VARIATIONS ON A THEME OF JOST AND PAIS

FRITZ GESZTESY, MARIUS MITREA, AND MAXIM ZINCHENKO

Abstract. We explore the extent to which a variant of a celebrated formula due to Jost and Pais, which reduces the Fredholm perturbation determinant associated with the Schrödinger operator on a half-line to a simple Wronski determinant of appropriate distributional solutions of the underlying Schrödinger equation, generalizes to higher dimensions. In this multi-dimensional extension the half-line is replaced by an open set $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, where Ω has a compact, nonempty boundary $\partial\Omega$ satisfying certain regularity conditions. Our variant involves ratios of perturbation determinants corresponding to Dirichlet and Neumann boundary conditions on $\partial\Omega$ and invokes the corresponding Dirichlet-to-Neumann map. As a result, we succeed in reducing a certain ratio of modified Fredholm perturbation determinants associated with operators in $L^2(\Omega; d^n x)$, $n \in \mathbb{N}$, to modified Fredholm determinants associated with operators in $L^2(\partial\Omega; d^{n-1}\sigma)$, $n \geq 2$.

Applications involving the Birman–Schwinger principle and eigenvalue counting functions are discussed.

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1. Introduction

To illustrate the reason behind the title of this paper, we briefly recall a celebrated result of Jost and Pais [47], who proved in 1951 a spectacular reduction of the Fredholm determinant associated with the Birman–Schwinger kernel of a one-dimensional Schrödinger operator on a half-line, to a simple Wronski determinant of distributional solutions of the underlying Schrödinger equation. This Wronski determinant also equals the so-called Jost function of the corresponding half-line Schrödinger operator. In this paper we prove a certain multi-dimensional variant of this result.

To describe the result due to Jost and Pais [47], we need a few preparations. Denoting by $H^D_{0,+}$ and $H^N_{0,+}$ the one-dimensional Dirichlet and Neumann Laplacians in $L^2((0,\infty); dx)$, and assuming

$$V \in L^1((0,\infty); dx), \tag{1.1}$$

1.1

we introduce the perturbed Schrödinger operators H^D_+ and H^N_+ in $L^2((0,\infty); dx)$ by

$$\begin{aligned} &H^D_+ f = -f'' + Vf, \\ &f \in \operatorname{dom}\big(H^D_+\big) = \{g \in L^2((0,\infty); dx) \,|\, g, g' \in AC([0,R]) \text{ for all } R > 0, \\ &\qquad\qquad g(0) = 0,\, (-g'' + Vg) \in L^2((0,\infty); dx)\}, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &H^N_+ f = -f'' + Vf, \\ &f \in \operatorname{dom}\big(H^N_+\big) = \{g \in L^2((0,\infty); dx) \,|\, g, g' \in AC([0,R]) \text{ for all } R > 0, \\ &\qquad\qquad g'(0) = 0,\, (-g'' + Vg) \in L^2((0,\infty); dx)\}. \end{aligned} \tag{1.3}$$

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Thus, H_+^D and H_+^N are self-adjoint if and only if V is real-valued, but since the latter restriction plays no special role in our results, we will not assume real-valuedness of V throughout this paper.

A fundamental system of solutions $\phi_+^D(z,\cdot)$, $\theta_+^D(z,\cdot)$, and the Jost solution $f_+(z,\cdot)$ of

$$-\psi''(z,x)+V\psi(z,x)=z\psi(z,x),\quad z\in\mathbb{C}\backslash\{0\},\ x\geq 0, \tag{1.4}$$

are then introduced via the standard Volterra integral equations

$$\phi_+^D(z,x)=z^{-1/2}\sin(z^{1/2}x)+\int_0^x dx'\,z^{-1/2}\sin(z^{1/2}(x-x'))V(x')\phi_+^D(z,x'), \tag{1.5}$$

$$\theta_+^D(z,x)=\cos(z^{1/2}x)+\int_0^x dx'\,z^{-1/2}\sin(z^{1/2}(x-x'))V(x')\theta_+^D(z,x'), \tag{1.6}$$

$$f_+(z,x)=e^{iz^{1/2}x}-\int_x^\infty dx'\,z^{-1/2}\sin(z^{1/2}(x-x'))V(x')f_+(z,x'), \tag{1.7}$$

$$z\in\mathbb{C}\backslash\{0\},\ \mathrm{Im}(z^{1/2})\geq 0,\ x\geq 0.$$

In addition, we introduce

$$u=\exp(i\arg(V))|V|^{1/2},\quad v=|V|^{1/2},\ \text{ so that }\ V=u\,v, \tag{1.8}$$

and denote by I_+ the identity operator in $L^2((0,\infty);dx)$. Moreover, we denote by

$$W(f,g)(x)=f(x)g'(x)-f'(x)g(x),\quad x\geq 0, \tag{1.9}$$

the Wronskian of f and g, where $f,g\in C^1([0,\infty))$. We also use the standard convention to abbreviate (with a slight abuse of notation) the operator of multiplication in $L^2((0,\infty);dx)$ by an element $f\in L^1_{\rm loc}((0,\infty);dx)$ (and similarly in the higher-dimensional context later) by the same symbol f (rather than M_f, etc.). For additional notational conventions we refer to the paragraph at the end of this introduction.

Then, the following results hold:

Theorem 1.1. *Assume $V\in L^1((0,\infty);dx)$ and let $z\in\mathbb{C}\backslash[0,\infty)$ with $\mathrm{Im}(z^{1/2})>0$. Then,*

$$\overline{u\big(H_{0,+}^D-zI_+\big)^{-1}v},\ \overline{u\big(H_{0,+}^N-zI_+\big)^{-1}v}\in\mathcal{B}_1(L^2((0,\infty);dx)) \tag{1.10}$$

and

$$\begin{aligned}\det\Big(I_+ + \overline{u\big(H_{0,+}^D-zI_+\big)^{-1}v}\Big)&=1+z^{-1/2}\int_0^\infty dx\,\sin(z^{1/2}x)V(x)f_+(z,x)\\&=W(f_+(z,\cdot),\phi_+^D(z,\cdot))=f_+(z,0),\end{aligned} \tag{1.11}$$

$$\begin{aligned}\det\Big(I_+ + \overline{u\big(H_{0,+}^N-zI_+\big)^{-1}v}\Big)&=1+iz^{-1/2}\int_0^\infty dx\,\cos(z^{1/2}x)V(x)f_+(z,x)\\&=-\frac{W(f_+(z,\cdot),\theta_+^D(z,\cdot))}{iz^{1/2}}=\frac{f_+'(z,0)}{iz^{1/2}}.\end{aligned} \tag{1.12}$$

Equation (1.11) is the modern formulation of the classical result due to Jost and Pais [47] (cf. the detailed discussion in [33]). Performing calculations similar to Section 4 in [33] for the pair of operators $H_{0,+}^N$ and H_+^N, one obtains the analogous result (1.12). For similar considerations in the context of finite interval problems, we refer to Dreyfus and Dym [23] and Levit and Smilansky [54].

We emphasize that (1.11) and (1.12) exhibit the remarkable fact that the Fredholm determinant associated with trace class operators in the infinite-dimensional space $L^2((0,\infty);dx)$ is reduced to a simple Wronski determinant of $\mathbb{C}$-valued distributional solutions of (1.4). This fact goes back to Jost and Pais [47] (see also [33], [67], [68], [69, Sect. 12.1.2], [87], [88, Proposition 5.7], and the extensive literature cited in these references). The principal aim of this paper is to explore the extent to which this fact may generalize to higher dimensions $n\in\mathbb{N}$, $n\geq 2$. While a straightforward

generalization of (1.11), (1.12) appears to be difficult, we will next derive a formula for the ratio of such determinants which indeed permits a direct extension to higher dimensions.

For this purpose we introduce the boundary trace operators γ_D (Dirichlet trace) and γ_N (Neumann trace) which, in the current one-dimensional half-line situation, are just the functionals,

$$\gamma_D\colon \begin{cases} C([0,\infty)) \to \mathbb{C}, \\ \qquad g \mapsto g(0), \end{cases} \qquad \gamma_N\colon \begin{cases} C^1([0,\infty)) \to \mathbb{C}, \\ \qquad h \mapsto -h'(0). \end{cases} \tag{1.13}$$

In addition, we denote by $m_{0,+}^D$, m_+^D, $m_{0,+}^N$, and m_+^N the Weyl–Titchmarsh m-functions corresponding to $H_{0,+}^D$, H_+^D, $H_{0,+}^N$, and H_+^N, respectively, that is,

$$m_{0,+}^D(z) = iz^{1/2}, \qquad m_{0,+}^N(z) = -\frac{1}{m_{0,+}^D(z)} = iz^{-1/2}, \tag{1.14}$$

$$m_+^D(z) = \frac{f_+'(z,0)}{f_+(z,0)}, \quad m_+^N(z) = -\frac{1}{m_+^D(z)} = -\frac{f_+(z,0)}{f_+'(z,0)}. \tag{1.15}$$

We briefly recall the spectral theoretic significance of m_+^D in the special case where V is real-valued: It is a Herglotz function (i.e., it maps the open complex upper half-plane $\mathbb{C}_+$ analytically into itself) and the measure $d\rho_+^D$ in its Herglotz representation is then the spectral measure of the operator H_+^D and hence encodes all spectral information of H_+^D. Similarly, m_+^D also encodes all spectral information of H_+^N since $-1/m_+^D = m_+^N$ is also a Herglotz function and the measure $d\rho_+^N$ in its Herglotz representation represents the spectral measure of the operator H_+^N. In particular, $d\rho_+^D$ (respectively, $d\rho_+^N$) uniquely determine V a.e. on $(0,\infty)$ by the inverse spectral approach of Gelfand and Levitan [29] or Simon [86], [35] (see also Remling [81] and Section 6 in the survey [30]).

Then we obtain the following result for the ratio of the perturbation determinants in (1.11) and (1.12):

Theorem 1.2. *Assume $V \in L^1((0,\infty);dx)$ and let $z \in \mathbb{C}\backslash\sigma(H_+^D)$ with $\operatorname{Im}(z^{1/2}) > 0$. Then,*

$$\begin{aligned} &\frac{\det\Big(I_+ + \overline{u\big(H_{0,+}^N - zI_+\big)^{-1}v}\Big)}{\det\Big(I_+ + \overline{u\big(H_{0,+}^D - zI_+\big)^{-1}v}\Big)} \\ &\quad = 1 - \Big(\overline{\gamma_N(H_+^D - zI_+)^{-1}V\big[\gamma_D(H_{0,+}^N - \overline{z}I_+)^{-1}\big]^*}\Big) \qquad (1.16) \\ &\quad = \frac{W(f_+(z),\phi_+^N(z))}{iz^{1/2}W(f_+(z),\phi_+^D(z))} = \frac{f_+'(z,0)}{iz^{1/2}f_+(z,0)} = \frac{m_+^D(z)}{m_{0,+}^D(z)} = \frac{m_{0,+}^N(z)}{m_+^N(z)}. \qquad (1.17) \end{aligned}$$

At first sight it may seem unusual to even attempt to derive (1.16) in the one-dimensional context since (1.17) already yields the reduction of a Fredholm determinant to a simple Wronski determinant. However, we will see in Section 4 (cf. Theorem 4.2) that it is precisely (1.16) that permits a natural extension to dimensions $n \in \mathbb{N}$, $n \geq 2$. Moreover, the latter is also instrumental in proving the analog of (1.17) in terms of Dirichlet-to-Neumann maps (cf. Theorem 4.3).

The proper multi-dimensional generalizations to Schrödinger operators in $L^2(\Omega;d^nx)$, corresponding to an open set $\Omega \subset \mathbb{R}^n$ with compact, nonempty boundary $\partial\Omega$, more precisely, the proper operator-valued generalization of the Weyl–Titchmarsh function $m_+^D(z)$ is then given by the Dirichlet-to-Neumann map, denoted by $M_\Omega^D(z)$. This operator-valued map indeed plays a fundamental role in our extension of (1.17) to the higher-dimensional case. In particular, under Hypothesis 2.6 on Ω and V (which regulates smoothness properties of $\partial\Omega$ and L^p-properties of V), we will prove the following multi-dimensional extension of (1.16) and (1.17) in Section 4:

t1.3 **Theorem 1.3.** *Assume Hypothesis* h2.6 *and let* $k \in \mathbb{N}$, $k \geq p$ *and* $z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$. *Then,*

$$\frac{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}v}\Big)}{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big)}$$

$$= \det_k\Big(I_{\partial\Omega} - \overline{\gamma_N\big(H_\Omega^D - zI_\Omega\big)^{-1}V\big[\gamma_D(H_{0,\Omega}^N - \overline{z}I_\Omega)^{-1}\big]^*}\Big)e^{\mathrm{tr}(T_k(z))} \tag{1.18}$$ 1.18

$$= \det_k\big(M_\Omega^D(z)M_{0,\Omega}^D(z)^{-1}\big)e^{\mathrm{tr}(T_k(z))}. \tag{1.19}$$ 1.19

Here, $\det_k(\cdot)$ denotes the modified Fredholm determinant in connection with $\mathcal{B}_k$ perturbations of the identity and $T_k(z)$ is some trace class operator. In particular, $T_2(z)$ is given by

$$T_2(z) = \overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}V\big(H_\Omega^D - zI_\Omega\big)^{-1}V\big[\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\big]^*}, \tag{1.20}$$

where I_Ω and $I_{\partial\Omega}$ represent the identity operators in $L^2(\Omega; d^nx)$ and $L^2(\partial\Omega; d^{n-1}\sigma)$, respectively (with $d^{n-1}\sigma$ denoting the surface measure on $\partial\Omega$). The sudden appearance of the term $\exp(\mathrm{tr}(T_k(z)))$ in (1.18) and (1.19), when compared to the one-dimensional case, is due to the necessary use of the modified determinant $\det_k(\cdot)$ in Theorem t1.3.

We note that the multi-dimensional extension (1.18) of (1.16), under the stronger hypothesis $V \in L^2(\Omega; d^nx)$, $n = 2, 3$, first appeared in GLMZ05 [32]. However, the present results in Theorem t1.3 go decidedly beyond those in [32] in the following sense: (i) the class of domains Ω permitted by Hypothesis h2.6 (actually, Hypothesis h2.1) is greatly enlarged as compared to GLMZ05 [32]; (ii) the multi-dimensional extension (1.19) of (1.17) invoking Dirichlet-to-Neumann maps is a new (and the most significant) result in this paper; (iii) while GLMZ05 [32] focused on dimensions $n = 2, 3$, we now treat the general case $n \in \mathbb{N}$, $n \geq 2$; (iv) we provide an application involving eigenvalue counting functions at the end of Section s4; (v) we study a representation of the product formula for modified Fredholm determinants, which should be of independent interest, at the beginning of Section s4.

The principal reduction in Theorem t1.3 reduces (a ratio of) modified Fredholm determinants associated with operators in $L^2(\Omega; d^nx)$ on the left-hand side of (1.18) to modified Fredholm determinants associated with operators in $L^2(\partial\Omega; d^{n-1}\sigma)$ on the right-hand side of (1.18) and especially, in (1.19). This is the analog of the reduction described in the one-dimensional context of Theorem t1.2, where Ω corresponds to the half-line $(0, \infty)$ and its boundary $\partial\Omega$ corresponds to the one-point set $\{0\}$. As a result, the ratio of determinants on the left-hand side of (1.16) associated with operators in $L^2((0, \infty); dx)$ is reduced to ratios of Wronskians and Weyl–Titchmarsh functions on the right-hand side of (1.16) and in (1.17).

Finally, we briefly list most of the notational conventions used throughout this paper. Let $\mathcal{H}$ be a separable complex Hilbert space, $(\cdot,\cdot)_\mathcal{H}$ the scalar product in $\mathcal{H}$ (linear in the second factor), and $I_\mathcal{H}$ the identity operator in $\mathcal{H}$. Next, let T be a linear operator mapping (a subspace of) a Banach space into another, with $\mathrm{dom}(T)$ and $\mathrm{ran}(T)$ denoting the domain and range of T. The closure of a closable operator S is denoted by $\overline{S}$. The kernel (null space) of T is denoted by $\ker(T)$. The spectrum and resolvent set of a closed linear operator in $\mathcal{H}$ will be denoted by $\sigma(\cdot)$ and $\rho(\cdot)$. The Banach spaces of bounded and compact linear operators in $\mathcal{H}$ are denoted by $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}_\infty(\mathcal{H})$, respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by $\mathcal{B}_k(\mathcal{H})$, $k \in \mathbb{N}$. Analogous notation $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $\mathcal{B}_\infty(\mathcal{H}_1, \mathcal{H}_2)$, etc., will be used for bounded, compact, etc., operators between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. In addition, $\mathrm{tr}(T)$ denotes the trace of a trace class operator $T \in \mathcal{B}_1(\mathcal{H})$ and $\det_p(I_\mathcal{H} + S)$ represents the (modified) Fredholm determinant associated with an operator $S \in \mathcal{B}_k(\mathcal{H})$, $k \in \mathbb{N}$ (for $k = 1$ we omit the subscript 1). Moreover, $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2$ denotes the continuous embedding of the Banach space $\mathcal{X}_1$ into the Banach space $\mathcal{X}_2$.

For general references on the theory of (modified) Fredholm determinants we refer, for instance, to [24, Sect. XI.9], [37, Ch. Chs. IX–XI], [38, Ch. Sect. 4.2], [80, Sect. XIII.17], [85], and [88, Ch. 9].

2. Schrödinger Operators with Dirichlet and Neumann boundary conditions

s2

In this section we primarily focus on various properties of Dirichlet, $H^D_{0,\Omega}$, and Neumann, $H^N_{0,\Omega}$, Laplacians in $L^2(\Omega; d^n x)$ associated with open sets $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, introduced in Hypothesis 2.1 below. In particular, we study mapping properties of $\big(H^{D,N}_{0,\Omega} - zI_\Omega\big)^{-q}$, $q \in [0,1]$ (with I_Ω the identity operator in $L^2(\Omega; d^n x)$) and trace ideal properties of the maps $f\big(H^{D,N}_{0,\Omega} - zI_\Omega\big)^{-q}$, $f \in L^p(\Omega; d^n x)$, for appropriate $p \geq 2$, and $\gamma_N \big(H^D_{0,\Omega} - zI_\Omega\big)^{-r}$, and $\gamma_D \big(H^N_{0,\Omega} - zI_\Omega\big)^{-s}$, for appropriate $r > 3/4$, $s > 1/4$, with γ_N and γ_D being the Neumann and Dirichlet boundary trace operators defined in (2.2) and (2.3).

At the end of this section we then introduce the Dirichlet and Neumann Schrödinger operators H^D_Ω and H^N_Ω in $L^2(\Omega; d^n x)$, that is, perturbations of the Dirichlet and Neumann Laplacians $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$ by a potential V satisfying Hypothesis 2.6.

We start with introducing our assumptions on the set Ω:

h2.1

Hypothesis 2.1. *Let $n \in \mathbb{N}$, $n \geq 2$, and assume that $\Omega \subset \mathbb{R}^n$ is an open set with a compact, nonempty boundary $\partial\Omega$. In addition, we assume that one of the following three conditions holds:*
(i) Ω is of class $C^{1,r}$ for some $1/2 < r < 1$;
(ii) Ω is convex;
(iii) Ω is a Lipschitz domain satisfying a uniform exterior ball condition (UEBC).

We note that while $\partial\Omega$ is assumed to be compact, Ω may be unbounded in connection with conditions *(i)* or *(iii)*. For more details in this context we refer to Appendix A.

First, we introduce the boundary trace operator γ^0_D (Dirichlet trace) by

$$\gamma^0_D \colon C(\overline{\Omega}) \to C(\partial\Omega), \quad \gamma^0_D u = u|_{\partial\Omega}. \tag{2.1}$$

Then there exists a bounded, linear operator γ_D (cf. [57, Theorem 3.38]),

$$\gamma_D \colon H^s(\Omega) \to H^{s-(1/2)}(\partial\Omega) \hookrightarrow L^2(\partial\Omega; d^{n-1}\sigma), \quad 1/2 < s < 3/2, \tag{2.2}$$

2.2

whose action is compatible with that of γ^0_D. That is, the two Dirichlet trace operators coincide on the intersection of their domains. We recall that $d^{n-1}\sigma$ denotes the surface measure on $\partial\Omega$ and we refer to Appendix A for our notation in connection with Sobolev spaces.

Next, we introduce the operator γ_N (Neumann trace) by

$$\gamma_N = \nu \cdot \gamma_D \nabla \colon H^{s+1}(\Omega) \to L^2(\partial\Omega; d^{n-1}\sigma), \quad 1/2 < s < 3/2, \tag{2.3}$$

2.3

where ν denotes outward pointing normal unit vector to $\partial\Omega$. It follows from (2.2) that γ_N is also a bounded operator.

Given Hypothesis 2.1, we introduce the self-adjoint and nonnegative Dirichlet and Neumann Laplacians $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$ associated with the domain Ω as follows,

$$H^D_{0,\Omega} = -\Delta, \quad \operatorname{dom}\big(H^D_{0,\Omega}\big) = \{u \in H^2(\Omega) \,|\, \gamma_D u = 0\}, \tag{2.4}$$

2.4

$$H^N_{0,\Omega} = -\Delta, \quad \operatorname{dom}\big(H^N_{0,\Omega}\big) = \{u \in H^2(\Omega) \,|\, \gamma_N u = 0\}. \tag{2.5}$$

2.5

A detailed discussion of $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$ is provided in Appendix A.

l2.2

Lemma 2.2. *Assume Hypothesis 2.1. Then the operators $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$ introduced in (2.4) and (2.5) are nonnegative and self-adjoint in $L^2(\Omega; d^n x)$ and the following boundedness properties hold for all $q \in [0,1]$ and $z \in \mathbb{C}\backslash[0,\infty)$,*

$$\big(H^D_{0,\Omega} - zI_\Omega\big)^{-q}, \, \big(H^N_{0,\Omega} - zI_\Omega\big)^{-q} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^{2q}(\Omega)\big). \tag{2.6}$$

2.6

The fractional powers in (2.6) (and in subsequent analogous cases) are defined via the functional calculus implied by the spectral theorem for self-adjoint operators.

As explained in Appendix A (cf. particularly Lemma A.2), the key ingredients in proving Lemma 2.2 are the inclusions

$$\operatorname{dom}\big(H_{0,\Omega}^D\big) \subset H^2(\Omega), \quad \operatorname{dom}\big(H_{0,\Omega}^N\big) \subset H^2(\Omega) \tag{2.7}$$

and methods based on real interpolation spaces.

For the remainder of this paper we agree to the simplified notation that the operator of multiplication by the measurable function f in $L^2(\Omega; d^n x)$ is again denoted by the symbol f.

The next result is an extension of [32, Lemma 6.8] and aims at an explicit discussion of the z-dependence of the constant c appearing in estimate (6.48) of [32].

Lemma 2.3. *Assume Hypothesis 2.1 and let $2 \leq p$, $(n/2p) < q \leq 1$, $f \in L^p(\Omega; d^n x)$, and $z \in \mathbb{C}\backslash[0,\infty)$. Then,*

$$f\big(H_{0,\Omega}^D - zI_\Omega\big)^{-q},\, f\big(H_{0,\Omega}^N - zI_\Omega\big)^{-q} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big), \tag{2.8}$$

and for some $c > 0$ (independent of z and f)

$$\begin{aligned}
&\big\|f\big(H_{0,\Omega}^D - zI_\Omega\big)^{-q}\big\|^2_{\mathcal{B}_p(L^2(\Omega;d^n x))} \\
&\quad \leq c\bigg(1 + \frac{|z|^{2q}+1}{\operatorname{dist}\big(z, \sigma\big(H_{0,\Omega}^D\big)\big)^{2q}}\bigg)\|(|\cdot|^2 - z)^{-q}\|^2_{L^p(\mathbb{R}^n;d^n x)}\|f\|^2_{L^p(\Omega;d^n x)}, \\
&\big\|f\big(H_{0,\Omega}^N - zI_\Omega\big)^{-q}\big\|^2_{\mathcal{B}_p(L^2(\Omega;d^n x))} \\
&\quad \leq c\bigg(1 + \frac{|z|^{2q}+1}{\operatorname{dist}\big(z, \sigma\big(H_{0,\Omega}^N\big)\big)^{2q}}\bigg)\|(|\cdot|^2 - z)^{-q}\|^2_{L^p(\mathbb{R}^n;d^n x)}\|f\|^2_{L^p(\Omega;d^n x)}.
\end{aligned} \tag{2.9}$$

Proof. We start by noting that under the assumption that Ω is a Lipschitz domain, there is a bounded extension operator $\mathcal{E}$,

$$\mathcal{E} \in \mathcal{B}\big(H^s(\Omega), H^s(\mathbb{R}^n)\big) \text{ such that } (\mathcal{E}u)|_\Omega = u, \quad u \in H^s(\Omega), \tag{2.10}$$

for all $s \in \mathbb{R}$ (see, e.g., [82]). Next, for notational convenience, we denote by $H_{0,\Omega}$ either one of the operators $H_{0,\Omega}^D$ or $H_{0,\Omega}^N$ and by $\mathcal{R}_\Omega$ the restriction operator

$$\mathcal{R}_\Omega \colon \begin{cases} L^2(\mathbb{R}^n; d^n x) \to L^2(\Omega; d^n x), \\ \qquad u \mapsto u|_\Omega. \end{cases} \tag{2.11}$$

Moreover, we introduce the following extension $\tilde{f}$ of f,

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^n\backslash\Omega, \end{cases} \quad \tilde{f} \in L^p(\mathbb{R}^n; d^n x). \tag{2.12}$$

Then,

$$f(H_{0,\Omega} - zI_\Omega)^{-q} = \mathcal{R}_\Omega \tilde{f}(H_0 - zI)^{-q}(H_0 - zI)^q \mathcal{E}(H_{0,\Omega} - zI_\Omega)^{-q}, \tag{2.13}$$

where (for simplicity) I denotes the identity operator in $L^2(\mathbb{R}^n; d^n x)$ and H_0 denotes the nonnegative self-adjoint operator

$$H_0 = -\Delta, \quad \operatorname{dom}(H_0) = H^2(\mathbb{R}^n) \tag{2.14}$$

in $L^2(\mathbb{R}^n; d^n x)$.

Let $g \in L^2(\Omega; d^n x)$ and define $h = (H_{0,\Omega} - zI_\Omega)^{-q} g$, then by Lemma A.2, $h \in H^{2q}(\Omega) \subset L^2(\Omega; d^n x)$. Using the spectral theorem for the nonnegative self-adjoint operator $H_{0,\Omega}$ in $L^2(\Omega; d^n x)$, one computes,

$$\begin{aligned} \|h\|^2_{L^2(\Omega;d^n x)} &= \big\|(H_{0,\Omega} - zI_\Omega)^{-q} g\big\|^2_{L^2(\Omega;d^n x)} \\ &= \int_{\sigma(H_{0,\Omega})} |\lambda - z|^{-2q} \big(dE_{H_{0,\Omega}}(\lambda) g, g\big)_{L^2(\Omega;d^n x)} \\ &\leq \operatorname{dist}(z, \sigma(H_{0,\Omega}))^{-2q} \, \|g\|^2_{L^2(\Omega;d^n x)} \end{aligned} \tag{2.15}$$

and since $(H_{0,\Omega} + I_\Omega)^{-q} \in \mathcal{B}(L^2(\Omega; d^n x), H^{2q}(\Omega))$,

$$\begin{aligned} \|h\|^2_{H^{2q}(\Omega)} &= \big\|(H_{0,\Omega} + I_\Omega)^{-q} (H_{0,\Omega} + I_\Omega)^q h\big\|^2_{H^{2q}(\Omega)} \leq c \, \|(H_{0,\Omega} + I_\Omega)^q h\|^2_{L^2(\Omega;d^n x)} \\ &= c \int_{\sigma(H_{0,\Omega})} |\lambda + 1|^{2q} \big(dE_{H_{0,\Omega}}(\lambda) h, h\big)_{L^2(\Omega;d^n x)} \\ &\leq 2c \int_{\sigma(H_{0,\Omega})} \big(|\lambda - z|^{2q} + |z+1|^{2q}\big) \big(dE_{H_{0,\Omega}}(\lambda) h, h\big)_{L^2(\Omega;d^n x)} \\ &= 2c\big(\|(H_{0,\Omega} - zI_\Omega)^q h\|^2_{H^{2q}(\Omega)} + |z+1|^{2q} \, \|h\|^2_{L^2(\Omega;d^n x)} \big) \\ &\leq 2c\big(1 + |z+1|^{2q} \operatorname{dist}(z, \sigma(H_{0,\Omega}))^{-2q}\big) \, \|g\|^2_{L^2(\Omega;d^n x)} \,, \end{aligned} \tag{2.16}$$

where $E_{H_{0,\Omega}}(\cdot)$ denotes the family of spectral projections of $H_{0,\Omega}$. Moreover, utilizing the representation of $(H_0 - zI)^q$ as the operator of multiplication by $\big(|\xi|^2 - z\big)^q$ in the Fourier space $L^2(\mathbb{R}^n; d^n \xi)$, and the fact that by (2.10)

$$\mathcal{E} \in \mathcal{B}\big(H^{2q}(\Omega), H^{2q}(\mathbb{R}^n)\big) \cap \mathcal{B}\big(L^2(\Omega; d^n x), L^2(\mathbb{R}^n; d^n x)\big), \tag{2.17}$$

one computes

$$\begin{aligned} \|(H_0 - zI)^q \mathcal{E} h\|^2_{L^2(\mathbb{R}^n;d^n x)} &= \int_{\mathbb{R}^n} d^n \xi \, \big||\xi|^2 - z\big|^{2q} \, |(\widehat{\mathcal{E}h})(\xi)|^2 \\ &\leq 2 \int_{\mathbb{R}^n} d^n \xi \big(|\xi|^{4q} + |z|^{2q}\big) |(\widehat{\mathcal{E}h})(\xi)|^2 \\ &\leq 2\big(\|\mathcal{E}h\|^2_{H^{2q}(\mathbb{R}^n)} + |z|^{2q} \, \|\mathcal{E}h\|^2_{L^2(\mathbb{R}^n;d^n x)} \big) \\ &\leq 2c\big(\|h\|^2_{H^{2q}(\Omega)} + |z|^{2q} \, \|h\|^2_{L^2(\Omega;d^n x)} \big). \end{aligned} \tag{2.18}$$

Combining the estimates (2.15), (2.16), and (2.18), one obtains

$$(H_0 - zI)^q \mathcal{E} (H_{0,\Omega} - zI_\Omega)^{-q} \in \mathcal{B}\big(L^2(\Omega; d^n x), L^2(\mathbb{R}^n; d^n x)\big) \tag{2.19}$$

and the following norm estimate with some constant $c > 0$,

$$\big\|(H_0 - zI)^q \mathcal{E} (H_{0,\Omega} - zI_\Omega)^{-q}\big\|^2_{\mathcal{B}(L^2(\Omega;d^n x), L^2(\mathbb{R}^n;d^n x))} \leq c + \frac{c(|z|^{2q} + 1)}{\operatorname{dist}(z, \sigma(H_{0,\Omega}))^{2q}}. \tag{2.20}$$

Next, by [Si05, 88, Theorem 4.1] (or [RS79, 79, Theorem XI.20]) one obtains

$$\tilde{f}(H_0 - zI)^{-q} \in \mathcal{B}_p\big(L^2(\mathbb{R}^n; d^n x)\big) \tag{2.21}$$

and

$$\begin{aligned} \big\|\tilde{f}(H_0 - zI)^{-q}\big\|_{\mathcal{B}_p(L^2(\mathbb{R}^n;d^n x))} &\leq c \, \|(|\cdot|^2 - z)^{-q}\|_{L^p(\mathbb{R}^n;d^n x)} \|\tilde{f}\|_{L^p(\mathbb{R}^n;d^n x)} \\ &= c \, \|(|\cdot|^2 - z)^{-q}\|_{L^p(\mathbb{R}^n;d^n x)} \|f\|_{L^p(\Omega;d^n x)}. \end{aligned} \tag{2.22}$$

Thus, (2.8) follows from (2.13), (2.19), (2.21), and (2.9) follows from (2.13), (2.20), and (2.22). □

Next we recall certain mapping properties of powers of the resolvents of Dirichlet and Neumann Laplacians multiplied by the Neumann and Dirichlet boundary trace operators, respectively:

l2.4 **Lemma 2.4.** *Assume Hypothesis 2.1 and let $\varepsilon > 0$, $z \in \mathbb{C}\backslash[0,\infty)$. Then,*

$$\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-(3+\varepsilon)/4}, \gamma_D\big(H_{0,\Omega}^N - zI_\Omega\big)^{-(1+\varepsilon)/4} \in \mathcal{B}\big(L^2(\Omega;d^nx), L^2(\partial\Omega;d^{n-1}\sigma)\big). \tag{2.23}$$ 2.22

As in [32, Lemma 6.9], Lemma 2.4 follows from Lemma 2.2 and from (2.2) and (2.3).

c2.5 **Corollary 2.5.** *Assume Hypothesis 2.1 and let $f_1 \in L^{p_1}(\Omega;d^nx)$, $p_1 \geq 2$, $p_1 > 2n/3$, $f_2 \in L^{p_2}(\Omega;d^nx)$, $p_2 > 2n$, and $z \in \mathbb{C}\backslash[0,\infty)$. Then, denoting by f_1 and f_2 the operators of multiplication by functions f_1 and f_2 in $L^2(\Omega;d^nx)$, respectively, one has*

$$\overline{\gamma_D\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} f_1} \in \mathcal{B}_{p_1}\big(L^2(\Omega;d^nx), L^2(\partial\Omega;d^{n-1}\sigma)\big), \tag{2.24}$$ 2.25

$$\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} f_2} \in \mathcal{B}_{p_2}\big(L^2(\Omega;d^nx), L^2(\partial\Omega;d^{n-1}\sigma)\big) \tag{2.25}$$ 2.26

and for some $c_j(z) > 0$ (independent of f_j), $j = 1, 2$,

$$\Big\| \overline{\gamma_D\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} f_1} \Big\|_{\mathcal{B}_{p_1}(L^2(\Omega;d^nx), L^2(\partial\Omega;d^{n-1}\sigma))} \leq c_1(z) \|f_1\|_{L^{p_1}(\Omega;d^nx)}, \tag{2.26}$$ 2.27

$$\Big\| \overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} f_2} \Big\|_{\mathcal{B}_{p_2}(L^2(\Omega;d^nx), L^2(\partial\Omega;d^{n-1}\sigma))} \leq c_2(z) \|f_2\|_{L^{p_2}(\Omega;d^nx)}. \tag{2.27}$$ 2.28

As in [32, Corollary 6.10], Corollary 2.5 follows from Lemmas 2.3 and 2.4.

Finally, we turn to our assumptions on the potential V and the corresponding definition of Dirichlet and Neumann Schrödinger operators H_Ω^D and H_Ω^N in $L^2(\Omega;d^nx)$:

h2.6 **Hypothesis 2.6.** *Suppose that Ω satisfies Hypothesis 2.1 and assume that $V \in L^p(\Omega;d^nx)$ for some p satisfying $p > 4/3$ in the case $n = 2$, and $p > n/2$ in the case $n \geq 3$.*

Assuming Hypothesis 2.6, we next introduce the perturbed operators H_Ω^D and H_Ω^N in $L^2(\Omega;d^nx)$ by alluding to abstract perturbation results summarized in Appendix B as follows: Let V, u, and v denote the operators of multiplication by functions V, $u = \exp(i\arg(V))|V|^{1/2}$, and $v = |V|^{1/2}$ in $L^2(\Omega;d^nx)$, respectively. Since $u, v \in L^{2p}(\Omega;d^nx)$, Lemma 2.3 yields

$$u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1/2}, \ \overline{\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1/2} v} \in \mathcal{B}_{2p}\big(L^2(\Omega;d^nx)\big), \quad z \in \mathbb{C}\backslash[0,\infty), \tag{2.28}$$ 2.31

$$u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1/2}, \ \overline{\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1/2} v} \in \mathcal{B}_{2p}\big(L^2(\Omega;d^nx)\big), \quad z \in \mathbb{C}\backslash[0,\infty), \tag{2.29}$$ 2.32

and hence, in particular,

$$\operatorname{dom}(u) = \operatorname{dom}(v) \supseteq H^1(\Omega) \supset H^2(\Omega) \supset \operatorname{dom}\big(H_{0,\Omega}^N\big), \tag{2.30}$$

$$\operatorname{dom}(u) = \operatorname{dom}(v) \supseteq H^1(\Omega) \supseteq H_0^1(\Omega) \supset \operatorname{dom}\big(H_{0,\Omega}^D\big). \tag{2.31}$$

Thus, operators $H_{0,\Omega}^D$, $H_{0,\Omega}^N$, u, and v satisfy Hypothesis B.1(i). Moreover, (2.28) and (2.29) imply

$$\overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} v}, \ \overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} v} \in \mathcal{B}_p\big(L^2(\Omega;d^nx)\big), \quad z \in \mathbb{C}\backslash[0,\infty), \tag{2.32}$$ 2.35

which verifies Hypothesis B.1(ii) for $H_{0,\Omega}^D$ and $H_{0,\Omega}^N$. Utilizing (2.9) in Lemma 2.3 with $-z > 0$ sufficiently large, such that the $\mathcal{B}_{2p}$-norms of the operators in (2.28) and (2.29) are less than 1, and hence the $\mathcal{B}_p$-norms of the operators in (2.32) are less than 1, one also verifies Hypothesis B.1(iii). Thus, applying Theorem B.2 one obtains the densely defined, closed operators H_Ω^D and H_Ω^N (which are extensions of $H_{0,\Omega}^D + V$ on $\operatorname{dom}\big(H_{0,\Omega}^D\big) \cap \operatorname{dom}(V)$ and $H_{0,\Omega}^N + V$ on $\operatorname{dom}\big(H_{0,\Omega}^N\big) \cap \operatorname{dom}(V)$, respectively). In particular, the resolvent of H_Ω^D (respectively, H_Ω^N) is explicitly given by the analog of (B.5) in terms of the resolvent of $H_{0,\Omega}^D$ (respectively, $H_{0,\Omega}^N$) and the factorization $V = uv$.

We note in passing that (2.6)–(2.9), (2.22), (2.23), (2.25)–(2.27), (2.28), (2.31), (2.32), (2.35), etc., extend of course to all z in the resolvent set of the corresponding operators $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$.

3. Dirichlet and Neumann boundary value problems and Dirichlet-to-Neumann maps

s3

This section is devoted to Dirichlet and Neumann boundary value problems associated with the Helmholtz differential expression $-\Delta - z$ as well as the corresponding differential expression $-\Delta + V - z$ in the presence of a potential V, both in connection with the open set Ω. In addition, we provide a detailed discussion of Dirichlet-to-Neumann, $M^D_{0,\Omega}$, M^D_Ω, and Neumann-to-Dirichlet maps, $M^N_{0,\Omega}$, M^N_Ω, in $L^2(\partial\Omega; d^{n-1}\sigma)$.

Denote by

$$\widetilde{\gamma}_N : \big\{u \in H^1(\Omega) \,\big|\, \Delta u \in \big(H^1(\Omega)\big)^*\big\} \to H^{-1/2}(\partial\Omega) \tag{3.1}$$

3.0

a weak Neumann trace operator defined by

$$\langle \widetilde{\gamma}_N u, \phi\rangle = \int_\Omega d^n x\, \nabla u(x)\cdot\nabla\Phi(x) + \langle \Delta u, \Phi\rangle \tag{3.2}$$

3.1a

for all $\phi \in H^{1/2}(\partial\Omega)$ and $\Phi \in H^1(\Omega)$ such that $\gamma_D\Phi = \phi$. We note that this definition is independent of the particular extension Φ of ϕ, and that $\widetilde{\gamma}_N$ is a bounded extension of the Neumann trace operator γ_N defined in (2.3). For more details we refer to equations (A.14)–(A.17).

We start with the Helmholtz Dirichlet and Neumann boundary value problems:

t3.1

Theorem 3.1. *Suppose Ω is an open Lipschitz domain with a compact nonempty boundary $\partial\Omega$. Then for every $f \in H^1(\partial\Omega)$ and $z \in \mathbb{C}\backslash\sigma\big(H^D_{0,\Omega}\big)$ the following Dirichlet boundary value problem,*

$$\begin{cases} (-\Delta - z)u^D_0 = 0 \text{ on } \Omega, \quad u^D_0 \in H^{3/2}(\Omega), \\ \gamma_D u^D_0 = f \text{ on } \partial\Omega, \end{cases} \tag{3.3}$$

3.1

has a unique solution u^D_0 satisfying $\widetilde{\gamma}_N u^D_0 \in L^2(\partial\Omega; d^{n-1}\sigma)$. Moreover, there exist constants $C^D = C^D(\Omega, z) > 0$ such that

$$\|u^D_0\|_{H^{3/2}(\Omega)} \le C^D \|f\|_{H^1(\partial\Omega)}. \tag{3.4}$$

3.3a

Similarly, for every $g \in L^2(\partial\Omega; d^{n-1}\sigma)$ and $z \in \mathbb{C}\backslash\sigma\big(H^N_{0,\Omega}\big)$ the following Neumann boundary value problem,

$$\begin{cases} (-\Delta - z)u^N_0 = 0 \text{ on } \Omega, \quad u^N_0 \in H^{3/2}(\Omega), \\ \widetilde{\gamma}_N u^N_0 = g \text{ on } \partial\Omega, \end{cases} \tag{3.5}$$

3.2

has a unique solution u^N_0. Moreover, there exist constants $C^N = C^N(\Omega, z) > 0$ such that

$$\|u^N_0\|_{H^{3/2}(\Omega)} \le C^N \|g\|_{L^2(\partial\Omega; d^{n-1}\sigma)}. \tag{3.6}$$

3.4a

In addition, (3.1)–(3.6) imply that the following maps are bounded

$$\big[\gamma_N\big(\big(H^D_{0,\Omega} - zI_\Omega\big)^{-1}\big)^*\big]^* : H^1(\partial\Omega) \to H^{3/2}(\Omega), \quad z \in \mathbb{C}\backslash\sigma\big(H^D_{0,\Omega}\big), \tag{3.7}$$

3.4b

$$\big[\gamma_D\big(\big(H^N_{0,\Omega} - zI_\Omega\big)^{-1}\big)^*\big]^* : L^2(\partial\Omega; d^{n-1}\sigma) \to H^{3/2}(\Omega), \quad z \in \mathbb{C}\backslash\sigma\big(H^N_{0,\Omega}\big). \tag{3.8}$$

3.4c

Finally, the solutions u^D_0 and u^N_0 are given by the formulas

$$u^D_0(z) = -\big(\gamma_N\big(H^D_{0,\Omega} - \overline{z}I_\Omega\big)^{-1}\big)^* f, \tag{3.9}$$

3.3

$$u^N_0(z) = \big(\gamma_D\big(H^N_{0,\Omega} - \overline{z}I_\Omega\big)^{-1}\big)^* g. \tag{3.10}$$

3.4

Proof. It follows from Theorem 9.3 in [Mi96, 61] that the boundary value problems,

$$\begin{cases} (\Delta + z)u_0^D = 0 \text{ on } \Omega, \quad \mathcal{N}(\nabla u_0^D) \in L^2(\partial\Omega; d^{n-1}\sigma), \\ \gamma_D u_0^D = f \in H^1(\partial\Omega) \text{ on } \partial\Omega \end{cases} \tag{3.11}$$

and

$$\begin{cases} (\Delta + z)u_0^N = 0 \text{ on } \Omega, \quad \mathcal{N}(\nabla u_0^N) \in L^2(\partial\Omega; d^{n-1}\sigma), \\ \widetilde{\gamma}_N u_0^N = g \in L^2(\partial\Omega; d^{n-1}\sigma) \text{ on } \partial\Omega, \end{cases} \tag{3.12}$$

have unique solutions for all $z \in \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big)$ and $z \in \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big)$, respectively, satisfying natural estimates. Here $\mathcal{N}(\cdot)$ denotes the non-tangential maximal function (cf. [JK95, 46], [Mi96, 61])

$$(\mathcal{N}w)(x) = \sup_{y\in\Gamma(x)} |w(y)|, \quad x \in \partial\Omega, \tag{3.13}$$

where w is a locally bounded function and $\Gamma(x)$ is a nontangential approach region with vertex at x, that is, for some fixed constant $C > 1$ one has

$$\Gamma(x) = \{y \in \Omega \mid |x - y| < C \operatorname{dist}(y, \partial\Omega)\}. \tag{3.14}$$

In the case of a bounded domain Ω, it follows from Corollary 5.7 in [JK95, 46] that for any harmonic function v in Ω,

$$\mathcal{N}(\nabla v) \in L^2(\partial\Omega; d^{n-1}\sigma) \text{ if and only if } v \in H^{3/2}(\Omega), \tag{3.15}$$

accompanied with natural estimates. For any solution u of the Helmholtz equation $(\Delta + z)u = 0$ on a bounded domain Ω, one can introduce the harmonic function

$$v(x) = u(x) + z \int_\Omega d^n y\, E_n(x - y)u(y), \quad x \in \Omega, \tag{3.16}$$

such that $\mathcal{N}(\nabla u) \in L^2(\partial\Omega; d^{n-1}\sigma)$ if and only if $\mathcal{N}(\nabla v) \in L^2(\partial\Omega; d^{n-1}\sigma)$, and $u \in H^{3/2}(\Omega)$ if and only if $v \in H^{3/2}(\Omega)$. (Again, natural estimates are valid in each case.) Here E_n denotes the fundamental solution of the Laplace equation in $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$,

$$E_n(x) = \begin{cases} \frac{1}{2\pi}\ln(|x|), & n = 2, \\ \frac{1}{n(2-n)\omega_{n-1}}|x|^{2-n}, & n \geq 3, \end{cases}, \quad x \in \mathbb{R}^n\backslash\{0\}, \tag{3.17}$$

with ω_{n-1} denoting the area of the unit sphere in $\mathbb{R}^n$. The equivalence in (3.15) extends from harmonic functions to all functions u satisfying the Helmholtz equation, $(\Delta + z)u = 0$ on a bounded domain Ω,

$$\mathcal{N}(\nabla u) \in L^2(\partial\Omega; d^{n-1}\sigma) \text{ if and only if } u \in H^{3/2}(\Omega). \tag{3.18}$$

Thus, in the case of a bounded domain Ω, (3.3) and (3.5) follow from (3.11), (3.12), and (3.18). Moreover, one has the chain of estimates

$$\|u_0^D\|_{H^{3/2}(\Omega)} \leq C_1\big[\big\|\mathcal{N}\big(\nabla u_0^D\big)\big\|_{L^2(\partial\Omega; d^{n-1}\sigma)} + \|u_0^D\|_{L^2(\Omega; d^n x)}\big] \leq C_2\|f\|_{H^1(L^2(\partial\Omega; d^{n-1}\sigma))} \tag{3.19}$$

for some constants $C_k > 0$, $k = 1, 2$. In the case of an unbounded domain Ω, one first obtains (3.18) for $\Omega \cap B$, where B is a sufficiently large ball containing $\partial\Omega$. Then, since $z \in \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big) = \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big) = \mathbb{C}\backslash[0, \infty)$ (since now Ω contains the exterior of a ball in $\mathbb{R}^n$), one exploits the exponential decay of solutions of the Helmholtz equation to extend (3.18) from $\Omega \cap B$ to Ω. This, together with (3.11) and (3.12), yields (3.3) and (3.5).

Next, we turn to the proof of (3.9) and (3.10). We note that by Lemma 2.4,

$$\gamma_N\big(H_{0,\Omega}^D - \overline{z}I_\Omega\big)^{-1}, \gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1}\sigma)\big), \tag{3.20}$$

and hence

$$\big(\gamma_N\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big)^*,\big(\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big)^*\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\sigma),L^2(\Omega;d^nx)\big). \tag{3.21}$$ 3.21a

Then, denoting by u_0^D and u_0^N the unique solutions of (3.3) and (3.5), respectively, and using Green's formula, one computes

$$\begin{aligned}\big(u_0^D,v\big)_{L^2(\Omega;d^nx)}&=\big(u_0^D,(-\Delta-\overline{z})\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\Omega;d^nx)}\\&=\big((-\Delta-z)u_0^D,\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\Omega;d^nx)}\\&\quad+\big(\widetilde{\gamma}_Nu_0^D,\gamma_D\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\sigma)}\\&\quad-\big(\gamma_Du_0^D,\gamma_N\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\sigma)}\\&=-\big(f,\gamma_N\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\sigma)}\\&=-\big(\big(\gamma_N\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big)^*f,v\big)_{L^2(\Omega;d^nx)}\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\big(u_0^N,v\big)_{L^2(\Omega;d^nx)}&=\big(u_0^N,(-\Delta-\overline{z})\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\Omega;d^nx)}\\&=\big((-\Delta-z)u_0^N,\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\Omega;d^nx)}\\&\quad+\big(\widetilde{\gamma}_Nu_0^N,\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\sigma)}\\&\quad-\big(\gamma_Du_0^N,\gamma_N\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\sigma)}\\&=\big(g,\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}v\big)_{L^2(\partial\Omega;d^{n-1}\sigma)}\\&=\big(\big(\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big)^*g,v\big)_{L^2(\Omega;d^nx)}\end{aligned} \tag{3.23}$$

for any $v\in L^2(\Omega;d^nx)$. This proves (3.9) and (3.10) with the operators involved understood in the sense of (3.21). Granted (3.4) and (3.6), one finally obtains (3.7) and (3.8). □

We temporarily strengthen our hypothesis on V and introduce the following assumption:

h3.2 **Hypothesis 3.2.** *Suppose the set Ω satisfies Hypothesis 2.1 and assume that $V\in L^p(\Omega;d^nx)$ for some $p>2$ if $n=2,3$ and $p\geq 2n/3$ if $n\geq 4$.*

By employing a perturbative approach, we now extend Theorem 3.1 in connection with the Helmholtz differential expression $-\Delta-z$ on Ω to the case of a Schrödinger differential expression $-\Delta+V-z$ on Ω.

t3.3 **Theorem 3.3.** *Assume Hypothesis 3.2. Then for every $f\in H^1(\partial\Omega)$ and $z\in\mathbb{C}\backslash\sigma\big(H^D_\Omega\big)$ the following Dirichlet boundary value problem,*

$$\begin{cases}(-\Delta+V-z)u^D=0 \text{ on } \Omega, \quad u^D\in H^{3/2}(\Omega),\\ \gamma_Du^D=f \text{ on } \partial\Omega,\end{cases} \tag{3.24}$$ 3.9

has a unique solution u^D satisfying $\widetilde{\gamma}_Nu^D\in L^2(\partial\Omega;d^{n-1}\sigma)$. Moreover, there exist constants $C^D=C^D(\Omega,z)>0$ such that

$$\|u^D\|_{H^{3/2}(\Omega)}\leq C^D\|f\|_{H^1(\partial\Omega)}. \tag{3.25}$$ 3.9a

Similarly, for every $g \in L^2(\partial\Omega; d^{n-1}\sigma)$ and $z \in \mathbb{C}\backslash\sigma\big(H_\Omega^N\big)$ the following Neumann boundary value problem,

$$\begin{cases} (-\Delta + V - z)u^N = 0 \text{ on } \Omega, \quad u^N \in H^{3/2}(\Omega), \\ \widetilde{\gamma}_N u^N = g \text{ on } \partial\Omega, \end{cases} \tag{3.26}$$

has a unique solution u^N. Moreover, there exist constants $C^N = C^N(\Omega, z) > 0$ such that

$$\|u^N\|_{H^{3/2}(\Omega)} \le C^N \|g\|_{L^2(\partial\Omega; d^{n-1}\sigma)}. \tag{3.27}$$

In addition, (3.24)–(3.27) imply that the following maps are bounded

$$\big[\gamma_N\big(\big(H_\Omega^D - zI_\Omega\big)^{-1}\big)^*\big]^* : H^1(\partial\Omega) \to H^{3/2}(\Omega), \quad z \in \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.28}$$

$$\big[\gamma_D\big(\big(H_\Omega^N - zI_\Omega\big)^{-1}\big)^*\big]^* : L^2(\partial\Omega; d^{n-1}\sigma) \to H^{3/2}(\Omega), \quad z \in \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big). \tag{3.29}$$

Finally, the solutions u^D and u^N are given by the formulas

$$u^D(z) = -\big[\gamma_N\big(\big(H_\Omega^D - zI_\Omega\big)^{-1}\big)^*\big]^* f, \tag{3.30}$$

$$u^N(z) = \big[\gamma_D\big(\big(H_\Omega^N - zI_\Omega\big)^{-1}\big)^*\big]^* g. \tag{3.31}$$

Proof. We temporarily assume that $z \in \mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_\Omega^D\big)\big)$ in the case of the Dirichlet problem and $z \in \mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^N\big) \cup \sigma\big(H_\Omega^N\big)\big)$ in the context of the Neumann problem. Uniqueness of solutions follows from the fact that $z \notin \sigma(H_\Omega^D)$ and $z \notin \sigma(H_\Omega^N)$, respectively.

Next, we will show that the functions

$$u^D(z) = u_0^D(z) - \big(H_\Omega^D - zI_\Omega\big)^{-1} V u_0^D(z), \tag{3.32}$$

$$u^N(z) = u_0^N(z) - \big(H_\Omega^N - zI_\Omega\big)^{-1} V u_0^N(z), \tag{3.33}$$

with u_0^D, u_0^N given by Theorem 3.1, satisfy (3.30) and (3.31), respectively. Indeed, it follows from Theorem 3.1 that $u_0^D, u_0^N \in H^{3/2}(\Omega)$ and $\widetilde{\gamma}_N u_0^D \in L^2(\partial\Omega; d^{n-1}\sigma)$. Using the Sobolev embedding theorem

$$H^{3/2}(\Omega) \hookrightarrow L^q(\Omega; d^n x) \text{ for all } q \ge 2 \text{ if } n = 2, 3 \text{ and } 2 \le q \le 2n/(n-3) \text{ if } n \ge 4,$$

and the fact that $V \in L^p(\Omega; d^n x)$, $p > 2$ if $n = 2, 3$ and $p \ge 2n/3$ if $n \ge 4$, one concludes that $Vu_0^D, Vu_0^N \in L^2(\Omega; d^n x)$, and hence (3.32) and (3.33) are well-defined. Moreover, it follows from Lemma 2.3 that $V\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}, V\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big)$, and hence

$$\big[I_\Omega + V\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big]^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \quad z \in \mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_\Omega^D\big)\big), \tag{3.34}$$

$$\big[I_\Omega + V\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big]^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \quad z \in \mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^N\big) \cup \sigma\big(H_\Omega^N\big)\big), \tag{3.35}$$

by applying Theorem B.3. Thus, by (2.4) and (2.5),

$$\big(H_\Omega^D - zI_\Omega\big)^{-1} V u_0^D = \big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big[I_\Omega + V\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big]^{-1} V u_0^D \in H^2(\Omega), \tag{3.36}$$

$$\big(H_\Omega^N - zI_\Omega\big)^{-1} V u_0^N = \big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big[I_\Omega + V\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big]^{-1} V u_0^N \in H^2(\Omega), \tag{3.37}$$

and hence $u^D, u^N \in H^{3/2}(\Omega)$ and $\widetilde{\gamma}_N u^D \in L^2(\partial\Omega; d^{n-1}\sigma)$. Moreover,

$$\begin{aligned} (-\Delta + V - z)u^D &= (-\Delta - z)u_0^D + Vu_0^D - (-\Delta + V - z)\big(H_\Omega^D - zI_\Omega\big)^{-1} V u_0^D \\ &= Vu_0^D - I_\Omega V u_0^D = 0, \end{aligned} \tag{3.38}$$

$$\begin{aligned} (-\Delta + V - z)u^N &= (-\Delta - z)u_0^N + Vu_0^N - (-\Delta + V - z)\big(H_\Omega^N - zI_\Omega\big)^{-1} V u_0^N \\ &= Vu_0^N - I_\Omega V u_0^N = 0, \end{aligned} \tag{3.39}$$

and by (2.4), (2.5) and (3.15), (3.16) one also obtains,

$$\begin{aligned}\gamma_D u^D &= \gamma_D u_0^D - \gamma_D \big(H_\Omega^D - zI_\Omega\big)^{-1} V u_0^D \\ &= f - \gamma_D \big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} \big[I_\Omega + V\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big]^{-1} V u_0^D = f, \quad (3.40)\\ \widetilde{\gamma}_N u^N &= \widetilde{\gamma}_N u_0^N - \widetilde{\gamma}_N \big(H_\Omega^N - zI_\Omega\big)^{-1} V u_0^N \\ &= g - \gamma_N \big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} \big[I_\Omega + V\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big]^{-1} V u_0^N = g. \quad (3.41)\end{aligned}$$

Finally, (3.11) and (3.12) follow from (3.3), (3.4), (3.10), (3.13), (3.14), and the resolvent identity,

$$\begin{aligned}u^D(z) &= \big[I_\Omega - \big(H_\Omega^D - zI_\Omega\big)^{-1} V\big]\big[-\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big]^* f \\ &= -\big[\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big[I_\Omega - \big(H_\Omega^D - zI_\Omega\big)^{-1} V\big]^*\big]^* f \\ &= -\big[\gamma_N\big(\big(H_\Omega^D - zI_\Omega\big)^{-1}\big)^*\big]^* f, \quad (3.42)\\ u^N(z) &= \big[I_\Omega - \big(H_\Omega^N - zI_\Omega\big)^{-1} V\big]\big[\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big]^* g \\ &= \big[\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big[I_\Omega - \big(H_\Omega^N - zI_\Omega\big)^{-1} V\big]^*\big]^* g \\ &= \big[\gamma_D\big(\big(H_\Omega^N - zI_\Omega\big)^{-1}\big)^*\big]^* g. \quad (3.43)\end{aligned}$$

Analytic continuation with respect to z then permits one to remove the additional condition $z \notin \sigma\big(H_{0,\Omega}^D\big)$ in the case of the Dirichlet problem, and the additional condition $z \notin \sigma\big(H_{0,\Omega}^N\big)$ in the context of the Neumann problem. □

Assuming Hypothesis 2.1, we now introduce the Dirichlet-to-Neumann map $M_{0,\Omega}^D(z)$ associated with $(-\Delta - z)$ on Ω, as follows,

$$M_{0,\Omega}^D(z)\colon \begin{cases} H^1(\partial\Omega) \to L^2(\partial\Omega; d^{n-1}\sigma), \\ \qquad f \mapsto -\widetilde{\gamma}_N u_0^D, \end{cases} \quad z \in \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \quad (3.44)$$ 3.20

where u_0^D is the unique solution of

$$(-\Delta - z)u_0^D = 0 \text{ on } \Omega, \quad u_0^D \in H^{3/2}(\Omega), \quad \gamma_D u_0^D = f \text{ on } \partial\Omega, \quad (3.45)$$

Similarly, assuming Hypothesis 3.2, we introduce the Dirichlet-to-Neumann map $M_\Omega^D(z)$, associated with $(-\Delta + V - z)$ on Ω, by

$$M_\Omega^D(z)\colon \begin{cases} H^1(\partial\Omega) \to L^2(\partial\Omega; d^{n-1}\sigma), \\ \qquad f \mapsto -\widetilde{\gamma}_N u^D, \end{cases} \quad z \in \mathbb{C}\backslash\sigma\big(H_\Omega^D\big), \quad (3.46)$$ 3.22

where u^D is the unique solution of

$$(-\Delta + V - z)u^D = 0 \text{ on } \Omega, \quad u^D \in H^{3/2}(\Omega), \quad \gamma_D u^D = f \text{ on } \partial\Omega. \quad (3.47)$$

By Theorems 3.1 and 3.3 one obtains

$$M_{0,\Omega}^D(z), M_\Omega^D(z) \in \mathcal{B}\big(H^1(\partial\Omega), L^2(\partial\Omega; d^{n-1}\sigma)\big). \quad (3.48)$$

In addition, assuming Hypothesis 2.1, we introduce the Neumann-to-Dirichlet map $M_{0,\Omega}^N(z)$ associated with $(-\Delta - z)$ on Ω, as follows,

$$M_{0,\Omega}^N(z)\colon \begin{cases} L^2(\partial\Omega; d^{n-1}\sigma) \to H^1(\partial\Omega), \\ \qquad g \mapsto \gamma_D u_0^N, \end{cases} \quad z \in \mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big), \quad (3.49)$$ 3.24

where u_0^N is the unique solution of

$$(-\Delta - z)u_0^N = 0 \text{ on } \Omega, \quad u_0^N \in H^{3/2}(\Omega), \quad \widetilde{\gamma}_N u_0^N = g \text{ on } \partial\Omega, \quad (3.50)$$

Similarly, assuming Hypothesis h3.2 3.2, we introduce the Neumann-to-Dirichlet map $M_\Omega^N(z)$ associated with $(-\Delta+V-z)$ on Ω by

$$M_\Omega^N(z)\colon \begin{cases} L^2(\partial\Omega;d^{n-1}\sigma)\to H^1(\partial\Omega), \\ \qquad g\mapsto \gamma_D u^N, \end{cases} \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^N\big), \tag{3.51}$$ 3.26

where u^N is the unique solution of

$$(-\Delta+V-z)u^N=0 \text{ on } \Omega, \quad u^N\in H^{3/2}(\Omega), \quad \widetilde{\gamma}_N u^N=g \text{ on } \partial\Omega. \tag{3.52}$$

Again, by Theorems t3.1 3.1 and t3.3 3.3 one obtains

$$M_{0,\Omega}^N(z), M_\Omega^N(z)\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\sigma),H^1(\partial\Omega)\big). \tag{3.53}$$

Moreover, under the assumption of Hypothesis h2.1 2.1 for $M_{0,\Omega}^D(z)$ and $M_{0,\Omega}^N(z)$, and under the assumption of Hypothesis h3.2 3.2 for $M_\Omega^D(z)$ and $M_\Omega^N(z)$, one infers the following equalities:

$$M_{0,\Omega}^N(z)=-M_{0,\Omega}^D(z)^{-1}, \quad z\in\mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^D\big)\cup\sigma\big(H_{0,\Omega}^N\big)\big), \tag{3.54}$$ 3.28

$$M_\Omega^N(z)=-M_\Omega^D(z)^{-1}, \quad z\in\mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big)\cup\sigma\big(H_\Omega^N\big)\big), \tag{3.55}$$ 3.29

and

$$M_{0,\Omega}^D(z)=\widetilde{\gamma}_N\big[\gamma_N\big(\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}\big)^*\big]^*, \quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.56}$$ 3.30

$$M_\Omega^D(z)=\widetilde{\gamma}_N\big[\gamma_N\big(\big(H_\Omega^D-zI_\Omega\big)^{-1}\big)^*\big]^*, \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^D\big), \tag{3.57}$$ 3.31

$$M_{0,\Omega}^N(z)=\gamma_D\big[\gamma_D\big(\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}\big)^*\big]^*, \quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big), \tag{3.58}$$ 3.32

$$M_\Omega^N(z)=\gamma_D\big[\gamma_D\big(\big(H_\Omega^N-zI_\Omega\big)^{-1}\big)^*\big]^*, \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^N\big). \tag{3.59}$$ 3.33

The representations (3.56)–(3.59) provide a convenient point of departure for proving the operator-valued Herglotz property of M_Ω^D and M_Ω^N. We will return to this topic in a future paper.

Next, we note that the above formulas (3.56)–(3.59) may be used as alternative definitions of the Dirichlet-to-Neumann and Neumann-to-Dirichlet maps. In particular, we will next use (3.57) and (3.59) to extend the above definition of the operators $M_\Omega^D(z)$ and $M_\Omega^N(z)$ to a more general setting. This is done in the following two lemmas.

l3.4 **Lemma 3.4.** *Assume Hypothesis h2.6 2.6. Then the following boundedness properties hold:*

$$\gamma_N\big(H_\Omega^D-zI_\Omega\big)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\partial\Omega;d^{n-1}\sigma)\big), \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^D\big), \tag{3.60}$$ 3.38a

$$\gamma_D\big(H_\Omega^N-zI_\Omega\big)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^1(\partial\Omega)\big), \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^N\big), \tag{3.61}$$ 3.39a

$$\big[\gamma_N\big(\big(H_\Omega^D-zI_\Omega\big)^{-1}\big)^*\big]^*\in\mathcal{B}\big(H^1(\partial\Omega),H^{3/2}(\Omega)\big), \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^D\big), \tag{3.62}$$ 3.40a

$$\big[\gamma_D\big(\big(H_\Omega^N-zI_\Omega\big)^{-1}\big)^*\big]^*\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\sigma),H^{3/2}(\Omega)\big), \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^N\big). \tag{3.63}$$ 3.41a

Moreover, the operators $M_\Omega^D(z)$ in (3.57) and $M_\Omega^N(z)$ in (3.59) remain well-defined and satisfy

$$M_\Omega^D(z)\in\mathcal{B}\big(H^1(\partial\Omega),L^2(\partial\Omega;d^{n-1}\sigma)\big), \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^D\big), \tag{3.64}$$ 3.42a

$$M_\Omega^N(z)\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\sigma),H^1(\partial\Omega)\big), \quad z\in\mathbb{C}\backslash\sigma\big(H_\Omega^N\big). \tag{3.65}$$ 3.43a

In particular, $M_\Omega^N(z)$, $z\in\mathbb{C}\backslash\sigma\big(H_\Omega^N\big)$, are compact operators in $L^2(\partial\Omega;d^{n-1}\sigma)$.

Proof. We temporarily assume that $z\in\mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^D\big)\cup\sigma\big(H_\Omega^D\big)\big)$ in the case of Dirichlet Laplacian and that $z\in\mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^N\big)\cup\sigma\big(H_\Omega^N\big)\big)$ in the context of Neumann Laplacian.

Next, let u,v and $\widetilde{u},\widetilde{v}$ denote the following factorizations of the perturbation V,

$$V(x)=u(x)v(x), \quad u(x)=\exp(i\arg(V(x)))|V(x)|^{1/2}, \quad v(x)=|V(x)|^{1/2}, \tag{3.66}$$ 3.44a

$$V(x)=\widetilde{u}(x)\widetilde{v}(x),\quad \widetilde{u}(x)=\exp(i\arg(V(x)))|V(x)|^{p/p_1},\quad \widetilde{v}(x)=|V(x)|^{p/p_2}, \tag{3.67}$$ 3.45a

where

$$p_1=\begin{cases}3p/2, & n=2,\\ 4p/3, & n\geq 3,\end{cases}\qquad p_2=\begin{cases}3p, & n=2,\\ 4p, & n\geq 3.\end{cases} \tag{3.68}$$ 3.46a

We note that Hypothesis h2.6 and (3.66), (3.67) imply

$$\widetilde{u}\in L^{p_1}(\Omega;d^nx),\ \widetilde{v}\in L^{p_2}(\Omega;d^nx),\ \text{ and }\ u,v\in L^{2p}(\Omega;d^nx). \tag{3.69}$$ 3.46b

It follows from the definition of the operators H_Ω^D and H_Ω^N and, in particular, from (B.5) that

$$\begin{aligned}\big(H_\Omega^D-zI_\Omega\big)^{-1}&=\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}-\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}v\Big[I_\Omega+\overline{u\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}v}\Big]^{-1}u\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}\\ &=\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}-\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}\widetilde{v}\Big[I_\Omega+\overline{\widetilde{u}\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}\widetilde{v}}\Big]^{-1}\widetilde{u}\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1},\end{aligned} \tag{3.70}$$ 3.47a

$$\begin{aligned}\big(H_\Omega^N-zI_\Omega\big)^{-1}&=\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}-\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}v\Big[I_\Omega+\overline{u\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}v}\Big]^{-1}u\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}\\ &=\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}-\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}\widetilde{v}\Big[I_\Omega+\overline{\widetilde{u}\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}\widetilde{v}}\Big]^{-1}\widetilde{u}\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}.\end{aligned} \tag{3.71}$$ 3.48a

Next, we establish a number of boundedness properties that will imply (3.60)–(3.65). First, note that it follows from Hypothesis 2.6 and (3.68) that $p_1=\frac{3}{2}p>2\geq 2n/3$, $p_2=3p>4$ for $n=2$ and $p_1=\frac{4}{3}p>2n/3$, $p_2=4p>2n$ for $n\geq 3$. Then, utilizing Lemma 2.3, one obtains

$$\widetilde{u}\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.72}$$ 3.51a

$$\widetilde{u}\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big), \tag{3.73}$$ 3.52a

$$\big(H_{0,\Omega}^D-zI_\Omega\big)^{-\frac{1-\varepsilon}{4}}\widetilde{v}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.74}$$ 3.53a

$$\big(H_{0,\Omega}^N-zI_\Omega\big)^{-\frac{1-\varepsilon}{4}}\widetilde{v}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big), \tag{3.75}$$ 3.54a

and, utilizing Lemma 2.2 and the inclusion (A.4), one obtains for $\varepsilon\in(0,1-2n/p_2)$,

$$\big(H_{0,\Omega}^D-zI_\Omega\big)^{-\frac{3+\varepsilon}{4}}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^{\frac{3+\varepsilon}{2}}(\Omega)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.76}$$ 3.54b

$$\big(H_{0,\Omega}^N-zI_\Omega\big)^{-\frac{3+\varepsilon}{4}}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^{\frac{3+\varepsilon}{2}}(\Omega)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big). \tag{3.77}$$ 3.54c

In addition,

$$\big(H_{0,\Omega}^D-zI_\Omega\big)^{-\frac{3+\varepsilon}{4}}:L^2(\Omega;d^nx)\to H^{\frac{3+\varepsilon}{2}}(\Omega)\hookrightarrow H^{3/2}(\Omega),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.78}$$ 3.55a

$$\big(H_{0,\Omega}^N-zI_\Omega\big)^{-\frac{3+\varepsilon}{4}}:L^2(\Omega;d^nx)\to H^{\frac{3+\varepsilon}{2}}(\Omega)\hookrightarrow H^{3/2}(\Omega),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big). \tag{3.79}$$ 3.56a

In particular, one concludes from (3.74)–(3.79) that

$$\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}\widetilde{v}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^{3/2}(\Omega)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.80}$$ 3.57a

$$\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}\widetilde{v}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^{3/2}(\Omega)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^N\big). \tag{3.81}$$ 3.58a

In addition, it follows from (3.74)–(3.79), the definition of γ_N (2.3), inclusion (A.4), and Lemma A.6 that

$$\gamma_N\big(H_{0,\Omega}^D-zI_\Omega\big)^{-1}\widetilde{v}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\partial\Omega;d^{n-1}\sigma)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H_{0,\Omega}^D\big), \tag{3.82}$$ 3.59a

$$\gamma_D\big(H_{0,\Omega}^N-zI_\Omega\big)^{-1}\widetilde{v}\in\mathcal{B}\big(L^2(\Omega;d^nx),H^1(\partial\Omega)\big),\quad z\in\mathbb{C}\backslash\sigma(H_{0,\Omega}^N). \tag{3.83}$$ 3.60a

Next, it follows from Theorem 3.1 that

$$\big[\gamma_N\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big]^*\in\mathcal{B}\big(H^1(\partial\Omega),H^{3/2}(\Omega)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H^D_{0,\Omega}\big), \tag{3.84}$$

$$\big[\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big]^*\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\sigma),H^{3/2}(\Omega)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H^N_{0,\Omega}\big). \tag{3.85}$$

Then, employing the Sobolev embedding theorem

$$H^{3/2}(\Omega)\hookrightarrow L^q(\Omega;d^nx) \tag{3.86}$$

with q satisfying $1/q=(1/2)-(1/p_1)>(1/2)-3/(2n)$, $n\geq 2$, and the fact that $\widetilde{u}\in L^{p_1}(\Omega;d^nx)$, one obtains the following boundedness properties from (3.84) and (3.85),

$$\widetilde{u}\big[\gamma_N\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big]^*\in\mathcal{B}\big(H^1(\partial\Omega),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H^D_{0,\Omega}\big), \tag{3.87}$$

$$\widetilde{u}\big[\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big]^*\in\mathcal{B}\big(L^2(\partial\Omega;d^{n-1}\sigma),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\sigma\big(H^N_{0,\Omega}\big). \tag{3.88}$$

Moreover, it follows from Theorem B.3 that the operators $\big[I_\Omega+\widetilde{u}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}\big]$ and $\big[I_\Omega+\widetilde{u}\big(H^N_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}\big]$ are boundedly invertible on $L^2(\Omega;d^nx)$ for $z\in\mathbb{C}\backslash\big(\sigma\big(H^D_{0,\Omega}\big)\cup\sigma\big(H^D_\Omega\big)\big)$ and $z\in\mathbb{C}\backslash\big(\sigma\big(H^N_{0,\Omega}\big)\cup\sigma\big(H^N_\Omega\big)\big)$, respectively, that is, the following operators are bounded,

$$\big[I_\Omega+\widetilde{u}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}\big]^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\big(\sigma\big(H^D_{0,\Omega}\big)\cup\sigma\big(H^D_\Omega\big)\big), \tag{3.89}$$

$$\big[I_\Omega+\widetilde{u}\big(H^N_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}\big]^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx),L^2(\Omega;d^nx)\big),\quad z\in\mathbb{C}\backslash\big(\sigma\big(H^N_{0,\Omega}\big)\cup\sigma\big(H^N_\Omega\big)\big). \tag{3.90}$$

Finally, combining (3.70)–(3.90), one obtains the assertions of Lemma 3.4 as follows: (3.60) follows from (3.47a), (3.51a), (3.59a), (3.82), (3.89); (3.61) follows from (3.47a), (3.57a), (3.65a), (3.87), (3.89); (3.62) follows from (3.48a), (3.52a), (3.60a), (3.68a), (3.40a), (3.70), (3.80), (3.87), (3.89); (3.63) follows from (3.48a), (3.58a), (3.66a), (3.68a), (3.41a), (3.71), (3.81), (3.88), (3.90);

Thus, by (3.44), (3.82), (3.87), and (3.89), we may introduce the operator

$$M^D_\Omega(z)=M^D_{0,\Omega}(z)-\gamma_N\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}\Big[I_\Omega+\overline{\widetilde{u}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}}\Big]^{-1}\widetilde{u}\big(\gamma_N\big(H^D_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big)^*, \tag{3.91}$$

and observe that it satisfies (3.64). In addition, (3.70) shows that (3.57) remains in effect under Hypothesis 2.6.

Similarly, by (3.49), (3.83), (3.88), and (3.90), we may introduce the operator

$$M^N_\Omega(z)=M^N_{0,\Omega}(z)-\gamma_D\big(H^N_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}\Big[I_\Omega+\overline{\widetilde{u}\big(H^N_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}}\Big]^{-1}\widetilde{u}\big(\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big)^*, \tag{3.92}$$

and observe that it satisfies (3.65). In addition, (3.71) shows that (3.59) remains in effect under Hypothesis 2.6. Moreover, since $H^1(\partial\Omega)$ embeds compactly into $L^2(\partial\Omega;d^{n-1}\sigma)$ (cf. (A.6) and [60, Proposition 2.4]), $M^N_\Omega(z)$, $z\in\mathbb{C}\backslash\sigma\big(H^N_\Omega\big)$, are compact operators in $L^2(\partial\Omega;d^{n-1}\sigma)$.

Finally, formulas (3.57) and (3.59) together with analytic continuation with respect to z then permit one to remove the additional restrictions $z\notin\sigma\big(H^D_{0,\Omega}\big)$ and $z\notin\sigma\big(H^N_{0,\Omega}\big)$, respectively. □

Actually, one can go a step further and allow an additional perturbation $V_1\in L^\infty(\Omega;d^nx)$ of H^D_Ω and H^N_Ω,

$$H^D_{1,\Omega}=H^D_\Omega+V_1,\quad \operatorname{dom}(H^D_{1,\Omega})=\operatorname{dom}(H^D_\Omega), \tag{3.93}$$

$$H^N_{1,\Omega}=H^N_\Omega+V_1,\quad \operatorname{dom}(H^N_{1,\Omega})=\operatorname{dom}(H^N_\Omega). \tag{3.94}$$

Defining the Dirichlet-to-Neumann and Neumann-to-Dirichlet operators $M^D_{1,\Omega}$ and $M^N_{1,\Omega}$ in an analogous fashion as in (3.57) and (3.59),

$$M^D_{1,\Omega}(z)=\widetilde{\gamma}_N\big[\gamma_N\big(\big(H^D_{1,\Omega}-zI_\Omega\big)^{-1}\big)^*\big]^*,\quad z\in\mathbb{C}\backslash\sigma\big(H^D_{1,\Omega}\big), \tag{3.95}$$

$$M^N_{1,\Omega}(z)=\gamma_D\big[\gamma_D\big(\big(H^N_{1,\Omega}-zI_\Omega\big)^{-1}\big)^*\big]^*,\quad z\in\mathbb{C}\backslash\sigma\big(H^N_{1,\Omega}\big), \tag{3.96}$$

one can then prove the following result:

Lemma 3.5. *Assume Hypothesis 2.6 and let $V_1 \in L^\infty(\Omega; d^n x)$. Then the operators $M^D_{1,\Omega}(z)$ and $M^N_{1,\Omega}(z)$ defined by (3.95) and (3.96) satisfy the following boundedness properties,*

$$M^D_{1,\Omega}(z) \in \mathcal{B}\big(H^1(\partial\Omega), L^2(\partial\Omega; d^{n-1}\sigma)\big), \quad z \in \mathbb{C}\backslash\sigma\big(H^D_{1,\Omega}\big), \tag{3.97}$$

$$M^N_{1,\Omega}(z) \in \mathcal{B}\big(L^2(\partial\Omega; d^{n-1}\sigma), H^1(\partial\Omega)\big), \quad z \in \mathbb{C}\backslash\sigma\big(H^N_{1,\Omega}\big). \tag{3.98}$$

Proof. We temporarily assume that $z \in \mathbb{C}\backslash\big(\sigma\big(H^D_\Omega\big) \cup \sigma\big(H^D_{1,\Omega}\big)\big)$ in the case of $M^D_{1,\Omega}$ and that $z \in \mathbb{C}\backslash\big(\sigma\big(H^N_\Omega\big) \cup \sigma\big(H^N_{1,\Omega}\big)\big)$ in the context of $M^N_{1,\Omega}$.

Next, using resolvent identities and (3.93), (3.94), one computes

$$\big(H^D_{1,\Omega} - zI_\Omega\big)^{-1} = \big(H^D_\Omega - zI_\Omega\big)^{-1} - \big(H^D_\Omega - zI_\Omega\big)^{-1}\Big[I_\Omega + V_1\big(H^D_\Omega - zI_\Omega\big)^{-1}\Big]^{-1} V_1\big(H^D_\Omega - zI_\Omega\big)^{-1}, \tag{3.99}$$

$$\big(H^N_{1,\Omega} - zI_\Omega\big)^{-1} = \big(H^N_\Omega - zI_\Omega\big)^{-1} - \big(H^N_\Omega - zI_\Omega\big)^{-1}\Big[I_\Omega + V_1\big(H^N_\Omega - zI_\Omega\big)^{-1}\Big]^{-1} V_1\big(H^N_\Omega - zI_\Omega\big)^{-1}, \tag{3.100}$$

and hence,

$$M^D_{1,\Omega} = M^D_\Omega - \gamma_N\big(H^D_\Omega - zI_\Omega\big)^{-1}\Big[I_\Omega + V_1\big(H^D_\Omega - zI_\Omega\big)^{-1}\Big]^{-1} V_1\big[\gamma_N\big(\big(H^D_\Omega - zI_\Omega\big)^{-1}\big)^*\big]^*, \tag{3.101}$$

$$M^N_{1,\Omega} = M^N_\Omega - \gamma_D\big(H^N_\Omega - zI_\Omega\big)^{-1}\Big[I_\Omega + V_1\big(H^N_\Omega - zI_\Omega\big)^{-1}\Big]^{-1} V_1\big[\gamma_D\big(\big(H^N_\Omega - zI_\Omega\big)^{-1}\big)^*\big]^*. \tag{3.102}$$

The assertions (3.97) and (3.98) now follow from (3.60)–(3.65) and the fact that by Theorem B.3, the operators $\big[I_\Omega + V_1\big(H^D_\Omega - zI_\Omega\big)^{-1}\big]$ and $\big[I_\Omega + V_1\big(H^N_\Omega - zI_\Omega\big)^{-1}\big]$ are boundedly invertible on $L^2(\Omega; d^n x)$ for all $z \in \mathbb{C}\backslash\big(\sigma\big(H^D_\Omega\big) \cup \sigma\big(H^D_{1,\Omega}\big)\big)$ and $z \in \mathbb{C}\backslash\big(\sigma\big(H^N_\Omega\big) \cup \sigma\big(H^N_{1,\Omega}\big)\big)$, respectively. Formulas (3.95) and (3.96) together with analytic continuation with respect to z then permit one to remove the additional restrictions $z \notin \sigma\big(H^D_\Omega\big)$ and $z \notin \sigma\big(H^N_\Omega\big)$, respectively. □

Weyl–Titchmarsh operators, in a spirit close to ours, have recently been discussed by Amrein and Pearson [2] in connection with the interior and exterior of a ball in $\mathbb{R}^3$ and potentials $V \in L^\infty(\mathbb{R}^3; d^3x)$. For additional literature on Weyl–Titchmarsh operators, relevant in the context of boundary value spaces (boundary triples, etc.), we refer, for instance, to [1], [5], [6], [9], [10], [11], [21], [22], [31], [39, Ch. 3], [55], [56], [58] [76], [77]. For applications of the Dirichlet-to-Neumann map to Borg–Levinson-type inverse spectral problems we refer to [17], [65], [73], [83], [91], [92] (see also [53] for an alternative approach based on the boundary control method). The inverse problem of detecting the number of connected components (i.e., the number of holes) in $\partial\Omega$ using the high-energy spectral asymptotics of the Dirichlet-to-Neumann map is studied in [43].

Next, we prove the following auxiliary result, which will play a crucial role in Theorem 4.3, the principal result of this paper.

Lemma 3.6. *Assume Hypothesis 2.6. Then the following identities hold,*

$$M^D_{0,\Omega}(z) - M^D_\Omega(z) = \overline{\widetilde{\gamma}_N\big(H^D_\Omega - zI_\Omega\big)^{-1} V\big[\gamma_N\big(\big(H^D_{0,\Omega} - zI_\Omega\big)^{-1}\big)^*\big]^*}, \quad z \in \mathbb{C}\backslash\big(\sigma\big(H^D_{0,\Omega}\big) \cup \sigma\big(H^D_\Omega\big)\big), \tag{3.103}$$

$$M^D_\Omega(z) M^D_{0,\Omega}(z)^{-1} = I_{\partial\Omega} - \overline{\widetilde{\gamma}_N\big(H^D_\Omega - zI_\Omega\big)^{-1} V\big[\gamma_D\big(\big(H^N_{0,\Omega} - zI_\Omega\big)^{-1}\big)^*\big]^*}, \quad z \in \mathbb{C}\backslash\big(\sigma\big(H^D_{0,\Omega}\big) \cup \sigma\big(H^D_\Omega\big) \cup \sigma\big(H^N_{0,\Omega}\big)\big). \tag{3.104}$$

Proof. Let $z \in \mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_\Omega^D\big)\big)$. Then (3.103) follows from (3.56), (3.57), and the resolvent identity

$$\begin{aligned} M_{0,\Omega}^D(z) - M_\Omega^D(z) &= \widetilde{\gamma}_N\big[\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} - \big(H_\Omega^D - zI_\Omega\big)^{-1}\big)^*\big]^* \\ &= \overline{\widetilde{\gamma}_N\big[\gamma_N\big(\big(H_\Omega^D - zI_\Omega\big)^{-1} V \big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big]^*} \\ &= \overline{\widetilde{\gamma}_N\big(H_\Omega^D - zI_\Omega\big)^{-1} V \big[\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big]^*}. \end{aligned} \tag{3.105}$$

Next, let $z \in \mathbb{C}\backslash\big(\sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$, then it follows from (3.54), (3.58), and (3.103) that

$$\begin{aligned} M_\Omega^D(z) M_{0,\Omega}^D(z)^{-1} &= I_{\partial\Omega} + \big(M_\Omega^D(z) - M_{0,\Omega}^D(z)\big) M_{0,\Omega}^D(z)^{-1} \\ &= I_{\partial\Omega} + \big(M_{0,\Omega}^D(z) - M_\Omega^D(z)\big) M_{0,\Omega}^N(z) \\ &= I_{\partial\Omega} + \overline{\widetilde{\gamma}_N\big(H_\Omega^D - zI_\Omega\big)^{-1} V \big[\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big]^*} \\ &\quad \times \gamma_D\big[\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big]^*. \end{aligned} \tag{3.106}$$

Let $g \in L^2(\partial\Omega; d^{n-1}\sigma)$. Then by Theorem 3.1,

$$u = \big[\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big]^* g \tag{3.107}$$

is the unique solution of

$$(-\Delta - z)u = 0 \text{ on } \Omega, \quad u \in H^{3/2}(\Omega), \quad \widetilde{\gamma}_N u = g \text{ on } \partial\Omega. \tag{3.108}$$

Setting $f = \gamma_D u \in H^1(\partial\Omega)$ and utilizing Theorem 3.1 once again, one obtains

$$\begin{aligned} u &= -\big[\gamma_N\big(H_{0,\Omega}^D - \overline{z} I_\Omega\big)^{-1}\big]^* f \\ &= -\big[\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big]^* \gamma_D\big[\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big]^* g. \end{aligned} \tag{3.109}$$

Thus, it follows from (3.107) and (3.109) that

$$\big[\gamma_N\big(\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\big)^*\big]^* \gamma_D\big[\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big]^* = -\big[\gamma_D\big(\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big)^*\big]^*. \tag{3.110}$$

Finally, insertion of (3.110) into (3.106) yields (3.104). □

It follows from (4.38)–(4.44), $\widetilde{\gamma}_N$ can be replaced by γ_N on the right-hand side of (3.103) and (3.104).

We note that the right-hand side (and hence the left-hand side) of (3.104) permits an analytic continuation to $z \in \sigma\big(H_{0,\Omega}^D\big)$ as long as $z \notin \big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$.

4. A Multi-Dimensional Variant of a Formula due to Jost and Pais

In this section we prove our multi-dimensional variants of the Jost and Pais formula as discussed in the introduction.

We start with an elementary comment on determinants which, however, lies at the heart of the matter of our multi-dimensional variant of the one-dimensional Jost and Pais result. Suppose $A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, $B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ with $AB \in \mathcal{B}_1(\mathcal{H}_2)$ and $BA \in \mathcal{B}_1(\mathcal{H}_1)$. Then,

$$\det(I_{\mathcal{H}_2} - AB) = \det(I_{\mathcal{H}_1} - BA). \tag{4.1}$$

Equation (4.1) follows from the fact that all nonzero eigenvalues of AB and BA coincide including their algebraic multiplicities. The latter fact, in turn, can be derived from the formula

$$A(BA - zI_{\mathcal{H}_1})^{-1} B = I_{\mathcal{H}_2} + z(AB - zI_{\mathcal{H}_2})^{-1}, \quad z \in \mathbb{C}\backslash(\sigma(AB) \cup \sigma(BA)) \tag{4.2}$$

(and its companion with A and B interchanged), as discussed in detail by Deift [19].

In particular, $\mathcal{H}_1$ and $\mathcal{H}_2$ may have different dimensions. Especially, one of them may be infinite and the other finite, in which case one of the two determinants in (4.1) reduces to a finite determinant. This case indeed occurs in the original one-dimensional case studied by Jost and Pais [47] as described in detail in [33] and the references therein. In the proof of Theorem 4.2 below, the role of $\mathcal{H}_1$ and $\mathcal{H}_2$ will be played by $L^2(\Omega; d^n x)$ and $L^2(\partial\Omega; d^{n-1}\sigma)$, respectively. In the context of KdV flows and reflectionless (i.e., generalizations of soliton-type) potentials represented as Fredholm determinants, a reduction of such determinants (in some cases to finite determinants) has also been studied by Kotani [52], relying on certain connections to stochastic analysis.

We start with an auxiliary lemma which is of independent interest in the area of modified Fredholm determinants.

Lemma 4.1. *Let $\mathcal{H}$ be a separable, complex Hilbert space, and assume $A, B \in \mathcal{B}_k(\mathcal{H})$ for some fixed $k \in \mathbb{N}$. Then there exists a polynomial $T_k(\cdot,\cdot)$ in A and B with $T_k(A,B) \in \mathcal{B}_1(\mathcal{H})$, such that the following formula holds*

$$\det{}_k((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) = \det{}_k(I_{\mathcal{H}} - A) \det{}_k(I_{\mathcal{H}} - B) e^{\operatorname{tr}(T_k(A,B))}. \tag{4.3}$$

Moreover, $T_k(\cdot,\cdot)$ is unique up to cyclic permutations of its terms, and an explicit formula for T_k may be derived from the representation

$$T_k(A,B) = \sum_{m=k}^{2k-2} P_m(A,B), \tag{4.4}$$

where $P_m(\cdot,\cdot)$, $m = 1, \dots, 2k-2$, denote homogeneous polynomials in A and B of degree m (i.e., each term of $P_m(A,B)$ contains precisely the total number m of A's and B's) that one obtains after rearranging the following expression in powers of t,

$$\sum_{j=1}^{k-1} \frac{1}{j} \big((tA + tB - t^2 AB)^j - (tA)^j - (tB)^j\big) = \sum_{m=1}^{2k-2} t^m P_m(A,B), \quad t \in \mathbb{R}. \tag{4.5}$$

In particular, computing $T_k(A,B)$ from (4.4) and (4.5), and subsequently using cyclic permutations to simplify the resulting expressions, then yields for the terms $T_k(A,B)$ in (4.3)

$$\begin{aligned}
T_1(A,B) &= 0,\\
T_2(A,B) &= -AB,\\
T_3(A,B) &= -A^2B - AB^2 + \frac{1}{2}ABAB,\\
T_4(A,B) &= -A^3B - AB^3 - \frac{1}{2}ABAB - A^2B^2 + A^2BAB + AB^2AB - \frac{1}{3}ABABAB,\\
T_5(A,B) &= -A^4B - AB^4 - A^3B^2 - A^2B^3 - A^2BAB - AB^2AB + A^3BAB + AB^3AB\\
&\quad + A^2B^2AB + A^2BAB^2 + \frac{2}{3}ABABAB + \frac{1}{2}A^2BA^2B + \frac{1}{2}AB^2AB^2\\
&\quad - A^2BABAB - AB^2ABAB + \frac{1}{4}ABABABAB, \quad \textit{etc.}
\end{aligned} \tag{4.6}$$

Proof. Suppose temporarily that $A, B \in \mathcal{B}_1(\mathcal{H})$. Then it follows from [88, Theorem 9.2] that

$$\det{}_k((I_{\mathcal{H}} - A)(I_{\mathcal{H}} - B)) = \det{}_k(I_{\mathcal{H}} - A) \det{}_k(I_{\mathcal{H}} - B) e^{\operatorname{tr}(\widetilde{T}_k(A,B))}, \tag{4.7}$$

where

$$\widetilde{T}_k(A,B) = \sum_{j=1}^{k-1} \frac{1}{j} \big((A + B - AB)^j - (A)^j - (B)^j\big), \tag{4.8}$$

and hence, by (4.5a)

$$\widetilde{T}_k(A,B)=\sum_{m=1}^{2k-2}P_m(A,B). \tag{4.9}$$ 4.9a

Since $\operatorname{tr}(\cdot)$ is linear and invariant under cyclic permutation of its argument, it remains to show that $T_k(A,B)$ in (4.4a) and $\widetilde{T}_k(A,B)$ in (4.9a) are equal up to cyclic permutations of their terms, that is, to show that $P_m(A,B)$ vanish for $m=1,\dots,k-1$ after a finite number of cyclic permutations of their terms.

Let $\widetilde{P}_m(\cdot,\cdot)$, $m\geq 1$, denote a sequence of polynomials in A and B, obtained after rearranging the following expression in powers of $t\in\mathbb{C}$,

$$\begin{aligned}&\ln((I_{\mathcal{H}}-tA)(I_{\mathcal{H}}-tB))-\ln(I_{\mathcal{H}}-tA)-\ln(I_{\mathcal{H}}-tB)\\ &\quad=\sum_{j=1}^{\infty}\frac{1}{j}\big((tA+tB-t^2AB)^j-(tA)^j-(tB)^j\big)=\sum_{m=1}^{\infty}t^m\widetilde{P}_m(A,B)\ \text{ for } |t| \text{ sufficiently small.}\end{aligned} \tag{4.10}$$ 4.10a

Then it follows from (4.5a) and (4.10a) that $P_m(A,B)=\widetilde{P}_m(A,B)$ for $m=1,\dots,k-1$, and hence, it suffices to show that $\widetilde{P}_m(A,B)$ vanish for $m=1,\dots,k-1$ after a finite number of cyclic permutations of their terms. The latter fact now follows from the Baker–Campbell–Hausdorff (BCH) formula as follows: First, assume $D,E\in\mathcal{B}(\mathcal{H})$, $\mathcal{H}$. Then,

$$e^{tD}e^{tE}=e^{tD+tE+F(t)}\ \text{ for } |t| \text{ sufficiently small,} \tag{4.11}$$ 4.12a

where $F(t)$ is given by a norm convergent infinite sum of certain repeated commutators involving D and E, as discussed, for instance, in [Su77] (cf. also [BC04]). Explicitly, F is of the form

$$F(t)=\sum_{\ell=2}^{\infty}t^{\ell}F_{\ell},\quad F_p=\frac{1}{p!}\bigg[\frac{d^p}{dt^p}\ln\bigg(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{t^{j+k}}{j!k!}D^jE^k\bigg)\bigg]\bigg|_{t=0},\quad p\in\mathbb{N},\ p\geq 2, \tag{4.12}$$

where

$$F_2=\frac{1}{2}[D,E],\ F_3=\frac{1}{6}[F_2,E-D],\ F_4=\frac{1}{12}[[F_2,D],E],\ \text{ etc.} \tag{4.13}$$

That each F_ℓ, $\ell\geq 2$, is indeed at most a finite sum of commutators follows from a formula derived by Dynkin (cf., e.g., [Bo89, eqs. (1)–(4)], [Ot91, eqs. (2.5), (2.6), (3.7), (3.8)]).

If in addition, $D,E\in\mathcal{B}_1(\mathcal{H})$, the expression for $F(t)$ is actually convergent in the $\mathcal{B}_1(\mathcal{H})$-norm for $|t|$ sufficiently small. Thus, $F(t)$ vanishes after a finite number of cyclic permutations of each of its coefficients F_n.

Next, setting $D=\ln(I_{\mathcal{H}}-tA)$, $E=\ln(I_{\mathcal{H}}-tB)$ and taking the natural logarithm in (4.12a) then implies

$$\ln((I_{\mathcal{H}}-tA)(I_{\mathcal{H}}-tB))-\ln(I_{\mathcal{H}}-tA)-\ln(I_{\mathcal{H}}-tB)=F(t) \tag{4.14}$$

and hence

$$\ln((I_{\mathcal{H}}-tA)(I_{\mathcal{H}}-tB))-\ln(I_{\mathcal{H}}-tA)-\ln(I_{\mathcal{H}}-tB)=0 \tag{4.15}$$

after a finite number of cyclic permutations in each of the coefficients F_ℓ in $F(t)=\sum_{\ell=2}^{\infty}t^\ell F_\ell$. Thus, by (4.10a), each $\widetilde{P}_m(A,B)$, $m\geq 1$, vanishes after a finite number of cyclic permutations of its terms. Consequently, $P_m(A,B)$ vanish for $m=1,\dots,k-1$ after a finite number of cyclic permutations of their terms.

Finally, to remove the assumption $A,B\in\mathcal{B}_1(\mathcal{H})$, one uses a standard approximation argument of operators in $\mathcal{B}_k(\mathcal{H})$ by operators in $\mathcal{B}_1(\mathcal{H})$, together with the fact that both sides of (4.3a) are well-defined for $A,B\in\mathcal{B}_k(\mathcal{H})$. □

Next, we prove an extension of a result in [GLMZ05] to arbitrary space dimensions:

t4.1 **Theorem 4.2.** *Assume Hypothesis* h2.6 *2.6, let* $k \in \mathbb{N}$, $k \geq p$, *and* $z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$. *Then,*

$$\overline{\gamma_N\big(H_\Omega^D - zI_\Omega\big)^{-1} V\big[\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\big]^*} \in \mathcal{B}_p\big(L^2(\partial\Omega; d^{n-1}\sigma)\big) \subset \mathcal{B}_k\big(L^2(\partial\Omega; d^{n-1}\sigma)\big) \tag{4.16}$$

4.2

and

$$\begin{aligned} &\frac{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} v}\Big)}{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} v}\Big)} \\ &\quad = \det_k\Big(I_{\partial\Omega} - \overline{\gamma_N\big(H_\Omega^D - zI_\Omega\big)^{-1} V\big[\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\big]^*}\Big) \exp\big(\mathrm{tr}(T_k(z))\big). \end{aligned} \tag{4.17}$$

4.3

Here $T_k(z) \in \mathcal{B}_1\big(L^2(\partial\Omega; d^{n-1}\sigma)\big)$ *denotes one of the cyclic permutations of the polynomial* $T_k(\cdot,\cdot)$ *defined in Lemma* 14.1 *4.1 with the following choice of* $A = A_0(z)$ *and* $B = B_0(z)$, *with* $A_0(z)$ *and* $B_0(z)$ *given by*

$$\begin{aligned} A_0(z) &= \Big[\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{\widetilde{u}}}\Big]^* \overline{\gamma_N\big(H_\Omega^D - zI_\Omega\big)^{-1}\widetilde{v}} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big) \subset \mathcal{B}_k\big(L^2(\Omega; d^n x)\big), \\ B_0(z) &= -\overline{\widetilde{u}\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\widetilde{v}} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big) \subset \mathcal{B}_k\big(L^2(\Omega; d^n x)\big), \end{aligned} \tag{4.18}$$

4.4

and the functions u, v, $\widetilde{u}$, *and* $\widetilde{v}$ *are given by*

$$u(x) = \exp(i\arg(V(x)))|V(x)|^{1/2}, \quad v(x) = |V(x)|^{1/2}, \tag{4.19}$$

$$\widetilde{u}(x) = \exp(i\arg(V(x)))|V(x)|^{p/p_1}, \quad \widetilde{v}(x) = |V(x)|^{p/p_2}, \tag{4.20}$$

with

$$p_1 = \begin{cases} 3p/2, & n = 2, \\ 4p/3, & n \geq 3, \end{cases} \qquad p_2 = \begin{cases} 3p, & n = 2, \\ 4p, & n \geq 3, \end{cases} \tag{4.21}$$

4.6

and $V = uv = \tilde{u}\tilde{v}$. *In particular,*

$$T_2(z) = \overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} V\big(H_\Omega^D - zI_\Omega\big)^{-1} V\Big[\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\Big]^*} \in \mathcal{B}_1\big(L^2(\partial\Omega; d^{n-1}\sigma)\big). \tag{4.22}$$

4.5

Proof. From the outset we note that the left-hand side of (4.17) is well-defined by (2.32). Let $z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$ and note that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ for all $n \geq 2$, and hence $V = uv = \widetilde{u}\widetilde{v}$.

Next, we introduce

$$K_D(z) = -\overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} v}, \quad K_N(z) = -\overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} v} \tag{4.23}$$

4.7

(cf. (B.4)) and note that by Theorem tB.3 B.3

$$[I_\Omega - K_D(z)]^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \quad z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big)\big). \tag{4.24}$$

Then Lemma 14.1 4.1 with $A = \widetilde{A}_0(z)$ and $B = \widetilde{B}_0(z)$ defined by

$$\widetilde{A}_0(z) = I_\Omega - (I_\Omega - K_N(z))[I_\Omega - K_D(z)]^{-1} = (K_N(z) - K_D(z))[I_\Omega - K_D(z)]^{-1}, \tag{4.25}$$

4.10

$$\widetilde{B}_0(z) = K_D(z) = -\overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} v}, \tag{4.26}$$

4.10A

yields

$$\begin{aligned} &\frac{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} v}\Big)}{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} v}\Big)} = \frac{\det_k\big(I_\Omega - K_N(z)\big)}{\det_k\big(I_\Omega - K_D(z)\big)} \\ &\quad = \det_k\big(I_\Omega - (K_N(z) - K_D(z))[I_\Omega - K_D(z)]^{-1}\big) \exp\big(\mathrm{tr}(T_k(\widetilde{A}_0(z), \widetilde{B}_0(z)))\big), \end{aligned} \tag{4.27}$$

4.12

where $T_k(\cdot,\cdot)$ is the polynomial defined in (4.4a). Explicit formulas for the first few T_k are computed in (4.6a).

Next, temporarily suppose that $V \in L^p(\Omega; d^n x) \cap L^\infty(\Omega; d^n x)$. Using Lemma A.3 (an extension of a result of Nakamura [Na01, Lemma 6]) and Remark A.5 (cf. (A.29)), one finds

$$\begin{aligned} K_N(z) - K_D(z) &= \overline{u\big[\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} - \big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\big]v} \\ &= \overline{u\big[\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\big]^*}\,\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v} \\ &= \Big[\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{u}}\Big]^*\,\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}. \end{aligned} \tag{4.28}$$

Inserting (4.28) into (4.25) and utilizing (4.23) and the following resolvent identity which follows from (B.5),

$$\overline{\big(H_\Omega^D - zI_\Omega\big)^{-1}v} = \overline{\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big[I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big]^{-1}, \tag{4.29}$$

one obtains the following equality for $\widetilde{A}_0(z)$,

$$\widetilde{A}_0(z) = \Big[\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{u}}\Big]^*\,\overline{\gamma_N\big(H_\Omega^D - zI_\Omega\big)^{-1}v}. \tag{4.30}$$

Moreover, insertion of (4.28) into (4.27) yields

$$\begin{aligned} &\frac{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}v}\Big)}{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big)} \\ &\quad = \det_k\Big(I_\Omega - \Big[\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{u}}\Big]^*\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big[I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big]^{-1}\Big) \\ &\qquad \times \exp\big(\mathrm{tr}(T_k(\widetilde{A}_0(z), \widetilde{B}_0(z)))\big). \end{aligned} \tag{4.31}$$

Utilizing Corollary 2.5 with p_1 and p_2 as in (4.21), one finds

$$\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{u}} \in \mathcal{B}_{p_1}\big(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1}\sigma)\big), \tag{4.32}$$

$$\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v} \in \mathcal{B}_{p_2}\big(L^2(\Omega; d^n x), L^2(\partial\Omega; d^{n-1}\sigma)\big), \tag{4.33}$$

and hence,

$$\Big[\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{u}}\Big]^*\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big) \subset \mathcal{B}_k\big(L^2(\Omega; d^n x)\big), \tag{4.34}$$

$$\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big[\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{u}}\Big]^* \in \mathcal{B}_p\big(L^2(\partial\Omega; d^{n-1}\sigma)\big) \subset \mathcal{B}_k\big(L^2(\partial\Omega; d^{n-1}\sigma)\big). \tag{4.35}$$

Then, using the fact that

$$\Big[I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big]^{-1} \in \mathcal{B}\big(L^2(\Omega; d^n x)\big), \quad z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big)\big), \tag{4.36}$$

one applies the idea expressed in formula (4.1) and rearranges the terms in (4.31) as follows:

$$\begin{aligned} &\frac{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}v}\Big)}{\det_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big)} \\ &\quad = \det_k\Big(I_{\partial\Omega} - \overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big[I_\Omega + \overline{u\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}v}\Big]^{-1}\Big[\overline{\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\overline{u}}\Big]^*\Big) \\ &\qquad \times \exp\big(\mathrm{tr}(T_k(\widetilde{A}_0, \widetilde{B}_0))\big). \end{aligned} \tag{4.37}$$

Similarly, using the cyclicity property of $\mathrm{tr}(\cdot)$, one rearranges $T_k\big(\widetilde{A}_0(z),\widetilde{B}_0(z)\big)$ to get an operator on $L^2(\partial\Omega;d^{n-1}\sigma)$ which in the following we denote by $T_k(z)$. This is always possible since each term of $T_k\big(\widetilde{A}_0(z),\widetilde{B}_0(z)\big)$ has at least one factor of $\widetilde{A}_0(z)$. Then using equalities (4.4), (4.10A), (4.4A), and $uv=\tilde{u}\tilde{v}$, one concludes that $T_k(z)$ is a cyclic permutation of $T_k(A_0,B_0)$ with $A_0(z)$ and $B_0(z)$ given by (4.4). In particular, rearranging $T_2\big(\widetilde{A}_0(z),\widetilde{B}_0(z)\big)=-\widetilde{A}_0(z)\widetilde{B}_0(z)$ or equivalently $T_2(A_0(z),B_0(z))=-A_0(z)B_0(z)$, one obtains $T_2(z)=-\widetilde{B}_0(z)\widetilde{A}_0(z)=-B_0(z)A_0(z)$, and hence equality (4.8). Thus, (4.3), subject to the extra assumption $V\in L^p(\Omega;d^nx)\cap L^\infty(\Omega;d^nx)$, follows from (4.22) and (4.13a). (4.20) (4.29) (4.37)

Finally, assuming only $V\in L^p(\Omega;d^nx)$ and utilizing Theorem B.3, Lemma 2.3, and Corollary 2.5 once again, one obtains

$$\Big[I_\Omega+\overline{\widetilde{u}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}}\Big]^{-1}\in\mathcal{B}\big(L^2(\Omega;d^nx)\big), \tag{4.38}$$

$$\widetilde{u}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-p/p_1}\in\mathcal{B}_{p_1}\big(L^2(\Omega;d^nx)\big), \tag{4.39}$$

$$\widetilde{v}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-p/p_2}\in\mathcal{B}_{p_2}\big(L^2(\Omega;d^nx)\big), \tag{4.40}$$

$$\overline{\gamma_D\big(H^N_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{u}}\in\mathcal{B}_{p_1}\big(L^2(\Omega;d^nx),L^2(\partial\Omega;d^{n-1}\sigma)\big), \tag{4.41}$$

$$\overline{\gamma_N\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}}\in\mathcal{B}_{p_2}\big(L^2(\Omega;d^nx),L^2(\partial\Omega;d^{n-1}\sigma)\big), \tag{4.42}$$

and thus,

$$\overline{\widetilde{u}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}}\in\mathcal{B}_p\big(L^2(\Omega;d^nx)\big)\subset\mathcal{B}_k\big(L^2(\Omega;d^nx)\big). \tag{4.43}$$

Relations (4.24)–(4.43) together with the following resolvent identity that follows from (B.5),

$$\overline{\big(H^D_\Omega-zI_\Omega\big)^{-1}\widetilde{v}}=\overline{\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}}\Big[I_\Omega+\overline{\widetilde{u}\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}\widetilde{v}}\Big]^{-1}, \tag{4.44}$$

prove the $\mathcal{B}_k$-property (4.16), (4.18), and (4.22), and hence the left- and the right-hand sides of (4.17) are well-defined for $V\in L^p(\Omega;d^nx)$. Thus, using (2.9), (2.26), (2.27), the continuity of $\det{}_k(\cdot)$ with respect to the $\mathcal{B}_k$-norm $\|\cdot\|_{\mathcal{B}_k(L^2(\Omega;d^nx))}$, the continuity of $\mathrm{tr}(\cdot)$ with respect to the trace norm $\|\cdot\|_{\mathcal{B}_1(L^2(\Omega;d^nx))}$, and an approximation of $V\in L^p(\Omega;d^nx)$ by a sequence of potentials $V_j\in L^p(\Omega;d^nx)\cap L^\infty(\Omega;d^nx)$, $j\in\mathbb{N}$, in the norm of $L^p(\Omega;d^nx)$ as $j\uparrow\infty$, then extends the result from $V\in L^p(\Omega;d^nx)\cap L^\infty(\Omega;d^nx)$ to $V\in L^p(\Omega;d^nx)$. □

Given these preparations, we are ready for the principal result of this paper, the multi-dimensional analog of Theorem 1.2:

t4.2 **Theorem 4.3.** *Assume Hypothesis 2.6, let $k\in\mathbb{N}$, $k\geq p$, and $z\in\mathbb{C}\backslash\big(\sigma\big(H^D_\Omega\big)\cup\sigma\big(H^D_{0,\Omega}\big)\cup\sigma\big(H^N_{0,\Omega}\big)\big)$. Then,*

$$M^D_\Omega(z)M^D_{0,\Omega}(z)^{-1}-I_{\partial\Omega}=-\overline{\gamma_N\big(H^D_\Omega-zI_\Omega\big)^{-1}V\Big[\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\Big]^*}\in\mathcal{B}_k\big(L^2(\partial\Omega;d^{n-1}\sigma)\big) \tag{4.45}$$

and

$$\frac{\det{}_k\Big(I_\Omega+\overline{u\big(H^N_{0,\Omega}-zI_\Omega\big)^{-1}v}\Big)}{\det{}_k\Big(I_\Omega+\overline{u\big(H^D_{0,\Omega}-zI_\Omega\big)^{-1}v}\Big)}$$

$$=\det{}_k\Big(I_{\partial\Omega}-\overline{\gamma_N\big(H^D_\Omega-zI_\Omega\big)^{-1}V\big[\gamma_D\big(H^N_{0,\Omega}-\overline{z}I_\Omega\big)^{-1}\big]^*}\Big)\exp\big(\mathrm{tr}(T_k(z))\big) \tag{4.46}$$

$$=\det{}_k\big(M^D_\Omega(z)M^D_{0,\Omega}(z)^{-1}\big)\exp\big(\mathrm{tr}(T_k(z))\big) \tag{4.47}$$

with $T_k(z)$ defined in Theorem 4.1.

Proof. The result follows from combining Lemma 3.6 and Theorem 4.2. □

r4.4 **Remark 4.4.** Assume Hypothesis 2.6, let $k \in \mathbb{N}$, $k \geq p$, and $z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^N\big) \cup \sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$. Then,

$$M_{0,\Omega}^N(z)^{-1} M_\Omega^N(z) - I_{\partial\Omega} = \overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} V \Big[\gamma_D\Big(\big(H_\Omega^N - zI_\Omega\big)^{-1}\Big)^*\Big]^*} \in \mathcal{B}_k\big(L^2(\partial\Omega; d^{n-1}\sigma)\big) \tag{4.48}$$

4.32

and one can also prove the following analog of (4.46):

$$\begin{aligned}
&\frac{\det_k\Big(I_\Omega + \overline{u(H_{0,\Omega}^D - zI_\Omega)^{-1} v}\Big)}{\det_k\Big(I_\Omega + \overline{u(H_{0,\Omega}^N - zI_\Omega)^{-1} v}\Big)} \\
&\quad = \det_k\Big(I_{\partial\Omega} + \overline{\gamma_N (H_{0,\Omega}^D - zI_\Omega)^{-1} V \big[\gamma_D((H_\Omega^N - zI_\Omega)^{-1})^*\big]^*}\Big) \exp\big(\mathrm{tr}(T_k(z))\big), \tag{4.49} \\
&\quad = \det_k\big(M_{0,\Omega}^N(z)^{-1} M_\Omega^N(z)\big) \exp\big(\mathrm{tr}(T_k(z))\big) \tag{4.50}
\end{aligned}$$

4.33 4.34

where $T_k(z)$ denotes one of the cyclic permutations of the polynomial $T_k(A, B)$ defined in Lemma 4.1 with the following choice of $A = A_1(z)$ and $B = B_1(z)$,

$$\begin{aligned}
A_1(z) &= -\Big[\overline{\gamma_D\big(H_\Omega^N - \overline{z}I_\Omega\big)^{-1}\widetilde{\overline{u}}}\Big]^* \overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1}\widetilde{v}} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big) \subset \mathcal{B}_k\big(L^2(\Omega; d^n x)\big), \\
B_1(z) &= -\overline{\widetilde{u}\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1}\widetilde{v}} \in \mathcal{B}_p\big(L^2(\Omega; d^n x)\big) \subset \mathcal{B}_k\big(L^2(\Omega; d^n x)\big),
\end{aligned}$$

and the functions u, v, $\widetilde{u}$, and $\widetilde{v}$ are given by

$$u(x) = \exp(i \arg(V(x)))|V(x)|^{1/2}, \quad v(x) = |V(x)|^{1/2}, \tag{4.51}$$

$$\widetilde{u}(x) = \exp(i \arg(V(x)))|V(x)|^{p/p_1}, \quad \widetilde{v}(x) = |V(x)|^{p/p_2}, \tag{4.52}$$

with

$$p_1 = \begin{cases} 3p/2, & n = 2, \\ 4p/3, & n \geq 3, \end{cases} \qquad p_2 = \begin{cases} 3p, & n = 2, \\ 4p, & n \geq 3, \end{cases} \tag{4.53}$$

and $V = uv = \tilde{u}\tilde{v}$. In particular,

$$T_2(z) = -\overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} V \big(H_\Omega^N - zI_\Omega\big)^{-1} V \Big[\gamma_D\big(H_{0,\Omega}^N - \overline{z}I_\Omega\big)^{-1}\Big]^*}. \tag{4.54}$$

r4.5 **Remark 4.5.** It seems tempting at this point to turn to an abstract version of Theorem 4.3 using the notion of boundary value spaces (see, e.g., [5], [21], [22], [39, Ch. 3] and the references therein). However, the analogs of the necessary mapping and trace ideal properties as recorded in Sections 2 and 3 do not seem to be available at the present time for general self-adjoint extensions of a densely defined, closed symmetric operator (respectively, maximal accretive extensions of closed accretive operators) in a separable complex Hilbert space. For this reason we decided to start with the special, but important case of multi-dimensional Schrödinger operators.

A few comments are in order at this point:

The sudden appearance of the exponential term $\exp(\mathrm{tr}(T_k(z)))$ in (4.46), (4.47), and (4.48), when compared to the one-dimensional case, is due to the necessary use of the modified determinant $\det_k(\cdot)$, $k \geq 2$, in Theorems 4.2 and 4.3.

As mentioned in the introduction, the multi-dimensional extension (4.46) of (1.16), under the stronger hypothesis $V \in L^2(\Omega; d^n x)$, $n = 2, 3$, first appeared in [32]. However, the present results in Theorem 4.2 go decidedly beyond those in [32] in the following sense:

(i) the class of domains Ω permitted by Hypothesis 2.1 is substantiallly expanded as compared to

[32].
(*ii*) For $n = 2, 3$, the conditions on V satisfying Hypothesis 2.6 are now nearly optimal by comparison with the Sobolev inequality (cf. Chenev [18], Reed and Simon [78, Sect. IX.4], Simon [84, Sect. I.1]).
(*iii*) The multi-dimensional extension (4.47) of (1.17) invoking Dirichlet-to-Neumann maps is a new (and the most significant) result in this paper.
(*iv*) While the results in [32] were confined to dimensions $n = 2, 3$, all results in this paper are now derived in the general case $n \in \mathbb{N}$, $n \geq 2$.

The principal reduction in Theorem 4.3 reduces (a ratio of) modified Fredholm determinants associated with operators in $L^2(\Omega; d^n x)$ on the left-hand side of (4.46) to modified Fredholm determinants associated with operators in $L^2(\partial\Omega; d^{n-1}\sigma)$ on the right-hand side of (4.46) and especially, in (4.47). This is the analog of the reduction described in the one-dimensional context of Theorem 1.2, where Ω corresponds to the half-line $(0, \infty)$ and its boundary $\partial\Omega$ thus corresponds to the one-point set $\{0\}$.

In the context of elliptic operators on smooth k-dimensional manifolds, the idea of reducing a ratio of zeta-function regularized determinants to a calculation over the $k-1$-dimensional boundary has been studied by Forman [27]. He also pointed out that if the manifold consists of an interval, the special case of a pair of boundary points then permits one to reduce the zeta-function regularized determinant to the determinant of a finite-dimensional matrix. The latter case is of course an analog of the one-dimensional Jost and Pais formula mentioned in the introduction (cf. Theorems 1.1 and 1.2). Since then, this topic has been further developed in various directions and we refer, for instance, to Burghelea, Friedlander, and Kappeler [12], [13], [14], [15], Carron [16], Friedlander [28], Guillarmou and Guillopé [42], Müller [64], Okikiolu [70], [71], Park and Wojciechowski [74], [75], and the references therein.

Combining Theorems 4.3 and B.3 yields the following applications of (4.46) and (4.49):

Theorem 4.6. *Assume Hypothesis 2.6 and $k \in \mathbb{N}$, $k \geq p$.*
(*i*) *One infers that*

$$\begin{aligned} &\text{for all } z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big), \text{ one has } z \in \sigma\big(H_\Omega^N\big) \\ &\quad \text{if and only if } \det{}_k\Big(I_{\partial\Omega} - \overline{\gamma_N\big(H_\Omega^D - zI_\Omega\big)^{-1} V \big[\gamma_D\big(H_{0,\Omega}^N - \overline{z} I_\Omega\big)^{-1}\big]^*}\Big) = 0. \end{aligned} \tag{4.55}$$

(*ii*) *Similarly, one infers that*

$$\begin{aligned} &\text{for all } z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^N\big) \cup \sigma\big(H_{0,\Omega}^N\big) \cup \sigma\big(H_{0,\Omega}^D\big)\big), \text{ one has } z \in \sigma\big(H_\Omega^D\big) \\ &\quad \text{if and only if } \det{}_k\Big(I_{\partial\Omega} + \overline{\gamma_N\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1} V \big[\gamma_D\big(\big(H_\Omega^N - zI_\Omega\big)^{-1}\big)^*\big]^*}\Big) = 0. \end{aligned} \tag{4.56}$$

Proof. By the Birman–Schwinger principle, as discussed in Theorem B.3, for any $k \in \mathbb{N}$ such that $k \geq p$ and $z \in \mathbb{C}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$, one has

$$z \in \sigma\big(H_\Omega^N\big) \text{ if and only if } \det{}_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1} v}\Big) = 0. \tag{4.57}$$

Thus, (4.55) follows from (4.46). In the same manner, (4.56) follows from (4.49). □

We conclude with another application to eigenvalue counting functions in the case where H_Ω^D and H_Ω^N are self-adjoint and have purely discrete spectra (i.e., empty essential spectra). To set the stage we introduce the following assumptions:

Hypothesis 4.7. *In addition to assuming Hypothesis 2.6 suppose that V is real-valued and that H_Ω^D and H_Ω^N have purely discrete spectra.*

r4.8 **Remark 4.8.**
(*i*) Real-valuedness of V implies self-adjointness of H_Ω^D and H_Ω^N as noted in (B.11).
(*ii*) Since $\partial\Omega$ is assumed to be compact, purely discrete spectra of $H_{0,\Omega}^D$ and $H_{0,\Omega}^N$, that is, compactness of their resolvents (cf. [RS78, Sect. XIII.14]), is equivalent to Ω being bounded. Indeed, if Ω had an unbounded component, then one can construct Weyl sequences which would yield nonempty essential spectra of $H_{0,\Omega}^D$ and $H_{0,\Omega}^N$. On the other hand, $H_{0,\Omega}^D$ has empty essential spectrum for any bounded open set $\Omega \subset \mathbb{R}^n$ as discussed in the Corollary to [RS78, Theorem XIII.73]. Similarly, $H_{0,\Omega}^N$ has empty essential spectrum for any bounded open set Ω satisfying the segment property as discussed in Corollary 1 to [RS78, Theorem XIII.75]. Since any bounded Lipschitz domain satisfies the segment property (cf. [Gr85, Sect, 1.2.2]), any bounded domain Ω satisfying Hypothesis 2.1 yields a purely discrete spectrum of $H_{0,\Omega}^N$.
(*iii*) We recall that V is relatively form compact with respect to $H_{0,\Omega}^D$ and $H_{0,\Omega}^N$, that is,

$$v\big(H_{0,\Omega}^D - zI_\Omega\big)^{-1/2},\, v\big(H_{0,\Omega}^N - zI_\Omega\big)^{-1/2} \in \mathcal{B}_\infty\big(L^2(\Omega; d^n x)\big) \tag{4.58}$$

for all z in the resolvent sets of $H_{0,\Omega}^D$, respectively, $H_{0,\Omega}^N$ (in fact, much more is true as recorded in (2.28) and (2.29) since $\mathcal{B}_\infty$ can be replaced by $\mathcal{B}_{2p}$). By (3.47a) and (3.48a) this yields that the difference of the resolvents of H_Ω^D and H_Ω^N is compact (in fact, it even lies in $\mathcal{B}_p\big(L^2(\Omega; d^n x)\big)$). By a variant of Weyl's theorem (cf., e.g., [RS78, Theorem XIII.14]), one concludes that H_Ω^D and H_Ω^N have empty essential spectrum if and only if $H_{0,\Omega}^D$ and $H_{0,\Omega}^N$ have (cf. [RS78, Problem 39, p. 369]). Thus, by part (*ii*) of this remark, the assumption that H_Ω^D and H_Ω^N have purely discrete spectra in Hypothesis 4.7 can equivalently be replaced by the assumption that Ω is bounded (still assuming Hypothesis 2.6 and that V is real-valued).

Assuming Hypothesis 4.7, $k \in \mathbb{N}$, $k \geq p$, we introduce (cf. also [Ya07])

$$\xi_k(\lambda) = \begin{cases} \pi^{-1}\mathrm{Im}\Big(\ln\Big(\det{}_k\Big(I_\Omega + \overline{u(H_{0,\Omega} - \lambda I_\Omega)^{-1}v}\Big)\Big)\Big), & \lambda \in (e_0, \infty)\backslash(\sigma(H_\Omega) \cup \sigma(H_{0,\Omega})), \\ 0, & \lambda < e_0, \end{cases} \tag{4.59}$$ 4.57

where

$$e_0 = \inf(\sigma(H_\Omega), \sigma(H_{0,\Omega})), \tag{4.60}$$

and H_Ω and $H_{0,\Omega}$ temporarily abbreviate H_Ω^D and $H_{0,\Omega}^D$ in the case of Dirichlet boundary conditions on $\partial\Omega$ and H_Ω^N and $H_{0,\Omega}^N$ in the case of Neumann boundary conditions on $\partial\Omega$. Moreover, we subsequently agree to write $\xi_k^D(\cdot)$ and $\xi_k^N(\cdot)$ for $\xi(\cdot)$ in the case of Dirichlet and Neumann boundary conditions in $H_\Omega, H_{0,\Omega}$.

The branch of the logarithm in (4.59) has been fixed by putting $\xi_k(\lambda) = 0$ for λ in a neighborhood of $-\infty$. This is possible since

$$\lim_{\lambda \downarrow -\infty} \det{}_k\Big(I_\Omega + \overline{u(H_{0,\Omega} - \lambda I_\Omega)^{-1}v}\Big) = 1. \tag{4.61}$$ 4.58

Equation (4.61) in turn follows from Lemma 2.3 since

$$\lim_{\lambda \downarrow -\infty} \Big\|\overline{u(H_{0,\Omega} - \lambda I_\Omega)^{-1}v}\Big\|_{\mathcal{B}_k(L^2(\Omega; d^n x))} = 0 \tag{4.62}$$

by applying the dominated convergence theorem to $\|(|\cdot|^2 - \lambda)^{-1/2}\|_{L^{2p}(\mathbb{R}^n; d^n x)}^2$ as $\lambda \downarrow -\infty$ in (2.9) (replacing p by $2p$, q by $1/2$, f by u and v, etc.). Since $H_{0,\Omega}$ is self-adjoint in $L^2(\Omega; d^n x)$ with purely discrete spectrum, for any $\lambda_0 \in \mathbb{R}$, we obtain the norm convergent expansion

$$(H_{0,\Omega} - zI_\Omega)^{-1} \underset{z \to \lambda_0}{=} P_{0,\Omega,\lambda_0}(\lambda_0 - z)^{-1} + \sum_{k=0}^{\infty} (-1)^k S_{0,\Omega,\lambda_0}^{k+1}(\lambda_0 - z)^k, \tag{4.63}$$ 4.63

where P_{0,Ω,λ_0} denotes the Riesz projection associated with $H_{0,\Omega}$ and the point λ_0, and S_{0,Ω,λ_0} is given by

$$S_{0,\Omega,\lambda_0} = \lim_{z\to\lambda_0} (H_{0,\Omega} - zI_\Omega)^{-1}(I_\Omega - P_{0,\Omega,\lambda_0}), \tag{4.64}$$

with the limit taken in the topology of $\mathcal{B}(L^2(\Omega; d^n x))$. Hence, $S_{0,\Omega,\lambda_0} P_{0,\Omega,\lambda_0} = P_{0,\Omega,\lambda_0} S_{0,\Omega,\lambda_0} = 0$. If, in fact, λ_0 is a (necessarily discrete) eigenvalue of $H_{0,\Omega}$, then P_{0,Ω,λ_0} is the projection onto the corresponding eigenspace of $H_{0,\Omega}$ and the dimension of its range equals the multiplicity of the eigenvalue λ_0, denoted by

$$n_{0,\lambda_0} = \dim(\operatorname{ran}(P_{0,\Omega,\lambda_0})). \tag{4.65}$$

We recall that all eigenvalues of $H_{0,\Omega}$ are semisimple, that is, their geometric and algebraic multiplicities coincide, since $H_{0,\Omega}$ is assumed to be self-adjoint. If λ_0 is not in the spectrum of $H_{0,\Omega}$ then, of course, $P_{0,\Omega,\lambda_0} = 0$ and $n_{0,\lambda_0} = 0$. In exactly, the same manner, and in obvious notation, one then also obtains

$$(H_\Omega - zI_\Omega)^{-1} \underset{z\to\lambda_0}{=} P_{\Omega,\lambda_0}(\lambda_0 - z)^{-1} + \sum_{k=0}^{\infty} (-1)^k S_{\Omega,\lambda_0}^{k+1} (\lambda_0 - z)^k \tag{4.66}$$

4.66

and

$$n_{\lambda_0} = \dim(\operatorname{ran}(P_{\Omega,\lambda_0})). \tag{4.67}$$

In the following we denote half-sided limits by

$$f(x_+) = \lim_{\varepsilon\downarrow 0} f(x+\varepsilon), \quad f(x_-) = \lim_{\varepsilon\uparrow 0} f(x-\varepsilon), \quad x\in\mathbb{R}. \tag{4.68}$$

Moreover, we denote by $N_{H_\Omega}(\lambda)$ (respectively, $N_{H_{0,\Omega}}(\lambda)$), $\lambda\in\mathbb{R}$, the right-continuous function on $\mathbb{R}$ which counts the number of eigenvalues of H_Ω (respectively, $H_{0,\Omega}$) less than or equal to λ, counting multiplicities.

14.8 **Lemma 4.9.** *Assume Hypothesis* h4.7 *and let $k\in\mathbb{N}$, $k\geq p$. Then ξ_k equals a fixed integer on any open interval in $\mathbb{R}\backslash(\sigma(H_\Omega)\cup\sigma(H_{0,\Omega}))$. Moreover, for any $\lambda\in\mathbb{R}$,*

$$\xi_k(\lambda_+) - \xi_k(\lambda_-) = -(n_\lambda - n_{0,\lambda}), \tag{4.69}$$

4.69

and hence ξ_k is piecewise integer-valued on $\mathbb{R}$ and normalized to vanish on $(-\infty, e_0)$ such that

$$\xi_k(\lambda) = -[N_{H_\Omega}(\lambda) - N_{H_{0,\Omega}}(\lambda)], \quad \lambda\in\mathbb{R}\backslash(\sigma(H_\Omega)\cup\sigma(H_{0,\Omega})). \tag{4.70}$$

4.70

Proof. Introducing the unitary operator S in $L^2(\Omega; d^n x)$ of multiplication by the function $\operatorname{sgn}(V)$,

$$(Sf)(x) = \operatorname{sgn}(V(x)) f(x), \quad f\in L^2(\Omega; d^n x) \tag{4.71}$$

4.71

such that $Su = uS = v$, $Sv = vS = u$, $S^2 = I_\Omega$, one computes for $\lambda\in\mathbb{R}\backslash\sigma(H_{0,\Omega})$,

$$\begin{aligned}
\overline{\det{}_k\Big(I_\Omega + \overline{u(H_{0,\Omega} - \lambda I_\Omega)^{-1} v}\Big)} &= \det{}_k\Big(I_\Omega + \overline{v(H_{0,\Omega} - \lambda I_\Omega)^{-1} u}\Big) \\
&= \det{}_k\Big(I_\Omega + \overline{Su(H_{0,\Omega} - \lambda I_\Omega)^{-1} vS}\Big) \\
&= \det{}_k\Big(I_\Omega + \overline{u(H_{0,\Omega} - \lambda I_\Omega)^{-1} v}\Big),
\end{aligned} \tag{4.72}$$

4.72

that is, $\det_k\Big(I_\Omega + \overline{u(H_{0,\Omega} - \lambda I_\Omega)^{-1} v}\Big)$ is real-valued for $\lambda\in\mathbb{R}\backslash\sigma(H_{0,\Omega})$. (Here the bars either denote complex conjugation, or the operator closure, depending on the context in which they are used.) Together with the Birman–Schwinger principle as expressed in Theorem B.3, this proves that ξ_k equals a fixed integer on any open interval in $\mathbb{R}\backslash(\sigma(H_\Omega)\cup\sigma(H_{0,\Omega}))$.

Next, we note that for $z \in \mathbb{C}\backslash(\sigma(H_\Omega) \cup \sigma(H_{0,\Omega}))$,

$$-\frac{d}{dz}\ln\Big(\det{}_k\Big(I_\Omega + \overline{u\big(H_{0,\Omega} - zI_\Omega\big)^{-1}v}\Big)\Big) = \operatorname{tr}\bigg((H_\Omega - zI_\Omega)^{-1} - (H_{0,\Omega} - zI_\Omega)^{-1} - \sum_{\ell=1}^{k-1}(-1)^\ell \overline{(H_{0,\Omega} - zI_\Omega)^{-1}v}\Big[\overline{u(H_{0,\Omega} - zI_\Omega)^{-1}v}\Big]^{\ell-1} u(H_{0,\Omega} - zI_\Omega)^{-1}\bigg), \tag{4.73}$$

which represents just a slight extension of the result recorded in [Ya07, 102]. Insertion of (4.63) and (4.66) into (4.73) then yields that for any $\lambda_0 \in \mathbb{R}$,

$$-\frac{d}{dz}\ln\Big(\det{}_k\Big(I_\Omega + \overline{u(H_{0,\Omega} - zI_\Omega)^{-1}v}\Big)\Big) \underset{z\to\lambda_0}{=} \operatorname{tr}(P_{\Omega,\lambda_0} - P_{0,\Omega,\lambda_0})(\lambda_0 - z)^{-1} + \sum_{\ell=-k}^{\infty} c_\ell(\lambda_0 - z)^\ell$$

$$\underset{z\to\lambda_0}{=} [n_{\lambda_0} - n_{0,\lambda_0}](\lambda_0 - z)^{-1} + \sum_{\ell=-k}^{\infty} c_\ell(\lambda_0 - z)^\ell, \tag{4.74}$$

where

$$c_\ell \in \mathbb{R}, \ \ \ell \in \mathbb{Z},\, \ell \geq k, \ \text{and} \ \ c_{-1} = 0. \tag{4.75}$$

That $c_\ell \in \mathbb{R}$ is clear from the real-valuedness of V and the self-adjointness of H_Ω and $H_{0,\Omega}$ by expanding the $(\ell-1)$th power of $\overline{u(H_{0,\Omega} - zI_\Omega)^{-1}v}$ in (4.73). To demonstrate that c_{-1} actually vanishes, that is, that the term proportional to $(\lambda_0 - z)^{-1}$ cancels in the sum $\sum_{\ell=-k}^{\infty} c_\ell(\lambda_0 - z)^\ell$ in (4.74), we temporarily introduce $u_m = P_m u$, $v_m = vP_m$, where $\{P_m\}_{m\in\mathbb{N}}$ is a family of orthogonal projections in $L^2(\Omega; d^n x)$ satisfying

$$P_m^2 = P_m = P_m^*, \quad \dim(\operatorname{ran}(P_m)) = m, \quad \operatorname{ran}(P_m) \subset \operatorname{dom}(v), \ \ m \in \mathbb{N}, \quad \operatorname*{s-lim}_{m\uparrow\infty} P_m = I_\Omega, \tag{4.76}$$

where s-lim denotes the limit in the strong operator topology. (E.g., it suffices to choose P_m as appropriate spectral projections associated with $H_{0,\Omega}$.) In addition, we introduce $V_m = v_m u_m$ and the operator $H_{\Omega,m}$ in $L^2(\Omega; d^n x)$ by replacing V by V_m in H_Ω. Since

$$V_m = (vP_m)P_m(uP_m)^*, \tag{4.77}$$

one obtains that V_m is a trace class (in fact, finite rank) operator, that is,

$$V_m \in \mathcal{B}_1\big(L^2(\Omega; d^n x)\big), \quad m \in \mathbb{N}. \tag{4.78}$$

Moreover, since by (2.31) and (2.32),

$$u(H_{0,\Omega} - zI_\Omega)^{-1/2}, \overline{(H_{0,\Omega} - zI_\Omega)^{-1/2}v} \in \mathcal{B}_{2p}\big(L^2(\Omega; d^n x)\big), \quad z \in \mathbb{C}\backslash\sigma(H_{0,\Omega}), \tag{4.79}$$

one concludes that $\overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1}vP_m} = P_m\overline{u(H_{0,\Omega} - zI_\Omega)^{-1}v}P_m$, $m \in \mathbb{N}$, satisfies

$$\lim_{m\uparrow\infty} \big\|\overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1}vP_m} - u(H_{0,\Omega} - zI_\Omega)^{-1}v\big\|_{\mathcal{B}_p(L^2(\Omega;d^n x))} = 0, \quad z \in \mathbb{C}\backslash\sigma(H_{0,\Omega}), \tag{4.80}$$

$$\lim_{m\uparrow\infty} \big\|\overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-2}vP_m} - u(H_{0,\Omega} - zI_\Omega)^{-2}v\big\|_{\mathcal{B}_p(L^2(\Omega;d^n x))} = 0, \quad z \in \mathbb{C}\backslash\sigma(H_{0,\Omega}). \tag{4.81}$$

Applying the formula (cf. [Ya92, 101, p. 44])

$$\frac{d}{dz}\ln(\det{}_k(I_\mathcal{H} - A(z)) = -\operatorname{tr}\big((I_\mathcal{H} - A(z))^{-1}A(z)^{k-1}A'(z)\big), \quad z \in \mathcal{D}, \tag{4.82}$$

where $A(\cdot)$ is analytic in some open domain $\mathcal{D} \subseteq \mathbb{C}$ with respect to the $\mathcal{B}_k(\mathcal{H})$-norm, $\mathcal{H}$ a separable complex Hilbert space, one obtains for $z \in \mathbb{C}\backslash(\sigma(H_\Omega) \cup \sigma(H_{0,\Omega}))$,

$$-\frac{d}{dz}\ln\Big(\det{}_k\Big(I_\Omega + \overline{u(H_{0,\Omega} - zI_\Omega)^{-1}v}\Big)\Big)$$

$$= (-1)^k \mathrm{tr}\Big(\Big[I_\Omega + \overline{u(H_{0,\Omega} - zI_\Omega)^{-1} v} \Big]^{-1} \Big[\overline{u(H_{0,\Omega} - zI_\Omega)^{-1} v} \Big]^{k-1} \overline{u(H_{0,\Omega} - zI_\Omega)^{-2} v} \Big), \quad (4.83)$$ 4.81

$$- \frac{d}{dz} \ln\Big(\det_k \Big(I_\Omega + \overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big) \Big)$$
$$= (-1)^k \mathrm{tr}\Big(\Big[I_\Omega + \overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big]^{-1} \Big[\overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big]^{k-1} \quad (4.84)$$ 4.82
$$\times \overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-2} v P_m} \Big), \quad m \in \mathbb{N}.$$

Combining equations (4.78), (4.79) and (4.81), (4.82) then yields

$$\lim_{m \uparrow \infty} \frac{d}{dz} \ln\Big(\det_k \Big(I_\Omega + \overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big) \Big) = \frac{d}{dz} \ln\Big(\det_k \Big(I_\Omega + \overline{u(H_{0,\Omega} - zI_\Omega)^{-1} v} \Big) \Big),$$
$$z \in \mathbb{C} \backslash (\sigma(H_\Omega) \cup \sigma(H_{0,\Omega})). \quad (4.85)$$ 4.83

Because of (4.85), to prove that $c_{-1} = 0$ in (4.74) (as claimed in (4.75)), it suffices to replace V in (4.74) by V_m and prove that $c_{m,-1} = 0$ for all $m \in \mathbb{N}$ in the following equation analogous to (4.74),

$$- \frac{d}{dz} \ln\Big(\det{}_k \Big(I_\Omega + P_m \overline{u(H_{0,\Omega} - zI_\Omega)^{-1} v} P_m \Big) \Big)$$
$$\underset{z \to \lambda_0}{=} \mathrm{tr}(P_{\Omega,m,\lambda_0} - P_{0,\Omega,\lambda_0})(\lambda_0 - z)^{-1} + \sum_{\ell=-k}^{\infty} c_{m,\ell} (\lambda_0 - z)^\ell, \quad m \in \mathbb{N}, \quad (4.86)$$ 4.84

where

$$c_{m,\ell} \in \mathbb{R}, \ \ \ell \in \mathbb{Z}, \, \ell \geq k, \ \ m \in \mathbb{N}, \quad (4.87)$$ 4.85

and P_{Ω,m,λ_0} denotes the corresponding Riesz projection associated with $H_{\Omega,m}$ (obtained by replacing V by V_m in H_Ω) and the point λ_0.

Applying the analog of formula (4.73) to $H_{\Omega,m}$ (cf. again [102]), and noting that P_m has rank $m \in \mathbb{N}$, one concludes that for $z \in \mathbb{C} \backslash (\sigma(H_\Omega) \cup \sigma(H_{0,\Omega}))$,

$$- \frac{d}{dz} \ln\Big(\det{}_k \Big(I_\Omega + P_m \overline{u(H_{0,\Omega} - zI_\Omega)^{-1} v} P_m \Big) \Big) = - \frac{d}{dz} \ln\Big(\det{}_k \Big(I_\Omega + \overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big) \Big)$$
$$= \mathrm{tr}\Big((H_{\Omega,m} - zI_\Omega)^{-1} - (H_{0,\Omega} - zI_\Omega)^{-1}$$
$$- \sum_{\ell=1}^{k-1} (-1)^\ell \overline{(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big[\overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big]^{\ell-1} P_m u(H_{0,\Omega} - zI_\Omega)^{-1} \Big)$$
$$= \mathrm{tr}\big((H_{\Omega,m} - zI_\Omega)^{-1} - (H_{0,\Omega} - zI_\Omega)^{-1} \big) - \sum_{\ell=1}^{k-1} \frac{(-1)^\ell}{\ell} \frac{d}{dz} \mathrm{tr}\Big(\Big[\overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big]^\ell \Big) \quad (4.88)$$ 4.86
$$= \mathrm{tr}\big((H_{\Omega,m} - zI_\Omega)^{-1} - (H_{0,\Omega} - zI_\Omega)^{-1} \big)$$
$$+ \sum_{\ell=1}^{k-1} (-1)^\ell \mathrm{tr}\Big(\Big[\overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big]^{\ell-1} \overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-2} v P_m} \Big), \quad m \in \mathbb{N}.$$

Here we have used the fact that by (4.78),

$$- \frac{d}{dz} \ln\Big(\det \Big(I_\Omega + \overline{P_m u(H_{0,\Omega} - zI_\Omega)^{-1} v P_m} \Big) \Big) = \mathrm{tr}\big((H_{\Omega,m} - zI_\Omega)^{-1} - (H_{0,\Omega} - zI_\Omega)^{-1} \big), \quad (4.89)$$

for $z \in \mathbb{C}\backslash(\sigma(H_\Omega) \cup \sigma(H_{0,\Omega}))$, and that (cf. [Si05, Theorem 9.2])

$$\begin{aligned} \frac{d}{dz}\ln(\det_k(I_{\mathcal{H}} - B(z))) &= \frac{d}{dz}\ln(\det(I_{\mathcal{H}} - B(z))) + \sum_{\ell=1}^{k-1} \frac{1}{\ell}\frac{d}{dz}\mathrm{tr}\big(B(z)^\ell\big) \\ &= \frac{d}{dz}\ln(\det(I_{\mathcal{H}} - B(z))) + \sum_{\ell=1}^{k-1} \mathrm{tr}\big(B(z)^{\ell-1}B'(z)\big), \quad z \in \mathcal{D}, \end{aligned} \tag{4.90}$$

where $B(\cdot)$ is analytic in some open domain $\mathcal{D} \subseteq \mathbb{C}$ with respect to the $\mathcal{B}_1(\mathcal{H})$-norm (with $\mathcal{H}$ a separable complex Hilbert space).

The presence of the d/dz-term under the sum in (4.88) proves that the only $(\lambda_0 - z)^{-1}$-term in (4.86), respectively, (4.88), as $z \to \lambda_0$, must originate from the trace of the resolvent difference

$$\mathrm{tr}\big((H_{\Omega,m} - zI_\Omega)^{-1} - (H_{0,\Omega} - zI_\Omega)^{-1}\big) \underset{z\to\lambda_0}{=} \mathrm{tr}(P_{\Omega,m,\lambda_0} - P_{0,\Omega,\lambda_0})(\lambda_0 - z)^{-1} + O(1), \quad m \in \mathbb{N}. \tag{4.91}$$

Thus we have proved that

$$c_{m,-1} = 0, \quad m \in \mathbb{N}, \tag{4.92}$$

in (4.86). By (4.85) this finally proves

$$c_{-1} = 0 \tag{4.93}$$

in (4.74). Equations (4.74) and (4.75) then prove (4.69). Together with the paragraph following (4.72), this also proves (4.70). □

Given Lemma 4.9, Theorem 4.3 yields the following application to differences of Dirichlet and Neumann eigenvalue counting functions:

t4.9 **Theorem 4.10.** *Assume Hypothesis 4.7 and let $k \in \mathbb{N}$, $k \geq p$. Then, for all $\lambda \in \mathbb{R}\backslash\big(\sigma\big(H_\Omega^D\big) \cup \sigma\big(H_{0,\Omega}^D\big) \cup \sigma\big(H_{0,\Omega}^N\big)\big)$,*

$$\begin{aligned} \xi_k^N(\lambda) - \xi_k^D(\lambda) &= [N_{H_\Omega^D}(\lambda) - N_{H_{0,\Omega}^D}(\lambda)] - [N_{H_\Omega^N}(\lambda) - N_{H_{0,\Omega}^N}(\lambda)] \\ &= \pi^{-1}\mathrm{Im}\Big(\ln\Big(\det_k\Big(I_{\partial\Omega} - \overline{\gamma_N\big(H_\Omega^D - \lambda I_\Omega\big)^{-1}V\big[\gamma_D\big(H_{0,\Omega}^N - \lambda I_\Omega\big)^{-1}\big]^*}\Big)\Big)\Big) + \pi^{-1}\mathrm{Im}(\mathrm{tr}(T_k(\lambda))) \\ &= \pi^{-1}\mathrm{Im}\big(\ln\big(\det_k\big(M_\Omega^D(\lambda)M_{0,\Omega}^D(\lambda)^{-1}\big)\big)\big) + \pi^{-1}\mathrm{Im}(\mathrm{tr}(T_k(\lambda))) \end{aligned} \tag{4.94}$$

with T_k defined in Theorem 4.2.

Proof. This is now an immediate consequence of (4.46), (4.47), (4.59), and (4.70). □

Appendix A. Properties of Dirichlet and Neumann Laplacians

sA The purpose of this appendix is to recall some basic operator domain properties of Dirichlet and Neumann Laplacians on sets $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, satisfying Hypothesis 2.1. We will show that the methods developed in [GLMZ05] in the context of $C^{1,r}$-domains, $1/2 < r < 1$, in fact, apply to all domains Ω permitted in Hypothesis 2.1.

In this manuscript we use the following notation for the standard Sobolev Hilbert spaces ($s \in \mathbb{R}$),

$$H^s(\mathbb{R}^n) = \bigg\{U \in \mathcal{S}(\mathbb{R}^n)^* \,\big|\, \|U\|_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} d^n\xi \,\big|\widehat{U}(\xi)\big|^2\big(1 + |\xi|^{2s}\big) < \infty\bigg\}, \tag{A.1}$$

$$H^s(\Omega) = \{u \in C_0^\infty(\Omega)^* \,|\, u = U|_\Omega \text{ for some } U \in H^s(\mathbb{R}^n)\}, \tag{A.2}$$

$$H_0^s(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in the norm of } H^s(\Omega). \tag{A.3}$$

Here $C_0^\infty(\Omega)^*$ denotes the usual set of distributions on $\Omega\subseteq\mathbb{R}^n$, Ω open and nonempty, $\mathcal{S}(\mathbb{R}^n)^*$ is the space of tempered distributions on $\mathbb{R}^n$, and $\widehat{U}$ denotes the Fourier transform of $U\in\mathcal{S}(\mathbb{R}^n)^*$. It is then immediate that

$$H^{s_1}(\Omega)\hookrightarrow H^{s_0}(\Omega)\ \text{ for }\ -\infty<s_0\leq s_1<+\infty, \tag{A.4}$$ incl-xxx

continuously and densely.

Next, we recall the definition of a $C^{1,r}$-domain $\Omega\subseteq\mathbb{R}^n$, Ω open and nonempty, for convenience of the reader: Let $\mathcal{N}$ be a space of real-valued functions in $\mathbb{R}^{n-1}$. One calls a bounded domain $\Omega\subset\mathbb{R}^n$ of class $\mathcal{N}$ if there exists a finite open covering $\{\mathcal{O}_j\}_{1\leq j\leq N}$ of the boundary $\partial\Omega$ of Ω with the property that, for every $j\in\{1,...,N\}$, $\mathcal{O}_j\cap\Omega$ coincides with the portion of $\mathcal{O}_j$ lying in the overgraph of a function $\varphi_j\in\mathcal{N}$ (considered in a new system of coordinates obtained from the original one via a rigid motion). Two special cases are going to play a particularly important role in the sequel. First, if $\mathcal{N}$ is $\mathrm{Lip}\,(\mathbb{R}^{n-1})$, the space of real-valued functions satisfying a (global) Lipschitz condition in $\mathbb{R}^{n-1}$, we shall refer to Ω as being a Lipschitz domain; cf. [St70, p. 189], where such domains are called "minimally smooth". Second, corresponding to the case when $\mathcal{N}$ is the subspace of $\mathrm{Lip}\,(\mathbb{R}^{n-1})$ consisting of functions whose first-order derivatives satisfy a (global) Hölder condition of order $r\in(0,1)$, we shall say that Ω is of class $C^{1,r}$. The classical theorem of Rademacher of almost everywhere differentiability of Lipschitz functions ensures that, for any Lipschitz domain Ω, the surface measure $d^{n-1}\sigma$ is well-defined on $\partial\Omega$ and that there exists an outward pointing normal vector ν at almost every point of $\partial\Omega$. For a Lipschitz domain $\Omega\subset\mathbb{R}^n$ it is known that

$$\big(H^s(\Omega)\big)^*=H^{-s}(\Omega),\quad -1/2<s<1/2. \tag{A.5}$$ dual-xxx

See [Tr02] for this and other related properties.

Next, assume that $\Omega\subset\mathbb{R}^n$ is the domain lying above the graph of a function $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ of class $C^{1,r}$. Then for $0\leq s<1+r$, the Sobolev space $H^s(\partial\Omega)$ consists of functions $f\in L^2(\partial\Omega;d^{n-1}\sigma)$ such that $f(x',\varphi(x'))$, as a function of $x'\in\mathbb{R}^{n-1}$, belongs to $H^s(\mathbb{R}^{n-1})$. This definition is easily adapted to the case when Ω is a domain of class $C^{1,r}$ whose boundary is compact, by using a smooth partition of unity. Finally, for $-1-r<s<0$, we set $H^s(\partial\Omega)=\big(H^{-s}(\partial\Omega)\big)^*$. For additional background information in this context we refer, for instance, to [Au04], [Au06], [EE89] [3], [4], [25, Chs. V, VI], [Gr85] [40, Ch. 1], [Mc00] [57, Ch. 3], [Wl87] [100, Sect. I.4.2].

To see that $H^1(\partial\Omega)$ embeds compactly into $L^2(\partial\Omega;d^{n-1}\sigma)$ one can argue as follows: Given a Lipschitz domain Ω in $\mathbb{R}^n$, we recall that the Sobolev space $H^1(\partial\Omega)$ is defined as the collection of functions in $L^2(\partial\Omega;d^{n-1}\sigma)$ with the property that the norm of their tangential gradient belongs to $L^2(\partial\Omega;d^{n-1}\sigma)$. It is essentially well-known that an equivalent characterization is that $f\in H^1(\partial\Omega)$ if and only if the assignment $\mathbb{R}^{n-1}\ni x'\mapsto(\psi f)(x',\varphi(x'))$ is in $H^1(\mathbb{R}^{n-1})$ whenever $\psi\in C_0^\infty(\mathbb{R}^n)$ and $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$ is a Lipschitz function with the propery that if Σ is an appropriate rotation and translation of $\{(x',\varphi(x'))\in\mathbb{R}^n\,|\,x'\in\mathbb{R}^{n-1}\}$, then $\mathrm{supp}\,(\psi)\cap\partial\Omega\subset\Sigma$. This appears to be folklore, but a proof will appear in [MM07, Proposition 2.4].

From the latter characterization of $H^1(\partial\Omega)$ it follows that any property of Sobolev spaces (of order 1) defined in Euclidean domains, which are invariant under multiplication by smooth, compactly supported functions as well as composition by bi-Lipschitz diffeomorphisms, readily extends to the setting of $H^1(\partial\Omega)$ (via localization and pull-back). As a concrete example, for each Lipschitz domain Ω with compact boundary, one has

$$H^1(\partial\Omega)\hookrightarrow L^2(\partial\Omega;d^{n-1}\sigma)\quad\text{compactly.} \tag{A.6}$$ EQ1

Going a bit further, we say that a domain $\Omega\subset\mathbb{R}^n$ satisfies a *uniform exterior ball condition* (abbreviated by UEBC), if there exists $R>0$ with the following property: For each $x\in\partial\Omega$, there exists $y=y(x)\in\mathbb{R}^n$ such that

$$\overline{B(y,R)}\,\big\backslash\{x\}\subseteq\mathbb{R}^n\backslash\Omega\ \text{ and }\ x\in\partial B(y,R). \tag{A.7}$$ UEBC

We recall that any $C^{1,1}$-domain (i.e., the first-order partial derivatives of the functions defining the boundary are Lipschitz) satisfies a UEBC.

Assuming Hypothesis 2.1, we introduce the Dirichlet and Neumann Laplacians $\widetilde{H}^D_{0,\Omega}$ and $\widetilde{H}^N_{0,\Omega}$ associated with the domain Ω as the unique self-adjoint operators on $L^2(\Omega;d^nx)$ whose quadratic form equals $q(f,g)=\int_\Omega d^nx\,\overline{\nabla f}\cdot\nabla g$ with (form) domains given by $H^1_0(\Omega)$ and $H^1(\Omega)$, respectively. Then,

$$\begin{aligned}\operatorname{dom}\big(\widetilde{H}^D_{0,\Omega}\big)=\big\{u\in H^1_0(\Omega)\,\big|\,&\text{there exists } f\in L^2(\Omega;d^nx) \text{ such that}\\ &q(u,v)=(f,v)_{L^2(\Omega;d^nx)} \text{ for all } v\in H^1_0(\Omega)\big\},\end{aligned}\tag{A.8}$$

$$\begin{aligned}\operatorname{dom}\big(\widetilde{H}^N_{0,\Omega}\big)=\big\{u\in H^1(\Omega)\,\big|\,&\text{there exists } f\in L^2(\Omega;d^nx) \text{ such that}\\ &q(u,v)=(f,v)_{L^2(\Omega;d^nx)} \text{ for all } v\in H^1(\Omega)\big\},\end{aligned}\tag{A.9}$$

with $(\cdot,\cdot)_{L^2(\Omega;d^nx)}$ denoting the scalar product in $L^2(\Omega;d^nx)$. Equivalently, we introduce the densely defined closed linear operators

$$D=\nabla,\ \operatorname{dom}(D)=H^1_0(\Omega)\ \text{ and }\ N=\nabla,\ \operatorname{dom}(N)=H^1(\Omega)\tag{A.10}$$

from $L^2(\Omega;d^nx)$ to $L^2(\Omega;d^nx)^n$ and note that

$$\widetilde{H}^D_{0,\Omega}=D^*D\ \text{ and }\ \widetilde{H}^N_{0,\Omega}=N^*N.\tag{A.11}$$

For details we refer to [RS78, Sects. XIII.14, XIII.15]. Moreover, with div denoting the divergence operator,

$$\operatorname{dom}(D^*)=\big\{w\in L^2(\Omega;d^nx)^n\,\big|\,\operatorname{div}(w)\in L^2(\Omega;d^nx)\big\},\tag{A.12}$$

and hence,

$$\begin{aligned}\operatorname{dom}\big(\widetilde{H}^D_{0,\Omega}\big)&=\{u\in\operatorname{dom}(D)\,|\,Du\in\operatorname{dom}(D^*)\}\\ &=\big\{u\in H^1_0(\Omega)\,\big|\,\Delta u\in L^2(\Omega;d^nx)\big\}.\end{aligned}\tag{A.13}$$ domHD

One can also define the following bounded linear map

$$\begin{cases}\big\{w\in L^2(\Omega;d^nx)^n\,\big|\,\operatorname{div}(w)\in\big(H^1(\Omega)\big)^*\big\}\to H^{-1/2}(\partial\Omega)=\big(H^{1/2}(\partial\Omega)\big)^*\\ \qquad\qquad w\mapsto\nu\cdot w\end{cases}\tag{A.14}$$ A.11

by setting

$$\langle\nu\cdot w,\phi\rangle=\int_\Omega d^nx\,w(x)\cdot\nabla\Phi(x)+\langle\operatorname{div}(w)\,,\,\Phi\rangle\tag{A.15}$$ A.11a

whenever $\phi\in H^{1/2}(\partial\Omega)$ and $\Phi\in H^1(\Omega)$ is such that $\gamma_D\Phi=\phi$. Here the pairing $\langle\operatorname{div}(w)\,,\,\Phi\rangle$ in (A.15) is the natural one between functionals in $\big(H^1(\Omega)\big)^*$ and elements in $H^1(\Omega)$ (which, in turn, is compatible with the (bilinear) distributional pairing). It should be remarked that the above definition is independent of the particular extension $\Phi\in H^1(\Omega)$ of ϕ. Indeed, by linearity this comes down to proving that

$$\langle\operatorname{div}(w)\,,\,\Phi\rangle=-\int_\Omega d^nx\,w(x)\cdot\nabla\Phi(x)\tag{A.16}$$ ibp

if $w\in L^2(\Omega;d^nx)^n$ has $\operatorname{div}(w)\in\big(H^1(\Omega)\big)^*$ and $\Phi\in H^1(\Omega)$ has $\gamma_D\Phi=0$. To see this we rely on the existence of a sequence $\Phi_j\in C^\infty_0(\Omega)$ such that $\Phi_j\underset{j\uparrow\infty}{\to}\Phi$ in $H^1(\Omega)$. When Ω is a bounded Lipschitz domain, this is well-known (see, e.g., [JK95, Remark 2.7] for a rather general result of this nature), and this result is easily extended to the case when Ω is an unbounded Lipschitz domain with a compact boundary. Indeed, if $\xi\in C^\infty_0(B(0;2))$ is such that $\xi=1$ on $B(0;1)$ and $\xi_j(x)=\xi(x/j)$, $j\in\mathbb{N}$ (here $B(x_0;r_0)$ denotes the ball in $\mathbb{R}^n$ centered at $x_0\in\mathbb{R}^n$ of radius $r_0>0$), then $\xi_j\Phi\underset{j\uparrow\infty}{\to}\Phi$ in $H^1(\Omega)$ and matters are reduced to approximating $\xi_j\Phi$ in $H^1(B(0;2j)\cap\Omega)$ with test functions supported

in $B(0;2j)\cap\Omega$, for each fixed $j\in\mathbb{N}$. Since $\gamma_D(\xi_j\Phi)=0$, the result for bounded Lipschitz domains applies.

Returning to the task of proving (A.16), it suffices to prove a similar identity with Φ_j in place of Φ. This, in turn, follows from the definition of $\operatorname{div}(\cdot)$ in the sense of distributions and the fact that the duality between $\big(H^1(\Omega)\big)^*$ and $H^1(\Omega)$ is compatible with the duality between distributions and test functions.

Going further, one can introduce a (weak) Neumann trace operator $\widetilde{\gamma}_N$ as follows:

$$\widetilde{\gamma}_N\colon\big\{u\in H^1(\Omega)\,\big|\,\Delta u\in\big(H^1(\Omega)\big)^*\big\}\to H^{-1/2}(\partial\Omega),\quad \widetilde{\gamma}_N u=\nu\cdot\nabla u, \tag{A.17}$$

with the dot product understood in the sense of (A.14). We emphasize that the weak Neumann trace operator $\widetilde{\gamma}_N$ in (A.17) is a bounded extension of the operator γ_N introduced in (2.3). Indeed, to see that $\operatorname{dom}(\gamma_N)\subset\operatorname{dom}(\widetilde{\gamma}_N)$, we note that if $u\in H^{s+1}(\Omega)$ for some $1/2<s<3/2$, then $\Delta u\in H^{-1+s}(\Omega)=\big(H^{1-s}(\Omega)\big)^*\hookrightarrow\big(H^1(\Omega)\big)^*$, by (A.5) and (A.4). With this in hand, it is then easy to show that $\widetilde{\gamma}_N$ in (A.19) and γ_N in (2.3) agree (on the smaller domain), as claimed.

We now return to the mainstream discussion. From the above preamble it follows that

$$\operatorname{dom}(N^*)=\big\{w\in L^2(\Omega;d^nx)^n\,\big|\,\operatorname{div}(w)\in L^2(\Omega;d^nx)\text{ and }\nu\cdot w=0\big\}, \tag{A.18}$$

where the dot product operation is understood in the sense of (A.14). Consequently, with $\widetilde{H}^N_{0,\Omega}=N^*N$, we have

$$\begin{aligned}\operatorname{dom}\big(\widetilde{H}^N_{0,\Omega}\big)&=\{u\in\operatorname{dom}(N)\,|\,Nu\in\operatorname{dom}(N^*)\}\\&=\big\{u\in H^1(\Omega)\,\big|\,\Delta u\in L^2(\Omega;d^nx)\text{ and }\widetilde{\gamma}_N u=0\big\}.\end{aligned} \tag{A.19}$$

Next, we intend to recall that $H^D_{0,\Omega}=\widetilde{H}^D_{0,\Omega}$ and $H^N_{0,\Omega}=\widetilde{H}^N_{0,\Omega}$, where $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$ denote the operators introduced in (2.4) and (2.5), respectively. For this purpose one can argue as follows: Since it follows from the first Green's formula (cf., e.g., [57, Theorem 4.4]) that $H^D_{0,\Omega}\subseteq\widetilde{H}^D_{0,\Omega}$ and $H^N_{0,\Omega}\subseteq\widetilde{H}^N_{0,\Omega}$, it remains to show that $H^D_{0,\Omega}\supseteq\widetilde{H}^D_{0,\Omega}$ and $H^N_{0,\Omega}\supseteq\widetilde{H}^N_{0,\Omega}$. Moreover, it follows from comparing (2.4) with (A.13) and (2.5) with (A.19), that one needs only to show that $\operatorname{dom}\big(\widetilde{H}^D_{0,\Omega}\big)$, $\operatorname{dom}\big(\widetilde{H}^N_{0,\Omega}\big)\subseteq H^2(\Omega)$. This is the content of the next lemma.

Lemma A.1. *Assume Hypothesis 2.1. Then,*

$$\operatorname{dom}\big(\widetilde{H}^D_{0,\Omega}\big)\subset H^2(\Omega),\quad \operatorname{dom}\big(\widetilde{H}^N_{0,\Omega}\big)\subset H^2(\Omega). \tag{A.20}$$

Moreover,

$$H^D_{0,\Omega}=\widetilde{H}^D_{0,\Omega},\quad H^N_{0,\Omega}=\widetilde{H}^N_{0,\Omega}. \tag{A.21}$$

For $C^{1,r}$-domains Ω, $1/2<r<1$, Lemma A.1 was proved in [32, Appendix A]. For bounded convex domains Ω, $\operatorname{dom}\big(\widetilde{H}^D_{0,\Omega}\big)\subset H^2(\Omega)$ was shown by Kadlec [48] and Talenti [93] and $\operatorname{dom}\big(\widetilde{H}^N_{0,\Omega}\big)\subset H^2(\Omega)$ was proved by Grisvard and Ioss [41]. A unified approach to Dirichlet and Neumann problems in bounded convex domains, which also applies to bounded Lipschitz domains satisfying UEBC, has been presented by Mitrea [62]. The extension to domains Ω with a compact boundary satisfying UEBC then follows as described in the paragraph following (A.5). This establishes (A.20) and hence (A.21) as discussed after (A.19).

We note that Lemma A.1 also follows from [20, Theorem 8.2] in the case of C^2-domains Ω with compact boundary. This is proved in [20] by rather different methods and can be viewed as a generalization of the classical result for bounded C^2-domains.

As shown in [32, Lemma A.2], (A.20) and methods of real interpolation spaces yield the following key result (A.22) needed in the main body of this paper:

lA.2 **Lemma A.2.** *Assume Hypothesis 2.1 and let $q \in [0,1]$. Then for each $z \in \mathbb{C}\backslash[0,\infty)$, one has*

$$\big(H^D_{0,\Omega} - zI_\Omega\big)^{-q},\, \big(H^N_{0,\Omega} - zI_\Omega\big)^{-q} \in \mathcal{B}\big(L^2(\Omega; d^n x), H^{2q}(\Omega)\big). \tag{A.22}$$ new6.45

Finally, we recall an extension of a result of Nakamura [66, Lemma 6] from a cube in $\mathbb{R}^n$ to a Lipschitz domain Ω. This requires some preparation. First, we note that (A.17) and (A.15) yield the following Green formula

$$\langle \widetilde{\gamma}_N u, \gamma_D \Phi\rangle = \big(\overline{\nabla u}, \nabla\Phi\big)_{L^2(\Omega; d^n x)^n} + \langle \Delta u, \Phi\rangle, \tag{A.23}$$ wGreen

valid for any $u \in H^1(\Omega)$ with $\Delta u \in \big(H^1(\Omega)\big)^*$, and any $\Phi \in H^1(\Omega)$. The pairing on the left-hand side of (A.23) is between functionals in $\big(H^{1/2}(\partial\Omega)\big)^*$ and elements in $H^{1/2}(\partial\Omega)$, whereas the last pairing on the right-hand side is between functionals in $\big(H^1(\Omega)\big)^*$ and elements in $H^1(\Omega)$. For further use, we also note that the adjoint of (2.2) maps boundedly as follows

$$\gamma_D^* : \big(H^{s-1/2}(\partial\Omega)\big)^* \to \big(H^s(\Omega)\big)^*, \quad 1/2 < s < 3/2. \tag{A.24}$$ ga*

Next, one observes that the operator $\big(\widetilde{H}^N_{0,\Omega} - zI_\Omega\big)^{-1}$, $z \in \mathbb{C}\backslash\sigma\big(\widetilde{H}^N_{0,\Omega}\big)$, originally defined as

$$\big(\widetilde{H}^N_{0,\Omega} - zI_\Omega\big)^{-1} : L^2(\Omega; d^n x) \to L^2(\Omega; d^n x), \tag{A.25}$$ fukcH

can be extended to a bounded operator, mapping $\big(H^1(\Omega)\big)^*$ into $L^2(\Omega; d^n x)$. Specifically, since $\big(\widetilde{H}^N_{0,\Omega} - \overline{z}I_\Omega\big)^{-1} : L^2(\Omega; d^n x) \to \mathrm{dom}\big(\widetilde{H}^N_{0,\Omega}\big)$ is bounded and since the inclusion $\mathrm{dom}\big(\widetilde{H}^N_{0,\Omega}\big) \hookrightarrow H^1(\Omega)$ is bounded, we can naturally view $\big(\widetilde{H}^N_{0,\Omega} - \overline{z}I_\Omega\big)^{-1}$ as an operator

$$\big(\widehat{H}^N_{0,\Omega} - \overline{z}I_\Omega\big)^{-1} \colon L^2(\Omega; d^n x) \to H^1(\Omega) \tag{A.26}$$

mapping in a linear, bounded fashion. Consequently, for its adjoint, we have

$$\big(\big(\widehat{H}^N_{0,\Omega} - \overline{z}I_\Omega\big)^{-1}\big)^* : \big(H^1(\Omega)\big)^* \to L^2(\Omega; d^n x), \tag{A.27}$$ fukcH-bis

and it is easy to see that this latter operator extends the one in (A.25). Hence, there is no ambiguity in retaining the same symbol, that is, $\big(\widetilde{H}^N_{0,\Omega} - \overline{z}I_\Omega\big)^{-1}$, both for the operator in (A.27) as well as for the operator in (A.25). Similar considerations and conventions apply to $\big(\widetilde{H}^D_{0,\Omega} - zI_\Omega\big)^{-1}$.

Given these preparations, we now state without proof (and for the convenience of the reader) the following result proven in [32, Lemma A.3] (an extension of a result proven in [66]).

lA.3 **Lemma A.3.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open Lipschitz domain and let $z \in \mathbb{C}\backslash\big(\sigma\big(\widetilde{H}^D_{0,\Omega}\big) \cup \sigma\big(\widetilde{H}^N_{0,\Omega}\big)\big)$. Then, on $L^2(\Omega; d^n x)$,*

$$\big(\widetilde{H}^D_{0,\Omega} - zI_\Omega\big)^{-1} - \big(\widetilde{H}^N_{0,\Omega} - zI_\Omega\big)^{-1} = \big(\widetilde{H}^N_{0,\Omega} - zI_\Omega\big)^{-1} \gamma_D^* \widetilde{\gamma}_N \big(\widetilde{H}^D_{0,\Omega} - zI_\Omega\big)^{-1}, \tag{A.28}$$ Na1

where γ_D^* is an adjoint operator to γ_D in the sense of (A.24)

rA.4 **Remark A.4.** While it is tempting to view γ_D as an unbounded but densely defined operator on $L^2(\Omega; d^n x)$ whose domain contains the space $C_0^\infty(\Omega)$, one should note that in this case its adjoint γ_D^* is not densely defined: Indeed (cf. [32, Remark A.4]), $\mathrm{dom}(\gamma_D^*) = \{0\}$ and hence γ_D is not a closable linear operator in $L^2(\Omega; d^n x)$.

rA.5 **Remark A.5.** In the case of domains Ω satisfying Hypothesis 2.1, Lemma A.1 implies that the operators $\widetilde{H}^D_{0,\Omega}$ and $\widetilde{H}^N_{0,\Omega}$ coincide with the operators $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$, respectively, and hence one can use the operators $H^D_{0,\Omega}$ and $H^N_{0,\Omega}$ in Lemma A.3. Moreover, since $\mathrm{dom}\big(H^D_{0,\Omega}\big) \subset H^2(\Omega)$, one can also replace $\widetilde{\gamma}_N$ by γ_N (cf. (2.3)) in Lemma A.3. In particular,

$$\begin{aligned}\big(H^D_{0,\Omega} - zI_\Omega\big)^{-1} - \big(H^N_{0,\Omega} - zI_\Omega\big)^{-1} = \big[\gamma_D\big(H^N_{0,\Omega} - \overline{z}I_\Omega\big)^{-1}\big]^* \gamma_N \big(H^D_{0,\Omega} - zI_\Omega\big)^{-1},\\ z \in \mathbb{C}\backslash\big(\sigma\big(H^D_{0,\Omega}\big) \cup \sigma\big(H^N_{0,\Omega}\big)\big),\end{aligned} \tag{A.29}$$ Na1-bis

a result exploited in the proof of Theorem 4.2 (cf. (4.28)).

Finally, we prove the following result used in the proof of Lemma 3.4.

Lemma A.6. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open Lipschitz domain with a compact, nonempty boundary $\partial\Omega$. Then the Dirichlet trace operator γ_D satisfies the following property (see also (2.2)),*

$$\gamma_D \in \mathcal{B}\big(H^{(3/2)+\varepsilon}(\Omega), H^1(\partial\Omega)\big), \quad \varepsilon > 0. \tag{A.30}$$

Proof. First, we recall one of the equivalent definitions of $H^1(\partial\Omega)$, specifically,

$$H^1(\partial\Omega) = \big\{ f \in L^2(\partial\Omega; d^{n-1}\sigma) \,\big|\, \partial f/\partial\tau_{j,k} \in L^2(\partial\Omega; d^{n-1}\sigma),\ j,k = 1,\dots,n \big\}, \tag{A.31}$$

where $\partial/\partial\tau_{k,j} = \nu_k\partial_j - \nu_j\partial_k$, $j,k = 1,\dots,n$, is a tangential derivative operator (cf. (A.33)), or equivalently,

$$H^1(\partial\Omega) = \bigg\{ f \in L^2(\partial\Omega; d^{n-1}\sigma) \,\bigg|\, \text{there exists a constant } c > 0 \text{ such that for every } v \in C_0^\infty(\mathbb{R}^n),$$
$$\bigg| \int_{\partial\Omega} d^{n-1}\sigma f\, \partial v/\partial\tau_{j,k} \bigg| \leq c \, \|v\|_{L^2(\partial\Omega; d^{n-1}\sigma)},\ j,k = 1,\dots,n \bigg\}. \tag{A.32}$$

Next, let $u \in H^{(3/2)+\varepsilon}(\Omega)$, $v \in C_0^\infty(\mathbb{R}^n)$, and $u_i \in C^\infty(\overline{\Omega}) \hookrightarrow H^{(3/2)+\varepsilon}(\Omega)$, $i \in \mathbb{N}$, be a sequence of functions approximating u in $H^{(3/2)+\varepsilon}(\Omega)$. It follows from (2.2) and (A.4) that $\gamma_D u, \gamma_D(\nabla u) \in L^2(\partial\Omega; d^{n-1}\sigma)$. Introducing the tangential derivative operator $\partial/\partial\tau_{k,j} = \nu_k\partial_j - \nu_j\partial_k$, $j,k = 1,\dots,n$, one has

$$\int_{\partial\Omega} d^{n-1}\sigma \, \frac{\partial h_1}{\partial\tau_{j,k}} h_2 = -\int_{\partial\Omega} d^{n-1}\sigma \, h_1 \frac{\partial h_2}{\partial\tau_{j,k}}, \quad h_1, h_2 \in H^{1/2}(\partial\Omega). \tag{A.33}$$

Utilizing (A.33), one computes for all $j,k = 1,\dots,n$,

$$\bigg| \int_{\partial\Omega} d^{n-1}\sigma \, \gamma_D u \frac{\partial v}{\partial\tau_{j,k}} \bigg| = \bigg| \lim_{i\to\infty} \int_{\partial\Omega} d^{n-1}\sigma \, u_i \frac{\partial v}{\partial\tau_{j,k}} \bigg| = \bigg| \lim_{i\to\infty} \int_{\partial\Omega} d^{n-1}\sigma \, v \frac{\partial u_i}{\partial\tau_{j,k}} \bigg| \tag{A.34}$$
$$\leq c \bigg| \lim_{i\to\infty} \int_{\partial\Omega} d^{n-1}\sigma \, v \, \gamma_D(\nabla u_i) \bigg| \leq c \, \|\gamma_D(\nabla u)\|_{L^2(\partial\Omega; d^{n-1}\sigma)} \, \|v\|_{L^2(\partial\Omega; d^{n-1}\sigma)}.$$

Thus, it follows from (A.32) and (A.34) that $\gamma_D u \in H^1(\partial\Omega)$. □

Appendix B. Abstract Perturbation Theory

The purpose of this appendix is to summarize some of the abstract perturbation results in [32] which where motivated by Kato's pioneering work [49] (see also [44], [51]) as they are needed in this paper.

We introduce the following set of assumptions.

Hypothesis B.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable, complex Hilbert spaces.*
(i) Suppose that $H_0 \colon \operatorname{dom}(H_0) \to \mathcal{H}$, $\operatorname{dom}(H_0) \subseteq \mathcal{H}$ is a densely defined, closed, linear operator in $\mathcal{H}$ with nonempty resolvent set,

$$\rho(H_0) \neq \emptyset, \tag{B.1}$$

$A \colon \operatorname{dom}(A) \to \mathcal{K}$, $\operatorname{dom}(A) \subseteq \mathcal{H}$ a densely defined, closed, linear operator from $\mathcal{H}$ to $\mathcal{K}$, and $B \colon \operatorname{dom}(B) \to \mathcal{K}$, $\operatorname{dom}(B) \subseteq \mathcal{H}$ a densely defined, closed, linear operator from $\mathcal{H}$ to $\mathcal{K}$ such that

$$\operatorname{dom}(A) \supseteq \operatorname{dom}(H_0), \quad \operatorname{dom}(B) \supseteq \operatorname{dom}(H_0^*). \tag{B.2}$$

In the following we denote

$$R_0(z) = (H_0 - zI_{\mathcal{H}})^{-1}, \quad z \in \rho(H_0). \tag{B.3}$$

(ii) Assume that for some (and hence for all) $z \in \rho(H_0)$, the operator $-AR_0(z)B^*$, defined on $\operatorname{dom}(B^*)$, has a bounded extension in $\mathcal{K}$, denoted by $K(z)$,

$$K(z) = -\overline{AR_0(z)B^*} \in \mathcal{B}(\mathcal{K}). \tag{B.4}$$

(iii) Suppose that $1 \in \rho(K(z_0))$ for some $z_0 \in \rho(H_0)$.
(iv) Assume that $K(z) \in \mathcal{B}_\infty(\mathcal{K})$ for all $z \in \rho(H_0)$.

Next, following Kato [49], one introduces

$$R(z) = R_0(z) - \overline{R_0(z)B^*}[I_\mathcal{K} - K(z)]^{-1}AR_0(z), \quad z \in \{\zeta \in \rho(H_0) \,|\, 1 \in \rho(K(\zeta))\}. \tag{B.5}$$

Theorem B.2 ([32]). *Assume Hypothesis B.1(i)–(iii) and suppose $z \in \{\zeta \in \rho(H_0) \,|\, 1 \in \rho(K(\zeta))\}$. Then, $R(z)$ introduced in (B.5) defines a densely defined, closed, linear operator H in $\mathcal{H}$ by*

$$R(z) = (H - zI_\mathcal{H})^{-1}. \tag{B.6}$$

In addition,

$$AR(z), BR(z)^* \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \tag{B.7}$$

and

$$R(z) = R_0(z) - \overline{R(z)B^*}AR_0(z) \tag{B.8}$$

$$= R_0(z) - \overline{R_0(z)B^*}AR(z). \tag{B.9}$$

Moreover, H is an extension of $(H_0 + B^*A)|_{\operatorname{dom}(H_0)\cap\operatorname{dom}(B^*A)}$ (the latter intersection domain may consist of $\{0\}$ only),

$$H \supseteq (H_0 + B^*A)|_{\operatorname{dom}(H_0)\cap\operatorname{dom}(B^*A)}. \tag{B.10}$$

Finally, assume that H_0 is self-adjoint in $\mathcal{H}$. Then H is also self-adjoint if

$$(Af, Bg)_\mathcal{K} = (Bf, Ag)_\mathcal{K} \text{ for all } f, g \in \operatorname{dom}(A) \cap \operatorname{dom}(B). \tag{B.11}$$

In the case where H_0 is self-adjoint, Theorem B.2 is due to Kato [49] in this abstract setting.

The next result is an abstract version of the celebrated Birman–Schwinger principle relating eigenvalues λ_0 of H and the eigenvalue 1 of $K(\lambda_0)$:

Theorem B.3 ([32]). *Assume Hypothesis B.1 and let $\lambda_0 \in \rho(H_0)$. Then,*

$$Hf = \lambda_0 f, \quad 0 \neq f \in \operatorname{dom}(H) \text{ implies } K(\lambda_0)g = g \tag{B.12}$$

where, for fixed $z_0 \in \{\zeta \in \rho(H_0) \,|\, 1 \in \rho(K(\zeta))\}$, $z_0 \neq \lambda_0$,

$$0 \neq g = [I_\mathcal{K} - K(z_0)]^{-1}AR_0(z_0)f = (\lambda_0 - z_0)^{-1}Af. \tag{B.13}$$

Conversely,

$$K(\lambda_0)g = g, \quad 0 \neq g \in \mathcal{K} \text{ implies } Hf = \lambda_0 f, \tag{B.14}$$

where

$$0 \neq f = -\overline{R_0(\lambda_0)B^*}g \in \operatorname{dom}(H). \tag{B.15}$$

Moreover,

$$\dim(\ker(H - \lambda_0 I_\mathcal{H})) = \dim(\ker(I_\mathcal{K} - K(\lambda_0))) < \infty. \tag{B.16}$$

In particular, let $z \in \rho(H_0)$, then

$$z \in \rho(H) \text{ if and only if } 1 \in \rho(K(z)). \tag{B.17}$$

In the case where H_0 and H are self-adjoint, Theorem B.3 is due to Konno and Kuroda [51].

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Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: fritz@math.missouri.edu
URL: http://www.math.missouri.edu/personnel/faculty/gesztesyf.html

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
E-mail address: marius@math.missouri.edu
URL: http://www.math.missouri.edu/personnel/faculty/mitream.html

Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA
E-mail address: maxim@caltech.edu
URL: http://math.caltech.edu/∼maxim