Resonances of the confined hydrogenoïd ion and the Dicke effect in non-relativistic quantum electrodynamics

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Abstract

We pursue the spectral analysis of the model describing a system of one dynamical nucleus and one electron together with the quantized electromagnetic field. We impose an ultraviolet cutoff and assume that the fine-structure constant is sufficiently small. Then we prove that the unperturbed eigenvalues turn into resonances when the nucleus and the electron are coupled to the radiation field. Furthermore, a method to calculate the resonances up to all orders in the coupling parameters is given. This analysis is related to the Lamb-Dicke effect.

Contents

1 Introduction and statements of results

In this paper, we pursue the spectral analysis of the model studied in [AF]. This model describes a hydrogenoïd ion confined by its center of mass, and is used in theoretical physics to explain the Lamb-Dicke effect (see [CTDRG]). It is also related to the Mössbauer effect (see [CTDRG]). Our purpose is to present a mathematically rigorous aspect of this phenomenon.

We consider a system of one nucleus and one electron together with the quantized electromagnetic field. Here the nucleus is *dynamical*, so that, as compared to a model with fixed nucleus, the kinetic energy of the atom or ion is modified when it emits or absorbs a photon. If it is assumed that the center of mass motion of the atomic system is free, this modification is the sum of a term due to the Doppler effect, and another term that is the "recoil energy" (see [CTDRG]). Now, if the center of mass of the atomic system is assumed to be confined by a suitable external potential, the energy associated with the center of mass motion can take only discrete values. Then, during an emission or absorption process for a photon, the energy of the emitted or absorbed photon depends on the value of the energy associated with the center of mass motion, respectively before and after the process. This situation allows to explain the Dicke effect (see [CTDRG], exercise 11).

In [AF], we obtained the existence of a ground state for the Hamiltonian describing the present model, *for all values* of the fine-structure constant. Furthermore, when instead of fixing the nucleus, one only assume that the center of mass of the atom or ion is confined, new intense rays appear in the scattering spectrum of the physical system (see [CTDRG]). Thus, some resonances depending on the confining potential with a very small imaginary part should appear in the spectrum of the Hamiltonian. The aim of our paper is to prove this, assuming here that the fine-structure constant is sufficiently small. In addition, our proof shall provide a method to localize the resonances up to all orders in the fine-structure constant.

Let us formally write the Hamiltonian that describes the system that we consider as:

$$H_U^V = \sum_{j=1,2} \frac{1}{2m_j}(p_j - q_j A(x_j))^2 + H_f + U(R) + V(r). \tag{1}$$

Here, x_j, q_j, $p_j = -i\nabla_j$, m_j denote respectively the position, the charge, the momentum and the mass of the particle j (the electron and the nucleus). The position of the center of mass R, and the variable r are defined by

$$R := \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad , \quad r := x_1 - x_2. \tag{2}$$

Moreover, A is the quantized electromagnetic vector potential in the Coulomb gauge, H_f is the free photon energy field, U is a confining potential acting on the center of mass of the atomic system, and V denotes the Coulomb potential. An ultraviolet cutoff at a scale Λ is imposed on

A for some arbitrary but finite $\Lambda > 0$. Note that the units have been chosen such that $\hbar = c = 1$, where c stands for the velocity of light. Besides, for the sake of simplicity, the spin of the electron is not taken into account.

We emphasize that the differences between the model described by H_U^V and a model with a static nucleus are significant. Since we consider here a dynamical nucleus, we have to impose some confinement, otherwise the Hamiltonian of the system would be translation invariant. However, in the model that we study and that is used in [CTDRG] to explain the Dicke effect, U confines only the center of mass; the nucleus and the electron themselves are not confined. Actually, the interaction between the nucleus and the electron takes place exclusively through the attractive Coulomb potential V. Thus, one can imagine some states where the nucleus and the electron are localized very far from each other, and where, yet, the energy associated with the center of mass motion is low.

Mathematically, as we shall see below, the method to study the existence of resonances for H_U^V has to be significantly modified as compared to a similar model where the nuclei would be treated as static.

We denote by H_0 the unperturbed Hamiltonian, that is H_U^V where the coupling parameters $q_1 := q, q_2 := -Zq$ are put to 0. The main result of this paper is theorem 1.2 below which shows that the eigenvalues of H_0 turn into resonances when q_1, q_2 become $\neq 0$. This follows the strategy developed in [BFS1,2,3] and [BCFS], based on a renormalization group analysis. As stated above, the main difference between the models studied in these papers and our model is the presence of the potential U in our model, which *confines* the center of mass. This confining potential imposes some modifications of the proofs which are not straightforward. We shall only reproduce here the new aspects of the proofs and often refer to [BFS1,2,3] or [BCFS].

The assumptions on the confining potential U that we require are stated in subsection 2.2. We denote by $(E_j)_{j\geq 0}$ the non-decreasing sequence of eigenvalues of $p^2/2\mu + V$ and by $(e_j)_{j\geq 0}$ the non-decreasing sequence of eigenvalues of $P^2/2M + U$, where

$$\begin{aligned} M &:= m_1 + m_2 \quad , \quad \mu := \frac{m_1 m_2}{m_1 + m_2} \quad ; \\ P &:= p_1 + p_2 \quad , \quad \frac{p}{\mu} := \frac{p_1}{m_1} - \frac{p_2}{m_2}. \end{aligned} \tag{3}$$

Our main contributions are the following two points:

∗ First, we give a rigorous definition of the family $H_U^V(\theta)$ obtained through complex scaling (see subsection 3.1). Note that the potential U imposes a noticeably different definition of the family $H_U^V(\theta)$, as compared to the one given is [BFS3]. We shall have to use quadratic forms in a way related to the one in [AF], and we shall show that $H_U^V(\theta)$ is an analytic family of type

(B) on a disc $D(0, \theta_0)$, for a sufficiently small θ_0, with explicit form domain

$$Q(H_U^V(\theta)) = Q(p_1^2 + p_2^2) \cap Q(U) \cap Q(H_f). \tag{4}$$

Together with a result obtained in [BFS3], this shall imply:

Theorem 1.1
Assume that $g := (q^2\Lambda)^{3/2}$ is sufficiently small, where Λ denotes the parameter of the ultraviolet cutoff. Assume moreover that the hypothesis $(\mathcal{H}_{\Gamma_{l,n}})$ related to the Fermi golden rule (see subsection 3.2) is fulfilled for all (l,n) such that $(l,n) \neq (0,0)$ and $E_l + e_n < e_0$.
Then the spectrum of H_U^V is absolutely continuous in the interval $]E_0 + e_0 + O(g), e_0 - O(g)[$. In particular, the unperturbed eigenvalues $E_l + e_n$ disappear when the nucleus and the electron are coupled to the photons.

∗ Next, we prove that the confining potential U allows to perform a *renormalization group analysis*. At this point, we have to transform H_U^V through a Power-Zienau-Wooley transformation (see [CTDRG2]), and to impose, in the resulting interaction term $\widetilde{A}$, a spatial cutoff in the variable r that restricts the electron position to finite distances from the nucleus position. Note that in [BFS1], a similar spatial cutoff is imposed, which restricts the electrons positions to finite distance from the static nuclei. We denote by $\widetilde{H}_U^V$ the resulting Hamiltonian, and $\widetilde{H}_0$ denotes the operator obtained when the coupling between the two "particles" (the nucleus and the electron) and the photons vanish. We shall see that the eigenvalues $E_l + e_n$ of H_0 are slightly shifted by the transformation: we denote by $\widetilde{E}_l + e_n$ the corresponding eigenvalues of $\widetilde{H}_0$.

To get the result, we follow the strategy of [BFS2] and use *the smooth Feshbach* map defined in [BCFS]. However, the proof is not straightforward: we have to modify carefully the hypotheses 2. and 3. stated in [BFS2] in such a way that, on one hand, they are well adapted to tour model, and on the other hand, they are still sufficient to perform a renormalization group analysis. More precisely, our requirements about the interacting part of the Hamiltonian are stated in subsection 4.1; they are denoted by $(\mathcal{H}_{-1/2})$ and $(\mathcal{H}_{1/2})$. In particular, the fact that U confines the center of mass shall be essential: this is reflected in our choice of hypothesis $(\mathcal{H}_2)$ stated in subsection 2.2. We shall prove:

Proposition 1.1
Assume that U fulfills hypotheses $(\mathcal{H}_0)$, $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ stated in subsection 2.2. Denote by $\widetilde{H}_U^V(\theta)$ the analytic family of type (B) obtained from $\widetilde{H}_U^V$ through complex scaling. Then $\widetilde{H}_U^V(\theta)$ fulfills hypotheses $(\mathcal{H}_{-1/2})$ and $(\mathcal{H}_{1/2})$ stated in subsection 4.1.

Together with the results obtained in [BFS2] and [BCFS], this proposition shall allow us to prove that we can perform a renormalization group analysis starting from $\widetilde{H}_U^V$, which shall yield:

Theorem 1.2

Fix $E_l + e_n$ an eigenvalue of H_0 such that $(l,n) \neq 0$ and $E_l + e_n < e_0$. Pick some $\rho_0 > 0$ sufficiently small and assume that the coupling parameter $g > 0$ is also sufficiently small. Let δ be the distance from $\widetilde{E}_l + e_n$ to the other eigenvalues of $\widetilde{H}_0$ and pick $\theta = \eta + i\nu$ in $D(0,\theta_0)$ such that $\rho_0 \leq \delta \sin(\nu/2) < 1$.

Then the spectrum of $\widetilde{H}_U^V(\theta)$ in the disc $D_{\rho_0/2} := D(\widetilde{E}_l + e_n, \rho_0/2)$ around $\widetilde{E}_l + e_n$ is located as follows:

$$\sigma\left(\widetilde{H}_U^V(\theta)\right) \cap D_{\rho_0/2} \subset E_{l,n}(\theta) + K_{l,n}(\theta) \tag{5}$$

where $E_{l,n}(\theta)$ is an eigenvalue of $\widetilde{H}_U^V(\theta)$ and where $K_{l,n}(\theta)$ is a complex domain such that, for some $\tau > 1$ and $0 < \mathrm{C} < 1$:

$$K_{l,n}(\theta) \subset \left\{\widetilde{E}_l + e_n + e^{-i\nu}a + b, 0 \leq a \leq 1, |b| \leq \mathrm{C}a^\tau\right\}. \tag{6}$$

In particular, $E_{l,n}(\theta)$ is independent of θ.

The paper is organized as follows: in section 2, we define our model, we precise our notations and we state the hypotheses that we have to require on the confining potential U. In section 3, we define precisely $H_U^V(\theta)$ and we prove that it is an analytic family of type (B); this yields theorem 1.1. Finally in section 4, we show how the renormalization group method developed in [BFS1,2] and [BCFS] can be applied to our model. This gives theorem 1.2.

2 Definition and assumptions on the model

2.1 Definition of H_U^V and notations

Here, we only give a formal definition of the Hamiltonian H_U^V. We refer the reader to [AF] for a precise definition using quadratic forms. Let us write our Hamiltonian as

$$H_U^V := \sum_{j=1,2} \frac{1}{2m_j}(p_j - q_j A_j)^2 + H_f + U + V. \tag{7}$$

This operator acts in the Hilbert space $\mathrm{L}^2(\mathbb{R}^6) \otimes \mathcal{F}_s \simeq \mathrm{L}^2(\mathbb{R}^6; \mathcal{F}_s)$, where $\mathcal{F}_s$ denotes the symmetric Fock space of transversally polarized photons over $\mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$, that is

$$\mathcal{F}_s(\mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)) = \mathbb{C} \oplus \bigoplus_{n \geq 1} S_n \otimes_{k=1}^n \mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2). \tag{8}$$

Here S_n denotes the symmetrization of the n components in the tensor product $\otimes_{k=1}^n \mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$. The vector potential A_j in the Coulomb gauge is defined by

$$A_j := \int_{\mathbb{R}^6}^{\oplus} A(x_j) dX, \tag{9}$$

with $X = (x_1, x_2)$ and

$$A(x) := \frac{1}{2\pi} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\widehat{\chi}_\Lambda(k)}{\sqrt{|k|}} \varepsilon_\lambda(k) \left(\widehat{a}^*_\lambda(k) e^{-ik.x} + \widehat{a}_\lambda(k) e^{ik.x} \right) dk. \tag{10}$$

Recall that $\widehat{a}^*_\lambda(k)$ and $\widehat{a}_\lambda(k)$ are the usual creation and annihilation operators satisfying the Canonical Commutation Rules (in the sense of operator-valued distributions)

$$[\widehat{a}_\lambda(k), \widehat{a}^*_{\lambda'}(k')] = \delta_{\lambda\lambda'} \delta(k - k'),$$
$$[\widehat{a}_\lambda(k), \widehat{a}_{\lambda'}(k')] = [\widehat{a}^*_\lambda(k), \widehat{a}^*_{\lambda'}(k')] = 0.$$

$\varepsilon_1(k)$ and $\varepsilon_2(k)$ are the orthonormal polarization vectors that we choose as

$$\varepsilon_1(k) := \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad , \quad \varepsilon_2(k) := \frac{k}{|k|} \wedge \varepsilon_1(k). \tag{11}$$

Moreover, Λ is the parameter of the ultraviolet cutoff, and $\widehat{\chi}_\Lambda$ is a real smooth function depending only on $|k|$, which is equal to 1 in the ball $B(0, Z^2 q^4 \Lambda/2)$ and which vanishes outside the ball $B(0, Z^2 q^4 \Lambda)$, where we have assumed that $q_1 = q$ and $q_2 = -Zq$ (here q denotes the electron charge, and Z is the number of protons in the nucleus). Note that we define $\widehat{\chi}_\Lambda$ on the ball $B(0, Z^2 q^4 \Lambda)$ instead of $B(0, \Lambda)$ because of the unitary transformation $\mathcal{U}_1$ that we shall apply below.
The free field energy operator H_f is defined by

$$H_f := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| \widehat{a}^*_\lambda(k) \widehat{a}_\lambda(k) dk. \tag{12}$$

Finally, we write the Coulomb potential V as

$$V(r) := -Zq^2 \frac{\mathrm{C}}{|r|}. \tag{13}$$

As in [BFS1], we proceed to a change of units in order to exhibit the perturbative character of the problem. More precisely, we consider the unitary operator $\mathcal{U}_1$ that dilates the electron position and the photons momenta through $(x_j, k) \mapsto (x_j/Zq^2, Z^2 q^4 k)$. Then we set $H_1 := \mathcal{U}_1 H_U^V \mathcal{U}_1^*$, which leads to:

$$\frac{1}{Z^2 q^4} H_1 = \sum_{j=1,2} \frac{1}{2m_j} \left(p_j - q_j Z q^2 \widetilde{A}_j (Zq^2 \cdot) \right)^2 + H_f + \widetilde{V} + \widetilde{U}.$$

$\widetilde{A}(x)$ denotes $A(x)$ where $\widehat{\chi}_\Lambda$ is replaced by $\widehat{\chi}_\Lambda(Z^2 q^4 \cdot)$, $\widetilde{V}(r) := -\mathrm{C}/|r|$, and $\widetilde{U}(R) := \frac{1}{Z^2 q^4} U(R/Zq^2)$. We redefine $\widehat{\chi}_\Lambda(k) := \widehat{\chi}_\Lambda(Z^2 q^4 k)$, $V(r) := \widetilde{V}(r)$ and $U(R) := \widetilde{U}(R)$, so that the new Hamiltonian, still denoted by H_U^V, that we have to consider, is:

$$H_U^V = \sum_{j=1,2} \frac{1}{2m_j} \left(p_j - q_j Z q^2 A_j (Zq^2 \cdot) \right)^2 + H_f + U + V.$$

Now, we can write H_U^V as $H_U^V := H_0 + W_g$, where H_0 is the unperturbed Hamiltonian defined by

$$H_0 := \sum_{j=1,2} \frac{p_j^2}{2m_j} + U + V + H_f, \tag{14}$$

W_g is the coupling between the two particles (the nucleus and the electron) and the photons, defined by

$$W_g := H_U^V - H_0, \tag{15}$$

and $g := (q^2\Lambda)^{3/2}$ is the perturbative parameter, that we shall assume to be sufficiently small in the sequel. Let us describe more precisely the perturbation W_g: we write

$$W_g := gW_1 + g^2 W_2, \tag{16}$$

with

$$\begin{aligned}
W_1 &:= \sum_{j=1,2} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \left[G_{1,0}^j(k,\lambda) \otimes \widehat{a}_\lambda^*(k) + G_{0,1}^j(k,\lambda) \otimes \widehat{a}_\lambda(k) \right] dk, \\
W_2 &:= \sum_{j=1,2} \sum_{\lambda=1,2} \sum_{\lambda'=1,2} \int_{\mathbb{R}^6} \Big[G_{2,0}^j(k,\lambda;k',\lambda') \otimes \widehat{a}_\lambda^*(k)\widehat{a}_{\lambda'}^*(k') \\
&\qquad + G_{0,2}^j(k,\lambda;k',\lambda') \otimes \widehat{a}_\lambda(k)\widehat{a}_{\lambda'}(k') \\
&\qquad + G_{1,1}^j(k,\lambda;k',\lambda') \otimes \widehat{a}_\lambda^*(k)\widehat{a}_{\lambda'}(k') \Big] dk dk' + \Lambda_0^j,
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
G_{1,0}^1(k,\lambda) = G_{0,1}^1(k,\lambda)^* &= \frac{iZ}{2m_1\Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(k)}{2\pi\sqrt{|k|}} e^{-iZq^2k.x_1} \varepsilon_\lambda(k).\nabla_{x_1}, \\
G_{1,0}^2(k,\lambda) = G_{0,1}^2(k,\lambda)^* &= \frac{-iZ^2}{2m_2\Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(k)}{2\pi\sqrt{|k|}} e^{-iZq^2k.x_2} \varepsilon_\lambda(k).\nabla_{x_2},
\end{aligned} \tag{18}$$

$$\begin{aligned}
G_{2,0}^1(k,\lambda;k',\lambda') &= G_{0,2}^1(k,\lambda;k',\lambda')^* \\
&= \frac{Z^2}{2m_1\Lambda^3} \frac{\widehat{\chi}_\Lambda(k)\widehat{\chi}_\Lambda(k')}{4\pi^2\sqrt{|k||k'|}} \varepsilon_\lambda(k).\varepsilon_{\lambda'}(k') e^{-iZq^2k.x_1} e^{-iZq^2k'.x_1}, \\
G_{2,0}^2(k,\lambda;k',\lambda') &= G_{0,2}^2(k,\lambda;k',\lambda')^* \\
&= \frac{Z^4}{2m_2\Lambda^3} \frac{\widehat{\chi}_\Lambda(k)\widehat{\chi}_\Lambda(k')}{4\pi^2\sqrt{|k||k'|}} \varepsilon_\lambda(k).\varepsilon_{\lambda'}(k') e^{-iZq^2k.x_2} e^{-iZq^2k'.x_2},
\end{aligned} \tag{19}$$

$$\begin{aligned}
G_{1,1}^1(k,\lambda;k',\lambda') &= \frac{Z^2}{2m_1\Lambda^3} \frac{\widehat{\chi}_\Lambda(k)\widehat{\chi}_\Lambda(k')}{2\pi^2\sqrt{|k||k'|}} \varepsilon_\lambda(k).\varepsilon_{\lambda'}(k') e^{-iZq^2k.x_1} e^{iZq^2k'.x_1}, \\
G_{1,1}^2(k,\lambda;k',\lambda') &= \frac{Z^4}{2m_2\Lambda^3} \frac{\widehat{\chi}_\Lambda(k)\widehat{\chi}_\Lambda(k')}{2\pi^2\sqrt{|k||k'|}} \varepsilon_\lambda(k).\varepsilon_{\lambda'}(k') e^{-iZq^2k.x_2} e^{iZq^2k'.x_2},
\end{aligned} \tag{20}$$

and finally

$$\Lambda_0^1 = \frac{Z^2}{4\pi^2 m_1 \Lambda^3} \int_{\mathbb{R}^3} \frac{\widehat{\chi}_\Lambda(k)}{|k|} dk \quad , \quad \Lambda_0^2 = \frac{Z^4}{4\pi^2 m_2 \Lambda^3} \int_{\mathbb{R}^3} \frac{\widehat{\chi}_\Lambda(k)}{|k|} dk. \tag{21}$$

2.2 Hypotheses on the confining potential

Recall that the constants M, μ, the variables R, P of the center of mass and the relative variables r, p are defined in (2)-(3).

Let us state the assumptions on the confining potential U that we require in this paper. The first hypothesis, $(\mathcal{H}_0)$, insures that U is confining and well behaves; it is needed in [AF] in order to prove the existence of a ground state for H_U^V. Furthermore, $(\mathcal{H}_0)$ insures that $\mathrm{C}_0^\infty(\mathbb{R}^3)$ is a core for $P^2/2M + U$. The second hypothesis, $(\mathcal{H}_1)$, is related to complex scaling and is needed to verify that $H_U^V(\theta)$ is an analytic family of type (B) (see subsection 3.1). Finally, the last one, $(\mathcal{H}_2)$, is required in section 4 where we prove the existence of resonances for H_U^V. More precisely, we set:

$$(\mathcal{H}_0) \begin{cases} (i) & U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3), \\ (ii) & \inf(U(R)) > -\infty \text{ and } U^- \text{ is compactly supported,} \\ (iii) & P^2/2M + U \text{ has a non-degenerate ground state } \phi > 0 \text{ with energy } e_0 < 0, \\ & \text{and there exists } \gamma \text{ such that } |\phi(R)| \le \gamma e^{-|R|/\gamma}. \end{cases}$$

Before stating $(\mathcal{H}_1)$, we recall that $Q(A)$ denotes the domain of the quadratic form associated with the self-adjoint operator A. For $\theta \in D(0, \theta_0)$ where θ_0 is sufficiently small, let us define the quadratic form $q_{e^{-2\theta}\frac{P^2}{2M} + U(e^\theta \cdot)}$ on $Q(P^2/2M + U)$ by

$$q_{e^{-2\theta}\frac{P^2}{2M} + U(e^\theta \cdot)}(\phi, \psi) := \frac{e^{-2\theta}}{2M} \int_{\mathbb{R}^3} (\overline{P\phi})(R)(P\psi)(R) dR + \int_{\mathbb{R}^3} U(e^\theta R)\overline{\phi}(R)\psi(R) dR. \tag{22}$$

Note that henceforth, we assume that U is an analytic function defined on

$$\left\{ z \in \mathbb{C}^3, z = e^{i\nu} x, |\nu| \le \theta_0, x \in \mathbb{R}^3 \right\}.$$

Besides, the fact that $q_{e^{-2\theta}\frac{P^2}{2M} + U(e^\theta \cdot)}$ is well-defined is guaranteed by the following hypothesis:

$$(\mathcal{H}_1) \begin{cases} (i) & \text{There exists } C(\theta_0) \in \mathbb{C} \text{ such that } C(\theta_0) \underset{\theta_0 \to 0}{\to} 0 \text{ and} \\ & \forall \theta_0 \in D(0, \theta_0), \forall R \in \mathbb{R}^3, \left| e^{2\theta} U(e^\theta R) - U(R) \right| \le C(\theta_0) |U(R)|, \\ (ii) & \forall \Psi \in Q(\frac{P^2}{2M} + U), \theta \mapsto q_{e^{-2\theta}\frac{P^2}{2M} + U(e^\theta \cdot)}(\Psi, \Psi) \text{ is analytic on } D(0, \theta_0). \end{cases}$$

Finally we require:

$$(\mathcal{H}_2) \quad U(R) \ge c_0 R^2 - c_1 \text{ for all } R \in \mathbb{R}, \text{ where } c_0, c_1 > 0.$$

Note that hypothesis $(\mathcal{H}_2)$ implies that the spectrum of $P^2/2M+U$ is purely discrete.
We shall henceforth assume that U fulfills the three hypotheses $(\mathcal{H}_0)$, $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$. For instance, we see that the potential U defined by $U(R)=c_0R^2-c_1$ (where $c_0,c_1>0$) fulfils these three hypotheses.

Let us describe the spectrum of the unperturbed Hamiltonian H_0. First, the spectrum of $p^2/2\mu+V$ consists of an infinity of eigenvalues $E_0<E_1<\cdots<E_n<\cdots<0$ and of the essential spectrum $[0;\infty[$. Next, according to hypothesis $(\mathcal{H}_2)$, we can write the spectrum of $P^2/2M+U$ as $e_0<e_1<\cdots<e_n<\ldots$. Finally, it is well-known that the spectrum of H_f consists of the simple eigenvalue 0, and the half-axis $[0;\infty[$ as absolutely continuous spectrum. Thus, the equality

$$\sigma(\mathcal{H}_0)=\sigma(p^2/2\mu+V)+\sigma(P^2/2M+U)+\sigma(H_f) \tag{23}$$

shows that the continuous spectrum of H_0 consists on one hand of the union of branches $[E_l+e_n,\infty[$ where E_l+e_n are eigenvalues, and on the other hand of the union of branches $[e_n,\infty[$ (see fig. 1).

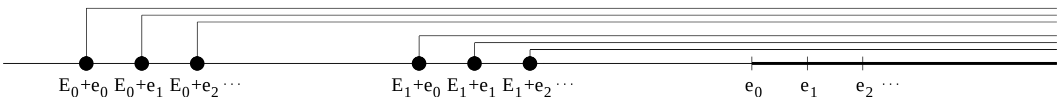

Figure 1: **Spectrum of the unperturbed Hamiltonian** H_0

3 Complex scaling and instability of excited states

In this section, in the same way as in [BFS3], we shall scale the electron and nucleus positions through

$$x_j\mapsto e^{\theta}x_j, \tag{24}$$

and the photons momenta through

$$k\mapsto e^{-\theta}k. \tag{25}$$

The scaling parameter θ will be assumed to lie in a disc $D(0,\theta_0)\subset\mathbb{C}$, with a sufficiently small radius θ_0. Yet, we notice that for θ real, the transformations (24)-(25) determine a unitary operator $\mathcal{U}_\theta$ such that:

$$\begin{aligned}
&* \quad \mathcal{U}_\theta\left(\frac{p^2}{2\mu}+V\right)\mathcal{U}_\theta^*=e^{-2\theta}\frac{p^2}{2\mu}+e^{-\theta}V,\\
&* \quad \mathcal{U}_\theta\left(\frac{P^2}{2M}+U\right)\mathcal{U}_\theta^*=e^{-2\theta}\frac{P^2}{2M}+U(e^{\theta}\cdot),\\
&* \quad \mathcal{U}_\theta H_f\mathcal{U}_\theta^*=e^{-\theta}H_f.
\end{aligned} \tag{26}$$

Moreover, $\mathcal{U}_\theta W_g \mathcal{U}_\theta^*$ is W_g where the operator-valued functions $G^j_{m,n}$ are replaced by $G^j_{m,n}(\theta)$; we get $G^j_{m,n}(\theta)$ by adding a factor $e^{-2\theta}$ and replacing $\widehat{\chi}_\Lambda(k)$ by $\widehat{\chi}_\Lambda(e^{-\theta}k)$ in $G^j_{m,n}$. For instance:

$$G^1_{1,0}(k,\lambda,\theta) = G^1_{0,1}(k,\lambda,\overline{\theta})^* = e^{-2\theta}\frac{iZ}{2m_1\Lambda^{3/2}}\frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}}e^{-iZq^2k.x_1}\varepsilon_\lambda(k).\nabla_{x_1}. \tag{27}$$

This complex scaling will allow us to prove theorem 3.1, that is to say that the unperturbed eigenstates associated with the eigenvalues $E_l + e_n$ become unstable when the coupling W_g is added.

In the first subsection, we define precisely the analytic family $H^V_U(\theta)$.

3.1 Definition and analyticity of $H^V_U(\theta)$

Since the Hamiltonian H^V_U is defined in [AF] using quadratic forms, we shall prove here that, through complex scaling, the family $H^V_U(\theta)$ is *analytic of type* (B). Recall that we require hypotheses $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$ (hypothesis $(\mathcal{H}_2)$ is not necessary in this section); then, we shall get the result with the help of the following sequence of lemmata.

Lemma 3.1 *Define the quadratic form q_θ on $\mathrm{H}^1(\mathbb{R}^3)$ by*

$$q_\theta(\phi,\psi) := \frac{e^{-2\theta}}{2\mu}(p\phi,p\psi) - e^{-\theta}\left((V^-)^{1/2}\phi,(V^-)^{1/2}\psi\right). \tag{28}$$

Then, for a sufficiently small θ_0, q_θ is a strictly m-sectorial quadratic form on $D(0,\theta_0)$, with form domain $\mathrm{H}^1(\mathbb{R}^3)$. Moreover, $\theta \mapsto e^{-2\theta}p^2/2\mu + e^{-\theta}V$ is analytic of type (B) *on $D(0,\theta_0)$.*

Proof First, the quadratic form q_θ given in (28) is well-defined since $\mathrm{H}^1(\mathbb{R}^3) \subset D((V^-)^{1/2})$: actually, we even know that for all $a > 0$, there exists $b > 0$ such that

$$\left((V^-)^{1/2}\psi,(V^-)^{1/2}\psi\right) \le a(p\psi,p\psi) + b(\psi,\psi),$$

for any ψ in $\mathrm{H}^1(\mathbb{R}^3)$. This inequality shows that q_θ is closed on $\mathrm{H}^1(\mathbb{R}^3)$ for all θ in $D(0,\theta_0)$ since, picking ψ_n in $\mathrm{H}^1(\mathbb{R}^3)$ such that $\psi_n \to \psi$ and $q_\theta(\psi_n-\psi_m,\psi_n-\psi_m) \to 0$, we have

$$|q_\theta(\psi_n-\psi_m,\psi_n-\psi_m)| \ge \left(\frac{\left|e^{-2\theta}\right|}{2\mu} - a\left|e^{-\theta}\right|\right)\|p(\psi_n-\psi_m)\|^2 - \left|e^{-\theta}\right| b\|\psi_n-\psi_m\|^2.$$

Hence, if a is chosen so small that $\left|e^{-2\theta}\right|/2\mu - a\left|e^{-\theta}\right| > 0$ for all θ in $D(0,\theta_0)$, then $\|p(\psi_n - \psi_m)\| \to 0$, so that $\psi \in \mathrm{H}^1(\mathbb{R}^3)$.
Now, it is easy to see that for all $\theta \in D(0,\theta_0)$, $\left|\arg[e^{2\theta}q_\theta(\psi,\psi) + (E_0+1)(\psi,\psi)]\right| \le \theta_1$, with

$0 < \theta_1 < \pi/2$, provided θ_0 is chosen sufficiently small.
Thus, q_θ is strictly m-sectorial on $D(0,\theta_0)$, with domain $\mathrm{H}^1(\mathbb{R}^3)$, and we denote the strictly m-sectorial operator associated with this quadratic form by $e^{-2\theta}p^2/2\mu + e^{-\theta}V$.
Finally, since $\theta \mapsto q_\theta(\psi,\psi)$ is analytic on $D(0,\theta_0)$ for all $\psi \in \mathrm{H}^1(\mathbb{R}^3)$, we conclude that $e^{-2\theta}p^2/2\mu + e^{-\theta}V$ is an analytic family of type (B) on $D(0,\theta_0)$. □

Remark 3.1 *The strictly m-sectorial operator associated with the quadratic form defined in* (28) *is* $e^{-2\theta}p^2/2\mu + e^{-\theta}V$*, with domain* $\mathrm{H}^2(\mathbb{R}^3)$.

Proof To determine the domain of the operator $e^{-2\theta}p^2/2\mu+e^{-\theta}V$ defined in the previous lemma, we note that we could have defined it directly on $\mathrm{H}^2(\mathbb{R}^3)$ by

$$\left(e^{-2\theta}\frac{p^2}{2\mu} + e^{-\theta}V\right)\psi := e^{-2\theta}\frac{p^2}{2\mu}\psi + e^{-\theta}V\psi.$$

Next, seeing that this operator is m-sectorial (in the sense that it is closed and $\Re(\psi,(p^2/2\mu + e^{\theta}V)\psi) \geq 0$ for all $\psi \in \mathrm{H}^2(\mathbb{R}^3)$), we could have associated with it a strictly m-sectorial quadratic form $\hat{q}_\theta$. Then $\mathrm{H}^2(\mathbb{R}^3)$ is a form core for $\hat{q}_\theta$, and it is easy to see that it is also a form core for q_θ. Hence we have two closed quadratic form that coincide on a domain that is a form core for the two of them, so that $q_\theta = \hat{q}_\theta$, and in particular $D(e^{-2\theta}p^2/2\mu + e^{-\theta}V) = \mathrm{H}^2(\mathbb{R}^3)$. □

Lemma 3.2 *Pick θ_0 sufficiently small. Then the quadratic form defined in* (22) *is strictly m-sectorial and the family $\theta \mapsto e^{-2\theta}P^2/2M + U(e^{\theta}\cdot)$ is analytic of type* (B) *on $D(0,\theta_0)$ with form domain $Q(P^2/2M + U)$.*

Proof First, we have to show that $q_{e^{-2\theta}\frac{P^2}{2M}+U(e^{\theta}\cdot)}$ is closed on $Q(P^2/2M+U)$. To this end, pick ψ_n in $Q(P^2/2M+U)$ such that $\psi_n \to \psi$ and $q_{e^{-2\theta}\frac{P^2}{2M}+U(e^{\theta}\cdot)}(\psi_n-\psi_m,\psi_n-\psi_m) \to 0$. Then we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\frac{1}{2M}\overline{P(\psi_n-\psi_m)}(R)P(\psi_n-\psi_m)(R)dR \\
&+(1-C(\theta_0))\int_{\mathbb{R}^3}U^+(R)(\overline{\psi_n-\psi_m})(R)(\psi_n-\psi_m)(R)dR \\
&-(1+C(\theta_0))\inf(U)\|\psi_n-\psi_m\|^2 \\
&\leq \left|e^{2\theta}q_{e^{-2\theta}\frac{P^2}{2M}+U(e^{\theta}\cdot)}(\psi_n-\psi_m,\psi_n-\psi_m)\right| \underset{n,m\to\infty}{\to} 0,
\end{aligned}$$

where $C(\theta_0)$ is defined in hypothesis $(\mathcal{H}_1)$.
Thus, $\psi \in Q(P^2/2M+U)$ and the quadratic form $q_{e^{-2\theta}\frac{P^2}{2M}+U(e^{\theta}\cdot)}$ is closed.
Now, we note that for all $\psi \in Q(P^2/2M+U)$,

$$\arg\left[e^{2\theta}q_{e^{-2\theta}\frac{P^2}{2M}+U(e^{\theta}\cdot)}(\psi,\psi) + (1+2\inf(U))(\psi,\psi)\right] \leq \theta_2,$$

with $0 < \theta_2 < \pi/2$ provided θ_0 is chosen sufficiently small.
Hence, $q_{e^{-2\theta}\frac{P^2}{2M}+U(e^\theta\cdot)}$ is a strictly m-sectorial quadratic form for any $\theta \in D(0,\theta_0)$. We denote by $e^{-2\theta}P^2/2M + U(e^\theta\cdot)$ the m-sectorial operator associated with it. Then, the hypothesis $(\mathcal{H}_1)(ii)$ shows that $\theta \mapsto e^{-2\theta}P^2/2M + U(e^\theta\cdot)$ is an analytic family of type (B) on $D(0,\theta_0)$, with form domain $Q(P^2/2M + U)$. □

Now, we want to define $H_{at}(\theta) := \big(e^{-2\theta}p^2/2\mu + e^{-\theta}V\big) \otimes I + I \otimes \big(e^{-2\theta}P^2/2M + U(e^\theta\cdot)\big)$ in such a way that $\sigma(H_{at}(\theta)) = \sigma(e^{-2\theta}p^2/2\mu + e^{-\theta}V) + \sigma(e^{-2\theta}P^2/2M + U(e^\theta\cdot))$. It suffices to apply Ichinose's lemma (see [RS4]), which we do now:

Lemma 3.3 *Let $H_{at}(\theta)$ denote the closure of $\big(e^{-2\theta}p^2/2\mu + e^{-\theta}V\big)\otimes I + I\otimes\big(e^{-2\theta}P^2/2M + U(e^\theta\cdot)\big)$ on $D(e^{-2\theta}p^2/2\mu + e^{-\theta}V) \otimes D(e^{-2\theta}P^2/2M + U(e^\theta\cdot)) \subset \mathrm{L}^2(\mathbb{R}^3) \otimes \mathrm{L}^2(\mathbb{R}^3)$. Then, for a sufficiently small θ_0, $H_{at}(\theta)$ is an analytic family of type* (B) *on* $D(0,\theta_0)$ *with form domain $Q(p_1^2 + p_2^2) \cap Q(U^+)$. Moreover, $\sigma(H_{at}(\theta)) = \sigma(e^{-2\theta}p^2/2\mu + e^{-\theta}V) + \sigma(e^{-2\theta}P^2/2M + U(e^\theta\cdot))$.*

Proof In order to apply Ichinose's lemma, we have to verify that we can suitably choose the sectors associated with the m-sectorial operators $e^{-2\theta}p^2/2\mu + e^{-\theta}V$ and $e^{-2\theta}P^2/2M + U(e^\theta\cdot)$. More precisely, it is easy to see that

$$S_{-E_0-1;-2\mathrm{Im}\theta;\theta_1} := \{z, -2\mathrm{Im}\theta - \theta_1 \leq \arg(z - E_0 - 1) \leq -2\mathrm{Im}\theta + \theta_1\}$$

is a sector for $e^{-2\theta}p^2/2\mu + e^{-\theta}V$.
Similarly, using hypothesis $(\mathcal{H}_1)$, we can see that

$$S_{-2\inf(U)-1;-2\mathrm{Im}\theta;\theta_2} := \{z, -2\mathrm{Im}\theta - \theta_2 \leq \arg(z - 2\inf(U) - 1) \leq -2\mathrm{Im}\theta + \theta_2\}$$

is a sector for $e^{-2\theta}P^2/2M + U(e^\theta\cdot)$.
Thus, we can apply Ichinose's lemma, which yields

$$\sigma(H_{at}(\theta)) = \sigma(e^{-2\theta}p^2/2\mu + e^{-\theta}V) + \sigma(e^{-2\theta}P^2/2M + U(e^\theta\cdot)).$$

Now, we define the quadratic form $q_{H_{at}(\theta)}$ on $Q(p_1^2 + p_2^2) \cap Q(U^+)$ by

$$\begin{aligned} q_{H_{at}(\theta)}(\phi,\psi) := \frac{e^{-2\theta}}{2m}\int_{\mathbb{R}^6}(\overline{p\phi})(r,R)(p\psi)(r,R)drdR + e^{-\theta}\int_{\mathbb{R}^6}V(r)\overline{\phi}(r,R)\psi(r,R)drdR \\ + \frac{e^{-2\theta}}{2M}\int_{\mathbb{R}^6}(\overline{P\phi})(r,R)(P\psi)(r,R)drdR + \int_{\mathbb{R}^6}U(e^\theta R)\overline{\phi}(r,R)\psi(r,R)drdR. \end{aligned} \tag{29}$$

Using hypotheses $(\mathcal{H}_0)$ and $(\mathcal{H}_1)$, we can see that $q_{H_{at}(\theta)}$ is well defined, that it is closed on $Q(p_1^2+p_2^2)\cap Q(U^+)$, that $\mathrm{C}_0^\infty(\mathbb{R}^6)$ is a form core for $q_{H_{at}(\theta)}$, and finally that $q_{H_{at}(\theta)}$ is a strictly m-sectorial quadratic form. Thus, there is a unique operator $\widetilde{H}_{at}(\theta)$ associated with this quadratic form such that

a) $\widetilde{H}_{at}(\theta)$ is closed ,
b) $D(\widetilde{H}_{at}(\theta)) \subset Q(q_{H_{at}}(\theta))$ and for all $\phi,\psi \in D(\widetilde{H}_{at}(\theta)), q_{H_{at}(\theta)}(\phi,\psi) = (\phi, \widetilde{H}_{at}(\theta)\psi)$,
c) $D(\widetilde{H}_{at}(\theta)^*) \subset Q(q_{H_{at}}(\theta))$ and for all $\phi,\psi \in D(\widetilde{H}_{at}(\theta)^*), q_{H_{at}(\theta)}(\phi,\psi) = (\widetilde{H}_{at}(\theta)^*\phi,\psi)$

(see [RS1], theorem VIII.16). But, using the fact that $H_{at}(\theta)^* = H_{at}(\overline{\theta})$, one can verify that $H_{at}(\theta)$ fulfills these three properties, so that we have $H_{at}(\theta) = \widetilde{H}_{at}(\theta)$; in particular, $Q(H_{at}(\theta)) = Q(p_1^2 + p_2^2) \cap Q(U^+)$.
To conclude, with the help of the hypothesis $(\mathcal{H}_1)(ii)$, we get that $H_{at}(\theta)$ is an analytic family of type (B) on $D(0, \theta_0)$ with form domain $Q(p_1^2 + p_2^2) \cap Q(U^+)$. $\square$

Remark 3.2 *We note here an inclusion that will be useful in the sequel:*

$$D(H_{at}) \subset D(H_{at}(\theta)). \tag{30}$$

Proof Pick ϕ in $D(H_{at})$, $\|\phi\| = 1$. Then for all ψ in $D(H_{at}(\theta)^*)$, we have

$$\begin{aligned}
&|(H_{at}(\theta)^*\psi, \phi)| = \left|q_{H_{at}(\theta)}(\psi, \phi)\right| \\
&\le \left| \frac{e^{-2\theta}}{2m} \int_{\mathbb{R}^6} (\overline{p\phi})(r,R)(p\psi)(r,R)drdR + e^{-\theta} \int_{\mathbb{R}^6} V(r)\overline{\phi}(r,R)\psi(r,R)drdR \right. \\
&\quad \left. + \frac{e^{-2\theta}}{2M} \int_{\mathbb{R}^6} (\overline{P\phi})(r,R)(P\psi)(r,R)drdR + \int_{\mathbb{R}^6} U(R)\overline{\phi}(r,R)\psi(r,R)drdR \right| \\
&\quad + C(\theta_0)\|\phi\|\|\psi\| \\
&= \left| \frac{e^{-2\theta}}{2m}(p^2\phi, \psi) + e^{-\theta}(V\phi, \psi) + \frac{e^{-2\theta}}{2M}(P\phi, P\psi) + (U^-\phi, \psi) + ((U^+)^{1/2}\phi, (U^+)^{1/2})\psi) \right| \\
&\quad + C(\theta_0)\|\phi\|\|\psi\| \\
&\le \mathrm{Cste}\|\psi\|,
\end{aligned}$$

so that $\phi \in D(H_{at}(\theta))$. $\square$

The next step in the definition of $H_U^V(\theta)$ is the construction of $H_0(\theta) := H_{at}(\theta) \otimes I + I \otimes e^{-\theta} H_f$, in such a way that $\sigma(H_0(\theta)) = \sigma(H_{at}(\theta)) + \sigma(e^{-\theta} H_f)$. The method to do this is similar to what was done to define $H_{at}(\theta)$ in lemma 3.3, so we do not reproduce the proof.

Lemma 3.4 *Let $H_0(\theta)$ denote the closure of $H_{at}(\theta) \otimes I + I \otimes e^{-\theta} H_f$ on $D(H_{at}(\theta)) \otimes D(e^{-\theta} H_f) \subset \mathrm{L}^2(\mathbb{R}^6) \otimes \mathcal{F}_s$. Then, for a sufficiently small θ_0, $H_0(\theta)$ is an analytic family of type* (B) *on $D(0, \theta_0)$ with form domain $Q(p_1^2 + p_2^2) \cap Q(U^+) \cap Q(H_f)$. Moreover, $\sigma(H_0(\theta)) = \sigma(H_{at}(\theta)) + \sigma(e^{-\theta} H_f)$.*

Proof Note that we have $Q(e^{-\theta} H_f) = Q(H_f)$ and for all $\psi \in Q(H_f)$,

$$q_{e^{-\theta} H_f}(\psi, \psi) := e^{-\theta} q_{H_f}(\psi, \psi) \in e^{-\theta} \mathbb{R}_+.$$

Thus, it is easy to follow the proof of lemma 3.3 in order to define $H_0(\theta)$ with the properties stated above. $\square$

Finally, we have to define the perturbative part $W_g(\theta)$. We recall from [AF] that W_g is given as a quadratic form, that we denote by $\hat{q}_{W_g}$, defined on $Q(p_1^2+p_2^2)\cap Q(H_f)$ by

$$\hat{q}_{W_g}(\Phi,\Psi) = -\sum_{j=1,2} q_j\left[(p_j\Phi, A_j\Psi) + (A_j\Phi, p_j\Psi)\right] + \sum_{j=1,2} q_j^2 (A_j\Phi, A_j\Psi). \tag{31}$$

Moreover, we have

$$\left|\hat{q}_{W_g}(\Phi,\Phi)\right| \leq C(g) q_{H_0^0}(\Phi,\Phi) + b(\Phi,\Phi), \tag{32}$$

for all $\Phi \in Q(p_1^2+p_2^2)\cap Q(H_f)$, where $C(g), b > 0$ and $C(g)\to 0$ as $g \to 0$. Note that we have written H_0^0 for H_U^V where we have made $U = V = 0$ (see the proof of lemma 2.1 in [AF]).

Now, we recall that, when θ is real, $W_g(\theta) := \mathcal{U}_\theta W_g \mathcal{U}_\theta^*$ is obtained by adding some factors $e^{-2\theta}$ and replacing $\widehat{\chi}_\Lambda(k)$ by $\widehat{\chi}_\Lambda(e^{-\theta}k)$ in W_g. Henceforth, we assume that the following assumption on $\widehat{\chi}_\Lambda$ is fulfilled:

$$(\mathcal{H}_{\widehat{\chi}_\Lambda})\left\{\begin{array}{l}\widehat{\chi}_\Lambda \text{ is an analytic function defined on } \left\{z\in\mathbb{C}^3, z = e^{i\nu}x, |\nu|\leq\theta_0, x\in\mathbb{R}^3\right\}\\ \text{and supported on some compact } \mathrm{K}_\Lambda.\end{array}\right.$$

Then, there is a natural definition for $W_g(\theta)$, when θ is complex, as a quadratic form $q_{W_g(\theta)}(\Phi,\Psi)$ defined on $Q(p_1^2+p_2^2)\cap Q(H_f)\cap Q(U^+)$. For instance, the term associated with $G_{1,0}^1(k,\lambda;\theta)$ (see (27)) is given by

$$g e^{-2\theta}\frac{iZ}{2m_1\Lambda^{3/2}}\sum_{\lambda=1,2}\int_{\mathbb{R}^3}\frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}}e^{-iZq^2k.x_1}\left(\varepsilon_\lambda(k).\nabla_{x_1}\Phi, \widehat{a}_\lambda^*(k)\Psi\right)dk. \tag{33}$$

Then, in the same way as for (32), it is easy to show:

Lemma 3.5 *$q_{W_g(\theta)}$ is relatively bounded with respect to $q_{H_0(\theta)}$ with relative bound strictly less than 1 provided that g is sufficiently small.*

Proof We want to show that there exists $0 < a < 1$ and $0 < b$ such that

$$\left|q_{W_g(\theta)}(\Phi,\Phi)\right| \leq a\left|q_{H_0(\theta)}(\Phi,\Phi)\right| + b(\Phi,\Phi). \tag{34}$$

We write the proof for the term given in (33); the terms associated with the other functions $G_{m,n}^j$ can be treated in the same way. Using hypothesis $(\mathcal{H}_{\widehat{\chi}_\Lambda})$, we have:

$$\begin{aligned}&\left|g e^{-2\theta}\frac{iZ}{2m_1\Lambda^{3/2}}\sum_{\lambda=1,2}\int_{\mathbb{R}^3}\frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}}e^{-iZq^2k.x_1}\left(\varepsilon_\lambda(k).\nabla_{x_1}\Phi, \widehat{a}_\lambda^*(k)\Phi\right)dk\right|\\ &\leq g\frac{Z}{2\pi m_1\Lambda^{3/2}}\sum_{\lambda=1,2}\int_{\mathrm{K}_\Lambda}\frac{\mathrm{Cste}}{2\pi\sqrt{|k|}}\frac{1}{2}\left[(\nabla_{x_1}\Phi,\nabla_{x_1}\Phi) + (\widehat{a}_\lambda^*(k)\Phi, \widehat{a}_\lambda^*(k)\Phi)\right]dk,\end{aligned}$$

Then, in the same way as in (32), the last expression is bounded by $C'(g)q_{H_0^0}(\Phi,\Phi) + b'(\Phi,\Phi)$, with $C'(g), b' > 0$ and $C'(g)\to 0$ as $g\to 0$. From there, we easily conclude that

$$\begin{aligned}&\left|g e^{-2\theta}\frac{iZ}{2m_1\Lambda^{3/2}}\sum_{\lambda=1,2}\int_{\mathbb{R}^3}\frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}}e^{-iZq^2k.x_1}\left(\varepsilon_\lambda(k).\nabla_{x_1}\Phi, \widehat{a}_\lambda^*(k)\Phi\right)dk\right|\\ &\leq a\left|q_{H_0(\theta)}(\Phi,\Phi)\right| + b''(\Phi,\Phi),\end{aligned}$$

with $b'' > 0$ and $0 < a < 1$ provided g is sufficiently small. □

Remark 3.3 *The previous lemma does not give a meaning to the* operator $W_g(\theta)$. *Yet, we will have to work with it in the sequel so that we have to precise here what we call* $W_g(\theta)$. *Recalling the expression* (16)-(21) *of* W_g, *we define* $W_g(\theta)$ *on the domain* $D(p_1^2 + p_2^2) \cap D(H_f)$ *by*

$$W_g(\theta) := e^{-2\theta} g W_1(\theta) + e^{-2\theta} g^2 W_2(\theta), \tag{35}$$

where $W_1(\theta)$ *and* $W_2(\theta)$ *denote respectively* W_1 *and* W_2 *where the function* $\widehat{\chi}_\Lambda$ *is replaced by* $\widehat{\chi}_\Lambda(e^{-\theta}\cdot)$. *Note that it is easy to verify that* $W_g(\theta)$ *is well defined on* $D(p_1^2 + p_2^2) \cap D(H_f)$.

To conclude, as a corollary of lemmata 3.1-3.5, we write the result stated at the beginning of this subsection:

Proposition 3.1 *Assume that* θ_0 *and* g *are sufficiently small. Define* $H_U^V(\theta)$ *as the strictly m-sectorial operator associated with the quadratic form*

$$q_{H_U^V(\theta)} := q_{H_0(\theta)} + q_{W_g(\theta)}. \tag{36}$$

Then, $H_U^V(\theta)$ *is an analytic family of type (B) on* $D(0, \theta_0)$ *with form domain* $Q(p_1^2+p_2^2) \cap Q(U^+) \cap Q(H_f)$.

Proof With our definitions of U and $\widehat{\chi}_\Lambda$, it is easy to see that for all Φ in $Q(p_1^2 + p_2^2) \cap Q(U^+) \cap Q(H_f)$, $\theta \mapsto q_{H_U^V(\theta)}(\Phi, \Phi)$ is analytic on $D(0, \theta_0)$.
We conclude with lemmata 3.1-3.5. □

3.2 The spectrum of H_U^V in a neighborhood of $E_j + e_k$

In this subsection, we shall show that the spectrum of H_U^V in a neighborhood of the unperturbed eigenvalues $E_j + e_k$ is absolutely continuous. The proof follows the one in [BFS3], so we shall not reproduce all the details here. In lemma 3.6, we derive a useful estimate. Next we describe precisely the spectrum of the unperturbed Hamiltonian $H_0(\theta)$, and we state the main result of this section. We refer the reader to [BFS3] for the detail of the proof.
Let us begin with a lemma whose proof differs from the one in [BFS3] because of our use of quadratic forms:

Lemma 3.6 *For* $\theta \in D(0, \theta_0)$, *define* $\Delta H_{at}(\theta) := H_{at}(\theta) - H_{at}$. *Then there exists a positive constant* $b(\theta)$ *such that* $b(\theta) \to 0$ *as* $\theta \to 0$ *and*

$$\left\| \Delta H_{at}(\theta)(H_{at} \pm i)^{-1} \right\| \leq b(\theta). \tag{37}$$

Proof First, we note that, by remark 3.2, the operator $\Delta H_{at}(\theta)(H_{at} \pm i)^{-1}$ is well defined on $L^2(\mathbb{R}^6)$. Then we have

$$\begin{aligned}\left\|\Delta H_{at}(\theta)(H_{at} \pm i)^{-1}\right\| &= \sup_{\substack{\phi \in Q(H_{at}), \psi \in L^2(\mathbb{R}^6) \\ \|\phi\|, \|\psi\|=1}} \left|(\phi, \Delta H_{at}(\theta)(H_{at} \pm i)^{-1}\psi)\right| \\ &= \sup_{\substack{\phi \in Q(H_{at}), \psi \in L^2(\mathbb{R}^6) \\ \|\phi\|, \|\psi\|=1}} \left|q_{H_{at}(\theta)}(\phi, (H_{at} \pm i)^{-1}\psi) - q_{H_{at}}(\phi, (H_{at} \pm i)^{-1}\psi)\right|.\end{aligned}$$

Next, using (29) and hypothesis $(\mathcal{H}_1)$, we get

$$\begin{aligned}&\left|q_{H_{at}(\theta)}(\phi, (H_{at} \pm i)^{-1}\psi) - q_{H_{at}}(\phi, (H_{at} \pm i)^{-1}\psi)\right| \\ &\leq \|\phi\| \left\| \left[|e^{-2\theta} - 1| (\frac{p^2}{2\mu} + \frac{P^2}{2M}) + |e^{-\theta} - 1||V| + |U(e^{\theta}R) - U(R)| \right] (H_{at} \pm i)^{-1}\psi \right\| \\ &\leq b(\theta)\|\phi\|\|\psi\|,\end{aligned}$$

where $b(\theta)$ is a positive constant which goes to 0 as $\theta \to 0$. This yields (37). □

Let us now describe the spectrum of $H_{at}(\theta)$. Well-known results about complex scaling show that the spectrum of $e^{-2\theta}p^2/2\mu + e^{-\theta}V$ is given by

$$\sigma(e^{-2\theta}p^2/2\mu + e^{-\theta}V) = \sigma_{pp}(p^2/2\mu + V) \cup \left\{e^{-2\theta}\mu | \mu \in [0, \infty[\right\}. \tag{38}$$

Besides, assuming hypothesis $(\mathcal{H}_2)$ and using the fact that $e^{-2\theta}P^2/2M + U(e^{\theta}\cdot)$ is analytic of type (B), we have

$$\sigma(e^{-2\theta}P^2/2M + U(e^{\theta}\cdot)) = \sigma(P^2/2M + U). \tag{39}$$

Thus, by lemma 3.3, we get the spectrum of $H_{at}(\theta)$ (see fig. 2).

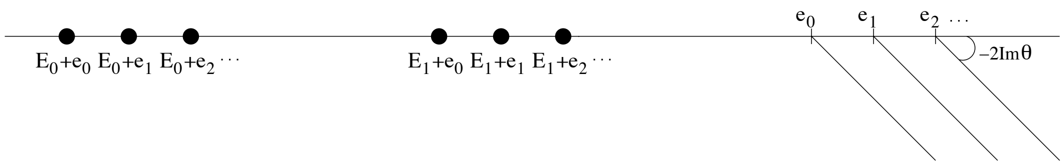

Figure 2: **Spectrum of the complex dilated atomic Hamiltonian** $H_{at}(\theta)$

We want to investigate the spectrum of H_U^V in a neighborhood of the isolated eigenvalues of H_{at}. Fix $l \geq 0$, $n \geq 0$ with $(l, n) \neq (0, 0)$ and consider the eigenvalue $E_l + e_n$ of $H_0(\theta)$. We assume moreover that $E_l + e_n < e_0$.

We set $R_{l,n} := \text{dist}(E_l + e_n, \sigma(H_{at}) \backslash \{E_l + e_n\})$; then, we shall use the projection $P_{at,l,n}(\theta)$ onto the eigenspace of $H_{at}(\theta)$ corresponding to $E_l + e_n$, defined by

$$P_{at,l,n}(\theta) := \frac{i}{2\pi} \int_{|z-(E_l+e_n)|=R_{j,k}/2} \frac{dz}{H_{at}(\theta) - z} \quad , \quad \overline{P}_{at,l,n}(\theta) := 1 - P_{at,l,n}(\theta). \tag{40}$$

As in [BFS3], we define the matrices

$$Z_{l,n}^{\mathrm{od}} := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} P_{at,l,n} G_{0,1}(k,\lambda) \overline{P}_{at,l,n} \left[H_{at} - (E_l + e_n) + |k| - i0\right]^{-1}$$

$$\overline{P}_{at,l,n} G_{1,0}(k,\lambda) P_{at,l,n} dk, \tag{41}$$

$$Z_{l,n}^{d} := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} P_{at,l,n} G_{0,1}(k,\lambda) P_{at,l,n} G_{1,0}(k,\lambda) P_{at,l,n} \frac{dk}{|k|},$$

where $G_{m,n}(k,\lambda) := \sum_{j=1,2} G_{m,n}^{j}(k,\lambda)$.
For $\theta \in D(0,\theta_0)$, we set

$$Z_{l,n}^{\mathrm{od}}(\theta) := \mathcal{U}_\theta Z_{l,n}^{\mathrm{od}} \mathcal{U}_\theta^{-1} \quad , \quad Z_{l,n}^{\mathrm{d}}(\theta) := \mathcal{U}_\theta Z_{l,n}^{\mathrm{d}} \mathcal{U}_\theta^{-1} \quad , \quad Z_{l,n}(\theta) := Z_{l,n}^{\mathrm{d}}(\theta) - Z_{l,n}^{\mathrm{od}}(\theta). \tag{42}$$

Let $\Gamma_{l,n} := \min\left\{\sigma(\mathrm{Im}(Z_{l,n}^{\mathrm{od}}))\right\}$; then for the needs of the proof, we have to require the following hypothesis, related to the Fermi golden rule:

$$(\mathcal{H}_{\Gamma_{l,n}}) \quad \Gamma_{l,n} > 0 \text{ is a simple eigenvalue of } \mathrm{Im}(Z_{l,n}^{\mathrm{od}}).$$

Let $\phi_{l,n,0}$ be the normalized eigenvector corresponding to the eigenvalue $\Gamma_{l,n}$ of $\mathrm{Im}(Z_{l,n}^{\mathrm{od}})$; we define

$$\Delta E_{l,n} := \mathrm{Re}\left[(\phi_{l,n0}, Z_{l,n}\phi_{l,n,0})\right]. \tag{43}$$

Let $\mathcal{S}_{l,n}(\varepsilon, C)$, $\mathcal{R}_{l,n}(\varepsilon, C)$ denote the following sets:

$$\begin{aligned} \mathcal{S}_{l,n}(\varepsilon, C) &:= E_l + e_n + g^2(\Delta E_{l,n} - i\Gamma_{l,n}) - i\mathcal{Q}_{l,n} \\ \mathcal{R}_{l,n}(\varepsilon, C) &:= \mathcal{S}_{l,n}(\varepsilon, C) + e^{-\theta}\mathbb{R}_+ + D(0, Cg^{2+\varepsilon}), \end{aligned} \tag{44}$$

where ε is a small positive constant, $C \in \mathbb{R}_+$ and $\mathcal{Q}_{l,n} := \{z \in \mathbb{C} | -\mu \le \arg(z) \le \mu\}$ (for some $0 < \mu < \pi/2$) is such that

$$\{(\phi, Z_{l,n}\phi), \|\phi\| = 1\} \subset \Delta E_{l,n} - i\Gamma_{l,n} - i\mathcal{Q}_{l,n}. \tag{45}$$

Finally, the set $\mathcal{A}_{l,n}(\varepsilon)$ is defined by

$$\mathcal{A}_{l,n}(\varepsilon) := \mathcal{I}_{l,n}(R_{l,n}/2) - i[-g^{2-\varepsilon}, \infty[, \tag{46}$$

where $\mathcal{I}_{l,n}(R_{l,n}/2)$ is the interval $]E_l + e_n - R_{l,n}/2; E_l + e_n + R_{l,n}/2[$ (see fig. 3).

Now, we state the main result of this subsection, derived from [BFS3]:

Theorem 3.1 *Let $0 < \varepsilon < 1/3$ and fix $\theta = i\nu$ in $D(0,\theta_0)$ with $\nu > 0$. Assume that $g > 0$ is sufficiently small. Assume moreover that $(\mathcal{H}_{\Gamma_{l,n}})$ is fulfilled. Then there exists C in $\mathbb{R}_+$ such that*

$$\mathcal{A}_{l,n}(\varepsilon) \backslash \mathcal{R}_{l,n}(\varepsilon, C) \subset \rho(H_U^V(\theta)), \tag{47}$$

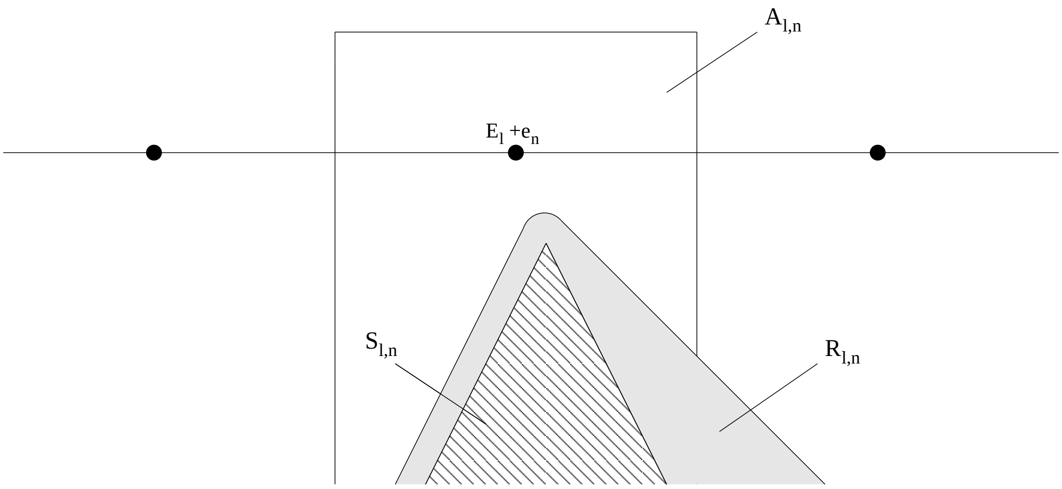

Figure 3: **The resolvent set of** $H_U^V(\theta)$ **:** $\mathcal{A}_{l,n}\backslash\mathcal{R}_{l,n} \subset \rho(H_U^V(\theta))$ **(see theorem** 3.1**)**

where $\rho(H_U^V(\theta))$ *denotes the resolvent set of* $H_U^V(\theta)$*.*
Since we have showed that the family $H_U^V(\theta)$ *is analytic of type* (B) *on* $D(0,\theta_0)$*,* (47) *implies that, for g sufficiently small, the spectrum of* H_U^V *is absolutely continuous in* $\mathcal{I}_{l,n}(R_{l,n}/2)$*.*

Proof The proof of this theorem follows step by step the one in [BFS3]. The main difficulty is to prove the existence of the *Feshbach operator* $\mathcal{F}_{P(\theta)}(H_U^V(\theta)-z)$ defined by

$$\begin{aligned}\mathcal{F}_{P(\theta)}(H_U^V(\theta)-z) :=& P(\theta)\left[H_U^V(\theta)-z\right]P(\theta)\\ &- P(\theta)W_g(\theta)\overline{P}(\theta)\left(\overline{P}(\theta)H_U^V(\theta)\overline{P}(\theta)-z\right)^{-1}\overline{P}(\theta)W_g(\theta)P(\theta),\end{aligned} \tag{48}$$

where $P(\theta) := P_{at,l,n}(\theta)\otimes \mathbf{1}_{H_f<\rho_0}$, $\overline{P}(\theta) := 1-P(\theta)$, and $\rho_0 := g^{2-2\varepsilon}$.
We refer to proposition 4.2 where we give an outline of the proof leading to the existence of $\mathcal{F}_{P(\theta)}(H_U^V(\theta)-z)$ (we shall work with the *smooth Feshbach map* (see [BCFS]) in proposition 4.2, but the same proof would hold for $\mathcal{F}_{P(\theta)}(H_U^V(\theta)-z)$). □

4 Renormalization group and existence of resonances

In the previous section, we showed that the spectrum of the Hamiltonian H_U^V is absolutely continuous in each interval $]E_l+e_n-R_{l,n}/2; E_l+e_n+R_{l,n}/2[$ such that $(l,n)\neq(0,0)$ and $E_l+e_n<e_0$ (under the condition that $(\mathcal{H}_{\Gamma_{l,n}})$ is fulfilled). More generally, the same arguments imply that the spectrum of H_U^V is absolutely continuous in the interval $]E_0+e_0+O(g), e_0-O(g)[$. In this section, under the same assumptions $(\mathcal{H}_{\Gamma_{l,n}})$, we shall show that the eigenvalues of H_0 turn into resonances when the perturbation W_g is added. In other words, we shall prove that

there exists complex eigenvalues $E_{l,n,k}(g)$ (with $1 \leq k \leq p$) of the complex dilated Hamiltonian $H_U^V(\theta)$ such that

$$E_{l,n,k}(g) \underset{g \to 0}{\to} E_l + e_n. \tag{49}$$

Here, p denotes the multiplicity of $E_l + e_n$ as an eigenvalue of H_0.

In order to get this result, we shall apply the renormalization group method developed in [BFS1,2]. Note that here, we appeal to the method developed in [BCFS] based on the smooth Feshbach map. Our aim is to define some conditions that are fulfilled by our model, and which allow to perform the renormalization group method. We emphasize that this point is not straightforward because the hypotheses stated in [BFS2] are not fulfilled in our case; we have to use carefully the fact that U confines the center of mass of the atomic system (hypothesis $(\mathcal{H}_2)$). Yet, we only give here the modifications that we have to bring and refer to [BFS1,2] or [BCFS] for the rest of the proof.

In the first subsection, we state the hypotheses that we require and verify that they are fulfilled by the Hamiltonian describing our model. We need to proceed to a Power-Zienau-Woolley transformation on H_U^V, and to impose, in a way similar to [BFS1], a spatial cutoff in the variable r in the resulting interaction term $\widetilde{W}_g$.

In the second subsection, we prove that we can apply the smooth Feshbach map defined in [BCFS] to the transformed Hamiltonian $\widetilde{H}_U^V(\theta)$.

Finally, in the last subsection, we show that the hypotheses stated in the first subsection allow to obtain, thanks to a renormalization group transformation, the existence of resonances for $\widetilde{H}_U^V$.

4.1 Conditions to perform a renormalization group analysis

Recall the definition of the dilated operator-valued functions $G^j_{m,n}(k,\lambda;\theta)$ given in (16)-(21), (27) and (33). In particular, $G^j_{m,n}(k,\lambda;\theta)$ is well defined as a quadratic form $q_{G^j_{m,n}(k,\lambda;\theta)}$ on $Q(p_1^2+p_2^2) \cap Q(U^+)$, but we have not yet defined $G^j_{m,n}(k,\lambda;\theta)$ as an operator. However, it is easy to see that, for $m+n=1$, $G^j_{m,n}(k,\lambda;\theta)$ is a well defined operator on the domain $D([p_1^2+p_2^2]^{1/2})$, and that, for $m+n=2$, $G^j_{m,n}(k,\lambda;k',\lambda';\theta)$ is well defined on $\mathrm{L}^2(\mathbb{R}^6)$.
Besides, with the help of the assumption $(\mathcal{H}_{\widehat{\chi}_\Lambda})$, we note that the map $\theta \mapsto q_{G^j_{m,n}(k,\lambda;\theta)}(\phi,\phi)$ is analytic on $D(0,\theta_0)$ for all $\phi \in Q(p_1^2+p_2^2) \cap Q(U^+)$.
Now, for the needs of the proof, we require the following hypotheses:

$$
(\mathcal{H}_{-1/2})\begin{cases} \text{There exists a non-negative function } J_{-1/2}(k) \text{ such that:} \\ (i) \quad \sup_{|\theta|\le\theta_0} \||H_{at}+i|^{-1/4}G_{m,n}((k,\lambda);\theta)|H_{at}+i|^{-1/4}\| \le J_{-1/2}(k) \text{ for } m+n=1, \\ (ii) \quad \sup_{|\theta|\le\theta_0} \|G_{m,n}((k,\lambda);(k',\lambda');\theta)\| \le J_{-1/2}(k)J_{-1/2}(k') \text{ for } m+n=2, \\ (iii) \quad \left[\int_{\mathbb{R}^3}\left(1+|k|^{-1}\right)J_{-1/2}(k)^2dk\right]^{1/2} := \Lambda_{-1/2} < \infty. \end{cases}
$$

$$
(\mathcal{H}_{1/2})\begin{cases} \text{There exists a non-negative function } J_{1/2}(k) \text{ such that:} \\ (i) \quad \sup_{|\theta|\le\theta_0} \||H_{at}+i|^{-1/2}G_{m,n}((k,\lambda);\theta)|H_{at}+i|^{-1/2}\| \le J_{1/2}(k) \text{ for } m+n=1, \\ (ii) \quad \sup_{|\theta|\le\theta_0} \||H_{at}+i|^{-1/2}G_{m,n}((k,\lambda);(k',\lambda');\theta)|H_{at}+i|^{-1/2}\| \le J_{1/2}(k)J_{1/2}(k'), \\ \qquad \sup_{|\theta|\le\theta_0} \||H_{at}+i|^{-1/4}G_{m,n}((k,\lambda);(k',\lambda');\theta)|H_{at}+i|^{-1/4}\| \le J_{1/2}(k)J_{-1/2}(k'), \\ \qquad \text{for } m+n=2, \text{ where } J_{-1/2} \text{ is defined in hypothesis } (\mathcal{H}_{-1/2}), \\ (iii) \quad \sup_{k\in\mathbb{R}^3}\left\{|k|^{\frac{1}{2}(1-\beta)}J_{1/2}(k)\right\} := \Lambda_\beta < \infty \text{ for some } \beta > 0. \end{cases}
$$

Note that we have used the notation $G_{m,n} := \sum_{j=1,2} G^j_{m,n}$. Besides, in hypothesis $(\mathcal{H}_{1/2})(ii)$, it is assumed that $G_{m,n}$ is symmetric under the permutation of the variables (k,λ) and (k',λ'). The first hypothesis, $(\mathcal{H}_{-1/2})$, is sufficient to get the existence of the Feshbach operator $\mathcal{F}_{P(\theta)}$ that we define below. However, we have to require the second hypothesis $(\mathcal{H}_{1/2})$ in order to prove that all the parts of the perturbation W_g are irrelevant under renormalization.

Now, it is easy to see that H_U^V fulfills hypothesis $(\mathcal{H}_{-1/2})$ with

$$
J_{-1/2}(k) \le \mathrm{Cste}\mathbf{1}_{\mathrm{K}_\Lambda}(k)|k|^{-1/2},
$$

where $\mathbf{1}_{\mathrm{K}_\Lambda}$ denotes the characteristic function of the compact K_Λ defined in hypothesis $(\mathcal{H}_{\widehat{\chi}_\Lambda})$. However, because of the fact that $G^j_{m,n}$ behaves like $|k|^{-1/2}$ near 0, we can see that hypothesis $(\mathcal{H}_{1/2})$ is not fulfilled by H_U^V.

To face this problem, we begin with performing the Power-Zienau-Woolley transformation on H_U^V. More precisely, we define a unitary operator $\mathcal{T}$ by

$$
\mathcal{T} = \int_{\mathbb{R}^6}^{\oplus} \mathcal{T}(X)dX \quad \text{ with } \quad \mathcal{T}(X) = e^{-i\sum_{j=1,2} q_j Zq^2 x_j . A(0)}. \tag{50}
$$

Then we have $\widehat{b}_\lambda(k,X) := \mathcal{T}(X)\widehat{a}_\lambda(k)\mathcal{T}^*(X) = \widehat{a}_\lambda(k) - iw_\lambda(k,X)$, with

$$
w_\lambda(k,X) = \frac{1}{2\pi}\frac{\widehat{\chi}_\Lambda(k)}{|k|^{1/2}}\varepsilon_\lambda(k).\sum_{j=1,2} q_j Zq^2 x_j, \tag{51}
$$

and the unitary equivalent Hamiltonian $\widetilde{H}_U^V$ is

$$
\widetilde{H}_U^V := \mathcal{T}H_U^V\mathcal{T}^* = \sum_{j=1,2}\frac{1}{2m_j}(p_j - q_jZq^2\widetilde{A}_j(Zq^2\cdot))^2 + \widetilde{H}_f + U + V, \tag{52}
$$

with $\widetilde{A}_j = \int_{\mathbb{R}^6}^{\oplus} \widetilde{A}_j(X)dX$, $\widetilde{H}_f = \int_{\mathbb{R}^6}^{\oplus} \widetilde{H}_f(X)dX$, and

$$\widetilde{A}_j(X) = A(x_j) - A(0) \quad , \quad \widetilde{H}_f(X) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| \widehat{b}_\lambda^*(k,X) \widehat{b}_\lambda(k,X) dk. \tag{53}$$

Developing (52), we can write

$$\widetilde{H}_U^V = \widetilde{H}_{at} + H_f + g\widetilde{W}_1 + g^2 \widetilde{W}_2,$$

with

$$\begin{aligned} \widetilde{H}_{at} =& H_{at} + g^2 \frac{Z^2}{\Lambda^3} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\widehat{\chi}_\Lambda(k)^2}{4\pi^2} (\varepsilon_\lambda(k).\widetilde{r})^2 dk \\ &+ 2g^2 \frac{Z^2}{\Lambda^3} \int_{\mathbb{R}^3} \frac{\widehat{\chi}_\Lambda(k)^2}{\pi^2|k|} \left[\frac{1}{2m_1} \sin^2(Zq^2 k.x_1/2) + \frac{Z}{2m_2} \sin^2(Zq^2 k.x_2/2) \right] dk, \end{aligned} \tag{54}$$

and

$$\widetilde{W}_1 = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \left[\widetilde{G}_{1,0}(k,\lambda) \otimes \widehat{a}_\lambda^*(k) + \widetilde{G}_{0,1}(k,\lambda) \otimes \widehat{a}_\lambda(k) \right] dk, \tag{55}$$

where

$$\begin{aligned} \widetilde{G}_{1,0}(k,\lambda) &= \widetilde{G}_{0,1}(k,\lambda)^* \\ &= \frac{iZ}{2m_1 \Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(k)}{2\pi\sqrt{|k|}} \left(e^{-iZq^2 k.x_1} - 1 \right) \varepsilon_\lambda(k).\nabla_{x_1} \\ &\quad - \frac{iZ^2}{2m_2 \Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(k)}{2\pi\sqrt{|k|}} \left(e^{-iZq^2 k.x_2} - 1 \right) \varepsilon_\lambda(k).\nabla_{x_2} \\ &\quad - \frac{iZ}{\Lambda^{3/2}} \frac{|k|^{1/2} \widehat{\chi}_\Lambda(k)}{2\pi} \varepsilon_\lambda(k).\widetilde{r}. \end{aligned} \tag{56}$$

Note that we have set $\widetilde{r} := x_1 - Zx_2$. Moreover, $\widetilde{W}_2$ is W_2 where the terms $e^{\pm iZq^2 k.x_j}$ are replaced by $e^{\pm iZq^2 k.x_j} - 1$ in $\widetilde{G}_{m,n}$, for $m+n=2$.
We can see that the new atomic Hamiltonian $\widetilde{H}_{at}$ is well defined as a closed quadratic form on $Q(p_1^2+p_2^2) \cap Q(U^+) \cap Q(\widetilde{r}^2) \cap Q(H_f)$ provided hypothesis $(\mathcal{H}_2)$ is fulfilled. In particular, we can use the fact that

$$\sin^2(Zq^2 k.x_j/2) \leq \frac{Zq^2}{2} |k||x_j| \leq \left(\frac{Zq^2}{2} |k| \right) (a_1 \widetilde{r}^2 + a_2 R^2),$$

where a_1 and a_2 are positive constants, in order to show that the last term in (54) is relatively bounded with respect to the first two terms.
Then, in the same way as for $H_U^V(\theta)$, we can see that the complex dilated Hamiltonian $\widetilde{H}_U^V(\theta)$ is an analytic family of type (B) on $D(0,\theta_0)$, with form domain $Q(p_1^2+p_2^2) \cap Q(U^+) \cap Q(\widetilde{r}^2) \cap Q(H_f)$.
In particular, the quadratic form $q_{\widetilde{W}_g(\theta)}$ is well defined on $Q(\widetilde{H}_U^V(\theta))$; as an operator, we can define $\widetilde{W}_g(\theta)$ on the domain $D(p_1^2+p_2^2) \cap D(x_1^2+x_2^2) \cap D(H_f)$.

Now, the behavior of $\widetilde{G}_{m,n}(k,\lambda)$ is better than the one of $G_{m,n}(k,\lambda)$ near 0, since we have

$$\left|e^{\pm iZq^2k.x_j}-1\right|\leq Zq^2|k||x_j|.$$

However, we only have:

$$\left\|\left|\widetilde{H}_{at}+i\right|^{-1/4}\widetilde{G}_{m,n}(k,\lambda;\theta)\left|\widetilde{H}_{at}+i\right|^{-1/4}\right\|\leq\frac{\mathrm{Cste}}{g}\mathbf{1}_{\mathrm{K}_\Lambda}(k)|k|^{1/2},$$

for $m+m=1$, because of the last term in (56) which is relatively bounded with respect to $|\widetilde{H}_{at}+i|^{1/2}/g$ only. This appears to be a problem, because we require that all the terms of the perturbation $\widetilde{W}_g:=g\widetilde{W}_1+g^2\widetilde{W}_2$ are small as compared to the unperturbed Hamiltonian $\widetilde{H}_0:=\widetilde{H}_{at}+H_f$, when g is small.
To avoid this difficulty, in a way similar to what is done in [BFS1], we consider the simplified model where a spatial cutoff is imposed on $\widetilde{W}_g$, which restricts the position of the electron to finite distances from the position of the nucleus. More precisely, we replace $\widetilde{W}_g$ by $\widetilde{W}_{g;\mathrm{reg}}$ where

$$\widetilde{W}_{g;\mathrm{reg}}:=\chi_{r_0}(r)\widetilde{W}_g. \tag{57}$$

Here χ_{r_0} is a smooth function which is equal to 1 in the ball $B(0,r_0/2)$ and which vanishes outside the ball $B(0,r_0)$. r_0 is arbitrary large but finite.
Then, we define the Hamiltonian $\widetilde{H}^V_{U;\mathrm{reg}}$ by

$$\widetilde{H}^V_{U;\mathrm{reg}}:=\widetilde{H}_0+\widetilde{W}_{g;\mathrm{reg}}, \tag{58}$$

and we can verify that:

Proposition 4.1 *Assume that hypothesis ($\mathcal{H}_2$) holds. Then $\widetilde{H}^V_{U;\mathrm{reg}}$ fulfills hypotheses ($\mathcal{H}_{-1/2}$) and ($\mathcal{H}_{1/2}$) (where H_{at} and $G_{m,n}$ are replaced respectively by $\widetilde{H}_{at}$ and $\widetilde{G}_{m,n;\mathrm{reg}}:=\chi_{r_0}\widetilde{G}_{m,n}$), with*

$$J_{-1/2}(k):=\mathrm{Cste}\mathbf{1}_{\mathrm{K}_\Lambda}(k)|k|^{-1/2}\quad,\quad J_{1/2}(k):=\mathrm{Cste}\mathbf{1}_{\mathrm{K}_\Lambda}(k)|k|^{1/2}, \tag{59}$$

where $\mathbf{1}_{\mathrm{K}_\Lambda}$ denotes the characteristic function of the compact K_Λ defined in hypothesis ($\mathcal{H}_{\widehat{\chi}_\Lambda}$).

Proof We write the proof for $\widetilde{G}_{1,0;\mathrm{reg}}$; the other terms $\widetilde{G}_{m,,n;\mathrm{reg}}$ can be treated in the same way. Notice that

$$\begin{aligned}
&\widetilde{G}_{1,0;\mathrm{reg}}(k,\lambda;\theta)=\widetilde{G}_{0,1;\mathrm{reg}}(k,\lambda;\overline{\theta})^*\\
&=\chi_{r_0}(r)e^{-2\theta}\frac{iZ}{2m_1\Lambda^{3/2}}\frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}}\left(e^{-iZq^2k.x_1}-1\right)\varepsilon_\lambda(k).\nabla_{x_1}\\
&\quad-\chi_{r_0}(r)e^{-2\theta}\frac{iZ^2}{2m_2\Lambda^{3/2}}\frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}}\left(e^{-iZq^2k.x_2}-1\right)\varepsilon_\lambda(k).\nabla_{x_2}\\
&\quad-\chi_{r_0}(r)e^{-2\theta}\frac{iZ}{\Lambda^{3/2}}\frac{|k|^{1/2}\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi}\varepsilon_\lambda(k).\widetilde{r}.
\end{aligned} \tag{60}$$

First, using the fact that $\left|e^{\pm iZq^2 k.x_j} - 1\right| \le 2$, we get

$$\Bigg\| \left|\widetilde{H}_{at} + i\right|^{-1/4} \chi_{r_0}(r) \Bigg[e^{-2\theta} \frac{iZ}{2m_1 \Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}} \left(e^{-iZq^2 k.x_1} - 1\right) \varepsilon_\lambda(k).\nabla_{x_1}$$

$$- e^{-2\theta} \frac{iZ^2}{2m_2 \Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}} \left(e^{-iZq^2 k.x_2} - 1\right) \varepsilon_\lambda(k).\nabla_{x_2} \Bigg] \left|\widetilde{H}_{at} + i\right|^{-1/4} \Bigg\|$$

$$\le \mathrm{Cste} \mathbf{1}_{\mathrm{K}_\Lambda}(k) |k|^{-1/2}.$$

Moreover, using that $\chi_{r_0}(r)|\widetilde{r}| \le \mathrm{Cste}(1 + |R|)$ and hypothesis $(\mathcal{H}_2)$, we get

$$\Bigg\| \left|\widetilde{H}_{at} + i\right|^{-1/4} \chi_{r_0}(r) e^{-2\theta} \frac{iZ}{\Lambda^{3/2}} \frac{|k|^{1/2} \widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi} \varepsilon_\lambda(k).\widetilde{r} \left|\widetilde{H}_{at} + i\right|^{-1/4} \Bigg\| \le \mathrm{Cste} \mathbf{1}_{\mathrm{K}_\Lambda}(k) |k|^{1/2}.$$

Thus, we see that hypothesis $(\mathcal{H}_{-1/2})$ is fulfilled with

$$J_{-1/2}(k) = \mathrm{Cste} \mathbf{1}_{\mathrm{K}_\Lambda}(k) |k|^{-1/2}.$$

Next, we use the fact that $\left|e^{\pm iZq^2 k.x_j} - 1\right| \le Zq^2 |k||x_j|$ together with $\chi_{r_0}(r)|x_j| \le \mathrm{Cste}(1 + |R|)$ and hypothesis $(\mathcal{H}_2)$, which yields

$$\Bigg\| \left|\widetilde{H}_{at} + i\right|^{-1/2} \chi_{r_0}(r) \Bigg[e^{-2\theta} \frac{iZ}{2m_1 \Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}} \left(e^{-iZq^2 k.x_1} - 1\right) \varepsilon_\lambda(k).\nabla_{x_1}$$

$$- e^{-2\theta} \frac{iZ^2}{2m_2 \Lambda^{3/2}} \frac{\widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi\sqrt{|k|}} \left(e^{-iZq^2 k.x_2} - 1\right) \varepsilon_\lambda(k).\nabla_{x_2} \Bigg] \left|\widetilde{H}_{at} + i\right|^{-1/2} \Bigg\|$$

$$\le \mathrm{Cste} \mathbf{1}_{\mathrm{K}_\Lambda}(k) |k|^{1/2}.$$

We have again

$$\Bigg\| \left|\widetilde{H}_{at} + i\right|^{-1/2} \chi_{r_0}(r) e^{-2\theta} \frac{iZ}{\Lambda^{3/2}} \frac{|k|^{1/2} \widehat{\chi}_\Lambda(e^{-\theta}k)}{2\pi} \varepsilon_\lambda(k).\widetilde{r} \left|\widetilde{H}_{at} + i\right|^{-1/2} \Bigg\| \le \mathrm{Cste} \mathbf{1}_{\mathrm{K}_\Lambda}(k) |k|^{1/2}.$$

Hence $(\mathcal{H}_{1/2})$ is fulfilled with

$$J_{1/2}(k) = \mathrm{Cste} \mathbf{1}_{\mathrm{K}_\Lambda}(k) |k|^{1/2},$$

and $\beta = 1$. □

Henceforth, we redefine $\widetilde{W}_g := \widetilde{W}_{g;\mathrm{reg}}$ and $\widetilde{H}_U^V := \widetilde{H}_{U;\mathrm{reg}}^V$.

To conclude with this subsection, we describe the spectrum of the unperturbed Hamiltonian $\widetilde{H}_0(\theta)$. We assume for the sake of simplicity that we are dealing with the hydrogen atom, that is $Z = 1$. Then (54) implies

$$\begin{aligned} \widetilde{H}_{at} &= \frac{p^2}{2\mu} + V + \mathrm{C} g^2 r^2 + \frac{P^2}{2M} + U \\ &\quad + 2g^2 \frac{1}{\Lambda^3} \int_{\mathbb{R}^3} \frac{\widehat{\chi}_\Lambda(k)^2}{\pi^2 |k|} \left[\frac{1}{2m_1} \sin^2(q^2 k.x_1/2) + \frac{1}{2m_2} \sin^2(q^2 k.x_2/2) \right] dk, \end{aligned} \tag{61}$$

where C is positive.
We can see that the spectrum of $p^2/2\mu + V + \mathrm{C}g^2r^2$ is discrete: the eigenvalues E_l of $p^2/2\mu + V$ are slightly shifted to become $\widetilde{E}_l$, and the continuous spectrum of $p^2/2\mu + V$ turns into a non-decreasing sequence of discrete eigenvalues $\widetilde{E}_l^c(g)$, $l \geq 1$, such that

$$\widetilde{E}_l^c(g) - \widetilde{E}_{l-1}^c(g) \underset{g \to 0}{\to} 0.$$

Thus the spectrum of $p^2/2\mu + V + \mathrm{C}g^2r^2 + P^2/2M + U$ is purely discrete, and the same holds for $\widetilde{H}_{at}$: its eigenvalues are $\widetilde{E}_l + e_n$ and $\widetilde{E}_l^c + e_n$, slightly perturbed by the last term of (61). We still denote by $\widetilde{E}_l + e_n$ and $\widetilde{E}_l^c + e_n$ the eigenvalues of $\widetilde{H}_{at}$.
To conclude, we obtain the spectrum of $\widetilde{H}_{at}$ as described in figure 4.

Figure 4: **Spectrum of the new atomic Hamiltonian $\widetilde{H}_{at}$ for $Z = 1$**

We notice that, if $Z > 1$, we can not use the same argument to say that the essential spectrum of $p^2/2\mu + V$ turns into discrete spectrum. However, we still see that the eigenvalues $E_l + e_n$ of H_{at} are slightly shifted to become eigenvalues of $\widetilde{H}_{at}$. They are still denoted by $\widetilde{E}_l + e_n$. Since we only study such eigenvalues in the sequel (we shall not study the behavior of $\widetilde{E}_l^c + e_n$), the case $Z > 1$ can be treated in the same way as the case $Z = 1$.

4.2 The smooth Feshbach map applied to $\widetilde{H}_U^V(\theta)$

Let us fix $(l, n) \neq (0, 0)$ such that $E_l + e_n < e_0$; by a translation, we can assume that $\widetilde{E}_l + e_n$ is located at 0, and for the sake of simplicity, we assume moreover that this eigenvalue is non-degenerate. The case of a degenerate eigenvalue with finite multiplicity could be treated in a similar manner. Note however that the results of this subsection does not hold for $\widetilde{E}_l^c + e_n$ (obtained in the case $Z = 1$) because these eigenvalues are separated from each other only by a distance $O(g)$.
Now, pick $\theta := i\nu$, with $\nu > 0$. Then we begin with some definitions.

Let $\delta := \mathrm{dist}(0, \sigma(\widetilde{H}_{at})\backslash\{0\}) > 0$. As in (40), we define the projection onto the subspace spanned by 0 by:

$$P_{at}(\theta) := \frac{i}{2\pi} \int_{|z|=\delta/2} \frac{dz}{\widetilde{H}_{at}(\theta) - z} \quad , \quad \overline{P}_{at}(\theta) := 1 - P_{at}(\theta). \tag{62}$$

The numbers $Z_0^{\rm od}$ and $Z_0^{\rm d}$ are defined as in (41):

$$\begin{aligned}
Z_0^{\rm od} &:= \sum_{\lambda=1,2}\int_{\mathbb{R}^3} P_{at}\widetilde{G}_{0,1}(k,\lambda)\overline{P}_{at}\left[\widetilde{H}_{at}+|k|-i0\right]^{-1}\overline{P}_{at}\widetilde{G}_{1,0}(k,\lambda)P_{at}dk,\\
Z_0^{\rm d} &:= \sum_{\lambda=1,2}\int_{\mathbb{R}^3} P_{at}\widetilde{G}_{0,1}(k,\lambda)P_{at}\widetilde{G}_{1,0}(k,\lambda)P_{at}\frac{dk}{|k|},
\end{aligned} \tag{63}$$

where we used the notations $P_{at} := P_{at}(0)$, $\overline{P}_{at} := \overline{P}_{at}(0)$.
Setting $\Gamma_0 := \mathrm{Im}(Z_0^{\rm od})$, we note that we shall require in the sequel the following hypothesis in order to prove that 0 turns into a resonance when $\widetilde{W}_g$ is added:

$$(\mathcal{H}_{\Gamma_0}) \quad \Gamma_0 > 0.$$

Assume that ν is sufficiently small. Then, for any $0 < \rho_0 \le \delta\sin(\nu/2) < 1$, we define the functions of H_f, $\chi_{\rho_0}(H_f)$ and $\overline{\chi}_{\rho_0}(H_f)$, by

$$\chi_{\rho_0}(H_f) := \sin\left[\frac{\pi}{2}\Theta(H_f/\rho_0)\right] \quad , \quad \overline{\chi}_{\rho_0}(H_f) := \sqrt{1-\chi_{\rho_0}^2(H_f)} = \cos\left[\frac{\pi}{2}\Theta(H_f/\rho_0)\right], \tag{64}$$

where $\Theta \in \mathrm{C}_0^\infty([0,\infty[;[0,1])$ is such that $\Theta = 1$ on $[0,3/4[$ and $\Theta = 0$ on $[1,\infty[$.
Next we use (62) and (64) to define:

$$P(\theta) := P_{at}(\theta)\otimes\chi_{\rho_0}(H_f) \quad , \quad \overline{P}(\theta) := P_{at}(\theta)\otimes\overline{\chi}_{\rho_0}(H_f) + \overline{P}_{at}(\theta)\otimes\mathbf{1}. \tag{65}$$

Note that (65) implies $P(\theta)^2 + \overline{P}(\theta)^2 = 1$.

As a first step of our analysis, we want to decimate the degrees of freedom corresponding to the photons of energy $\ge \rho_0$. To this end, we apply the smooth Feshbach map $\mathcal{F}_{P(\theta)}$ to the Feshbach pair $(\widetilde{H}_U^V(\theta)-z, \widetilde{H}_0(\theta)-z)$ (see [BCFS]), where z is a spectral parameter that we assume to lie in the disc $D_{\rho_0/2} := \{z\in\mathbb{C}, |z|\le\rho_0/2\}$:

$$\begin{aligned}
\mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta)-z, \widetilde{H}_0(\theta)-z) &:= (\widetilde{H}_0(\theta)-z) + P(\theta)\widetilde{W}_g(\theta)P(\theta)\\
&- P(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\left[(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\right]^{-1}\overline{P}(\theta)\widetilde{W}_g(\theta)P(\theta).
\end{aligned} \tag{66}$$

This operator is well-defined provided that $(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)$ is invertible on $\mathrm{Ran}\overline{P}(\theta)$, and that

$$\begin{aligned}
&\left|(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\right|^{-1/2}\mathcal{U}^{-1}\overline{P}(\theta)\widetilde{W}_g(\theta)P(\theta),\\
&P(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\left|(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\right|^{-1/2}
\end{aligned} \tag{67}$$

extend to bounded operators on $\mathrm{L}^2(\mathbb{R}^6)\otimes\mathcal{F}_s$ and $\mathrm{Ran}(\overline{P}(\theta))$ respectively. Here

$$(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta) = \mathcal{U}\left|(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\right| \tag{68}$$

denotes the polar decomposition of $(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)$ on $\mathrm{Ran}(\overline{P}(\theta))$.

The key property of the Feshbach map is that it is isospectral in the following sense: $\widetilde{H}_U^V(\theta)-z$ is invertible if and only if $\mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta)-z,\widetilde{H}_0(\theta)-z)$ is invertible on $\mathrm{Ran}(P(\theta))$; moreover, 0 is an eigenvalue of $\widetilde{H}_U^V(\theta)-z$ with multiplicity k if and only if 0 is an eigenvalue of

$$\mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta)-z,\widetilde{H}_0(\theta)-z)\Big|_{\mathrm{Ran}(P(\theta))},$$

with the same multiplicity k (see [BCFS]).

The following proposition shows that $\mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta)-z,\widetilde{H}_0(\theta)-z)$ is well-defined for all $z\in D_{\rho_0/2}$:

Proposition 4.2 *Pick z in $D_{\rho_0/2}$ with $0<\rho_0\le\delta\sin(\nu/2)<1$, and assume that $g{\rho_0}^{-1/2}$ is sufficiently small: $g{\rho_0}^{-1/2}\ll\nu^2$.*
Then $(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)$ is invertible on $\mathrm{Ran}(\overline{P}(\theta))$, and the operators given in (67) extend to bounded operators on $\mathrm{L}^2(\mathbb{R}^6)\otimes\mathcal{F}_s$ and $\mathrm{Ran}(\overline{P}(\theta))$ respectively.

Proof The proof follows the one in [BFS3] with some slight modifications. We sketch the proof and give some intermediate results.

First, using the assumption that $|z|\le\rho_0/2\le\delta\sin(\nu/2)/2$, we easily see that

$$H_0^{\overline{P}(\theta)}:=\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}(\widetilde{H}_0(\theta)-z)\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}\tag{69}$$

is invertible on $\mathrm{Ran}(\overline{P}(\theta))$. We denote by $H_0^{\overline{P}(\theta)}=\mathcal{U}_0\left|H_0^{\overline{P}(\theta)}\right|$ the polar decomposition of $H_0^{\overline{P}(\theta)}$ on $\mathrm{Ran}(\overline{P}(\theta))$.

Then we write

$$\begin{aligned}&\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}\left[(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\right]\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}\\&=\mathcal{U}_0\left|H_0^{\overline{P}(\theta)}\right|^{1/2}\left[1+\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\mathcal{U}_0^{-1}\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\right]\left|H_0^{\overline{P}(\theta)}\right|^{1/2}.\end{aligned}\tag{70}$$

Thus, to prove that $(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)$ is invertible on $\mathrm{Ran}(\overline{P}(\theta))$, it is sufficient to show that

$$\left\|\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\mathcal{U}_0^{-1}\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\right\|<1.\tag{71}$$

To this end, we write:

$$\begin{aligned}&\left\|\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\mathcal{U}_0^{-1}\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\right\|\\&\le\left\|\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\mathcal{U}_0^{-1}\overline{P}(\theta)\left|B_\theta(\rho_0)\right|^{1/2}\right\|\times\left\|\left|B_\theta(\rho_0)\right|^{-1/2}\widetilde{W}_g(\theta)\left|B_\theta(\rho_0)\right|^{-1/2}\right\|\\&\quad\times\left\|\left|B_\theta(\rho_0)\right|^{1/2}\overline{P}(\theta)\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\right\|,\end{aligned}\tag{72}$$

where $B_\theta(\rho_0)$ is defined by

$$B_\theta(\rho_0):=\widetilde{H}_0(\theta)+e^{-\theta}\rho_0=\widetilde{H}_{at}(\theta)+e^{-\theta}(H_f+\rho_0).\tag{73}$$

Using the fact that $\mathcal{U}_0$ is an isometry which commutes with $H_0^{\overline{P}(\theta)}$ (since $H_0^{\overline{P}(\theta)}$ is invertible), we get:

$$\left\| \left| H_0^{\overline{P}(\theta)} \right|^{-1/2} \mathcal{U}_0^{-1} \overline{P}(\theta) \left| B_\theta(\rho_0) \right|^{1/2} \right\| = \left\| \left| B_\theta(\rho_0) \right|^{1/2} \overline{P}(\theta) \left| H_0^{\overline{P}(\theta)} \right|^{-1/2} \right\| \leq \frac{\mathrm{Cste}}{\nu^{1/2}}. \tag{74}$$

Here, we used a result similar to what we derived in lemma 3.6:

$$\left\| \Delta \widetilde{H}_{at} (\widetilde{H}_{at} \pm i)^{-1} \right\| \leq b(\theta), \tag{75}$$

where $b(\theta)$ is a positive constant that goes to 0 as $\theta \to 0$, and where $\Delta\widetilde{H}_{at}$ is defined by $\Delta\widetilde{H}_{at} := \widetilde{H}_{at}(\theta) - \widetilde{H}_{at}$.
Furthermore, as in [BFS3], we can see that

$$\left\| \left| B_\theta(\rho_0) \right|^{-1/2} \widetilde{W}_g(\theta) \left| B_\theta(\rho_0) \right|^{-1/2} \right\| \leq \frac{\mathrm{Cste}}{\nu} g {\rho_0}^{-1/2}. \tag{76}$$

Let us make a few remarks about this last estimate. There is a difficulty because, with our definition using quadratic forms, $\widetilde{W}_g(\theta)$ is not defined *a priori* as an operator on $D(|B_\theta(\rho_0)|^{1/2})$. However, the quadratic form $q_{\widetilde{W}_g(\theta)}$ is well defined on this domain. Following the calculus of [BFS3] and using the fact that $\widetilde{H}_U^V$ fulfills hypothesis $(\mathcal{H}_{-1/2})$, we can see that

$$\left| q_{\widetilde{W}_g(\theta)} (|B_\theta(\rho_0)|^{-1/2}\Phi, |B_\theta(\rho_0)|^{-1/2}\Psi) \right| \leq \frac{\mathrm{Cste}}{\nu} g {\rho_0}^{-1/2} \|\Phi\| \|\Psi\|, \tag{77}$$

for all $\Phi, \Psi \in \mathrm{L}^2(\mathbb{R}^6) \otimes \mathcal{F}_s$.
Note that we do not need to require the second hypothesis $(\mathcal{H}_{1/2})$ in order to prove the estimate (77). Actually, $(\mathcal{H}_{1/2})$ is not necessary to obtain the existence of $\mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta) - z, \widetilde{H}_0(\theta) - z)$; however, it will become essential in the renormalization procedure.
Now $|B_\theta(\rho_0)|^{-1/2} \widetilde{W}_g(\theta) |B_\theta(\rho_0)|^{-1/2}$ is clearly well defined on $|B_\theta(\rho_0)|^{1/2} D(\widetilde{W}_g(\theta))$. But $\mathrm{C}_0^\infty(\mathbb{R}^6) \otimes D_{\mathcal{S}} \subset D(\widetilde{W}_g(\theta))$ is a core for $|\widetilde{H}_0|^{1/2}$ (here we use the fact that $U^+ \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3)$ in hypothesis (H_0)), and we can see that it is also a core for $|B_\theta(\rho_0)|^{1/2}$. Thus $|B_\theta(\rho_0)|^{1/2} D(\widetilde{W}_g(\theta))$ is dense; then (77) shows that $|B_\theta(\rho_0)|^{-1/2} \widetilde{W}_g(\theta) |B_\theta(\rho_0)|^{-1/2}$ extends to a bounded operator on $\mathrm{L}^2(\mathbb{R}^6) \otimes \mathcal{F}_s$ that satisfies (76).
Inserting (73) and (76) into (72), we obtain:

$$\left\| \left| H_0^{\overline{P}(\theta)} \right|^{-1/2} \mathcal{U}_0^{-1} \overline{P}(\theta) \widetilde{W}_g(\theta) \overline{P}(\theta) \left| H_0^{\overline{P}(\theta)} \right|^{-1/2} \right\| \leq \frac{\mathrm{Cste}}{\nu^2} g {\rho_0}^{-1/2}.$$

Thus, provided that $g{\rho_0}^{-1/2} \ll \nu^2$, (71) is proved, so that $(\widetilde{H}_0(\theta) - z) + \overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)$ is invertible on $\mathrm{Ran}(\overline{P}(\theta))$.
The proof that the operators given in (67) extend to bounded operators on $\mathrm{L}^2(\mathbb{R}^6) \otimes \mathcal{F}_s$ and $\mathrm{Ran}(\overline{P}(\theta))$ respectively is similar. □

Now, inverting (70), we can write:

$$\begin{aligned}
&\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}\left[(\widetilde{H}_0(\theta)-z)+\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\right]^{-1}\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}\\
&=\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\sum_{n\geq 0}\left[-\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\mathcal{U}_0^{-1}\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\right]^n\left|H_0^{\overline{P}(\theta)}\right|^{-1/2}\mathcal{U}_0^{-1}\\
&=\left[H_0^{\overline{P}(\theta)}\right]^{-1}\sum_{n\geq 0}\left[-\overline{P}(\theta)\widetilde{W}_g(\theta)\overline{P}(\theta)\left[H_0^{\overline{P}(\theta)}\right]^{-1}\right]^n.
\end{aligned} \tag{78}$$

Using this, our aim in the next subsection will be to prove that the operator

$$\mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta)-z,\widetilde{H}_0(\theta)-z)\Big|_{\mathrm{Ran}(P(\theta))},$$

can be identified with an element of a suitably chosen Banach space. The main property of this Banach space that we shall require is that the operator coming from the interaction is irrelevant under renormalization.

4.3 Renormalization in a Banach space of Hamiltonians

Our purpose, in this subsection, is to prove that the eigenvalues $\widetilde{E}_l+e_n$ of $\widetilde{H}_0$, such that $(l,n)\neq(0,0)$ and $E_l+e_n<e_0$, turn into resonances when the interaction $\widetilde{W}_g$ is added. Recall that, to simplify, we assume that $\widetilde{E}_l+e_n$ is non-degenerate and located at 0.

Let us begin with the definition of the Banach space $\mathcal{W}^{\#}_{\geq 0}$. We set:

$$\mathcal{W}^{\#}_{\geq 0}:=\mathbb{C}\oplus\mathcal{T}\oplus\mathcal{W}^{\#}_{\geq 1}:=\mathbb{C}\oplus\mathcal{T}\oplus\bigoplus_{M+N\geq 1}\mathcal{W}^{\#}_{M,N}, \tag{79}$$

where

$$\mathcal{T}:=\left\{f\in\mathrm{C}^1([0,1]),f(0)=0,\|f\|_{\mathcal{T}}:=\sup_{\alpha\in[0,1]}|f'(\alpha)|<\infty\right\}, \tag{80}$$

and

$$\begin{aligned}
\mathcal{W}^{\#}_{M,N}:=\big\{f_{M,N}:[0,1]\times(B_1\times\{1,2\})^M\times(B_1\times\{1,2\})^N\to\mathbb{C}\text{ such that:}\\
&*\ f_{M,N}(\cdot;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)})\in\mathrm{C}^1([0,1])\text{ for every}\\
&\quad((k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)})\in(B_1\times\{1,2\})^{M+N},\\
&*\ f_{M,N}(\alpha;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)})\text{ is totally symmetric with respect to}\\
&\quad(k,\lambda)^{(M)}\text{ and }(\tilde{k},\tilde{\lambda})^{(N)},\\
&*\ \|f_{M,N}\|^{\#}:=\|f_{M,N}\|+\|\partial_\alpha f_{M,N}\|<\infty\big\}.
\end{aligned} \tag{81}$$

Here B_1 denotes the unit ball in $\mathbb{R}^3$, $\partial_\alpha f_{M,N}$ is the partial derivative of $f_{M,N}$ with respect to the first variable, and

$$\|f_{M,N}\|:=\sup_{[0,1]\times(B_1\times\{1,2\})^{M+N}}\left|f_{M,N}(\alpha;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)})\right|\prod_{i=1}^M|k_i|^{-1/2}\prod_{j=1}^N|\tilde{k}_j|^{-1/2}. \tag{82}$$

Note that we have used the notations

$$(k,\lambda)^{(M)} := ((k_1,\lambda_1),\dots,(k_M,\lambda_M)) \in (\mathbb{R}^3\times\{1,2\})^M,$$
$$(\tilde{k},\tilde{\lambda})^{(N)} := ((\tilde{k}_1,\tilde{\lambda}_1),\dots,(\tilde{k}_N,\tilde{\lambda}_N)) \in (\mathbb{R}^3\times\{1,2\})^N.$$

Next, the space $\mathcal{W}^{\#}_{\geq 1} := \{\underline{w} := (w_{M,N})_{M+N\geq 1}\}$ is equipped with the norm

$$\|\underline{w}\|^{\#}_{\xi,1} := \sum_{M+N\geq 1} \xi^{-(M+N)} \|w_{M,N}\|^{\#}, \tag{83}$$

where $0<\xi<1$ is a parameter that we will precise below.
Defining

$$\mathcal{W}^{\#}_{0,0} := \left\{ f_{0,0} \in \mathrm{C}^1([0,1]), \|f_{0,0}\|^{\#} := \|f_{0,0}\|_\infty + \|\partial_\alpha f_{0,0}\|_\infty < \infty \right\}, \tag{84}$$

we can see that there is a natural isomorphism between the Banach spaces $\mathbb{C}\oplus\mathcal{T}$ and $\mathcal{W}^{\#}_{0,0}$, so that we identify $\mathbb{C}\oplus\mathcal{T}$ and $\mathcal{W}^{\#}_{0,0}$.
Thus, we can write an element of $\mathcal{W}^{\#}_{\geq 0}$ as $\underline{w} := (w_{M,N})_{M+N\geq 0}$, and we equip $\mathcal{W}^{\#}_{\geq 0}$ with the norm

$$\|\underline{w}\|^{\#}_{\xi} := \sum_{M+N\geq 0} \xi^{-(M+N)} \|w_{M,N}\|^{\#}. \tag{85}$$

Now, we want to identify an element of $\mathcal{W}^{\#}_{\geq 0}$ with an operator on the Hilbert space

$$\mathcal{H}_{\mathrm{red}} := \mathbf{1}_{H_f<1}\mathcal{F}_s. \tag{86}$$

To this end, we define for $\underline{w}\in\mathcal{W}^{\#}_{\geq 0}$:

$$H(\underline{w}) := \sum_{M+N\geq 0} W_{M,N}(\underline{w}) := w_{0,0}(H_f)\mathbf{1}_{H_f<1} + \sum_{M+N\geq 1} W_{M,N}(\underline{w}), \tag{87}$$

where for $M+N\geq 1$:

$$\begin{aligned} W_{M,N}(\underline{w}) := \mathbf{1}_{H_f<1} \sum_{\substack{\lambda^{(M)}\in\{1,2\}^M \\ \tilde{\lambda}^{(N)}\in\{1,2\}^N}} \int_{B_1^{M+N}} & a^*_{\lambda^{(M)}}(k^{(M)}) w_{M,N}[H_f;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)}] \\ & a_{\tilde{\lambda}^{(N)}}(\tilde{k}^{(N)}) dk^{(M)} d\tilde{k}^{(N)} \mathbf{1}_{H_f<1}. \end{aligned} \tag{88}$$

Note that in the last equation, we have used the notations:

$$\begin{aligned} \lambda^{(M)} := (\lambda_1,\dots,\lambda_M) \quad &, \quad \tilde{\lambda}^{(N)} := (\tilde{\lambda}_1,\dots,\tilde{\lambda}_N), \\ a^*_{\lambda^{(M)}}(k^{(M)}) := \prod_{j=1}^M \widehat{a}^*_{\lambda_j}(k_j) \quad &, \quad a_{\tilde{\lambda}^{(N)}}(\tilde{k}^{(N)}) := \prod_{j=1}^N \widehat{a}_{\tilde{\lambda}_j}(\tilde{k}_j), \\ dk^{(M)} := dk_1\dots dk_M \quad &, \quad d\tilde{k}^{(N)} := d\tilde{k}_1\dots d\tilde{k}_N. \end{aligned} \tag{89}$$

The following proposition is proved in [BCFS]; it shows that any operator of $\mathcal{H}_{\mathrm{red}}$ written as in (87) can be identified with an element of the Banach space $\mathcal{W}^{\#}_{\geq 0}$.

Proposition 4.3 *Pick ξ such that $0 < \xi < 1$.*
Then, the map H defined in (87) is an injective embedding from $\mathcal{W}_{\geq 0}^{\#}$ into $\mathcal{B}[\mathcal{H}_{\mathrm{red}}]$, the set of bounded operators on $\mathcal{H}_{\mathrm{red}}$. Moreover, we have

$$\|H(\underline{w})\|_{\mathcal{H}_{\mathrm{red}}} \leq \|\underline{w}\|_{\xi}^{\#}. \tag{90}$$

To control the dependence of the operators that we shall study in the spectral parameter z, we introduce the Banach space

$$\mathcal{W}_{\geq 0} := \left\{ \underline{w}[\cdot] : D_{1/2} \to \mathcal{W}_{\geq 0}^{\#}, \underline{w}[\cdot] \text{ is analytic} \right\}, \tag{91}$$

where $D_{1/2}$ denotes the disc $\{z \in \mathbb{C}, |z| \leq 1/2\}$. It is equipped with the norm:

$$\|\underline{w}[\cdot]\|_{\xi} := \sup_{z \in D_{1/2}} \|\underline{w}[z]\|_{\xi}^{\#}. \tag{92}$$

Likewise, $H(\mathcal{W}_{\geq 0})$ denotes the Banach space:

$$H(\mathcal{W}_{\geq 0}) := \left\{ H(\underline{w}[\cdot]) : D_{1/2} \to H(\mathcal{W}_{\geq 0}^{\#}), H(\underline{w}[\cdot]) \text{ is analytic} \right\}, \tag{93}$$

with the norm

$$\|H(\underline{w}[\cdot])\| := \sup_{z \in D_{1/2}} \|H(\underline{w}[z])\|_{\mathcal{H}_{\mathrm{red}}}. \tag{94}$$

This is on this Banach space that the renormalization map constructed in [BCFS] acts.

Our first aim is to define an operator $H_{(0)}[z]$ which is isospectral to $\widetilde{H}_U^V(\theta)$, and which belongs to $H(\mathcal{W}_{\geq 0})$. In the same way as in [BFS1], we define, for any ξ in $D_{\rho_0/2}$,

$$\begin{aligned} \widetilde{H}_{\mathrm{eff}}[\xi] &:= e^{i\nu} \mathbf{1}_{H_f < \rho_0} \left\langle \mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta) - \xi, \widetilde{H}_0(\theta) - \xi)) + \xi \right\rangle_{at} \mathbf{1}_{H_f < \rho_0} \\ &= H_f \mathbf{1}_{H_f < \rho_0} + e^{i\nu} \langle \dots \rangle_{at}, \end{aligned} \tag{95}$$

where $\langle \dots \rangle_{at}$ denotes:

$$\left\langle P(\theta) \left[\widetilde{W}_g(\theta) - \widetilde{W}_g(\theta) \overline{P}(\theta) \left[(\widetilde{H}_0(\theta) - \xi) + \overline{P}(\theta) \widetilde{W}_g(\theta) \overline{P}(\theta) \right]^{-1} \overline{P}(\theta) \widetilde{W}_g(\theta) \right] P(\theta) \right\rangle_{at}, \tag{96}$$

and where, for a bounded operator A on $\mathrm{L}^2(\mathbb{R}^6) \otimes \mathcal{F}_s$, the operator $\langle A \rangle_{at}$ on $\mathcal{F}_s$ is defined as the operator associated with the bounded quadratic form

$$q_{\langle A \rangle_{at}}(\Phi, \Psi) := (\phi_0(\theta) \otimes \Phi, A \phi_0(\theta) \otimes \Psi). \tag{97}$$

Here, $\phi_0(\theta)$ denotes the normalized eigenstate associated with the non-degenerate eigenvalue 0 of $\widetilde{H}_{at}(\theta)$.
We see that $\widetilde{H}_{\mathrm{eff}}[\xi]$ defines an operator on $\mathbf{1}_{H_f < \rho_0} \mathcal{F}_s$. To obtain an operator on $\mathcal{H}_{\mathrm{red}}$, we scale the photons momenta through a unitary transformation $\mathcal{U}_{\rho_0}$ such that $k \mapsto \rho_0 k$. Then we set:

$$H_{\mathrm{eff}}[\xi] := \frac{1}{\rho_0} \mathcal{U}_{\rho_0} \widetilde{H}_{\mathrm{eff}}[\xi] \mathcal{U}_{\rho_0}^* = H_f \mathbf{1}_{H_f < 1} + \frac{e^{i\nu}}{\rho_0} \mathcal{U}_{\rho_0} \langle \dots \rangle_{at} \mathcal{U}_{\rho_0}^*, \tag{98}$$

for all $\xi \in D_{\rho_0/2}$.
Finally, to obtain an analytic family of operators $H_{(0)}[z]$ with $z \in D_{1/2}$, we map the spectral parameter through the transformation:

$$Z_{(0)} : D_{\rho_0/2} \to D_{1/2} \quad , \quad \xi \mapsto \frac{e^{i\nu}}{\rho_0}\xi. \tag{99}$$

Thus, we get a family $H_{(0)}[z]$ which is well defined as a family of bounded operators on $\mathcal{H}_{\text{red}}$ for all $z \in D_{1/2}$:

$$\begin{aligned} H_{(0)}[z] &:= H_{\text{eff}}[Z_{(0)}^{-1}(z)] \\ &= \frac{e^{i\nu}}{\rho_0}\mathbf{1}_{H_f<1}\mathcal{U}_{\rho_0}\left\langle \mathcal{F}_{P(\theta)}(\widetilde{H}_U^V(\theta) - Z_{(0)}^{-1}(z), \widetilde{H}_0(\theta) - Z_{(0)}^{-1}(z)))\right\rangle_{at}\mathcal{U}_{\rho_0}^*\mathbf{1}_{H_f<1} + z\mathbf{1}_{H_f<1}. \end{aligned} \tag{100}$$

We come now to the main theorem of this section, which, with the help of the results obtained in [BFS1], [BFS2] and [BCFS], will lead to the existence of resonances:

Theorem 4.1 *Let $\theta = i\nu$ with $\nu > 0$ sufficiently small; pick $\rho_0 > 0$ such that $\rho_0 \leq \delta\sin(\nu/2) < 1$. Assume in addition that $\widetilde{H}_U^V$ fulfills hypotheses $(\mathcal{H}_{-1/2})$ and $(\mathcal{H}_{1/2})$. Choose $\beta, \varepsilon > 0$.*
Then, for $g\rho_0^{-1/2}$ sufficiently small, $H_{(0)}[\cdot]$ belongs to $H(\mathcal{W}_{\geq 0})$; furthermore, defining $H_{(0)}[z] := H(\underline{w}_{(0)}[z])$, we have $\underline{w}_{(0)}[\cdot] \in B(\beta, \varepsilon)$, where

$$\begin{aligned} B(\beta,\varepsilon) := \Big\{ &\underline{w}[\cdot] = (E[\cdot], T[\cdot], (w_{M,N}[\cdot])_{M+N\geq 1}) \in \mathcal{W}_{\geq 0}, \\ &\sup_{z\in D_{1/2}} \|T[z,\alpha] - \alpha\|_{\mathcal{T}} \leq \beta, \ \sup_{z\in D_{1/2}} |E[z]| \leq \varepsilon, \ \sup_{z\in D_{1/2}} \|(w_{M,N}[z])_{M+N\geq 1}\|_\xi^\# \leq \varepsilon \Big\}. \end{aligned} \tag{101}$$

Note that the parameter ξ appearing in the definition of $\mathcal{W}_{\geq 0}$ is chosen such that $\xi \geq \mathrm{C}{\rho_0}^{1/2}$ where C denotes a positive real number.

Proof We sketch the proof and emphasize the main point which differs from [BFS1,2], [BCFS]. In particular, the requirement to hypothesis $(\mathcal{H}_{1/2})$ shall be essential here.
We begin with the operator $\widetilde{H}_{\text{eff}}[\xi]$, defined in (95) for $\xi \in D_{\rho_0/2}$, that we write as

$$\widetilde{H}_{\text{eff}}[\xi] = \mathbf{1}_{H_f<\rho_0}\left[\widetilde{E}_{\text{eff}}[\xi] + \widetilde{T}_{\text{eff}}[\xi; H_f] + \widetilde{W}_{\text{eff}}[\xi; H_f]\right]\mathbf{1}_{H_f<\rho_0}, \tag{102}$$

where

$$\begin{aligned} \widetilde{E}_{\text{eff}}[\xi] &:= \widetilde{w}_{0,0}^{\text{eff}}[\xi, 0], \\ \widetilde{T}_{\text{eff}}[\xi; H_f] &:= H_f + \widetilde{w}_{0,0}^{\text{eff}}[\xi; H_f] - \widetilde{w}_{0,0}^{\text{eff}}[\xi, 0], \\ \widetilde{W}_{\text{eff}}[\xi; H_f] &:= \sum_{M+N\geq 1} \widetilde{W}_{M,N}^{\text{eff}}[\xi; H_f], \end{aligned} \tag{103}$$

and

$$\widetilde{W}^{\mathrm{eff}}_{M,N}[\xi;H_f] := \sum_{\substack{\lambda^{(M)}\in\{1,2\}^M\\ \tilde{\lambda}^{(N)}\in\{1,2\}^N}} \int_{\mathbb{R}^{3(M+N)}} a^*_{\lambda^{(M)}}(k^{(M)})\widetilde{w}^{\mathrm{eff}}_{M,N}[\xi;H_f;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)}] \\ a_{\tilde{\lambda}^{(N)}}(\tilde{k}^{(N)})dk^{(M)}d\tilde{k}^{(N)}. \tag{104}$$

Then, assuming that g and ρ_0 are chosen as in proposition 4.2, the Pull-Through formula, Wick's theorem and (78) yield (see [BFS2]):

$$\begin{aligned}&\widetilde{w}^{\mathrm{eff}}_{M,N}[\xi;r;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)}]\\ &= e^{i\nu}\sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\substack{m_l+n_l+p_l+q_l=1,2\\ l=1,\dots,L}} \delta_{M,\sum_{l=1}^M m_l}\delta_{N,\sum_{l=1}^N n_l}\prod_{l=1}^{L}\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\\ &\qquad \left\{\widetilde{D}_L\left[\xi;H_f;\left\{W^{m_l,n_l}_{p_l,q_l};(k_l,\lambda_l)^{(m_l)};(\tilde{k}_l,\tilde{\lambda}_l)^{(n_l)}\right\}_{l=1}^{L};\left\{R_0^{\overline{P}}[H_f;\theta]\right\}_{l=1}^{L-1}\right]\right\}^{\mathrm{symm}}_{M,N},\end{aligned} \tag{105}$$

where $\left\{f\left[(k,\lambda)^{(m)};(\tilde{k},\tilde{\lambda})^{(n)}\right]\right\}^{\mathrm{symm}}_{m,n}$ denotes the symmetrization of f with respect to the variables $(k,\lambda)^{(m)}$ and $(\tilde{k},\tilde{\lambda})^{(n)}$:

$$\begin{aligned}&\left\{f\left[(k,\lambda)^{(m)},(\tilde{k},\tilde{\lambda})^{(n)}\right]\right\}^{\mathrm{symm}}_{m,n}\\ &:= \frac{1}{m!n!}\sum_{\pi\in S_m}\sum_{\tilde{\pi}\in S_n} f\left[(k_{\pi(1)},\lambda_{\pi(1)}),\dots,(k_{\pi(m)},\lambda_{\pi(m)});(\tilde{k}_{\tilde{\pi}(1)},\tilde{\lambda}_{\pi(1)}),\dots,(\tilde{k}_{\tilde{\pi}(n)},\tilde{\lambda}_{\pi(n)})\right],\end{aligned} \tag{106}$$

and

$$\begin{aligned}&\widetilde{D}_L\left[\xi;\alpha;\left\{W^{m_l,n_l}_{p_l,q_l};(k_l,\lambda_l)^{(m_l)};(\tilde{k}_l,\tilde{\lambda}_l)^{(n_l)}\right\}_{l=1}^{L};\left\{R_0^{\overline{P}}[H_f;\theta]\right\}_{l=1}^{L-1}\right]\\ &:= \prod_{l=1}^{L}(-g)^{m_l+n_l+p_l+q_l}\chi_{\rho_0}(\alpha+\tau_0)\Big(\phi_0(\theta)\otimes\Omega, W^{m_1,n_1}_{p_1,q_1}[(k_1,\lambda_1)^{(m_1)};(\tilde{k}_1,\tilde{\lambda}_1)^{(n_1)}]\\ &\qquad R_0^{\overline{P}}[H_f+\alpha+\tau_1;\theta]W^{m_2,n_2}_{p_2,q_2}[(k_2,\lambda_2)^{(m_2)};(\tilde{k}_2,\tilde{\lambda}_2)^{(n_2)}]\dots\\ &\qquad R_0^{\overline{P}}[H_f+\alpha+\tau_{L-1};\theta]W^{m_l,n_L}_{p_l,q_L}[(k_L,\lambda_L)^{(m_L)};(\tilde{k}_L,\tilde{\lambda}_L)^{(n_L)}]\phi_0(\theta)\otimes\Omega\Big)\chi_{\rho_0}(\alpha+\tau_L).\end{aligned} \tag{107}$$

We have set:

$$R_0^{\overline{P}}[H_f;\theta] := \overline{P}(\theta)\left[H_0^{\overline{P}}(\theta)\right]^{-1}\overline{P}(\theta) = \overline{P}(\theta)\left[\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}(\widetilde{H}_0(\theta)-z)\mathbf{1}_{\mathrm{Ran}(\overline{P}(\theta))}\right]^{-1}\overline{P}(\theta), \tag{108}$$

and

$$
\begin{aligned}
& W_{p_l,q_l}^{m_l,n_l}[(k_l,\lambda_l)^{(m_l)};(\tilde{k}_l,\tilde{\lambda}_l)^{(n_l)}] := \\
& \sum_{\substack{\mu_l^{(p_l)}\in\{1,2\}^{p_l} \\ \tilde{\mu}_l^{(q_l)}\in\{1,2\}^{q_l}}} \int_{\mathbb{R}^{3(p_l+q_l)}} \widetilde{G}_{m_l+p_l,n_l+q_l}\left[(k_l,\lambda_l)^{(m_l)},(y_l,\mu_l)^{(p_l)};(\tilde{k}_l,\tilde{\lambda}_l)^{(n_l)},(\tilde{y}_l,\tilde{\mu}_l)^{(q_l)};\theta\right] \\
& \qquad\qquad \otimes a^*_{\mu_l^{(p_l)}}(y_l^{(p_l)}) a_{\tilde{\mu}_l^{(q_l)}}(\tilde{y}_l^{(q_l)}) dy_l^{(p_l)} d\tilde{y}_l^{(q_l)}.
\end{aligned}
\tag{109}
$$

Besides,

$$
\tau_l := \sum_{j=1}^{l} |\tilde{k}_j^{(n_j)}| + \sum_{j=l+1}^{L} |k_j^{(m_j)}| := \sum_{j=1}^{l}\sum_{i=1}^{n_j} |\tilde{k}_j^i| + \sum_{j=l+1}^{L}\sum_{i=1}^{m_j} |k_j^i|. \tag{110}
$$

We come now to the main point whose proof differs from [BFS2]: it is the purpose of the following lemma.

Lemma 4.1 *Assume that $\widetilde{H}_U^V$ fulfills hypotheses $(\mathcal{H}_{-1/2})$ and $(\mathcal{H}_{1/2})$.*
Pick $(k,\lambda)^{(m)}$ in $(\mathbb{R}\times\{1,2\})^{3m}$ and $(\tilde{k},\tilde{\lambda})^{(n)}$ in $(\mathbb{R}\times\{1,2\})^{3n}$. Let ω, $\tilde{\omega}$ be two nonnegative real numbers; let $m,n,p,q\in\mathbb{N}$ be such that $1\le m+n+p+q\le 2$. Pick $0<\rho_0<1$. Then we have:

$$
\begin{aligned}
& \left\| |B_\theta(\rho_0+\omega)|^{-1/2} W_{p,q}^{m,n}[(k,\lambda)^{(m)};(\tilde{k},\tilde{\lambda})^{(n)}] |B_\theta(\rho_0+\tilde{\omega})|^{-1/2} \right\| \\
& \le \mathrm{Cste}\rho_0^{-\frac{1}{2}-\frac{1}{2}\delta_{p+q,0}} \prod_{j=1}^{m} J_{1/2}(k_j) \prod_{j=1}^{n} J_{1/2}(\tilde{k}_j),
\end{aligned}
\tag{111}
$$

where $B_\theta(\rho_0)$ is defined in (73)*.*

Proof Recall that $\theta := i\nu$ is fixed with $\nu > 0$ sufficiently small. We note the following two estimates which can be obtained in the same way as in [BFS3]:

$$
\left\| |B_\theta(\rho_0)|^{-1} \left(\widetilde{H}_{at}+i\right) \right\| \le \frac{\mathrm{Cste}}{\nu}\left(1+\frac{1}{\rho_0}\right), \tag{112}
$$

$$
\left\| |B_\theta(\rho_0)|^{-1} \left(H_f+\omega\right) \right\| \le \frac{\mathrm{Cste}}{\nu}\left(1+\frac{\omega}{\rho_0}\right), \tag{113}
$$

for all $\rho_0 > 0$ and $\omega \ge 0$.
We shall distinguish between the different cases $m+n=i, p+q=j$ such that $i,j\in\{0,1,2\}$ and $i+j=1,2$.
First, if $m+n=0$ and $p+q=1,2$, using that $\widetilde{H}_U^V$ fulfills hypothesis $(\mathcal{H}_{-1/2})$, in the same way as in (76), we can see that:

$$
\left\| |B_\theta(\rho_0+\omega)|^{-1/2} W_{p,q}^{0,0} |B_\theta(\rho_0+\tilde{\omega})|^{-1/2} \right\| \le \frac{\mathrm{Cste}}{\nu}\rho_0^{-1/2}. \tag{114}
$$

Assume now that $m+n=1,2$ and $p+q=0$. Using (112) and hypothesis $(\mathcal{H}_{1/2})$, we write:

$$\begin{aligned}
&\left\| |B_\theta(\rho_0+\omega)|^{-1/2} W_{0,0}^{m,n}[(k,\lambda)^{(m)};(\tilde{k},\tilde{\lambda})^{(n)}] |B_\theta(\rho_0+\tilde{\omega})|^{-1/2} \right\| \\
&\le \left\| |B_\theta(\rho_0+\omega)|^{-1/2} \left| \widetilde{H}_{at}+i \right|^{1/2} \right\| \\
&\quad \times \left\| \left| \widetilde{H}_{at}+i \right|^{-1/2} \widetilde{G}_{m,n}\left[(k,\lambda)^{(m)};(\tilde{k},\tilde{\lambda})^{(n)};\theta\right] \left| \widetilde{H}_{at}+i \right|^{-1/2} \right\| \\
&\quad \times \left\| \left| \widetilde{H}_{at}+i \right|^{1/2} |B_\theta(\rho_0+\tilde{\omega})|^{-1/2} \right\| \\
&\le \frac{\text{Cste}}{\nu^2} {\rho_0}^{-1} \times \prod_{j=1}^{m} J_{1/2}(k_j) \prod_{j=1}^{n} J_{1/2}(\tilde{k}_j)
\end{aligned}$$

Finally, we have to deal with the possibility $m+n=1$ and $p+q=1$. For instance, we assume that $p=m=1$ and $n=q=0$; the other cases can be treated in the same way. We compute:

$$\begin{aligned}
&\left\| |B_\theta(\rho_0+\omega)|^{-1/2} W_{1,0}^{1,0}[(k,\lambda)^{(m)}] |B_\theta(\rho_0+\tilde{\omega})|^{-1/2} \right\| \\
&= \left\| |B_\theta(\rho_0+\omega)|^{-1/2} \sum_{\mu=1,2} \int_{\mathbb{R}^3} \widetilde{G}_{2,0}\left[(k,\lambda);(y,\mu);\theta\right] \otimes \widehat{a}_\mu^*(y) dy |B_\theta(\rho_0+\tilde{\omega})|^{-1/2} \right\| \\
&\le \left[\sup_{\mu=1,2} \int_{\mathbb{R}^3} \frac{1}{|y|} \left\| |B_\theta(\rho_0+\omega)|^{-1/2} \widetilde{G}_{2,0}\left[(k,\lambda);(y,\mu);\theta\right] |B_\theta(\rho_0+\tilde{\omega}+|y|)|^{-1/2} (H_f+|y|)^{1/2} \right\|^2 dy \right]^{1/2} \\
&\quad \times \sup_{\|\Psi\|=1} \left[\sum_{\mu=1,2} \int_{\mathbb{R}^3} |y| \left\| (H_f+|y|)^{-1/2} \widehat{a}_\mu^*(y) \Psi \right\|^2 dy \right]^{1/2}
\end{aligned}$$

We can easily see that the second term in the last line is less than 1. As for the first term, using (112), (113) and the two hypotheses $(\mathcal{H}_{-1/2})$, $(\mathcal{H}_{1/2})$, we have:

$$\begin{aligned}
&\left\| |B_\theta(\rho_0+\omega)|^{-1/2} \widetilde{G}_{2,0}\left[(k,\lambda);(y,\mu);\theta\right] |B_\theta(\rho_0+\tilde{\omega}+|y|)|^{-1/2} (H_f+|y|)^{1/2} \right\| \\
&\le \left\| (H_f+|y|)^{1/4} |B_\theta(\rho_0+\omega)|^{-1/4} \right\| \times \left\| |B_\theta(\rho_0+\omega)|^{-1/4} |\widetilde{H}_{at}+i|^{1/4} \right\| \\
&\quad \times \left\| |\widetilde{H}_{at}+i|^{-1/4} \widetilde{G}_{2,0}\left[(k,\lambda);(y,\mu);\theta\right] |\widetilde{H}_{at}+i|^{-1/4} \right\| \\
&\quad \times \left\| |\widetilde{H}_{at}+i|^{1/4} |B_\theta(\rho_0+\tilde{\omega}+|y|)|^{-1/4} \right\| \times \left\| |B_\theta(\rho_0+\tilde{\omega}+|y|)|^{-1/4} (H_f+|y|)^{1/4} \right\| \\
&\le \frac{\text{Cste}}{\nu} \left[\left(1+\frac{|y|}{\rho_0}\right) \left(1+\frac{1}{\rho_0}\right) \left(1+\frac{1}{\rho_0+|y|}\right) \left(1+\frac{|y|}{\rho_0+|y|}\right) \right]^{1/4} J_{1/2}(k) J_{-1/2}(y) \\
&\le \frac{\text{Cste}}{\nu} {\rho_0}^{-1/2} (1+|y|)^{1/2} J_{1/2}(k) J_{-1/2}(y).
\end{aligned}$$

This yields:

$$\begin{aligned}
&\left\| |B_\theta(\rho_0+\omega)|^{-1/2} W_{1,0}^{1,0}[(k,\lambda)^{(m)}] |B_\theta(\rho_0+\tilde{\omega})|^{-1/2} \right\| \\
&\le \frac{\text{Cste}}{\nu} {\rho_0}^{-1/2} J_{1/2}(k) \left[\int_{\mathbb{R}^3} \left(1+\frac{1}{|y|}\right) J_{-1/2}(y)^2 dy \right]^{1/2} = \frac{\text{Cste}}{\nu} \Lambda_{-1/2} {\rho_0}^{-1/2} J_{1/2}(k).
\end{aligned}$$

Thus the proof of the lemma is complete. □

Back to the proof of theorem 4.1

To finish the proof of theorem 4.1, with the help of lemma 4.1, it suffices to follow [BFS2], with some simplifications due to the use of the smooth Feshbach map. Namely, using the identity $|B_\theta(\rho_0)|^{1/2}\phi_0(\theta)\otimes\Omega = {\rho_0}^{1/2}\phi_0(\theta)\otimes\Omega$ together with lemma 4.1 and the bound obtained as in (74):

$$\Big\| |B_\theta(\rho_0)| R_0^{\overline{P}}[H_f+\alpha+\mu_l;\theta] \Big\| \leq \frac{\mathrm{Cste}}{\nu}, \tag{115}$$

we get (see [BFS2], lemma 3.7)

$$\begin{aligned} &\left| \widetilde{D}_L \left[\xi;\alpha;\left\{ W_{p_l,q_l}^{m_l,n_l};(k_l,\lambda_l)^{(m_l)};(\tilde{k}_l,\tilde{\lambda}_l)^{(n_l)} \right\}_{l=1}^{L};\left\{ R_0^{\overline{P}}[H_f;\theta] \right\}_{l=1}^{L-1} \right] \right| \\ &\leq \prod_{l=1}^{L} (\mathrm{C}_1 g {\rho_0}^{-1/2})^{m_l+n_l+p_l+q_l} {\rho_0}^{1-\frac{1}{2}(M+N)} \mathrm{C}_2^{M+N} \prod_{j=1}^{M} J_{1/2}(k_j) \prod_{j=1}^{N} J_{1/2}(\tilde{k}_j). \end{aligned} \tag{116}$$

Here, C_1,C_2 denote positive real numbers depending respectively on ν,Λ and Λ.
Inserting (116) into (105), we can obtain:

$$\begin{aligned} &\left| \widetilde{w}_{M,N}^{\mathrm{eff}}[\xi;r;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)}] \right| \\ &\leq {\rho_0}^{1-\frac{1}{2}(M+N)} (\mathrm{C}_3 g {\rho_0}^{-1/2})^{M+N+2\delta_{M+N,0}} \prod_{j=1}^{M} J_{1/2}(k_j) \prod_{j=1}^{N} J_{1/2}(\tilde{k}_j) \end{aligned} \tag{117}$$

for some positive constant C_3.
Next, seeing the definition of the Banach space of Hamiltonians at the beginning of this subsection, we have to consider the derivative of $\widetilde{D}_L[\dots]$ with respect to α. To this end, we notice that:

$$\overline{R}_0[H_f;\theta] = \Big[\overline{P}_{at}(\theta)\widetilde{H}_{at}(\theta)\otimes\mathbf{1} + e^{-i\nu}\overline{P}_{at}(\theta)\otimes(H_f-z) \Big]^{-1} + P_{at}(\theta)\frac{\overline{\chi}_{\rho_0}(H_f)^2}{e^{-i\nu}H_f - z}, \tag{118}$$

which yields

$$\partial_{H_f}\overline{R}_0[H_f;\theta] = -e^{-i\nu}\overline{R}_0[H_f;\theta]^2 + P_{at}(\theta)\otimes\frac{2\overline{\chi}_{\rho_0}(H_f)\partial_{H_f}\overline{\chi}_{\rho_0}(H_f)}{e^{-i\nu}H_f - z}. \tag{119}$$

Using this together with Leibniz' rule and the bound

$$\Big\| |B_\theta(\rho_0)| R_0^{\overline{P}}[H_f+\alpha+\mu_l;\theta]^2 \Big\| \leq \frac{\mathrm{Cste}}{\nu}{\rho_0}^{-1}, \tag{120}$$

we obtain

$$\begin{aligned} &\left| \partial_\alpha \widetilde{D}_L \left[\xi;\alpha;\left\{ W_{p_l,q_l}^{m_l,n_l};(k_l,\lambda_l)^{(m_l)};(\tilde{k}_l,\tilde{\lambda}_l)^{(n_l)} \right\}_{l=1}^{L};\left\{ R_0^{\overline{P}}[H_f;\theta] \right\}_{l=1}^{L-1} \right] \right| \\ &\leq \prod_{l=1}^{L} (\mathrm{C}_1 g {\rho_0}^{-1/2})^{m_l+n_l+p_l+q_l} {\rho_0}^{-\frac{1}{2}(M+N)} \mathrm{C}_2^{M+N} \prod_{j=1}^{M} J_{1/2}(k_j) \prod_{j=1}^{N} J_{1/2}(\tilde{k}_j). \end{aligned} \tag{121}$$

Hence, inserting (121) into (105), we can get in the same way as in (117):

$$\begin{aligned}&\left|\partial_\alpha \widetilde{w}^{\mathrm{eff}}_{M,N}[\xi; r; (k,\lambda)^{(M)}; (\tilde{k},\tilde{\lambda})^{(N)}]\right|\\&\leq {\rho_0}^{-\frac{1}{2}(M+N)} (\mathrm{C}_3 g {\rho_0}^{-1/2})^{M+N+2\delta_{M+N,0}} \prod_{j=1}^{M} J_{1/2}(k_j) \prod_{j=1}^{N} J_{1/2}(\tilde{k}_j).\end{aligned} \tag{122}$$

To conclude, we come back to the operator $H_{(0)}[z]$ defined in (98)-(100). The identity

$$H_{(0)}[z] = \frac{e^{i\nu}}{\rho_0} \mathcal{U}_{\rho_0} \widetilde{H}_{\mathrm{eff}}[Z^{-1}_{(0)}(z)] \mathcal{U}^*_{\rho_0} \tag{123}$$

together with (103)-(104) implies that for any $z \in D_{1/2}$:

$$H_{(0)}[z] = \mathbf{1}_{H_f<1} \left[E_{(0)}[z] + T_{(0)}[z; H_f] + W_{(0)}[z; H_f]\right] \mathbf{1}_{H_f<1}, \tag{124}$$

with

$$\begin{aligned}&E_{(0)}[z] := w^{(0)}_{0,0}[z, 0],\\&T_{(0)}[z; H_f] := H_f + w^{(0)}_{0,0}[z; H_f] - w^{(0)}_{0,0}[z, 0],\\&W_{(0)}[z; H_f] := \sum_{M+N\geq 1} W^{(0)}_{M,N}[z; H_f],\end{aligned} \tag{125}$$

and

$$\begin{aligned}W^{(0)}_{M,N}[z; H_f] := \sum_{\substack{\lambda^{(M)}\in\{1,2\}^M\\ \tilde{\lambda}^{(N)}\in\{1,2\}^N}} \int_{\mathbb{R}^{3(M+N)}} & a^*_{\lambda^{(M)}}(k^{(M)}) w^{(0)}_{M,N}[z; H_f; (k,\lambda)^{(M)}; (\tilde{k},\tilde{\lambda})^{(N)}]\\ & a_{\tilde{\lambda}^{(N)}}(\tilde{k}^{(N)}) dk^{(M)} d\tilde{k}^{(N)};\end{aligned} \tag{126}$$

furthermore, for all $M+N \geq 0$, $w^{(0)}_{M,N}$ is related to $\widetilde{w}^{\mathrm{eff}}_{M,N}$ through

$$\begin{aligned}&w^{(0)}_{M,N}\left[z; H_f; (k,\lambda)^{(M)}; (\tilde{k},\tilde{\lambda})^{(N)}\right]\\&= {\rho_0}^{\frac{3}{2}(M+N)-1} \widetilde{w}^{\mathrm{eff}}_{M,N}\left[Z^{-1}_{(0)}(z); \rho_0 H_f; (\rho_0 k,\lambda)^{(M)}; (\rho_0\tilde{k},\tilde{\lambda})^{(N)}\right].\end{aligned} \tag{127}$$

Thus, (117) and (122) yield

$$\begin{aligned}&\left|w^{(0)}_{M,N}[z; \alpha; (k,\lambda)^{(M)}; (\tilde{k},\tilde{\lambda})^{(N)}]\right| + \left|\partial_\alpha w^{(0)}_{M,N}[z; \alpha; (k,\lambda)^{(M)}; (\tilde{k},\tilde{\lambda})^{(N)}]\right|\\&\leq 2{\rho_0}^{(M+N)} (\mathrm{C}_3 g {\rho_0}^{-1/2})^{M+N+2\delta_{M+N,0}} \prod_{j=1}^{M} J_{1/2}(\rho_0 k_j) \prod_{j=1}^{N} J_{1/2}(\rho_0 \tilde{k}_j)\\&\leq {\rho_0}^{\frac{3}{2}(M+N)} (\mathrm{C}_4 g {\rho_0}^{-1/2})^{M+N+2\delta_{M+N,0}} \prod_{j=1}^{M} |k_j|^{1/2} \prod_{j=1}^{N} |\tilde{k}_j|^{1/2},\end{aligned} \tag{128}$$

where C_4 denotes a positive real number.
In other words, $H_{(0)}[z] = H(\underline{w}_{(0)}[z])$ with $\underline{w}_{(0)} = \Big(E_{(0)}, T_{(0)}, (w^{(0)}_{M,N})_{M+N\geq 1}\Big)$ and

$$\begin{aligned}
&\sup_{z\in D_{1//2}} \left|E_{(0)}[z]\right| \leq C_4 g^2 {\rho_0}^{-1},\\
&\sup_{z\in D_{1//2}} \left\|T_{(0)}[z;\alpha] - \alpha\right\|_{\mathcal{T}} \leq 2C_4 g^2 {\rho_0}^{-1},\\
&\sup_{z\in D_{1//2}} \left\|(w^{(0)}_{M,N}[z])_{M+N\geq 1}\right\|^{\#}_{\xi} \leq \sum_{M+N\geq 1} (C_5 g {\rho_0}^{1/2})^{M+N} \leq C_5 g {\rho_0}^{1/2},
\end{aligned} \tag{129}$$

where $C_5 > 0$, and where we have chosen $\xi \geq \mathrm{Cste}{\rho_0}^{1/2}$.
Hence, the proof of the theorem is complete, except for the fact that $H_{(0)}[z]$ is analytic on $D_{1/2}$. But since we have $\partial_z \overline{R}_0[H_f;\theta] = -\overline{R}_0[H_f;\theta]^2$, one can see, as in (120)-(122), that $\left|\partial_z w^{(0)}_{M,N}[z;\alpha;(k,\lambda)^{(M)};(\tilde{k},\tilde{\lambda})^{(N)}]\right|$ is bounded, which gives the analyticity of $H_{(0)}[z]$. □

To sum up, we have showed that $H_{(0)}[\cdot]$ belongs to $H(B(\rho/8,\rho/8))$ for any $\rho > 0$, provided that ρ_0 is less than $\delta \sin\nu/2$ and that $g{\rho_0}^{-1/2}$ is sufficiently small. Moreover, by construction, $H_{(0)}[\cdot]$ is isospectral to the initial Hamiltonian $\widetilde{H}^V_U(\theta)$ in the sense that:

$$z \in \sigma(H_{(0)}[z]) \cap D_{1/2} \Leftrightarrow Z^{-1}_{(0)}(z) \in \sigma(\widetilde{H}^V_U(\theta)) \cap D_{\rho_0/2}. \tag{130}$$

Now, we appeal to the results proved in [BFS2] and [BCFS]: the renormalization transformation $\mathcal{R}_\rho$ is defined for any $\underline{w} = (E, T, (w_{M,N})_{M+N\geq 1})$ in $B(\rho/8,\rho/8)$ by:

$$\begin{aligned}
\mathcal{R}_\rho\left(H(\underline{w}[z])\right) - z := \rho^{-1}\mathcal{U}_\rho \mathcal{F}_{\chi_\rho(H_f)}\big[&H(\underline{w}[Z^{-1}(z)]) - Z^{-1}(z),\\
&E[Z^{-1}(z)] + T[Z^{-1}(z);H_f] - Z^{-1}(z)\big]\,\mathcal{U}^*_\rho,
\end{aligned} \tag{131}$$

where the map Z is defined by

$$\{\xi, |\xi - E[\xi]| \leq \rho/2\} \to D_{1/2} \quad , \quad \xi \mapsto \rho^{-1}\left(\xi - E[\xi]\right). \tag{132}$$

Then we have (see [BCFS]):

Theorem 4.2 *Fix $\rho := (16C_\Theta)^{-2}$ and $\xi := (4C_\Theta)^{-1}\rho^{1/2}$ where $C_\Theta \geq 1$ is a constant only depending on the smooth function Θ defined in (64). Then,*

$$\mathcal{R}_\rho : H\left(B(\beta,\varepsilon)\right) \to H\left(B\left(\beta + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)\right), \tag{133}$$

for any $0 < \beta \leq \varepsilon_0$, $0 < \varepsilon \leq \varepsilon_0$, where $\varepsilon_0 := (8C_\Theta)^{-1}\rho$.

Proof Note that the Banach space $\mathcal{W}_{\geq 0}$ that we have defined in this paper is different from the one defined in [BCFS]; indeed, we have used a supremum in the definition of the norm (82), where the authors used a L^2-norm in [BCFS]. This modification allows us to simplify some estimates, but this also requires that the proof of the present theorem to be modified. However, the modifications are straightforward, and we do not reproduce the proof here. □

To conclude, we state a result which provides the existence of resonances for the Hamiltonian $\widetilde{H}_U^V$; its proof can be found in [BFS2], [BCFS]. We define for $n \geq 1$:

$$H_{(n)}[\cdot] := H\left((E_{(n)}, T_{(n)}, (w_{M,N}^{(n)})_{M+N\geq 1})\right) := \mathcal{R}_\rho^n(H_{(0)}[\cdot]), \tag{134}$$

and, as in (132),

$$Z_{(n)} : \left\{z \in D_{1/2}, \left|z - E_{(n-1)}[z]\right| \leq \rho/2\right\} \to D_{1/2} \quad , \quad z \mapsto \frac{1}{\rho}\left(z - E_{(n-1)}[z]\right). \tag{135}$$

Notice that, choosing ρ, ξ, ε_0, β, ε as in theorem 4.2 and ρ_0 such that $0 < \rho_0 \leq \min(\rho, \delta \sin(\nu/2))$, we obtain from theorems 4.1 and 4.2:

$$H_{(n)}[\cdot] \in H\left(B(\beta + \varepsilon, \frac{\varepsilon}{2^n})\right), \tag{136}$$

for any $n \geq 0$, provided $g\rho_0^{-1/2}$ is sufficiently small.
Then it is proved that:

Theorem 4.3 *Fix ρ and ξ as in theorem 4.2. Pick $\rho_0 > 0$ sufficiently small and assume that $g > 0$ is also sufficiently small. Then, for all $\theta = i\nu \in D(0, \theta_0)$, where $\nu > 0$ is chosen such that $\rho_0 \leq \delta \sin(\nu/2)$, the spectrum of $\widetilde{H}_U^V(\theta)$ is located as follows (see fig. 5):*

$$\sigma\left(\widetilde{H}_U^V(\theta)\right) \cap D_{\rho_0/2} \subset E_{(\infty)} + K_{(\infty)}. \tag{137}$$

Here, $E_{(\infty)} := \lim_{n\to\infty} Z_{(0)}^{-1} \circ Z_{(1)}^{-1} \circ \cdots \circ Z_{(n)}^{-1}(0)$ is a simple eigenvalue of $\widetilde{H}_U^V(\theta)$, and $K_{(\infty)}$ denotes a complex domain contained in:

$$\left\{e^{-i\nu}a + b, 0 \leq a \leq 1, |b| \leq \mathrm{C}a^\tau\right\}, \tag{138}$$

where $\tau > 1$ and C is a positive constant which can be chosen strictly less than 1 provided g is sufficiently small.
Assuming moreover that $(\mathcal{H}_{\Gamma_0})$ is fulfilled, this implies that $E_{(\infty)}$ is a resonance for $\widetilde{H}_U^V$.

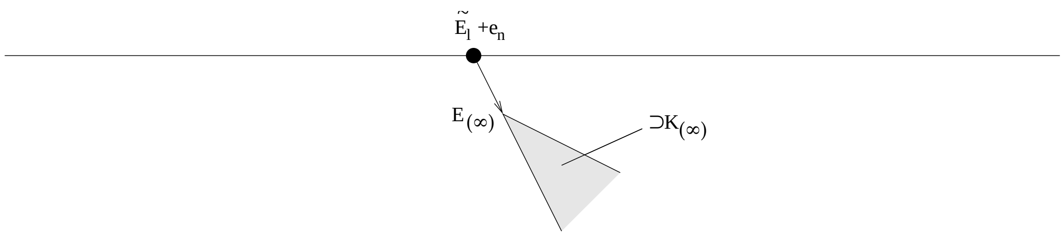

Figure 5: **The spectrum of $\widetilde{H}_U^V(\theta)$ around $\widetilde{E}_l + e_n$**

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