{T}^n+\rho_m)}$, $\left|(\mathrm{avg}\,\{A_m\}_\theta)^{-1}\right|$ such that, if

$$\gamma^{-4}\,\lambda_m\,\rho_m^{-(4\sigma+1)}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)} < \min(1\,,r_m), \tag{93}$$

then there is a parameterization $K_{m+1}\in\mathcal{A}\left(\rho_m/2,C^1\right)\cap\mathcal{N}$ such that $F(f_m,K_{m+1})=0$,

$$|K_{m+1}-K_m|_{C^0(\mathbb{T}^n+\rho_{m+1})}\le\lambda_m\,\gamma^{-2}\,\rho_m^{-2\,\sigma}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\;,$$

and

$$|DK_{m+1}-DK_m|_{C^0(\mathbb{T}^n+\rho_{m+1})}\le\lambda_m\,\gamma^{-2}\,\rho_m^{-(2\,\sigma+1)}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\;,$$

It turns out that λ_m depends in a polynomial way of the following quantities (see Remark 15 in [dlLGJV05]):

$$|f_m|_{C^2(\mathbb{U}_{\tau_m}+(2\,r_m\,\rho_m))}\;,\quad |DK_m|_{C^0(\mathbb{T}^n+\rho_m)}\,,\quad |N_m|_{C^0(\mathbb{T}^n+\rho_m)}\,,\quad \left|(\mathrm{avg}\,\{A_m\}_\theta)^{-1}\right|\,. \tag{94}$$

Let $\tilde\lambda$ be the constant obtained by replacing in the definition of $\tilde\lambda_m$ the quantities in (94), respectively, by

$$\kappa\,M_f,\quad 2\,r,\quad 2\,e\,|N|_{C^0(\mathbb{T}^n)}\,,\quad e\,\left|(\mathrm{avg}\,\{A\}_\theta)^{-1}\right|\,.$$

Assume that

$$\tilde\lambda\,\gamma^{-4}\,\rho_m^{-(4\sigma+1)}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)} < \min(1\,,r,\eta),$$

then using the estimates in A2(m) and A3(m) and $r<r_m<2\,r$, we have that (93) holds. In particular estimates (89) and (90) hold. Now we prove properties $A(m+1)$. First from (90) we have that if

$$2^m\,\tilde\lambda\,\gamma^{-2}\,\rho_m^{-(2\,\sigma+1)}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\;<\;r\,,$$

then

$$|DK_{m+1}|_{C^0(\mathbb{T}^n+\rho_{m+1})}\le r_m+\tilde\lambda\,\gamma^{-2}\,\rho_m^{-2\,\sigma+1}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\;\le r_m+2^{-m}\,r\,.$$

Hence A1(m+1) holds. Property A2(m+1) follows from (89) by assuming the following estimate

$$2^{m+1}\,\gamma^{-2}\,\tilde\lambda\,\rho_m^{-2\sigma}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)}<\eta\,.$$

Notice that A1(m+1) and A2(m+1) imply

$$K_{m+1}\left(\mathbb{T}^n+\rho_{m+1}\right)\subset\mathbb{U}_{\tau_{m+1}}+2\,r\,\rho_m\,,$$

so the composition $f_{m+1}\circ K_{m+1}$ is well defined on $\mathbb{T}^n+\rho_{m+1}$. Property A3(m+1) follows from Theorem 2. Now we prove A4(m+1). Using $|DK_m|\le r_m<2\,r$ and (90) we have

$$\left|DK_{m+1}(\theta)^\top DK_{m+1}(\theta)-N_n^{-1}\right|_{C^0(\mathbb{T}^n+\rho_{m+1})}\le 2\;(2\,r+1)\;\tilde\lambda\,\gamma^{-2}\;\rho_m^{-(2\,\sigma+1)}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\;,$$

then if $\hat\lambda\overset{\mathrm{def}}{=}\tilde\lambda\,2^3\,e\;(2\,r+1)\;|N|_{C^0(\mathbb{T}^n)}$ and

$$2^{m+1}\,\gamma^{-2}\,\hat\lambda\,\rho_m^{-(2\,\sigma+1)}\;|F(f_m,K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\;\le 1\,, \tag{95}$$

then we have that N_{m+1} exist and

$$|N_{m+1}-N_m|_{C^0(\mathbb{T}^n+\rho_{m+1})}\le|N_m|_{C^0(\mathbb{T}^n+\rho_m)}\;\hat\lambda\,\gamma^{-2}\;\rho_m^{-(2\,\sigma+1)}\;|F(f_m.K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\;, \tag{96}$$

where we have used $|N_m|_{C^0(\mathbb{T}^n+\rho_m)} < 2\,e\,|N|_{C^0(\mathbb{T}^n)}$, which follows from A4(m). Then using (95) we have

$$\begin{aligned} |N_{m+1}|_{C^0(\mathbb{T}^n+\rho_{m+1})} &\le |N_m|_{C^0(\mathbb{T}^n+\rho_m)} + |N_{m+1} - N_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \\ &\le |N_m|_{C^0(\mathbb{T}^n+\rho_m)} \left(1 + 2^{-(m+1)}\right) . \end{aligned}$$

From which the first estimate in A4(m+1) holds, let us now prove the second one. Define

$$P_{m+1} \stackrel{\text{def}}{=} DK_{m+1}\, N_{m+1},$$

then estimates (90) and (96) imply

$$|P_{m+1} - P_m|_{C^0(\mathbb{T}^n+\rho_{m+1})} \le \hat{\lambda}'\,\gamma^{-2}\,\rho_m^{-2(\sigma+1)}\; |F(f_m, K_m)|\; , \tag{97}$$

and

$$\left|J(K_{m+1})^{-1} - J(K_m)^{-1}\right|_{C^0(\mathbb{T}^n+\rho_{m+1})} \le \hat{\lambda}'\,\gamma^{-2}\,\rho_m^{-2\sigma}\; |F(f_m, K_m)|\; , \tag{98}$$

where $\hat{\lambda}'$ depends on r, $|N|_{C^0(\mathbb{T}^n)}$, $\left|J^{-1}\right|_{C^1(\mathbb{U})}$, $\tilde{\lambda}$ and $\hat{\lambda}$. Moreover, using property T6 of Theorem 2 we have

$$|f_{m+1} - f_m|_{C^0(\mathbb{U}+2\,r\,\rho_{m+1})} \le \kappa\, M_f\, \rho_m^{\mu-1}\, . \tag{99}$$

From estimates (89) and (99) and property T4 of Theorem 2 we have

$$|\,Df_{m+1}(K_{m+1}(\theta)) - Df_m(K_m(\theta))|_{C^0(\mathbb{T}^n)} \le \tilde{\tilde{\lambda}}\, \left(\rho_m^{\mu-2} + \gamma^{-2}\,\rho_m^{-(2\sigma+1)}\; |F(f_m, K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\right) \tag{100}$$

where we have used the Cauchy's estimates and

$$\tilde{\tilde{\lambda}} \stackrel{\text{def}}{=} \max\left(2\,\kappa\, M_f\, , \kappa\; |\,f|_{C^0(\mathbb{U})}\; (2\,r\,)^{-1}\; \tilde{\lambda}\right) .$$

Performing some computations and using (97), (98), and (100) we have that there exists a constant $\bar{\lambda}$, depending on σ, n, γ^{-4}, η, M_Ω, μ, $|f|_{C^\ell(\mathbb{U})}$, β, $|N|_{C^0(\mathbb{T}^n)}$, $\left|\operatorname{avg}\{A\}_\theta^{-1}\right|$, and M_f, such that

$$|A_{m+1} - A_m\,|_{C^0(\mathbb{T}^n)}\; \le\; \bar{\lambda}\, \left(\rho_m^{\mu-2} + \gamma^{-2}\,\rho_m^{-(2\sigma+1)}\; |F(f_m, K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\right) .$$

From which we have that if $\lambda \stackrel{\text{def}}{=} \bar{\lambda}\, \left|\operatorname{avg}\{A\}_\theta^{-1}\right|\, e$ and

$$2^{m+1}\,\lambda\, \left(\rho_m^{\mu-2} + \gamma^{-2}\,\rho_m^{-(2\sigma+1)}\; |F(f_m, K_m)|_{C^0(\mathbb{T}^n+\rho_m)}\right) \le 1\, ,$$

then, since $\left|\operatorname{avg}\{A_m\}_\theta^{-1}\right| \le \left|\operatorname{avg}\{A\}_\theta^{-1}\right|\, e$ (which follows from A4(m)), we have that $\operatorname{avg}\{A_{m+1}\}_\theta$ is invertible and

$$\left|\operatorname{avg}\{A_{m+1}\}_\theta^{-1}\right| \le \operatorname{avg}\{A_{m+1}\}_\theta\, \left(1 + 2^{-(m+1)}\right) ,$$

this proves A4(m+1). Finally using the equality $F(f_m, K_{m+1}) = 0$ and (99) we have

$$|\,F(f_{m+1}, K_{m+1})\,|_{C^0(\mathbb{T}^n+\rho_{m+1})} = |\,f_{m+1}\circ K_{m+1} - f_m \circ K_{m+1}\,|_{C^0(\mathbb{T}^n+\rho_{m+1})} \le \kappa\, M_f\, \rho_m^{\mu-1}\, .$$

□

Summarizing, from Lemma 46 we have that for t sufficiently large, $(T_t[f], S_t[K])$ is an analytic approximate invariant solution of equation (76), and Lemma 47 provides the iterative scheme to construct a sequence of analytic solutions (f_j, K_{j+1}) of of equation (76). Hence we have all the ingredients to apply the Moser's smoothing technique to prove Theorem 5.

Lemma 48. *Assume that the hypothesis of Theorem 5 hold. Let $4\sigma+2<\mu<\ell-1$, with $\ell\notin\mathbb{N}$, then there exist two positive constants c and $\rho^*<1$, depending μ, n, ℓ, σ, ζ, β, M_ω, $|f|_{C^\ell(\mathbb{U})}$, $|K|_{C^\ell(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, and $\left|(\operatorname{avg}\{A\}_\theta)^{-1}\right|$, such that: such that: given $0<\rho_1\le\rho^*$, if*

$$c\,\gamma^{-4}\,\rho_1^{-(4\,\sigma+1)}\;|F(f,K)|_{C^0(\mathbb{T}^n)}\;\le\min(1,r,\eta)\,, \tag{101}$$

then there exist two sequences of functions $\{f_m\}_{m\ge1}\subset\mathcal{A}(\mathbb{U}+2\,r\,\rho_m,C^2)$, and $\{K_m\}_{m\ge1}\subset\mathcal{A}(\mathbb{T}^n+\rho_m,C^1)$, with $\rho_m\stackrel{\text{def}}{=}2^{-(m-1)}\rho_1$, satisfying properties $A(m)$ of Lemma 47, and such that $f_m=T_{\tau_m}[f]$, with $\tau_m=\rho_m^{-1}$, and for $m\ge2$

$$|K_{m+1}-K_m|_{C^0(\mathbb{T}^n+\rho_{m+1})}\;\le\tilde c\,\gamma^{-2}\,\rho_m^{\mu-(2\,\sigma+1)}\,, \tag{102}$$

where $\tilde c$ is a constant depending on the same variables as c. Furthermore if $\mu-(2\sigma+1)\notin\mathbb{N}$, then the sequence $\{K_m\}_{m\ge1}$ converges to a function $K^*\in C^{\mu-(2\sigma+1)}(\mathbb{T}^n,\mathbb{U})$ such that

$$F(f,K^*)=0,$$

and

$$|K-K^*|_{C^\nu(\mathbb{T}^n)}\le M\,\gamma^{-2}\,\rho_1^{-(2\sigma+\nu)}\left(\rho_1^{\mu-1}+|F(f,K)|_{C^0(\mathbb{T}^n)}\right)\,,$$

for all $0\le\nu<\mu-(2\,\sigma+1)$, where M is a constant depending on the same variables as c.

Proof. Let λ, $\tilde\lambda$ be as in Lemma 47, let M_f and κ be as in Theorem 2, and let κ and $\hat M_{K,f}$ be as in part 5 of Lemma 46, define

$$c\stackrel{\text{def}}{=}2^\mu\,\kappa\,\max(4\,\lambda,\tilde\lambda)\,\max(1,\hat M_{K,f},\kappa\,M_f)\,. \tag{103}$$

Let t_8 be as in Lemma 46 and let $0<\rho^*<1$ be sufficiently small such that $\rho^*\le t_8^{-1}$ and such that following inequality holds:

$$c\,\gamma^{-4}\,(\rho^*)^{\mu-(4\sigma+2)}\;<\min(1,r,\eta)\,. \tag{104}$$

Let $0<\rho_1<\rho^*$ and define $\tau_1\stackrel{\text{def}}{=}\rho_1^{-1}$ and $f_1\stackrel{\text{def}}{=}T_{\tau_1}[f]$, and $K_1\stackrel{\text{def}}{=}S_{\tau_1}[K]$, then, because of Lemma 46, (f_1,K_1) satisfies properties A1(1)-A4(1) of Lemma 47. Moreover if (101) holds, then part 5 of Lemma 46, equation (103), and estimate (104) imply conditions (88) and (91) in Lemma 47 for $m=1$. Therefore, if $f_2=T_{2\,\tau_1}[f]$, Lemma 47 implies the existence of $K_2\in\mathcal{A}\left(\rho_2,C^1\right)$, with $\rho_2=\rho_1/2$, such that (f_2,K_2) satisfies properties A1(m+1)-A4(m+1) and estimate (92) in Lemma 47 for $m=1$. Moreover, estimate (89) and part 5 of Lemma 46 imply

$$|K_2-K_1|_{C^0(\mathbb{T}^n+\rho_2)}<\,c\,\gamma^{-2}\,\rho_1^{-2\sigma}\,\left(\rho_1^{\mu-1}+|F(f,K)|_{C^0(\mathbb{T}^n)}\right)\,.$$

Now assume that, for $m\ge2$ we have (f_m,K_m) satisfying properties A1(m)-A4(m) and estimate (92) in Lemma 47 for $(m-1)$. Performing some simple computations and using the definition of c in (103), and estimates (101), (104) one obtains that estimates (88) and (91) hold for m. Hence Lemma 47 can be iterated to obtain an analytic invariant torus K_m for f_m. Moreover, using estimates (89) and (92) one obtains (102).

The convergence of the sequence $\{K_m\}_{m\ge1}$ follows from the *Inverse Approximation Lemma* (see, for example, Lemma 2.2 in [Zeh75] or Lemma 6.14 in [BHS96]). Indeed, define $u_m\stackrel{\text{def}}{=}K_m-K_1$, then the following properties hold:

1. $u_m\in\mathcal{A}\left(\rho_m,C^1\right)$, for all $m\ge1$ and $u_1=0$.

2. $\sup_{m\ge2}\rho_m^{-\mu+(2\sigma+1)}\;|u_{m+1}-u_m|_{C^0(\mathbb{T}^n+\rho_{m+1})}\le\tilde c$.

3. If $0 \leq \nu < \mu - (2\sigma + 1)$, then

$$|u_m|_{C^\nu(\mathbb{T}^n)} \leq \sum_{j=1}^{m-1} \rho_{j+1}^{\nu} \, |K_{j+1} - K_j|_{C^0(\mathbb{T}^n + \rho_{j+1})}$$
$$\leq c\,\gamma^{-2}\, 2^{\nu}\, \rho_1^{-(\nu+2\sigma)} \left(\rho_1^{\mu-1} + |F(f,K)|_{C^0(\mathbb{T}^n)} \right) + \hat{c}\,\gamma^{-2}\, \rho_1^{\mu-(2\sigma+1)-\nu} ,$$

where $\hat{c} \stackrel{\text{def}}{=} \tilde{c}\; 2^{\mu-2\sigma} \sum_{j=2}^{\infty} 2^{-(\,\mu-(2\sigma+1)-\alpha\,)}$.

The Inverse Approximation Lemma implies the existence of a function $u^* \in C^{\mu-(2\sigma+1)}(\mathbb{T}^n, \mathbb{U})$ such that

$$\lim_{m\to\infty} |u^* - u_m|_{C^\nu(\mathbb{T}^n)} = 0 ,$$

for any $\nu < \mu - (2\,\sigma + 1)$. The proof of Lemma 48 is finished by defining $K^* \stackrel{\text{def}}{=} u^* + u_1$. □

3.2 Local uniqueness (Proof of Theorem 6)

Throughout this section we assume that the hypotheses of Theorem 6 hold. The proof of Theorem 6 we give here is rather standard, as it is proved in [Zeh75] it suffices to show that the operator $D_2F(f,K)$, with F defined in (77), has a *approximate left inverse* for each f fixed. In our context the existence of the approximate left inverse amounts to the uniqueness of the solutions of the following linear equation

$$D_2F(f,K)\Delta = Df(K(\theta))\Delta - \Delta \circ R_\omega = g(\theta) . \tag{105}$$

The uniqueness of equation (105) depends on the arithmetic properties of ω because the so called small divisors are involved. The following result is well known in KAM theory, for completeness we state it here, for a proof see [Rüs75, Rüs76a, Rüs76b].

Lemma 49. *Let $\omega \in D(\gamma,\sigma)$, for some $\gamma > 0$ and $\sigma > n$ and let $r > \sigma$ be not an integer. Let $h \in C^r(\mathbb{T}^n)$ be such that $\operatorname{avg}\{h\}_\theta = 0$, and assume that $r - \sigma \notin \mathbb{Z}$, then the linear difference equation*

$$u - u \circ R_\omega = h$$

has a unique zero average solution $u \in C^{r-\sigma}(\mathbb{T}^n)$. Moreover, the following holds:

$$|u|_{C^{r-\sigma}(\mathbb{T}^n)} \leq \kappa\,\gamma^{-1}\; |h|_{C^\sigma(\mathbb{T}^n)} ,$$

where κ is a constant depending on n, σ, and r.

Now we prove the uniqueness of the solution of (105).

Lemma 50. *Let $\omega \in D(\gamma,\sigma)$ for some $\gamma > 0$ and $\sigma > n$. Let $\ell > 2\,\sigma$ be such that $\ell, \ell-2\sigma \notin \mathbb{Z}$. Let $f \in \mathrm{Diff}^{\ell+1}(\mathbb{U})$ be symplectic. Assume that (f,K) is a solution of (76), with $K \in \mathcal{N} \cap C^{\ell+1}(\mathbb{T}^n, \mathbb{U})$ (see Definition 42). Then, for any $g \in C^\ell(\mathbb{T}^n, \mathbb{U})$ satisfying*

$$\operatorname{avg}\Big\{\; DK(\theta)^\top J(K(\theta)\,) g(\theta - \omega) \;\Big\}_\theta = 0 , \tag{106}$$

the linear equation (105) has a unique solution $\Delta \in C^{\ell-2\sigma}(\mathbb{T}^n)$, satisfying

$$\operatorname{avg}\{\, T(\theta)\Delta(\theta)\,\}_\theta = 0 ,$$

where

$$T(\theta) \stackrel{\text{def}}{=} N(\theta)^\top DK(\theta)^\top \Big\{ I_n - J(K)(\theta))^{-1} DK(\theta) N(\theta) DK(\theta)^\top J(K(\theta)) \Big\} , \tag{107}$$

with N defined as in N1 in Definition 42. Furthermore, the following estimate holds:

$$|\Delta|_{C^{\ell-2\sigma}(\mathbb{T}^n)} \leq \kappa\,\gamma^{-2}\,|g|_{C^{\ell}(\mathbb{T}^n)}\,,$$

where κ is a constant depending on n, σ, ℓ, $|N|_{C^{\ell}(\mathbb{T}^n)}$, $|K|_{C^{\ell+1}(\mathbb{T}^n)}$, and $\left|\operatorname{avg}\{A\}_\theta^{-1}\right|$, with A defined by (79).

Proof. Let $M(\theta)$ be the $2n\times 2n$-matrix valued function which first n-columns are the columns of $DK(\theta)$ and the last n-columns are the columns of $J(K(\theta))^{-1}DK(\theta)N(\theta)$:

$$M(\theta) = \left(\ DK(\theta) \quad \mid \quad J(K(\theta))^{-1}DK(\theta)N(\theta)\ \right).$$

It is clear that the components of M belong to $C^{\ell}(\mathbb{T}^n)$. In Section 4.2 of [dlLGJV05] it is proved that if K is a parameterization of an invariant torus for the symplectic map f, then:

1. M is invertible with inverse given by

$$M(\theta)^{-1} = \begin{pmatrix} T(\theta) \\ DK(\theta)^{\top}J(K(\theta)) \end{pmatrix}.$$

2. If $\Delta = M\,\xi$, then in the variable ξ the linear equation (105) becomes

$$\begin{aligned} \xi_1 - \xi_1\circ R_\omega &= T(\theta+\omega)\,g(\theta) - A(\theta)\xi_2 \\ \xi_2 - \xi_2\circ R_\omega &= DK(\theta+\omega)^{\top}J(K(\theta+\omega))\,g(\theta)\,. \end{aligned} \tag{108}$$

Notice that, by Lemma 49 and the assumption (106), there exists a unique zero average function $\tilde{\xi}_2$ satisfying

$$\tilde{\xi}_2 - \tilde{\xi}_2\circ R_\omega = DK(\theta+\omega)^{\top}J(K(\theta+\omega))\,g(\theta)\,.$$

The proof of Lemma 50 is finished by using Lemma 49 to find a unique solution of the triangular system (108) satisfying:

$$\begin{aligned} &\operatorname{avg}\{\xi_1\}_\theta = 0 \\ &\operatorname{avg}\{\xi_2\}_\theta = \operatorname{avg}\{A\}_\theta^{-1}\,\operatorname{avg}\left\{T(\theta+\omega)\,g(\theta) - A(\theta)\,\tilde{\xi}_2(\theta)\right\}_\theta\,. \end{aligned}$$

□

Lemma 51. *Let $\omega\in D(\gamma,\sigma)$, for some $\gamma>0$ and $\sigma>n$. Let $f\in \operatorname{Diff}^{\ell+1}(\mathbb{U})$ be symplectic. Assume that (f,K_1) and (f,K_2) satisfy (76), with $K_1,K_2\in\mathcal{N}\cap C^{\ell+1}(\mathbb{T}^n,\mathbb{U})$ (see Definition 42). Then, there exists a constant κ, depending on n, ℓ, $\left|J^{-1}\right|_{C^0(\mathbb{U})}$, $|K_1|_{C^2(\mathbb{T}^n)}$, $|K_2|_{C^1(\mathbb{T}^n)}$ $|N_2|_{C^0}$, with N_2 defined as in N1 in Definition 42 by replacing K with K_2, such that if*

$$\kappa\,|K_1-K_2|_{C^1(\mathbb{T}^n)} < 1\,, \tag{109}$$

then there exists $\theta_0\in\mathbb{R}^n$ such that

$$\operatorname{avg}\{T_2(\theta)\,(K_1\circ R_{\theta_0} - K_2)\}_\theta = 0\,, \tag{110}$$

where T_2 is defined by replacing K with K_2 in (107). Moreover, the following estimate holds:

$$|K_1\circ R_{\theta_0} - K_2|_{C^{\ell}(\mathbb{T}^n)} \leq \tilde{\kappa}\,|K_1-K_2|^{1-\alpha}_{C^0(\mathbb{T}^n)} + \tilde{\kappa}\,|K_1-K_2|_{C^{\ell}(\mathbb{T}^n)}\,, \tag{111}$$

where $0\leq\alpha<1$ is such that $\ell-\alpha\in\mathbb{N}$ and $\tilde{\kappa}$ is a constant depending on the same variables as κ and on $|K_1|_{C^{\ell+1}(\mathbb{T}^n)}$

Proof. Lemma 51 is consequence of the Implicit Function Theorem. Indeed, let $\mathcal{M}_{n\times n}(\mathbb{R})$ represent the space of $n \times n$ matrices with components in $\mathbb{R}$. Define

$$\begin{aligned} \Phi: \quad \mathbb{R}^n \times C^1(\mathbb{T}^n) \to &\quad \mathcal{M}_{n\times n}(\mathbb{R}) \\ (x, K) \quad \longrightarrow &\quad \operatorname{avg}\{\, T_2\,(K\circ R_x - K_2)\,\}_\theta \ , \end{aligned}$$

where T_2 is defined by (107) by replacing K with K_2. Notice that

$$\begin{aligned} &\Phi(0, K_2) = 0 \\ D_1\Phi(x,K)\,\Delta x &= \operatorname{avg}\{\, T_2(\theta)\, DK(\theta + x)\,\}_\theta\ \Delta x\,. \end{aligned}$$

Moreover, since $K_2(\mathbb{T}^n)$ is Lagrangian [dlLGJV05], from the definition of T_2 one easily verifies that $T_2(\theta)\, DK_2(\theta)\ = I_n$, this implies

$$D_1\Phi(x,K)\,|_{(x,K)=(0,K_2)} = I_n\,.$$

Hence the Implicit Function Theorem guarantees the existence of a constant κ as in Lemma 51 such that if (109) holds, then there is a $\theta_0 \in \mathbb{R}^n$ satisfying (110) and such that

$$|\,\theta_0\,| \le \kappa\ |\Phi(0, K_1)\,|\ \le \kappa\ |T_2|_{C^0(\mathbb{T}^n)}\ |K_1 - K_2|_{C^0(\mathbb{T}^n)}\ . \tag{112}$$

It is not difficult to prove the following estimate (see [dlLO99])

$$|K_1 \circ R_{\theta_0} - K_1|_{C^\ell(\mathbb{T}^n)} \le \tilde{\kappa}\, |K_1|_{C^{\ell+1}(\mathbb{T}^n)}\ |\,\theta_0\,|^{1-\alpha}\ , \tag{113}$$

where $0 < \alpha < 1$ is such that $\ell - \alpha \in \mathbb{N}$. Finally, estimate (111) follows from (112) and (113). $\square$

The proof of Theorem 6 is concluded using Lemma 50, Lemma 51 and Taylor's Theorem as follows. Assume that $|K_1 - K_2|_{C^\ell(\mathbb{T}^n)}$ is sufficiently small such that Lemma 51 holds, let θ_0 be as in Lemma 51. Define

$$\Delta(\theta) \stackrel{\text{def}}{=} K_1 \circ R_{\theta_0} - K_2\,.$$

Using that $(f, K_1 \circ R_{\theta_0})$ and (f, K_2) satisfy (76) and Lemma 51 we have

$$\begin{aligned} &D_2F(f, K_2)\,\Delta = \mathcal{R}(K_1 \circ R_{\theta_0}, K_2)\,, \\ &\operatorname{avg}\{T_2(\theta)\,\Delta(\theta)\}_\theta = 0\,, \end{aligned}$$

where F is as in (77), T_2 is as in Lemma 51, and

$$\mathcal{R}(K_1 \circ R_{\theta_0}, K_2)(\theta) = f \circ K_1 \circ R_{\theta_0} - f \circ K_2(\theta) - Df(K_2(\theta))\Delta(\theta)\,.$$

Then, from the Taylor's theorem and Lemma 50 we have the following estimate:

$$\begin{aligned} |\,\Delta|_{C^{\ell-2\sigma}} &\le \hat{\kappa}\,\gamma^{-2}\ |\mathcal{R}|_{C^\ell(\mathbb{T}^n)} \\ &\le \kappa\,\gamma^{-2}\ |\Delta|_{C^\ell(\mathbb{T}^n)}\ |\Delta|_{C^{\ell-2\sigma}(\mathbb{T}^n)}\ , \end{aligned}$$

from which and (111) we have that if $|K_1 - K_2|_{C^\ell(\mathbb{T}^n)}$ is sufficiently small such that

$$\kappa\,\gamma^{-2}\ |\Delta|_{C^\ell(\mathbb{T}^n)} < 1\,,$$

then $\Delta = 0$.

3.3 Bootstrap of regularity (Proof of Theorem 7)

Theorem 7 is a consequence of Theorem 4, Theorem 6, and the fact that, near to a finitely differentiable approximate solution (f, K) of (76) it is possible to obtain an analytic approximate solution of the same equation by means of the operators S_t and T_t of Theorem 2. More precisely, using Theorem 2 and Theorem 4 we prove that, under certain conditions, if (f, K) belongs to either $\mathrm{Diff}^{\ell}(\mathbb{U}) \times C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$ or $\mathcal{A}(\mathbb{U} + \varrho, C^{\ell_1}) \times C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$, with ℓ and ℓ_1 as in Theorem 7, then there exists a finitely differentiable parameterization of an n-dimensional torus K^* such that: *a) (f, K) is a solution of (76), b) K^* is close to K in certain norms*, and *c) K^* has the wished regularity.* Then Theorem 7 follows from the local uniqueness result Theorem 6.

Lemma 52. *Let γ, σ, ω, m, ℓ_1, and $\mathbb{U}$ be as in Theorem 7. Let (K, f) be a solution of (76) with $f \in \mathrm{Diff}^{\ell}(\mathbb{U})$ an exact symplectic diffeomorphism, and $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$. Let $\ell \in [\ell_1, m)$ and let $f \in C^{\ell}(\mathbb{U})$. Assume that hypothesis S1, S2, S3 (replacing ρ with ϱ in S3) in Theorem 5 hold. Then, for any $4\sigma + 2 < \mu < \ell - 1$, satisfying $\mu - (2\sigma + 1) \notin \mathbb{N}$, there is positive constant c, depending on μ, ℓ, ℓ_1, σ, ζ, M_Ω, $|f|_{C^{\ell}(\mathbb{U})}$, $|K|_{C^{\ell_1}(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, and $\left|(\mathrm{avg}\,\{A\}_\theta)^{-1}\right|$, such that for any $0 < \rho < 1$ satisfying*

$$c\,\gamma^{-4}\,\rho^{\mu-(4\sigma+2)} < \min(1, \beta, \eta), \tag{114}$$

there exists $K^* \in C^{\mu-2\sigma}(\mathbb{T}^n, \mathbb{U})$ satisfying N1 and N2 in Definition 42 and such that (f, K^*) is a solution of (76). Moreover, for any $0 \le \nu < \mu - (2\sigma + 1)$ the following estimate holds:

$$|K^* - K|_{C^{\nu}(\mathbb{T}^n)} \le \kappa\,\gamma^{-2}\,\rho^{\mu-(2\sigma+1+\nu)}\,.$$

for some positive constant κ.

Proof. The proof of Lemma 52 follows the same steps as the proof of Theorem 5 the only thing one has to be careful is that $f \in C^{\ell}(\mathbb{U})$ and $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$ (and not in $C^{\ell}(\mathbb{T}^n, \mathbb{U})$), hence we replace ℓ with ℓ_1 in the estimates of the norms involving the term $S_t[K]$. Moreover, the assumption $F(f, K) = 0$, with F as in (77), simplifies many estimates. $\square$

Lemma 53. *Let γ, σ, ω, m, ℓ_1, and $\mathbb{U}$ be as in Theorem 7. Let (K, f) be a solution of (76) with $f \in \mathcal{A}(\mathbb{U} + \varrho, C^{\ell_1})$, an exact symplectic diffeomorphism, and $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$. Assume that hypothesis S1, S2, S3 in Theorem 5 hold. Then, for any $4\,\sigma < \mu < \ell_1 - 1$, there is positive constant c, depending on n, μ, ℓ_1, σ, ζ, ϱ, β, M_Ω, $|f|_{C^{\ell_1}(\mathbb{U}+\varrho)}$, $|K|_{C^{\ell_1}(\mathbb{T}^n)}$, $|N|_{C^0(\mathbb{T}^n)}$, and $\left|(\mathrm{avg}\,\{A\}_\theta)^{-1}\right|$, such that for any $0 < \rho < 1$ satisfying*

$$c\,\gamma^{-4}\,\rho^{\mu-4\sigma} < \min(1, \varrho, \eta)\,, \tag{115}$$

there exists $K^* \in \mathcal{A}(\mathbb{T}^n + \rho/2, C^1)$ satisfying N1 and N2 in Definition 42 and such that (f, K^*) is a solution of (76). Moreover, for any $0 \le \nu < \mu - 2\,\sigma$, the following estimate holds:

$$|K^* - K|_{C^{\nu}(\mathbb{T}^n)} \le \kappa\,\left(\gamma^{-2}\,\rho^{\mu-(2\sigma+\nu)} + \rho^{\ell_1-\nu}\right)\,, \tag{116}$$

for some positive constant κ.

Proof. We prove Lemma 53 applying again the smoothing technique. Since f is already analytic we only smooth the parameterization $K \in C^{\ell_1}(\mathbb{T}^n, \mathbb{U})$ by using the smoothing operator S_t, defined in Section 2.1. Let $\kappa = \kappa(n, \ell_1, 1)$ be as in Proposition 20, and assume that t is sufficiently large so that

$$\kappa\,\beta\,t^{-\ell_1}\,|K|_{C^{\ell_1}(\mathbb{T}^n)} < \min\left(\varrho/2\,, \eta/2\right)\,, \tag{117}$$

then Proposition 20 implies $S_t[K](\mathbb{T}^n + t^{-1}) \subset \mathbb{U} + \varrho$, so that the composition $f \circ S_t[K]$ is well defined on $\mathbb{T}^n + t^{-1}$. Now, write

$$f \circ S_t[K] - S_t[K] \circ R_\omega = S_t[f] \circ S_t[K] - S_t[f \circ K]\,,$$

where we have used that (f, K) satisfies equation (76). Then, using Proposition 34 and Lemma 27 one has that for any $4\sigma < \mu < \ell_1$, there exists a constant $\tilde{c}$, depending on n, ℓ_1, β, μ, $|f|_{C^{\ell_1}(\mathbb{U}+\varrho)}$, and $|K|_{C^{\ell_1}(\mathbb{T}^n)}$ such that

$$|f \circ S_t[K] - S_t[K] \circ R_\omega|_{C^0(\mathbb{T}^n + t^{-1})} \leq \tilde{c}\, t^{-\mu}\,.$$

Hence for t satisfying (117) $(f, S_t[K])$ is an approximate solution of equation (76), with error bounded in (115). Moreover, it can be proved, as we did in Lemma 45, that for t sufficiently large, $S_t[K]$ satisfies N1 and N2 in Definition 42 and the estimates given in part 4 of Lemma 46. Hence, applying Theorem 4 to the analytic approximate solution $(f, S_t[K])$ one has that there is a positive constant c, depending on σ, n, β, μ, $|f|_{C^2(\mathbb{U}+\varrho)}$, ζ, M_ω, $|K|_{C^{\ell_1}(\mathbb{T}^n}$, $|N|_{C^0(\mathbb{T}^n+t^{-1})}$, and $\left|(\text{avg}\,\{A\}_\theta)^{-1}\right|$ such that, if $\rho = t^{-1}$, with t is sufficiently large so that (115) and (117), then there exists $K^* \in \mathcal{A}(\mathbb{T}^n + \rho/2, C^1)$ satisfying such that

$$|K^* - K|_{C^0(\mathbb{T}^n)} \leq \hat{c}\left(\gamma^{-2}\,\rho^{\mu-2\sigma} + \rho^{\ell_1}\right)\,,$$

Estimate (116) follows from the Cauchy's estimates. □

Now Theorem 7 follows easily from the local uniqueness formulated in Theorem 6, Lemma 52, and Lemma 53. Indeed, in the case that $f \in \text{Diff}^\ell(\mathbb{U})$ with $\ell \in [\ell_1, m) - \mathbb{Z}$, let K^* be as in Lemma 52. Fix $\nu \in (2\sigma, \mu - (2\sigma + 2)) \cap (2\sigma, \ell_1 - 1)$ such that $\nu, \nu - 2\sigma \notin \mathbb{Z}$, then $f \in \text{Diff}^{\nu+2}$, and $K, K^* \in C^{\nu+1}(\mathbb{T}^n, \mathbb{U}) \cap \mathcal{N}$. Assume that in (114) ρ is sufficiently small such that Theorem 6 holds, then $K = K^* \circ T_{\theta^*}$, for some $\theta^* \in \mathbb{R}^n$ and hence $K \in C^{\ell-2\sigma}(\mathbb{T}^n, \mathbb{U})$. The case $f \in \mathcal{A}(\mathbb{U} + \varrho, C^\ell)$ is proved similarly using Lemma 53 instead of Lemma 52 and fixing $\nu \in (2\sigma, \mu - 2\sigma - 1)$ such that $\nu, \nu - 2\sigma \notin \mathbb{Z}$ and applying Lemma 53 and Theorem 6.

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