Metastability in Interacting Nonlinear Stochastic Differential Equations II: Large-N Behaviour

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Abstract

We consider the dynamics of a periodic chain of N coupled overdamped particles under the influence of noise, in the limit of large N. Each particle is subjected to a bistable local potential, to a linear coupling with its nearest neighbours, and to an independent source of white noise. For strong coupling (of the order N^2), the system synchronises, in the sense that all oscillators assume almost the same position in their respective local potential most of the time. In a previous paper, we showed that the transition from strong to weak coupling involves a sequence of symmetry-breaking bifurcations of the system's stationary configurations, and analysed in particular the behaviour for coupling intensities slightly below the synchronisation threshold, for arbitrary N. Here we describe the behaviour for any positive coupling intensity γ of order N^2, provided the particle number N is sufficiently large (as a function of γ/N^2). In particular, we determine the transition time between synchronised states, as well as the shape of the "critical droplet" to leading order in $1/N$. Our techniques involve the control of the exact number of periodic orbits of a near-integrable twist map, allowing us to give a detailed description of the system's potential landscape, in which the metastable behaviour is encoded.

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1 Introduction

In this paper, we continue our analysis of the metastable dynamics of a periodic chain of coupled bistable elements, initiated in [BFG06a]. In contrast with similar models involving discrete on-site variables, or "spins", whose metastable behaviour has been studied extensively (see for instance [dH04, OV05]), our model involves continuous local variables, and is therefore described by a set of interacting stochastic differential equations.

The analysis of the metastable dynamics of such a system requires an understanding of its N-dimensional "potential landscape", in particular the number and location of its local minima and saddles of index 1. In the previous paper [BFG06a], we showed that the number of stationary configurations increases from 3 to 3^N as the coupling intensity γ decreases from a critical value γ_1 of order N^2 to 0. This transition from strong to weak coupling involves a sequence of successive symmetry-breaking bifurcations, and we analysed in particular the first of these bifurcations, which corresponds to desynchronisation.

In the present paper, we consider in more detail the behaviour for large particle number N. It turns out that a technique known as "spatial map" analysis allows us to obtain a precise control of the set of stationary points, for values of the coupling well below the synchronisation threshold. More precisely, given a strictly positive coupling intensity γ of order N^2, there is an integer $N_0(\gamma/N^2)$ such that we know precisely the number, location and type of the potential's stationary points for all $N \geqslant N_0(\gamma/N^2)$. This allows us to characterise the transition times and paths between metastable states for all these values of γ and N.

This paper is organised as follows. Section 2 contains the precise definition of our model, and the statement of all results. After introducing the model in Section 2.1 and describing general properties of the potential landscape in Section 2.2, we state in Section 2.3 the detailed results on number and location of stationary points for large N, and in Section 2.4 their consequences for the stochastic dynamics. Section 3 contains the proofs of these results. The proofs rely on the detailed analysis of the orbits of period N of a near-integrable twist map, which are in one-to-one correspondance with stationary points of the potential. Appendix A recalls some properties of Jacobi's elliptic functions needed in the analysis.

Acknowledgments

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2 Model and Results

2.1 Definition of the Model

Our model of interacting bistable systems perturbed by noise is defined by the following ingredients:

- The periodic one-dimensional lattice is given by $\Lambda = \mathbb{Z}/N\mathbb{Z}$, where $N \geqslant 2$ is the number of particles.
- To each site $i \in \Lambda$, we attach a real variable $x_i \in \mathbb{R}$, describing the position of the ith particle. The configuration space is thus $\mathcal{X} = \mathbb{R}^\Lambda$.
- Each particle feels a local bistable potential, given by

$$U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2 , \qquad x \in \mathbb{R} . \tag{2.1}$$

  The local dynamics thus tends to push the particle towards one of the two stable positions $x = 1$ or $x = -1$.
- Neighbouring particles in Λ are coupled via a discretised-Laplacian interaction, of intensity $\gamma/2$.
- Each site is coupled to an independent source of noise, of intensity $\sigma\sqrt{N}$ (this scaling is appropriate when studying the large-N behaviour for strong coupling, and is im-

material for small N). The sources of noise are described by independent Brownian motions $\{B_i(t)\}_{t\geqslant 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The system is thus described by the following set of coupled stochastic differential equations, defining a diffusion on $\mathcal{X}$:

$$\mathrm{d}x_i^\sigma(t) = f(x_i^\sigma(t))\,\mathrm{d}t + \frac{\gamma}{2}\big[x_{i+1}^\sigma(t) - 2x_i^\sigma(t) + x_{i-1}^\sigma(t)\big]\,\mathrm{d}t + \sigma\sqrt{N}\,\mathrm{d}B_i(t)\;, \tag{2.2}$$

where the local nonlinear drift is given by

$$f(x) = -\nabla U(x) = x - x^3\;. \tag{2.3}$$

For $\sigma = 0$, the system (2.2) is a gradient system of the form $\dot{x} = -\nabla V(x)$, with potential

$$V(x) = V_\gamma(x) = \sum_{i\in\Lambda} U(x_i) + \frac{\gamma}{4}\sum_{i\in\Lambda}(x_{i+1} - x_i)^2\;. \tag{2.4}$$

2.2 Potential Landscape and Metastability

The dynamics of the stochastic system depends essentially on the "potential landscape" V. More precisely, let

$$\mathcal{S} = \mathcal{S}(\gamma) = \{x \in \mathcal{X} \colon \nabla V_\gamma(x) = 0\} \tag{2.5}$$

denote the set of stationary points of the potential. A point $x \in \mathcal{S}$ is said to be of type (n_-, n_0, n_+) if the Hessian matrix of V in x has n_- negative, n_+ positive and $n_0 = N - n_- - n_+$ vanishing eigenvalues (counting multiplicity). For each $k = 0, \dots, N$, let $\mathcal{S}_k = \mathcal{S}_k(\gamma)$ denote the set of stationary points $x \in \mathcal{S}$ which are of type $(N-k, 0, k)$. For $k \geqslant 1$, these points are called *saddles of index* k, or simply k*-saddles*, while $\mathcal{S}_0$ is the set of strict local minima of V.

Understanding the dynamics for small noise essentially requires knowing the graph $\mathcal{G} = (\mathcal{S}_0, \mathcal{E})$, in which two vertices $x^\star, y^\star \in \mathcal{S}_0$ are connected by an edge if and only if there is a 1-saddle $s \in \mathcal{S}_1$ whose unstable manifolds converge to $x^\star$ and $y^\star$. The system behaves essentially like a Markovian jump process on $\mathcal{G}$. The mean transition time from $x^\star$ to $y^\star$ is of order e^{2H/σ^2}, where H is the potential difference between $x^\star$ and the lowest saddle leading to $y^\star$ (see [FW98], as well as [BFG06a] for more details).

It is easy to see that $\mathcal{S}$ always contains at least the three points

$$O = (0, \dots, 0)\;, \qquad I^\pm = \pm(1, \dots, 1)\;. \tag{2.6}$$

Depending on the value of γ, the origin O can be an N-saddle, or a k-saddle for any odd k. The points $I^\pm$ always belong to $\mathcal{S}_0$, in fact we have for all $\gamma > 0$

$$V_\gamma(x) > V_\gamma(I^+) = V_\gamma(I^-) = -\frac{N}{4} \quad \forall x \in \mathcal{X} \setminus \{I^-, I^+\}\;, \tag{2.7}$$

so that I^+ and I^- represent the most stable configurations of the system. The three points O, I^+ and I^- are the only stationary points belonging to the diagonal $\mathcal{D} = \{x \in \mathcal{X} \colon x_1 = x_2 = \dots = x_N\}$.

The potential $V(x)$ is invariant under the transformation group $G = G_N$ of order $4N$ (4 if $N = 2$), generated by the following three symmetries:

- the rotation around the diagonal given by $R(x_1, \dots, x_N) = (x_2, \dots, x_N, x_1)$;
- the mirror symmetry $S(x_1, \dots, x_N) = (x_N, \dots, x_1)$;
- the point symmetry $C(x_1, \dots, x_N) = -(x_1, \dots, x_N)$.

N	x	Type of symmetry
$4L$	A	$(x_1,\dots,x_L,x_L,\dots,x_1,-x_1,\dots,-x_L,-x_L,\dots,-x_1)$
	B	$(x_1,\dots,x_L,\dots,x_1,0,-x_1,\dots,-x_L,\dots,-x_1,0)$
$4L+2$	A	$(x_1,\dots,x_{L+1},\dots,x_1,-x_1,\dots,-x_{L+1},\dots,-x_1)$
	B	$(x_1,\dots,x_L,x_L\dots,x_1,0,-x_1,\dots,-x_L,-x_L,\dots,-x_1,0)$
$2L+1$	A	$(x_1,\dots,x_L,-x_L,\dots,-x_1,0)$
	B	$(x_1,\dots,x_L,x_L,\dots,x_1,x_0)$

Table 1. Symmetries of the stationary points bifurcating from the origin at $\gamma=\gamma_1$. The situation depends on whether N is odd (in which case we write $N=2L+1$) or even (in which case we write $N=4L$ or $N=4L+2$, depending on the value of $N \pmod 4$)). Points labeled A are 1-saddles near the desynchronisation bifurcation at $\gamma=\gamma_1$, those labeled B are 2-saddles (for odd N, this is actually a conjecture). More saddles of the same index are obtained by applying elements of the symmetry group G_N to A and B.

As a consequence, the set $\mathcal{S}$ of stationary points, as well as each $\mathcal{S}_k$, are invariant under the action of G. This has important consequences for the classification of stationary points.

In [BFG06a], we proved the following results:

- There is a critical coupling intensity
$$\gamma_1=\frac{1}{1-\cos(2\pi/N)} \tag{2.8}$$
such that for all $\gamma\geqslant\gamma_1$, the set of stationary points $\mathcal{S}$ consists of the three points O and $I^\pm$ only. The graph $\mathcal{G}$ has two vertices $I^\pm$, connected by a single edge.
- As γ decreases below γ_1, an even number of new stationary points bifurcate from the origin. Half of them are 1-saddles, while the others are 2-saddles. These points satisfy symmetries as shown in Table 1. The potential difference between $I^\pm$ and the 1-saddles behaves like $N(1/4-(\gamma_1-\gamma)^2/6)$ as $\gamma\nearrow\gamma_1$.
- New bifurcations of the origin occur for $\gamma=\gamma_M=(1-\cos(2\pi M/N))^{-1}$, with $2\leqslant M\leqslant N/2$, in which saddles of order higher than 2 are created.

The number of stationary points emerging from the origin at the desynchronisation bifurcation at $\gamma=\gamma_1$ depends on the parity of N. If N is even, there are exactly $2N$ new points (N saddles of index 1, and N saddles of index 2). If N is odd, we were only able to prove that the number of new stationary points is a multiple of $4N$, but formulated the conjecture that there are exactly $4N$ stationary points ($2N$ saddles of index 1, and $2N$ saddles of index 2). We checked this conjecture numerically for all N up to 101. As we shall see in the next section, the conjecture is also true for N sufficiently large.

2.3 Global Control of the Set of Stationary Points

We examine now the structure of the set $\mathcal{S}$ of stationary points for large particle number N, and large coupling intensity γ. More precisely, we introduce the rescaled coupling intensity
$$\widetilde{\gamma}=\frac{\gamma}{\gamma_1}=\gamma(1-\cos(2\pi/N))=\frac{2\pi}{N^2}\gamma\biggl[1+\mathcal{O}\biggl(\frac{1}{N^2}\biggr)\biggr]\;. \tag{2.9}$$

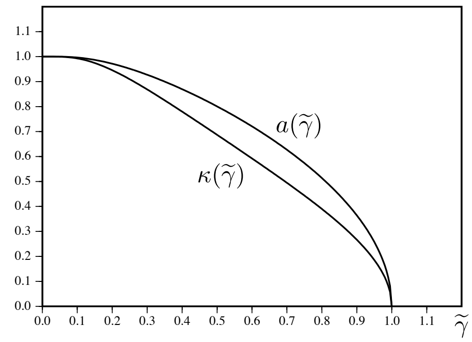

FIGURE 1. The functions $\kappa(\widetilde{\gamma})$ and $a(\widetilde{\gamma})$ appearing in the expressions (2.14) and (2.15) for the coordinates of A and B.

The desynchronisation bifurcation occurs for $\widetilde{\gamma} = \widetilde{\gamma}_1 := 1$. For $2 \leqslant M \leqslant N/2$, we also introduce scaled bifurcation values

$$\widetilde{\gamma}_M = \frac{\gamma_M}{\gamma_1} = \frac{1-\cos(2\pi/N)}{1-\cos(2\pi M/N)} = \frac{1}{M^2} + \mathcal{O}\biggl(\frac{1}{N^2}\biggr) . \tag{2.10}$$

For large N and $\widetilde{\gamma}$ bounded away from zero, we can get a precise control of the set of stationary points. Their coordinates will be given in terms of Jacobi elliptic functions and integrals[1], and involve function $\kappa(\widetilde{\gamma})$ and $a(\widetilde{\gamma})$, defined implicitly by the relations

$$\widetilde{\gamma} = \frac{\pi^2}{4\,\mathrm{K}(\kappa)^2(1+\kappa^2)} , \qquad a^2 = \frac{2\kappa^2}{1+\kappa^2} , \tag{2.11}$$

where $\mathrm{K}(\kappa)$ denotes Jacobi's elliptic integral of the first kind. In the following results, O_x denotes the group orbit of a point $x \in \mathcal{X}$ under G, that is, $O_x = \{gx \colon g \in G\}$.

Theorem 2.1. *There exists $N_1 < \infty$ such that when $N \geqslant N_1$ and $\widetilde{\gamma}_2 < \widetilde{\gamma} < \widetilde{\gamma}_1 = 1$, the set $\mathcal{S}$ of stationary points of the potential has cardinality*

$$|\mathcal{S}| = 3 + \frac{4N}{\gcd(N,2M)} = \begin{cases} 3+2N & \text{if } N \text{ is even} , \\ 3+4N & \text{if } N \text{ is odd} . \end{cases} \tag{2.12}$$

It can be decomposed as

$$\begin{aligned} \mathcal{S}_0 &= O_{I^+} = \{I^+, I^-\} , \\ \mathcal{S}_1 &= O_A , \\ \mathcal{S}_2 &= O_B , \\ \mathcal{S}_3 &= O_O = \{O\} , \end{aligned} \tag{2.13}$$

where the points $A = A(\widetilde{\gamma})$ and $B = B(\widetilde{\gamma})$ have the following properties:

- *If N is even, A and B satisfy the symmetries of Table 1, and have components given in terms of Jacobi's elliptic sine by*

$$\begin{aligned} A_j(\widetilde{\gamma}) &= a(\widetilde{\gamma}) \,\mathrm{sn}\biggl(\frac{4\,\mathrm{K}(\kappa(\widetilde{\gamma}))}{N}(j-\tfrac12), \kappa(\widetilde{\gamma})\biggr) + \mathcal{O}\biggl(\frac{1}{N}\biggr) , \\ B_j(\widetilde{\gamma}) &= a(\widetilde{\gamma}) \,\mathrm{sn}\biggl(\frac{4\,\mathrm{K}(\kappa(\widetilde{\gamma}))}{N}j, \kappa(\widetilde{\gamma})\biggr) + \mathcal{O}\biggl(\frac{1}{N}\biggr) . \end{aligned} \tag{2.14}$$

[1]For the reader's convenience, we recall the definitions and main properties of Jacobi's elliptic integrals and functions in Appendix A.

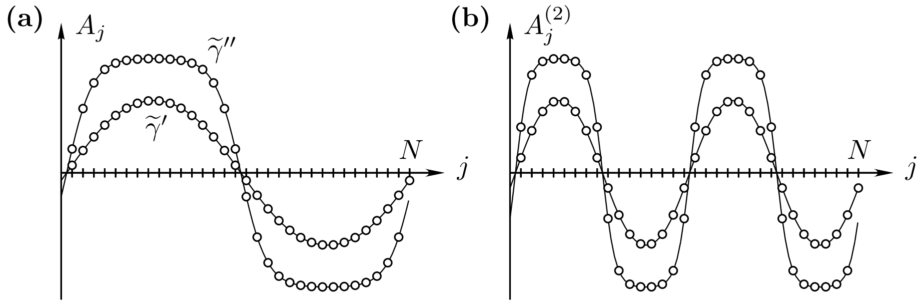

FIGURE 2. **(a)** Coordinates of the 1-saddles A in the case $N = 32$, shown for two different values of the coupling $\widetilde{\gamma}' > \widetilde{\gamma}''$. **(b)** Coordinates of the 3-saddles $A^{(2)}$ in the case $N = 32$, shown for the coupling intensities $\widetilde{\gamma}'/4$ and $\widetilde{\gamma}''/4$.

- *If N is odd, then A and B satisfy the symmetries of Table 1, and have components given by*

$$A_j(\widetilde{\gamma}) = a(\widetilde{\gamma})\,\mathrm{sn}\biggl(\frac{4\,\mathrm{K}(\kappa(\widetilde{\gamma}))}{N} j, \kappa(\widetilde{\gamma})\biggr) + \mathcal{O}\biggl(\frac{1}{N}\biggr)\,,$$
$$B_j(\widetilde{\gamma}) = a(\widetilde{\gamma})\,\mathrm{cn}\biggl(\frac{4\,\mathrm{K}(\kappa(\widetilde{\gamma}))}{N} j, \kappa(\widetilde{\gamma})\biggr) + \mathcal{O}\biggl(\frac{1}{N}\biggr)\,. \tag{2.15}$$

The potential difference between the 1*-saddles and the well bottoms (which is the same for all* 1*-saddles and well bottoms) satisfies*

$$\frac{V(A(\widetilde{\gamma})) - V(I^\pm)}{N} = \frac{1}{4} - \frac{1}{3(1+\kappa^2)}\biggl[\frac{2+\kappa^2}{1+\kappa^2} - 2\frac{\mathrm{E}(\kappa)}{\mathrm{K}(\kappa)}\biggr] + \mathcal{O}\biggl(\frac{\kappa^2}{N}\biggr)\,, \tag{2.16}$$

where $\kappa = \kappa(\widetilde{\gamma})$ and $\mathrm{E}(\kappa)$ denotes Jacobi's elliptic integral of the second kind.

For $\widetilde{\gamma} < \widetilde{\gamma}_2$, the stationary points A and B continue to exist, but new stationary points bifurcate from the origin O.

Theorem 2.2. *For any $M \geqslant 2$, there exists $N_M < \infty$ such that when $N \geqslant N_M$ and $\widetilde{\gamma}_{M+1} < \widetilde{\gamma} < \widetilde{\gamma}_M$, the set $\mathcal{S}$ of stationary points of the potential has cardinality*

$$|\mathcal{S}| = 3 + \sum_{m=1}^{M} \frac{4N}{\gcd(N, 2m)}\,. \tag{2.17}$$

It can be decomposed as

$$\begin{aligned}
\mathcal{S}_0 &= O_{I^+} = \{I^+, I^-\}\,, \\
\mathcal{S}_{2m-1} &= O_{A^{(m)}}\,, && m = 1,\dots,M\,, \\
\mathcal{S}_{2m} &= O_{B^{(m)}}\,, && m = 1,\dots,M\,, \\
\mathcal{S}_{2M+1} &= O_O = \{O\}\,,
\end{aligned} \tag{2.18}$$

where the points $A^{(m)} = A^{(m)}(\widetilde{\gamma})$ and $B^{(m)} = B^{(m)}(\widetilde{\gamma})$ have the following properties:

- *If $2m/\gcd(N, 2m)$ is odd, the components of $A^{(m)}$ and $B^{(m)}$ are given by*

$$A_j^{(m)}(\widetilde{\gamma}) = a(m^2\widetilde{\gamma})\,\mathrm{sn}\biggl(\frac{4\,\mathrm{K}(\kappa(m^2\widetilde{\gamma}))}{N} m\bigl(j - \tfrac{1}{2}\bigr), \kappa(m^2\widetilde{\gamma})\biggr) + \mathcal{O}\biggl(\frac{M}{N}\biggr)\,,$$
$$B_j^{(m)}(\widetilde{\gamma}) = a(m^2\widetilde{\gamma})\,\mathrm{sn}\biggl(\frac{4\,\mathrm{K}(\kappa(m^2\widetilde{\gamma}))}{N} mj, \kappa(m^2\widetilde{\gamma})\biggr) + \mathcal{O}\biggl(\frac{M}{N}\biggr)\,. \tag{2.19}$$

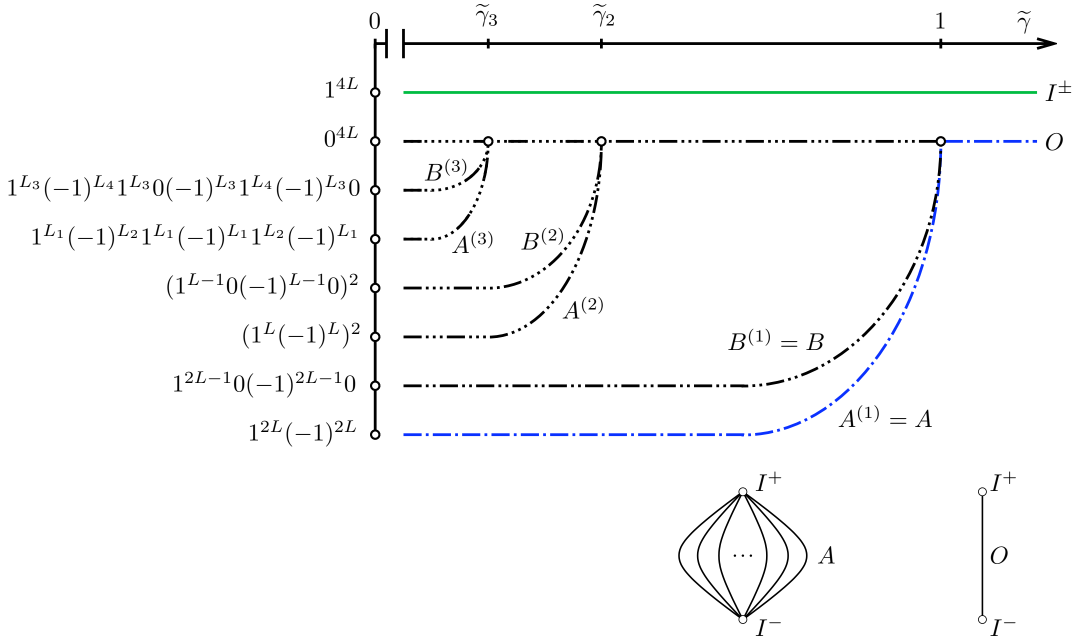

FIGURE 3. Partial bifurcation diagram for a case where $N = 4L$ is a multiple of four, and some associated graphs $\mathcal{G}$. Only one stationary point per orbit of the symmetry group G is shown. Dash–dotted curves with k dots represent k-saddles. The symbols at the left indicate the zero-coupling limit of the stationary points' coordinates, for instance $1^{2L}(-1)^{2L}$ stands for a point whose first $2L$ coordinates are equal to 1, and whose last $2L$ coordinates are equal to -1. The numbers associated with the branch created at $\widetilde{\gamma}_3$ are $L_1 = \lfloor 2L/3 \rfloor$, $L_2 = 2(L - L_1)$, $L_3 = \lfloor 2L/3 + 1/2 \rfloor$ and $L_4 = 2(L - L_3) - 1$ (in case N is a multiple of 12, there are more vanishing coordinates).

- *If $2m/\gcd(N, 2m)$ is even, the rôles of $A^{(m)}$ and $B^{(m)}$ are reversed.*

The potential difference between the saddles $A_j^{(m)}(\widetilde{\gamma})$ and the well bottoms satisfies a similar relation as (2.16), but with $\kappa = \kappa(m^2\widetilde{\gamma})$.

Note that the total number of stationary points accounted for by these results is of the order N^2, which is much less than the 3^N points present at zero coupling. Many additional stationary points thus have to be created as the rescaled coupling intensity $\widetilde{\gamma}$ decreases sufficiently (below order 1 in N), either by pitchfork-type second-order bifurcations of already existing points, or by saddle-node bifurcations. The existence of second-order bifurcations follows from stability arguments. For instance, for even N, the point $A(\widetilde{\gamma})$ converges to $(1, 1, \dots, 1, -1, -1, \dots, -1)$ as $\widetilde{\gamma} \to 0$, which is a local minimum of V instead of a 1-saddle. The A-branch thus has to bifurcate at least once as the coupling decreases to zero (Figure 4). For odd N, by contrast, the point $A(\widetilde{\gamma})$ converges to $(1, 1, \dots, 1, 0, -1, -1, \dots, -1)$ as $\widetilde{\gamma} \to 0$, which is also a 1-saddle. We thus expect that the point $A(\widetilde{\gamma})$ does not undergo any bifurcations for $0 \leqslant \widetilde{\gamma} < 1$ if N is odd.

2.4 Stochastic Case

We return now to the behaviour of the system of stochastic differential equations

$$\mathrm{d}x_i^\sigma(t) = f(x_i^\sigma(t))\,\mathrm{d}t + \frac{\gamma}{2}\big[x_{i+1}^\sigma(t) - 2x_i^\sigma(t) + x_{i-1}^\sigma(t)\big]\,\mathrm{d}t + \sigma\sqrt{N}\,\mathrm{d}B_i(t)\,. \tag{2.20}$$

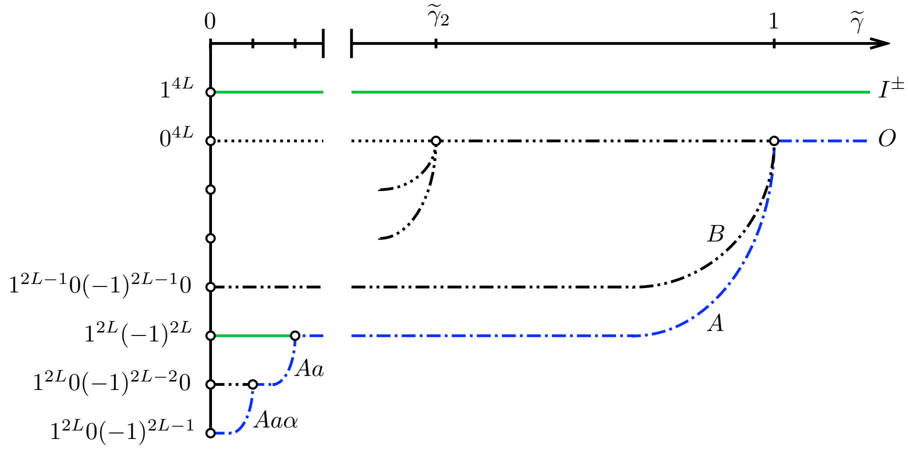

FIGURE 4. Partial bifurcation diagram for a case where $N = 4L$ is a multiple of four, showing the expected bifurcation behaviour of the critical 1-saddle in the zero-coupling limit.

Our main goal is to characterise the noise-induced transition from the configuration $I^- = (-1, -1, \dots, -1)$ to the configuration $I^+ = (1, 1, \dots, 1)$. In particular, we are interested in the time needed for this transition to occur, and by the shape of the critical configuration, i.e., the configuration of highest energy reached during the transition.

Since the probability of a stochastic process in continuous space hitting a given point is typically zero, we have to work with small neighbourhoods of the relevant configurations. Given a Borel set $\mathcal{A} \in \mathcal{X}$, and an initial condition $x_0 \in \mathcal{X} \setminus \mathcal{A}$, we denote by $\tau^{\text{hit}}(\mathcal{A})$ the *first-hitting time of* $\mathcal{A}$

$$\tau^{\text{hit}}(\mathcal{A}) = \inf\{t > 0 \colon x^\sigma(t) \in \mathcal{A}\} . \tag{2.21}$$

Similarly, for an initial condition $x_0 \in \mathcal{A}$, we denote by $\tau^{\text{exit}}(\mathcal{A})$ the *first-exit time from* $\mathcal{A}$

$$\tau^{\text{exit}}(\mathcal{A}) = \inf\{t > 0 \colon x^\sigma(t) \notin \mathcal{A}\} . \tag{2.22}$$

We fix radii $0 < r < R < 1/2$, and introduce random times

$$\begin{aligned}
\tau_+ &= \tau^{\text{hit}}(\mathcal{B}(I_+, r)) , \\
\tau_O &= \tau^{\text{hit}}(\mathcal{B}(O, r)) , \\
\tau_- &= \inf\{t > \tau^{\text{exit}}(\mathcal{B}(I^-, R)) \colon x_t \in \mathcal{B}(I^-, r)\} .
\end{aligned} \tag{2.23}$$

In [BFG06a, Theorem 2.7], we obtained in the synchronisation regime $\widetilde{\gamma} > 1$, for any initial condition $x_0 \in \mathcal{B}(I^-, r)$, any $N \geqslant 2$ and any $\delta > 0$,

$$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\bigl\{\mathrm{e}^{(1/2-\delta)/\sigma^2} < \tau_+ < \mathrm{e}^{(1/2+\delta)/\sigma^2}\bigr\} = 1 \tag{2.24}$$

and

$$\lim_{\sigma\to 0} \sigma^2 \log \mathbb{E}^{x_0}\{\tau_+\} = \frac{1}{2} . \tag{2.25}$$

This means that in the synchronisation regime, the transition between I^- and I^+ takes a time of the order $\mathrm{e}^{1/2\sigma^2}$. Furthermore,

$$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\bigl\{\tau_O < \tau_+ \bigm| \tau_+ < \tau_-\bigr\} = 1 , \tag{2.26}$$

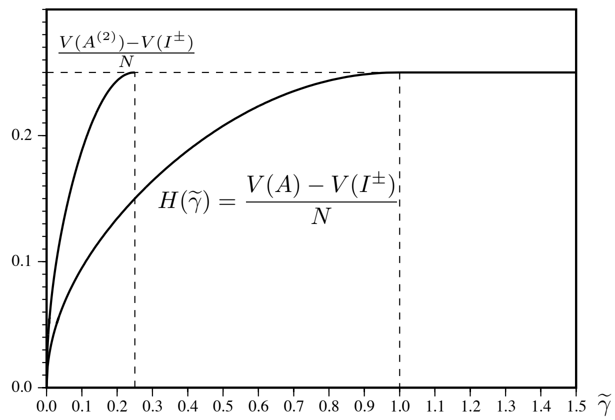

FIGURE 5. Value of the potential barrier height $H(\widetilde{\gamma}) = (V(A) - V(I^{\pm}))/N$ as a function of the rescaled coupling intensity $\widetilde{\gamma}$. For comparison, we also show the barrier height $(V(A^{(2)}) - V(I^{\pm}))/N$ for a stationary point of higher winding number $M = 2$.

meaning that during the transition, the system is likely to pass close the the origin, i.e., the origin is the critical configuration of the transition.

We can now prove a similar result in the desynchronised regime $\widetilde{\gamma} < 1$.

Theorem 2.3. *For $\widetilde{\gamma} < 1$, let*

$$H(\widetilde{\gamma}) = \frac{V(A(\widetilde{\gamma})) - V(I^{\pm})}{N} = \frac{1}{4} - \frac{1}{3(1+\kappa^2)}\left[\frac{2+\kappa^2}{1+\kappa^2} - 2\frac{\mathrm{E}(\kappa)}{\mathrm{K}(\kappa)}\right] + \mathcal{O}\left(\frac{\kappa^2}{N}\right), \qquad (2.27)$$

where $\kappa = \kappa(\widetilde{\gamma})$ is defined implicitly by (2.11)*. Fix an initial condition $x_0 \in \mathcal{B}(I^-, r)$. Then for any $0 < \widetilde{\gamma} < 1$, and any $\delta > 0$, there exists $N_0(\widetilde{\gamma})$ such that for all $N > N_0(\widetilde{\gamma})$,*

$$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\left\{\mathrm{e}^{(2H(\widetilde{\gamma})-\delta)/\sigma^2} < \tau_+ < \mathrm{e}^{(2H(\widetilde{\gamma})+\delta)/\sigma^2}\right\} = 1 \qquad (2.28)$$

and

$$\lim_{\sigma\to 0} \sigma^2 \log \mathbb{E}^{x_0}\left\{\tau_+\right\} = 2H(\widetilde{\gamma}) . \qquad (2.29)$$

Furthermore, let

$$\tau_A = \tau^{\mathrm{hit}}\Bigl(\bigcup_{g\in G} \mathcal{B}(gA, r)\Bigr) , \qquad (2.30)$$

where $A = A(\widetilde{\gamma})$ satisfies (2.14) *(or* (2.15) *if N is odd). Then for any $N > N_0(\widetilde{\gamma})$,*

$$\lim_{\sigma\to 0} \mathbb{P}^{x_0}\left\{\tau_A < \tau_+ \mid \tau_+ < \tau_-\right\} = 1 . \qquad (2.31)$$

The relations (2.28) and (2.29) mean that the transition time between the synchronised states I^- and I^+ is of order $\mathrm{e}^{2H(\widetilde{\gamma})/\sigma^2}$. Relation (2.31) implies that the set of critical configurations is given by the orbit of A.

The large-N limit of the rescaled potential difference $H(\widetilde{\gamma})$ is shown in Figure 5. The limiting function is increasing, with a discontinuous second-order derivative of at $\widetilde{\gamma} = 1$. For small $\widetilde{\gamma}$, $H(\widetilde{\gamma})$ grows like the square-root of $\widetilde{\gamma}$. This is compatible with the weak-coupling behaviour $H = (1/4 + 3/2\gamma + \mathcal{O}(\gamma^2))/N$ obtained in [BFG06a], if one takes into account the scaling of $\widetilde{\gamma}$.

The critical configuration, that is, the configuration with highest energy reached in the course of the transition from I^- to I^+, is any translate of the configuration shown in Figure 2a. If N is even, it has $N/2$ positive and $N/2$ negative coordinates, while for odd N, there are $(N-1)/2$ positive, one vanishing, and $(N-1)/2$ negative coordinates. The sites with positive and negative coordinates are always adjacent. The potential difference between the 1-saddles A and the 2-saddles B is actually very small, so that during a transition, the system may select at the last moment one of the different critical configurations, reflecting the fact that it becomes translation-invariant in the large-N limit.

3 Proofs

The stationary points of the potential satisfy the relation

$$f(x_n) + \frac{\gamma}{2}\big[x_{n+1} - 2x_n + x_{n-1}\big] = 0 , \tag{3.1}$$

where $f(x) = x - x^3$. Setting $v_n = x_n - x_{n-1}$ allows to rewrite this as the system

$$\begin{aligned} x_{n+1} &= x_n + v_{n+1} , \\ v_{n+1} &= v_n - 2\gamma^{-1} f(x_n) , \end{aligned} \tag{3.2}$$

in which n plays the rôle of "discrete time". It is easy to see that $(x_n, v_n) \mapsto (x_{n+1}, v_{n+1})$ is an area-preserving twist map ("twist" meaning that x_{n+1} is a monotonous function of v_n), for the study of which many tools are available [Mei92]. The stationary points of the potential for N particles are in one-to-one correspondence with the periodic orbits of period N of this map.

3.1 Symmetric Twist Map

The twist map (3.2) does not exploit the symmetries of the original system in an optimal way. In order to do so, it is more advantageous to introduce the variable

$$u_n = \frac{x_{n+1} - x_{n-1}}{2} \tag{3.3}$$

instead of v_n. Then a short computation shows that

$$\begin{aligned} x_{n+1} &= x_n + u_n - \gamma^{-1} f(x_n) , \\ u_{n+1} &= u_n - \gamma^{-1}\big[f(x_n) + f(x_{n+1})\big] . \end{aligned} \tag{3.4}$$

The map $T_1 : (x_n, u_n) \mapsto (x_{n+1}, u_{n+1})$ is also an area-preserving twist map. Although it looks more complicated than the map (3.2), it has the advantage that its inverse is obtained by changing the sign of u, namely

$$\begin{aligned} x_n &= x_{n+1} - u_{n+1} - \gamma^{-1} f(x_{n+1}) , \\ u_n &= u_{n+1} + \gamma^{-1}\big[f(x_{n+1}) + f(x_n)\big] . \end{aligned} \tag{3.5}$$

If we introduce the involutions

$$S_1 : (x, u) \mapsto (-x, u) \qquad \text{and} \qquad S_2 : (x, u) \mapsto (x, -u) , \tag{3.6}$$

then the map T_1 and its inverse are related by

$$T_1 \circ S_1 = S_1 \circ (T_1)^{-1} \qquad \text{and} \qquad T_1 \circ S_2 = S_2 \circ (T_1)^{-1} , \tag{3.7}$$

as a consequence of f being odd. This implies that the images of an orbit of the map under S_1 and S_2 are also orbits of the map.

3.2 Large-N Behaviour

For large N, it turns out to be useful to introduce the small parameter

$$\varepsilon = \sqrt{\frac{2}{\gamma}} = \sqrt{\frac{2}{\gamma_1 \widetilde{\gamma}}} = \frac{2\pi}{N\sqrt{\widetilde{\gamma}}}\left(1 + \mathcal{O}\left(\frac{1}{N^2}\right)\right) , \tag{3.8}$$

and the scaled variable $w = u/\varepsilon$. This transforms the map T_1 into a map $T_2 : (x_n, w_n) \mapsto (x_{n+1}, w_{n+1})$ defined by

$$\begin{aligned} x_{n+1} &= x_n + \varepsilon w_n - \frac{1}{2}\varepsilon^2 f(x_n) , \\ w_{n+1} &= w_n - \frac{1}{2}\varepsilon\big[f(x_n) + f(x_{n+1})\big] . \end{aligned} \tag{3.9}$$

T_2 is again an area-preserving twist map satisfying

$$T_2 \circ S_1 = S_1 \circ (T_2)^{-1} \qquad \text{and} \qquad T_2 \circ S_2 = S_2 \circ (T_2)^{-1} . \tag{3.10}$$

For small ε, we expect the orbits of this map to be close to those of the differential equation

$$\begin{aligned} \dot{x} &= w , \\ \dot{w} &= -f(x) , \end{aligned} \tag{3.11}$$

which is equivalent to the second-order equation $\ddot{x} = -f(x)$, describing the motion of a particle in the *inverted* double-well potential $-U(x)$. Solutions of (3.11) can be expressed in terms of Jacobi elliptic functions. Indeed, the function

$$C(x, w) = \frac{1}{2}(x^2 + w^2) - \frac{1}{4}x^4 \tag{3.12}$$

being a constant of motion, one sees that w satisfies

$$w = \pm\sqrt{(a(C)^2 - x^2)(b(C)^2 - x^2)/2} , \tag{3.13}$$

where

$$\begin{aligned} a(C)^2 &= 1 - \sqrt{1 - 4C} , \\ b(C)^2 &= 1 + \sqrt{1 - 4C} . \end{aligned} \tag{3.14}$$

This can be used to integrate the equation $\dot{x} = w$, yielding

$$\frac{b(C)}{\sqrt{2}}t = \mathrm{F}\left(\mathrm{Arcsin}\left(\frac{x(t)}{a(C)}\right), \kappa(C)\right) , \tag{3.15}$$

where $\kappa(C) = a(C)/b(C)$, and $\mathrm{F}(\phi, \kappa)$ denotes the incomplete elliptic integral of the first kind. The solution of the ODE can be written in terms of standard elliptic functions as

$$\begin{aligned} x(t) &= a(C)\,\mathrm{sn}\left(\frac{b(C)}{\sqrt{2}}t, \kappa(C)\right) , \\ w(t) &= \sqrt{2C}\,\mathrm{cn}\left(\frac{b(C)}{\sqrt{2}}t, \kappa(C)\right)\mathrm{dn}\left(\frac{b(C)}{\sqrt{2}}t, \kappa(C)\right) . \end{aligned} \tag{3.16}$$

3.3 Action–Angle Variables

We return now to the map T_2 defined in (3.9). The explicit solution of the continuous-time equation motivates the change of variables $\Phi_1 : (x,w) \mapsto (\varphi, C)$ given by

$$\begin{aligned}
\varphi &= \frac{\sqrt{2}}{b(C)} \operatorname{F}\biggl(\operatorname{Arcsin}\biggl(\frac{x}{a(C)}\biggr), \kappa(C)\biggr) , \\
C &= \frac{1}{2}(x^2 + w^2) - \frac{1}{4}x^4 .
\end{aligned} \tag{3.17}$$

One checks that Φ_1 is again area-preserving. The inverse Φ_1^{-1} is given by

$$\begin{aligned}
x &= a(C) \operatorname{sn}\biggl(\frac{b(C)}{\sqrt{2}}\varphi, \kappa(C)\biggr) , \\
w &= \sqrt{2C} \operatorname{cn}\biggl(\frac{b(C)}{\sqrt{2}}\varphi, \kappa(C)\biggr) \operatorname{dn}\biggl(\frac{b(C)}{\sqrt{2}}\varphi, \kappa(C)\biggr) .
\end{aligned} \tag{3.18}$$

The elliptic functions sn, cn and dn being periodic in their first argument, with period $4\operatorname{K}(\kappa)$, it is convenient to carry out a further area-preserving change of variables $\Phi_2 : (\varphi, C) \mapsto (\psi, I)$, defined by

$$\psi = \Omega(C)\varphi , \qquad I = h(C) , \tag{3.19}$$

where

$$\Omega(C) = \frac{b(C)}{\sqrt{2}} \frac{\pi}{2\operatorname{K}(\kappa(C))} , \qquad h(C) = \int_0^C \frac{\mathrm{d}C'}{\Omega(C')} . \tag{3.20}$$

Using the facts that C and $b = b(C)$ can be expressed as functions of $\kappa = \kappa(C)$ by $C = \kappa^2/(1+\kappa^2)^2$ and $b^2 = 2/(1+\kappa^2)$, one can check that

$$h(C) = \frac{4}{3\pi} \frac{(1+\kappa^2)\operatorname{E}(\kappa) - (1-\kappa^2)\operatorname{K}(\kappa)}{(1+\kappa^2)^{3/2}} \bigg|_{\kappa=\kappa(C)} \in \biggl[0, \frac{2\sqrt{2}}{3\pi}\biggr] . \tag{3.21}$$

We denote by $\Phi = \Phi_2 \circ \Phi_1$ the transformation $(x,w) \mapsto (\psi, I)$ and by $T = \Phi \circ T_2 \circ \Phi^{-1}$ the resulting map.

Proposition 3.1. *The map $T = T(\varepsilon)$ has the form*

$$\begin{aligned}
\psi_{n+1} &= \psi_n + \varepsilon\overline{\Omega}(I_n) + \varepsilon^3 f(\psi_n, I_n, \varepsilon) \qquad \pmod{2\pi} , \\
I_{n+1} &= I_n + \varepsilon^3 g(\psi_n, I_n, \varepsilon) ,
\end{aligned} \tag{3.22}$$

where $\overline{\Omega}(I) = \Omega(h^{-1}(I))$. The functions f and g are π-periodic in their first argument, and are real-analytic for $0 \leqslant I \leqslant h(1/4) - \mathcal{O}(\varepsilon^3)$. Furthermore, T satisfies the symmetries

$$T \circ \Sigma_1 = \Sigma_1 \circ T^{-1} \qquad \textit{and} \qquad T \circ \Sigma_2 = \Sigma_2 \circ T^{-1} , \tag{3.23}$$

where $\Sigma_1(\psi, I) = (-\psi, I)$ and $\Sigma_2(\psi, I) = (\pi - \psi, I)$.

Proof: First observe that Φ_1 and Φ are analytic whenever (x,w) is such that $C < 1/4$. The map T will thus be analytic whenever (ψ_n, I_n) is such that $C(x_n, w_n) < 1/4$ and $C(x_{n+1}, w_{n+1}) < 1/4$. A direct computation shows that

$$C(x_{n+1}, w_{n+1}) - C(x_n, w_n) = \frac{\varepsilon^3}{4}\Bigl[x_n w_n + 2x_n w_n^3 - 4x_n^3 w_n + 3x_n^5 w_n\Bigr] + \mathcal{O}(\varepsilon^4) . \tag{3.24}$$

This implies that $I_{n+1} = I_n + \mathcal{O}(\varepsilon^3)$, and allows to determine $g(\psi, I, 0)$. It also shows that T is analytic for $I_n < h(1/4) - \mathcal{O}(\varepsilon^3)$. Furthermore, writing $a_n = a(C(x_n, w_n))$, we see that (3.24) implies $a_{n+1} - a_n = \mathcal{O}(\varepsilon^3)$ and similarly for b_n, κ_n. This yields

$$\begin{aligned}
\varphi(x_{n+1}, w_{n+1}) - \varphi(x_n, w_n) &= \frac{\sqrt{2}}{b_n} \int_{x_n/a_n}^{x_{n+1}/a_n} \frac{\mathrm{d}u}{\sqrt{(1-\kappa_n^2 u^2)(1-u^2)}} + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon + \mathcal{O}(\varepsilon^3) \;, \qquad (3.25)
\end{aligned}$$

which implies the expression for ψ_{n+1}. We remark that the fact that T is area-preserving implies the relation

$$1 = \frac{\partial(\psi_{n+1}, I_{n+1})}{\partial(\psi_n, I_n)} = 1 + \varepsilon^3 \Big[\partial_\psi f(\psi, I, 0) + \partial_I g(\psi, I, 0) \Big] + \mathcal{O}(\varepsilon^4) \;, \qquad (3.26)$$

which allows to determine $f(\psi, I, 0)$. The fact that f and g are π-periodic in their first argument is a consequence of the fact that $T_2(-x, -w) = -T_2(x, w)$. Finally the relations (3.23) follow from the symmetries (3.10), with $\Sigma_i = \Phi \circ S_i \circ \Phi^{-1}$. □

A perturbation expansion at $I = 0$ shows in particular that

$$\overline{\Omega}(I) = 1 - \frac{3}{4} I + \mathcal{O}(I^2) \;. \qquad (3.27)$$

An important observation is that $\overline{\Omega}(I)$ is a monotonously decreasing function, taking values in $[0, 1]$. The monotonicity of $\overline{\Omega}$ makes T a *twist map* for sufficiently small ε, which has several important consequences on existence of periodic orbits.

We call *rotation number* of a periodic orbit of period N the quantity

$$\nu = \frac{1}{2\pi N} \left[\sum_{n=1}^{N} (\psi_{n+1} - \psi_n) \pmod{2\pi} \right] . \qquad (3.28)$$

Note that because of periodicity, ν is necessarily a rational number of the form $\nu = M/N$, for some positive integer M. We denote by $\mathbb{T}^N_\nu$ the set of points ψ in the torus $\mathbb{T}^N$ satisfying (3.28). It is sometimes more convenient to visualise $\mathbb{T}^N_\nu$ as the set of real N-tuples $(\psi_1, \dots, \psi_N)$ such that

$$\psi_1 < \psi_2 < \cdots < \psi_N < \psi_1 + 2\pi N \nu \;. \qquad (3.29)$$

In the sequel, we shall use the shorthand *stationary point with rotation number* ν instead of *stationary point corresponding to a periodic orbit of rotation number* ν.

The expression (3.22) for T implies that

$$\nu = \frac{\overline{\Omega}(I_0)}{2\pi} \varepsilon + \mathcal{O}(\varepsilon^2) \;. \qquad (3.30)$$

The following properties follow from the Poincaré–Birkhoff theorem, whenever $\varepsilon > 0$ is sufficiently small:

- For each positive integer M satisfying

$$M \leqslant \frac{N\varepsilon}{2\pi} (1 + \mathcal{O}(\varepsilon)) \;, \qquad (3.31)$$

  the twist map T admits at least two periodic orbits of period N and rotation number $\nu = M/N$. Note that Condition (3.31) is compatible with the fact that O bifurcates for $\gamma = \gamma_M$, $M = 1, 2, \dots, \lfloor N/2 \rfloor$.

- Any periodic orbit of period N of the map T is of the form

$$\begin{aligned} \psi_n &= \psi_0 + 2\pi\nu n + \mathcal{O}(\varepsilon^2) , \\ I_n &= \overline{\Omega}^{-1}\biggl(\frac{2\pi}{\varepsilon}\nu\biggr) + \mathcal{O}(\varepsilon^2) , \end{aligned} \tag{3.32}$$

for some ψ_0 and some $\nu = M/N$, where M is a positive integer satisfying (3.31).

Going back to original variables, we see that these periodic orbits are of the form

$$\begin{aligned} x_n &= a_n \operatorname{sn}\biggl(\frac{2\,\mathrm{K}(\kappa_n)}{\pi}\psi_n, \kappa_n\biggr) , \\ w_n &= \sqrt{2C_n} \operatorname{cn}\biggl(\frac{2\,\mathrm{K}(\kappa_n)}{\pi}\psi_n, \kappa_n\biggr) \operatorname{dn}\biggl(\frac{2\,\mathrm{K}(\kappa_n)}{\pi}\psi_n, \kappa_n\biggr) , \end{aligned} \tag{3.33}$$

where $a_n = a(C_n)$, $\kappa_n = \kappa(C_n)$ and

$$C_n = \Omega^{-1}\biggl(\frac{2\pi M}{N\varepsilon}\biggr) + \mathcal{O}(\varepsilon) = \Omega^{-1}\Bigl(M\sqrt{\widetilde{\gamma}}\Bigr) + \mathcal{O}(\varepsilon) . \tag{3.34}$$

This allows in particular to compute the value of the potential at the corresponding stationary point.

Proposition 3.2. *Let $\varepsilon > 0$ be sufficiently small, and let $x^\star$ be a stationary point of the potential V, corresponding to an orbit with rotation number $\nu = M/N$. Then*

$$\frac{V(x^\star)}{N} = -\frac{1}{3(1+\kappa^2)}\biggl[\frac{2+\kappa^2}{1+\kappa^2} - 2\frac{\mathrm{E}(\kappa)}{\mathrm{K}(\kappa)}\biggr] + \mathcal{O}(\varepsilon\kappa^2) , \tag{3.35}$$

where $\kappa = \kappa(C)$, and C satisfies $\Omega(C)^2 = M^2\widetilde{\gamma}$.

Proof: The expression (2.4) for the potential implies that

$$\begin{aligned} \frac{V(x^\star)}{N} &= \frac{1}{N}\sum_{n=1}^{N}\Bigl(U(x_n) + \frac{1}{2}w_n^2 + \mathcal{O}(\varepsilon^2)\Bigr) = \frac{1}{N}\sum_{n=1}^{N}(w_n^2 - C_n + \mathcal{O}(\varepsilon^2)) \\ &= \frac{C}{N}\sum_{n=1}^{N}\biggl[2\operatorname{cn}^2\biggl(\frac{2\,\mathrm{K}(\kappa)}{\pi}\psi_n, \kappa\biggr)\operatorname{dn}^2\biggl(\frac{2\,\mathrm{K}(\kappa)}{\pi}\psi_n, \kappa\biggr) - 1 + \mathcal{O}(\varepsilon)\biggr] \\ &= C\biggl[2\int_0^{2\pi}\operatorname{cn}^2\biggl(\frac{2\,\mathrm{K}(\kappa)}{\pi}\psi, \kappa\biggr)\operatorname{dn}^2\biggl(\frac{2\,\mathrm{K}(\kappa)}{\pi}\psi, \kappa\biggr)\,\mathrm{d}\psi - 1 + \mathcal{O}(\varepsilon)\biggr] , \end{aligned} \tag{3.36}$$

where $C = \Omega^{-1}(M\sqrt{\widetilde{\gamma}})$ and $\kappa = \kappa(C)$. The integral can then be computed using the change of variables $2\,\mathrm{K}(\kappa)\psi/\pi = \mathrm{F}(\phi, \kappa)$. Finally, recall that $C = \kappa^2/(1+\kappa^2)^2$. □

One can check that $V(x^\star)/N$ is a decreasing function of κ, which is itself a decreasing function of $M^2\widetilde{\gamma}$. As a consequence, $V(x^\star)/N$ is increasing in $M^2\widetilde{\gamma}$. This implies in particular that the potential is larger for larger winding numbers M.

Remark 3.3. The leading term in the expression (3.35) for the value of the potential is the same for all orbits of a given rotation number ν. Since stationary points of the potential of different index cannot be at exactly the same height, the difference has to be hidden in the error terms. In the previous paper [BFG06a], we showed that near the desynchronisation bifurcation, the potential difference between 1-saddles and 2-saddles is of order $(1-\widetilde{\gamma})^{N/2}$. For large N, we expect this difference to be exponentially small in $1/N$, owing to the fact that near-integrable maps of a form similar to (3.22) are known to admit adiabatic invariants to that order (cf. [BK96, Theorem 2]).

3.4 Generating Function

The fact that T is a twist map allows one to express I_n (and thus I_{n+1}) as a function of ψ_n and ψ_{n+1}. A *generating function* of T is a function $G(\psi_n, \psi_{n+1})$ such that

$$\partial_1 G(\psi_n, \psi_{n+1}) = -I_n \;, \qquad \partial_2 G(\psi_n, \psi_{n+1}) = I_{n+1} \;. \tag{3.37}$$

It is known that any area-preserving twist map admits a generating function, unique up to an additive constant. Since T depends on the parameter ε, the generating function G naturally also depends on ε. However, we will indicate this dependence only when we want to emphasize it.

Proposition 3.4. *The map T admits a generating function of the form*

$$G(\psi_1, \psi_2) = \varepsilon G_0\biggl(\frac{\psi_2-\psi_1}{\varepsilon}, \varepsilon\biggr) + 2\varepsilon^3 \sum_{p=1}^{\infty} \widehat{G}_p\biggl(\frac{\psi_2-\psi_1}{\varepsilon}, \varepsilon\biggr) \cos\bigl(p(\psi_1+\psi_2)\bigr) \;, \tag{3.38}$$

where the functions $G_0(u,\varepsilon)$ and $\widehat{G}_p(u,\varepsilon)$ are real-analytic for $u > \mathcal{O}(1/|\log\varepsilon|)$, and satisfy

$$\begin{aligned} G_0'(u,0) &= \overline{\Omega}^{-1}(u) \;, \\ \widehat{G}_p(u,0) &= \frac{1}{4p\pi} \int_0^{2\pi} g\bigl(\psi, \overline{\Omega}^{-1}(u), 0\bigr) \sin(-2p\psi) \,\mathrm{d}\psi \;. \end{aligned} \tag{3.39}$$

Proof: Fix $(\psi_2, I_2) = T(\psi_1, I_1)$. The fact that $T(\psi_1+\pi, I_1) = (\psi_2+\pi, I_2)$ implies

$$G(\psi_1+\pi, \psi_2+\pi) = G(\psi_1, \psi_2) + c \tag{3.40}$$

for some constant c. If we set $G(\psi_1,\psi_2) = \widetilde{G}(\psi_2-\psi_1, \psi_1+\psi_2)$, we thus have

$$\widetilde{G}(u, v+2\pi) = \widetilde{G}(u,v) + c \;. \tag{3.41}$$

This allows us to expand G as a Fourier series

$$G(\psi_1, \psi_2) = \sum_{p=-\infty}^{\infty} \widetilde{G}_p(\psi_2-\psi_1, \varepsilon)\, \mathrm{e}^{\mathrm{i}\, p(\psi_1+\psi_2)} + \frac{c}{2\pi}(\psi_1+\psi_2) \;. \tag{3.42}$$

Next we note that the symmetry (3.23) implies $T(-\psi_2, I_2) = (-\psi_1, I_1)$, and thus

$$\partial_1 G(\psi_1,\psi_2) = -\partial_2 G(-\psi_2, -\psi_1) \;. \tag{3.43}$$

Plugging (3.42) into this relation yields

$$c = 0 \qquad \text{and} \qquad \widetilde{G}_{-p}(u,\varepsilon) = \widetilde{G}_p(u,\varepsilon) \;, \tag{3.44}$$

which allows to represent G as a real Fourier series as well. Computing the derivatives $I_1 = -\partial_1 G(\psi_1,\psi_2)$ and $I_2 = \partial_2 G(\psi_1,\psi_2)$ yields

$$I_2 - I_1 = 2 \sum_{p=-\infty}^{\infty} \mathrm{i}\, p \widetilde{G}_p(\psi_2-\psi_1, \varepsilon)\, \mathrm{e}^{\mathrm{i}\, p(\psi_1+\psi_2)} \;, \tag{3.45}$$

which shows in particular that $\widetilde{G}_p(u,\varepsilon) = \mathcal{O}(\varepsilon^3)$ for $p \neq 0$, as a consequence of (3.22). This implies $I_1 = \widetilde{G}_0'(\psi_2-\psi_1, \varepsilon) + \mathcal{O}(\varepsilon^3)$, and thus $u = \widetilde{G}_0'(\varepsilon\overline{\Omega}(u) + \mathcal{O}(\varepsilon^3), \varepsilon)$. Renaming $\widetilde{G}_0(u,\varepsilon) = \varepsilon G_0(u/\varepsilon, \varepsilon)$ and $\widetilde{G}_p(u,\varepsilon) = \varepsilon^3 \widehat{G}_p(u/\varepsilon, \varepsilon)$ yields (3.38). Evaluating (3.45) for $\varepsilon = 0$ and taking the Fourier transform yields the expression (3.39) for $\widehat{G}_p(u,0)$. □

The relations (3.39) allow to determine the expression for the generating function of the map T, given by (3.22). In particular, one finds

$$G_0(u,0) = u\overline{\Omega}^{-1}(u) - \Omega^{-1}(u) , \tag{3.46}$$

so that

$$\begin{aligned} G_0(\Omega(C),0) &= h(C)\Omega(C) - C \\ &= -\frac{1}{3(1+\kappa^2)}\left[\frac{2+\kappa^2}{1+\kappa^2} - 2\frac{\mathrm{E}(\kappa)}{\mathrm{K}(\kappa)}\right] , \end{aligned} \tag{3.47}$$

with $\kappa = \kappa(C)$. Note that this quantity is identical with the leading term in the expression (3.35) for the average potential per site. This indicates that we have chosen the integration constant in the generating function in such a way that V and G_N take the same value on corresponding stationary points.

The main use of the generating function lies in the following fact. Consider the N-point function

$$G_N(\psi_1,\dots,\psi_N) = G(\psi_1,\psi_2) + G(\psi_2,\psi_3) + \dots + G(\psi_N,\psi_1+2\pi N\nu) , \tag{3.48}$$

defined on (a subset of) the set $\mathbb{T}_\nu^N$. The defining property (3.37) of the generating function implies that for any periodic orbit of period N of the map T, one has

$$\frac{\partial}{\partial\psi_n} G_N(\psi_1,\dots,\psi_N) = -I_n + I_n = 0 , \qquad \text{for } n = 1,\dots,N. \tag{3.49}$$

In other words, N-periodic orbits of T with rotation number ν are in one-to-one correspondence with stationary points of the N-point function G_N on $\mathbb{T}_\nu^N$.

The symmetries of the original potential imply that the N-point generating function satisfies the following relations on $\mathbb{T}_\nu^N$:

$$\begin{aligned} G_N(\psi_1,\dots,\psi_N) &= G_N(\psi_2,\dots,\psi_N,\psi_1+2\pi N\nu) , \\ G_N(\psi_1,\dots,\psi_N) &= G_N(-\psi_N,\dots,-\psi_1) , \\ G_N(\psi_1,\dots,\psi_N) &= G_N(\psi_1+\pi,\dots,\psi_N+\pi) . \end{aligned} \tag{3.50}$$

At this point, we are in the following situation. We have first transformed the initial problem of finding the stationary points of the potential V into the problem of finding periodic orbits of the map T_1, or, equivalently, of the map T. This problem in turn has been transformed into the problem of finding the stationary points of G_N. Obviously, the whole procedure is of interest only if the stationary points of G_N are easier to find and analyse than those of V. This, however, is the case here since the N-point function is a small perturbation of a function depending only on the differences $\psi_{n+1} - \psi_n$. In other words, G_N can be interpreted as the energy of a chain of particles with a uniform nearest-neighbour interaction, put in a weak external periodic potential.

3.5 Fourier Representation of the Generating Function

We fix $\nu = M/N$. Any stationary point of G_N on $\mathbb{T}_\nu^N$ admits a Fourier expansion of the form

$$\psi_n = 2\pi\nu n + \sum_{q=0}^{N-1} \hat{\psi}_q \omega^{qn} , \tag{3.51}$$

where $\omega = \mathrm{e}^{2\pi\,\mathrm{i}/N}$, and the Fourier coefficients are uniquely determined by

$$\hat{\psi}_q = \frac{1}{N}\sum_{n=1}^{N}\omega^{-qn}(\psi_n - 2\pi\nu n) = \overline{\hat{\psi}_{-q}} \; . \tag{3.52}$$

Note that $\hat{\psi}_q = \hat{\psi}_{q+N}$ for all q. Stationary points of G_N correspond to stationary points of the function $\overline{G}_N$, obtained by expressing G_N in terms of Fourier variables $(\hat{\psi}_0, \dots, \hat{\psi}_{N-1})$. In order to do this, it is convenient to write

$$\begin{aligned}\frac{\psi_{n+1}-\psi_n}{\varepsilon} &= \Delta + \varepsilon^2\alpha_n(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) \; , \\ \psi_n + \psi_{n+1} &= 2\hat{\psi}_0 + 2\pi\nu(2n+1) + \varepsilon^2\beta_n(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) \; ,\end{aligned} \tag{3.53}$$

where $\Delta = 2\pi\nu/\varepsilon$ and

$$\begin{aligned}\alpha_n(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) &= \frac{1}{\varepsilon^3}\sum_{q=1}^{N-1}\hat{\psi}_q(\omega^q - 1)\omega^{qn} \; , \\ \beta_n(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) &= \frac{1}{\varepsilon^2}\sum_{q=1}^{N-1}\hat{\psi}_q(\omega^q + 1)\omega^{qn} \; .\end{aligned} \tag{3.54}$$

Note that α_n is of order 1 in ε for any stationary point because of the expression (3.22) of the twist map. Taking the inverse Fourier transform shows that $|\hat{\psi}_q(\omega^q - 1)| = \mathcal{O}(\varepsilon^3)$ and $|\hat{\psi}_q| = \mathcal{O}(\varepsilon^2)$ for $q \neq 0$, and thus β_n is also of order 1.

Expressing G_N in Fourier variables yields the function

$$\overline{G}_N(\hat{\psi}_0,\dots,\hat{\psi}_{N-1}) = \sum_{p=-\infty}^{\infty}\mathrm{e}^{2\,\mathrm{i}\,p\hat{\psi}_0}\, g_p(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) \; , \tag{3.55}$$

where (we drop the ε-dependence of G_0 and $\widehat{G}_p$)

$$\begin{aligned} g_0(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) &= \varepsilon\sum_{n=1}^{N}G_0(\Delta + \varepsilon^2\alpha_n) \; , \\ g_p(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) &= \varepsilon^3\sum_{n=1}^{N}\widehat{G}_p(\Delta + \varepsilon^2\alpha_n)\omega^{pM(2n+1)}\,\mathrm{e}^{\mathrm{i}\,\varepsilon^2 p\beta_n} \qquad \text{for } p \neq 0 \; .\end{aligned} \tag{3.56}$$

We now examine the symmetry properties of the Fourier coefficients g_p. Table 2 shows how the Fourier variables transform under some symmetry transformations, where we only consider transformations leaving $\mathbb{T}^N_\nu$ invariant. As a consequence, the first two symmetries in (3.50) translate into

$$\begin{aligned} g_p(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) &= \omega^{2pM}g_p(\omega\hat{\psi}_1,\dots,\omega^{N-1}\hat{\psi}_{N-1}) \; , \\ g_p(\hat{\psi}_1,\dots,\hat{\psi}_{N-1}) &= \omega^{-2pM}g_{-p}(-\omega^{N-1}\hat{\psi}_{N-1},\dots,-\omega\hat{\psi}_1) \; .\end{aligned} \tag{3.57}$$

R	$x_j \mapsto x_{j+1}$	$\psi_n \mapsto \psi_{n+1}$	$\hat{\psi}_q \mapsto \omega^q\hat{\psi}_q + 2\pi\nu\delta_{q0}$
CS	$x_j \mapsto -x_{N+1-j}$	$\psi_n \mapsto -\psi_{N+1-n}$	$\hat{\psi}_q \mapsto -\omega^{-q}\hat{\psi}_{N-q} - 2\pi\nu(N+1)\delta_{q0}$
C	$x_j \mapsto -x_j$	$\psi_n \mapsto \psi_n + \pi$	$\hat{\psi}_q \mapsto \hat{\psi}_q + \pi\delta_{q0}$

Table 2. Effect of some symmetries on original variables, angle variables, and Fourier variables.

We now introduce new variables χ_q, $q \neq 0$, defined by

$$\chi_q = -\,\mathrm{i}\,\omega^{-q\hat{\psi}_0/2\pi\nu}\hat{\psi}_q = -\overline{\chi_{-q}} \; . \tag{3.58}$$

The χ_q are defined in such a way that they are real for stationary points satisfying, in original variables, the symmetry $x_j = -x_{n_0-j}$ for some n_0. For later convenience, we prefer to consider q as belonging to

$$\mathcal{Q} = \biggl\{-\biggl\lfloor\frac{N-1}{2}\biggr\rfloor,\dots,\biggl\lfloor\frac{N}{2}\biggr\rfloor\biggr\} \setminus \{0\} \tag{3.59}$$

rather than $\{1,\dots,N-1\}$. We set $\chi = \{\chi_q\}_{q\in\mathcal{Q}}$ and

$$\begin{aligned}\widetilde{G}_N(\hat{\psi}_0,\chi) &= \overline{G}_N(\hat{\psi}_0,\{\hat{\psi}_q = \mathrm{i}\,\omega^{q\hat{\psi}_0/2\pi\nu}\chi_q\}_{q\in\mathcal{Q}}) \\ &= \sum_{p=-\infty}^{\infty} \mathrm{e}^{2\,\mathrm{i}\,p\hat{\psi}_0}\,\tilde{g}_p(\hat{\psi}_0,\chi) \; ,\end{aligned} \tag{3.60}$$

where

$$\tilde{g}_p(\hat{\psi}_0,\chi) = g_p(\{\hat{\psi}_q = \mathrm{i}\,\omega^{q\hat{\psi}_0/2\pi\nu}\chi_q\}_{q\in\mathcal{Q}}) \; . \tag{3.61}$$

Lemma 3.5. *The function $\widetilde{G}_N(\hat{\psi}_0,\chi)$ is $2\pi\nu$-periodic in its first argument.*

Proof: By (3.57), we have

$$\tilde{g}_p(\hat{\psi}_0 + 2\pi\nu,\chi) = \omega^{-2pM}\tilde{g}_p(\hat{\psi}_0,\chi) \tag{3.62}$$

Since $\mathrm{e}^{2\,\mathrm{i}\,p\cdot 2\pi\nu} = \omega^{2pM}$, replacing $\hat{\psi}_0$ by $\hat{\psi}_0 + 2\pi\nu$ in (3.60) leaves $\widetilde{G}_N$ invariant. □

Since $\widetilde{G}_N$ also has period π, it has in fact period

$$\frac{\pi}{N}K \; , \qquad K = \gcd(N,2M) \; . \tag{3.63}$$

Our strategy now proceeds as follows:

1. Show that for each $\hat{\psi}_0$, and sufficiently small ε, the equations $\partial\widetilde{G}_N/\partial\chi_q = 0$, $q \in \mathcal{Q}$, admit exactly one solution $\chi = \chi^\star(\hat{\psi}_0)$.
2. Show that for $\chi = \chi^\star(\hat{\psi}_0)$, the equation $\partial\widetilde{G}_N/\partial\hat{\psi}_0 = 0$ is satisfied by exactly $4N/K$ values of $\hat{\psi}_0$.

3.6 Uniqueness of χ

We start by examining the conditions $\partial\widetilde{G}_N/\partial\chi_q = 0$, $q \in \mathcal{Q}$. It is useful to introduce the scaled variables

$$\rho_q = \rho_q(\chi) = -\frac{2}{\varepsilon^3}\chi_q \sin(\pi q/N) \tag{3.64}$$

and the function $\Gamma_\ell^{(a,b)}(\rho)$, $\rho = \{\rho_q\}_{q\in\mathcal{Q}}$, defined for $\ell \in \mathbb{Z}$ and $a,b \geqslant 0$ by

$$\Gamma_\ell^{(a,b)}(\rho) = \sum_{\substack{q_1,\dots,q_a\in\mathcal{Q}\\ q'_1,\dots,q'_b\in\mathcal{Q}}} 1_{\{\sum_i q_i + \sum_j q'_j = \ell\}} \prod_{i=1}^{a}\rho_{q_i} \prod_{j=1}^{b} \frac{\varepsilon}{\tan(\pi q'_j/N)}\rho_{q'_j} \; . \tag{3.65}$$

By convention, any term in the sum for which $q'_j = N/2$ for some j is zero, that is, we set $1/\tan(\pi/2) = 0$. A few elementary properties following immediately from this definition are:

- $\Gamma_\ell^{(0,0)}(\rho) = \delta_{\ell 0}$;
- $\Gamma_\ell^{(a,b)}(\rho) = 0$ for $|\ell| > (a+b)N/2$;
- If $\rho_q = 0$ for $q \notin K\mathbb{Z}$, then $\Gamma_\ell^{(a,b)}(\rho) = 0$ for $\ell \notin K\mathbb{Z}$;
- If $\rho'_q = \rho_{-q}$ for all q, then $\Gamma_\ell^{(a,b)}(\rho') = (-1)^b \Gamma_{-\ell}^{(a,b)}(\rho)$.

Lemma 3.6. *Let*

$$H_{p,q}(\Delta) = \widehat{G}'_p(\Delta) - \frac{\varepsilon p \widehat{G}_p(\Delta)}{\tan(\pi q/N)} , \tag{3.66}$$

with the convention that $H_{p,N/2}(\Delta) = \widehat{G}'_p(\Delta)$. For any stationary point, $\rho = \rho(\chi)$ satisfies the fixed-point equation

$$\rho = \mathcal{T}\rho = \rho^{(0)} + \Phi(\rho, \varepsilon) , \tag{3.67}$$

where the leading term is given by

$$\rho_q^{(0)} = \begin{cases} \dfrac{1}{G''_0(\Delta)} \displaystyle\sum_{k\in\mathbb{Z}\,:\, kN+q\in 2M\mathbb{Z}} (-1)^{k+1} \,\mathrm{e}^{\mathrm{i}\, k\hat{\psi}_0 N/M}\, H_{(kN+q)/2M,q}(\Delta) & \text{if } q \in K\mathbb{Z} , \\ 0 & \text{if } q \notin K\mathbb{Z} , \end{cases} \tag{3.68}$$

and the remainder is given by $\Phi_q(\rho,\varepsilon) = \Phi_q^{(1)}(\rho,\varepsilon) + \Phi_q^{(2)}(\rho,\varepsilon)$, with

$$\begin{aligned} \Phi_q^{(1)}(\rho,\varepsilon) &= \frac{1}{G''_0(\Delta)} \sum_{k\in\mathbb{Z}} (-1)^{k+1} \,\mathrm{e}^{\mathrm{i}\, k\hat{\psi}_0 N/M} \sum_{a\geqslant 1} \frac{\varepsilon^{2a}}{(a+1)!} G_0^{(a+2)}(\Delta) \Gamma_{kN+q}^{(a+1,0)}(\rho) , \\ \Phi_q^{(2)}(\rho,\varepsilon) &= \frac{1}{G''_0(\Delta)} \sum_{k\in\mathbb{Z}} (-1)^{k+1} \,\mathrm{e}^{\mathrm{i}\, k\hat{\psi}_0 N/M} \sum_{a+b\geqslant 1} \frac{\varepsilon^{2(a+b)}}{a!b!} \sum_{p\neq 0} H_{p,q}^{(a)}(\Delta) p^b \Gamma_{kN-2pM+q}^{(a,b)}(\rho) . \end{aligned} \tag{3.69}$$

PROOF: The definitions (3.54) of α_n and β_n imply, for any $a, b \geqslant 0$,

$$\begin{aligned} \alpha_n^a &= \sum_{q_1,\dots,q_a\in\mathcal{Q}} \prod_{i=1}^a \rho_{q_i} \omega^{q_i(n+1/2)} \,\mathrm{e}^{\mathrm{i}\,\hat{\psi}_0 q_i/M} , \\ \beta_n^b &= \sum_{q'_1,\dots,q'_b\in\mathcal{Q}} \prod_{j=1}^b \frac{-\mathrm{i}\,\varepsilon\rho_{q'_j}}{\tan(\pi q'_j/N)} \omega^{q'_j(n+1/2)} \,\mathrm{e}^{\mathrm{i}\,\hat{\psi}_0 q'_j/M} . \end{aligned} \tag{3.70}$$

It is more convenient to compute $\partial\widetilde{G}_N/\partial\rho_{-q}$ rather than $\partial\widetilde{G}_N/\partial\chi_q$. We thus have to compute the derivatives of $\tilde{g}_p$ with respect to ρ_{-q} for all p. For $p = 0$, we have

$$\frac{\partial\tilde{g}_0}{\partial\rho_{-q}} = \varepsilon^3 \sum_{n=1}^N G'_0(\Delta + \varepsilon^2\alpha_n) \frac{\partial\alpha_n}{\partial\rho_{-q}} , \tag{3.71}$$

where (3.70) shows that $\partial\alpha_n/\partial\rho_{-q} = \omega^{-q(n+1/2)} \,\mathrm{e}^{-\mathrm{i}\,\hat{\psi}_0 q/M}$. We expand $G'_0(\Delta + \varepsilon^2\alpha_n)$ into powers of ε^2, and plug in (3.70) again. In the resulting expression, the sum over n vanishes unless $\sum_i q_i - q$ is a multiple of N, say kN. This yields

$$\frac{\partial\tilde{g}_0}{\partial\rho_{-q}} = N\varepsilon^3 \sum_{k\in\mathbb{Z}} (-1)^k \,\mathrm{e}^{\mathrm{i}\, k\hat{\psi}_0 N/M} \sum_{a\geqslant 0} \frac{\varepsilon^{2a}}{a!} G_0^{(a+1)}(\Delta) \Gamma_{kN+q}^{(a,0)}(\rho) . \tag{3.72}$$

We consider the terms $a = 0$ and $a = 1$ separately:

- Since $\Gamma^{(0,0)}_{kN+q}(\rho) = \delta_{kN,-q}$ vanishes for all k, the sum actually starts at $a = 1$.
- The fact that $\Gamma^{(1,0)}_{\ell}(\rho)$ vanishes whenever $|\ell| > N/2$ implies that only the term $k = 0$ contributes, and yields a contribution proportional to $-\rho_q$.

Shifting by one unit the summation index a, we get

$$\frac{\partial \tilde g_0}{\partial \rho_{-q}} = N\varepsilon^5 \biggl[G_0''(\Delta)\rho_q + \sum_{k\in\mathbb{Z}} (-1)^k \,\mathrm{e}^{\mathrm{i}\, k\hat\psi_0 N/M} \sum_{a\geqslant 1} \frac{\varepsilon^{2a}}{(a+1)!} G_0^{(a+2)}(\Delta) \Gamma^{(a+1,0)}_{kN+q}(\rho) \biggr] \,. \qquad (3.73)$$

A similar computation for $p \neq 0$ shows that

$$\frac{\partial \tilde g_p}{\partial \rho_{-q}} \,\mathrm{e}^{2\,\mathrm{i}\, p\hat\psi_0} = N\varepsilon^5 \sum_{k\in\mathbb{Z}} (-1)^k \,\mathrm{e}^{\mathrm{i}\, k\hat\psi_0 N/M} \sum_{a,b\geqslant 0} \frac{\varepsilon^{2(a+b)}}{a!b!} H^{(a)}_{p,q}(\Delta) p^b \Gamma^{(a,b)}_{kN+q-2pM} \,. \qquad (3.74)$$

Solving the stationarity condition

$$0 = \frac{\partial \widetilde G_N}{\partial \rho_{-q}} = \sum_{p=-\infty}^{\infty} \mathrm{e}^{2\,\mathrm{i}\, p\hat\psi_0} \frac{\partial \tilde g_p}{\partial \rho_{-q}} \qquad (3.75)$$

with respect to ρ_q, and singling out the term $a = b = 0$ in (3.74) to give the leading term $\rho^{(0)}$ yields the result. □

Note the following symmetries, which follow directly from the definition of $\rho^{(0)}$ and the properties of $\Gamma^{(a,b)}_{\ell}$:

- For all $q \in \mathcal{Q}$, $\rho^{(0)}_{-q} = \rho^{(0)}_q$, because $H_{-p,-q}(\Delta) = H_{p,q}(\Delta)$, and thus $\rho^{(0)}_q \in \mathbb{R}$;
- If $\rho_q = 0$ for $q \notin K\mathbb{Z}$, then $\Phi_q(\rho,\varepsilon) = 0$ for $q \notin K\mathbb{Z}$;
- If $\rho'_q = \rho_{-q}$ for all q, then $\Phi_q(\rho',\varepsilon) = \Phi_{-q}(\rho,\varepsilon)$;

Remark 3.7. The condition $kN + q \in 2M\mathbb{Z}$, appearing in the definition of $\rho^{(0)}$, can only be fulfilled if $q \in N\mathbb{Z} + 2M\mathbb{Z} = K\mathbb{Z}$. If this is the case, set $N = nK$, $2M = mK$, $q = \ell K$, with n and m coprime. Then the condition becomes $mp - kn = \ell$. By Bezout's theorem, the general solution is given in terms of any particular solution (p_0, k_0) by

$$p = p_0 + nt \,, \qquad k = k_0 + mt \,, \qquad t \in \mathbb{Z} \,. \qquad (3.76)$$

Thus there will be exactly one p with $2|p| < n$. If N is very large, and M is fixed, then $n = N/K$ is also very large. Since the $\widehat G_p(\Delta)$, being Fourier coefficients of an analytic function, decrease exponentially fast in $|p|$, the sum in (3.68) will be dominated by the term with the lowest possible $|p|$.

We now introduce the following weighted norm on $\mathbb{C}^{\mathcal{Q}}$:

$$\|\rho\|_\lambda = \sup_{q\in\mathcal{Q}} \mathrm{e}^{\lambda|q|/2M} |\rho_q| \,, \qquad (3.77)$$

where $\lambda > 0$ is a free parameter. One checks that the functions $G_0(\Delta)$ and $\widehat G_p(\Delta)$ are analytic for $\operatorname{Re}\Delta > \mathcal{O}(1/\log|\varepsilon|)$. Thus it follows from Cauchy's theorem that there exist positive constants L_0, $r < \Delta - \mathcal{O}(1/\log|\varepsilon|)$ and λ_0 such that

$$|G_0^{(a)}(\Delta)| \leqslant L_0 \frac{a!}{r^a} \qquad \text{and} \qquad |\widehat G_p^{(a)}(\Delta)| \leqslant L_0 \frac{a!}{r^a} \,\mathrm{e}^{-\lambda_0|p|} \qquad (3.78)$$

for all $a \geqslant 0$ and $p \in \mathbb{Z}$. For sufficiently small ε, it is possible to choose $r = \Delta/2$.

Proposition 3.8. *There exist numerical constants $c_0, c_1 > 0$, such that for any $\lambda < \lambda_0$, and any N such that $N\,\mathrm{e}^{-\lambda_0 N/2M} \leqslant 1/2$, the estimates*

$$\|\mathcal{T}\rho\|_\lambda \leqslant \frac{c_1 L_0}{\Delta|G_0''(\Delta)|}\left[1 + \frac{M}{\Delta^3}\Big(\|\rho\|_\lambda + \varepsilon\Delta M\eta(\lambda_0,\lambda)\Big)\varepsilon\|\rho\|_\lambda\right], \tag{3.79}$$

$$\|\mathcal{T}\rho - \mathcal{T}\rho'\|_\lambda \leqslant \frac{c_1 L_0}{|G_0''(\Delta)|}\frac{M}{\Delta^4}\Big[\big(\|\rho\|_\lambda \vee \|\rho'\|_\lambda\big) + \varepsilon\Delta M\eta(\lambda_0,\lambda)\Big]\varepsilon\|\rho-\rho'\|_\lambda \tag{3.80}$$

hold with $\eta(\lambda_0,\lambda) = (\mathrm{e}^\lambda/\lambda_0) \vee (1/(\lambda_0-\lambda))$, provided ρ and ρ' satisfy

$$\varepsilon\big(\|\rho\|_\lambda \vee \|\rho'\|_\lambda\big) \leqslant c_0\frac{\Delta^2}{M}\left(1 \wedge \frac{\lambda_0-\lambda}{M} \wedge \frac{\lambda}{M}\right). \tag{3.81}$$

Proof: The lower bound

$$\frac{|\tan(\pi q/N)|}{\varepsilon} \geqslant \frac{\pi|q|}{N\varepsilon} = \frac{\Delta}{2M}|q| \tag{3.82}$$

directly implies

$$|H_{p,q}^{(a)}(\Delta)| \leqslant L_0\frac{a!}{r^{a+1}}\left[1 + \frac{2M|p|}{|q|}\right]\mathrm{e}^{-\lambda_0|p|}. \tag{3.83}$$

The assumption on N allows $|\rho_q|$ to be bounded by a geometric series of ratio smaller than 1/2, which is dominated by the term $k=0$, yielding

$$\|\rho^{(0)}\|_\lambda \leqslant \frac{c_2 L_0}{\Delta|G_0''(\Delta)|}\,\mathrm{e}^{-(\lambda_0-\lambda)|q|/2M} \leqslant \frac{c_2 L_0}{\Delta|G_0''(\Delta)|}, \tag{3.84}$$

where $c_2 > 0$ is a numerical constant. The fact that $\Gamma_\ell^{(a,b)}(\rho)$ contains less than N^{a+b-1} terms, together with (3.82), implies the bound

$$|\Gamma_\ell^{(a,b)}(\rho)| \leqslant N^{a+b-1}\left(\frac{2M}{\Delta}\right)^b \mathrm{e}^{-\lambda|\ell|/2M}\|\rho\|_\lambda^{a+b}. \tag{3.85}$$

Assuming that $\|\rho\|_\lambda \leqslant c_0\Delta^2/M\varepsilon$ for sufficiently small c_0, it is straightforward to obtain the estimate

$$\|\Phi^{(1)}(\rho,\varepsilon)\|_\lambda \leqslant \frac{c_3 L_0}{|G_0''(\Delta)|}\frac{2M}{\Delta^4}\varepsilon\|\rho\|_\lambda^2. \tag{3.86}$$

In the sequel, we assume that $q > 0$, since by symmetry of the norm under permutation of ρ_q and ρ_{-q} the same estimates will hold for $q < 0$. The norm of $\Phi^{(2)}(\rho,\varepsilon)$ is more delicate to estimate. We start by writing

$$|\Phi_q^{(2)}(\rho,\varepsilon)| \leqslant \frac{L_0}{|G_0''(\Delta)|}\frac{1}{N}\sum_{a+b\geqslant 1}\big(\varepsilon^2 N\|\rho\|_\lambda\big)^{a+b}\frac{1}{r^{a+1}}\left(\frac{2M}{\Delta}\right)^b S_q(b), \tag{3.87}$$

where

$$S_q(b) = \frac{1}{b!}\sum_{p\neq 0}|p|^b\left(1 + \frac{2M|p|}{q}\right)\mathrm{e}^{-(\lambda_0-\lambda)|p|}\sum_{k\in\mathbb{Z}}\exp\left\{-\frac{\lambda}{2M}(2M|p| + |kN+q-2Mp|)\right\}. \tag{3.88}$$

We decompose $S_q(b) = S_q^+(b) + S_q^-(b)$, where $S_q^+(b)$ and $S_q^-(b)$ contain, respectively, the sum over positive and negative p. In the sequel, we shall only treat the term $S_q^+(b)$. The sum over k in (3.88) is dominated by the term for which kN is the closest possible to $2Mp-q$, and can be bounded by a geometric series. The result for $p>0$ is

$$\sum_{k\in\mathbb{Z}} \exp\biggl\{-\frac{\lambda}{2M}(2Mp+|kN+q-2Mp|)\biggr\} \leqslant c_4\bigl(\mathrm{e}^{-\lambda p}\wedge \mathrm{e}^{-\lambda q/2M}\bigr) . \tag{3.89}$$

We now distinguish between two cases.

- If $q\leqslant 2M$, we bound the sum over k by $\mathrm{e}^{-\lambda p}$, yielding

$$S_q^+(b) \leqslant \frac{c_4}{b!}\frac{4M}{q}\sum_{p\geqslant 1} p^{b+1}\,\mathrm{e}^{-\lambda_0 p} \leqslant 4Mc_5\frac{b+1}{\lambda_0^b} . \tag{3.90}$$

  Since $\mathrm{e}^{\lambda q/2M}\leqslant \mathrm{e}^{\lambda}$, it follows

$$|\Phi_q^{(2)}(\rho,\varepsilon)| \leqslant \frac{c_6 L_0}{|G_0''(\Delta)|}\frac{2M^2\,\mathrm{e}^{\lambda}}{r^2\Delta\lambda_0}\varepsilon^2\|\rho\|_\lambda\,\mathrm{e}^{-\lambda q/2M} . \tag{3.91}$$

- If $q>2M$, we split the sum over p at $q/2M$. For $2Mp\leqslant q$, we bound $(1+2Mp/q)$ by 2 and the sum over k by $\mathrm{e}^{-\lambda q/2M}$. For $2Mp>q$, we bound the the sum over k by $\mathrm{e}^{-\lambda p}\leqslant \mathrm{e}^{-\lambda q/2M}$. This shows

$$S_q^+(b) \leqslant 2c_7M\frac{b+1}{(\lambda_0-\lambda)^b}\,\mathrm{e}^{-\lambda q/2M} , \tag{3.92}$$

  and thus

$$|\Phi_q^{(2)}(\rho,\varepsilon)| \leqslant \frac{c_8 L_0}{|G_0''(\Delta)|}\frac{2M^2}{r^2\Delta(\lambda_0-\lambda)}\varepsilon^2\|\rho\|_\lambda\,\mathrm{e}^{-\lambda q/2M} . \tag{3.93}$$

Now (3.93) and (3.91), together with (3.84) imply (3.79). The proof of (3.80) is similar, showing first the estimate

$$\biggl|\prod_{i=1}^a \rho_{q_i} - \prod_{i=1}^a \rho'_{q_i}\biggr| \leqslant a\bigl(\|\rho\|_\lambda\vee\|\rho'\|_\lambda\bigr)^{a-1}\,\mathrm{e}^{-\lambda\sum_{i=1}^a|q_i|/2M}\|\rho-\rho'\|_\lambda \tag{3.94}$$

by induction on a, and then

$$\bigl|\Gamma_\ell^{(a,b)}(\rho)-\Gamma_\ell^{(a,b)}(\rho')\bigr| \leqslant (a+b)\bigl[N(\|\rho\|_\lambda\vee\|\rho'\|_\lambda)\bigr]^{a+b-1}\biggl(\frac{2M}{\Delta}\biggr)^b\,\mathrm{e}^{-\lambda|\ell|/2M}\|\rho-\rho'\|_\lambda . \tag{3.95}$$

□

Corollary 3.9. *Fix $R_0>c_1L_0[\Delta|G_0''(\Delta)|]^{-1}$. For any $\Delta>0$ and any $\lambda<\lambda_0$, there exists an $\varepsilon_0=\varepsilon_0(\Delta,\lambda,\lambda_0,R_0)>0$ such that for all $\varepsilon<\varepsilon_0$, the map $\mathcal{T}$ admits a unique fixed point $\rho^\star$ in the ball $\mathcal{B}_\lambda(0,R_0)=\{\rho\in\mathbb{C}^{\mathcal{Q}}\colon\|\rho\|_\lambda<R_0\}$. Furthermore, the fixed point satisfies*

- *$\rho_q^\star=0$ whenever $q\notin K\mathbb{Z}$;*
- *$\rho_{-q}^\star=\rho_q^\star$, and thus $\rho_q^\star\in\mathbb{R}$ for all q.*

Proof: Estimate (3.79) for $\|\mathcal{T}\rho\|_\lambda$ implies that if

$$\varepsilon \leqslant \frac{R_0}{\Delta M\eta(\lambda_0,\lambda)}\wedge\frac{\Delta^3}{2MR_0^2}\biggl(\frac{\Delta|G_0''(\Delta)|}{c_1L_0}R_0-1\biggr) , \tag{3.96}$$

then $\mathcal{T}(\mathcal{B}_\lambda(0,R_0)) \subset \mathcal{B}_\lambda(0,R_0)$. If in addition

$$\varepsilon \leqslant c_0 \frac{\Delta^2}{MR_0}\left(1 \wedge \frac{\lambda_0-\lambda}{M} \wedge \frac{\lambda}{M}\right) , \tag{3.97}$$

then Estimate (3.80) for $\|\mathcal{T}\rho - \mathcal{T}\rho'\|_\lambda$ applies for $\rho, \rho' \in \mathcal{B}_\lambda(0,R_0)$. It is then immediate to check that $\mathcal{T}$ is a contracting in $\mathcal{B}_\lambda(0,R_0)$, as a consequence of (3.96). Thus the existence of a unique fixed point in that ball follows by Banach's contraction lemma. Finally, the assertions on the properties of $\rho^\star$ follow from the facts that they are true for $\rho^{(0)}$, that they are preserved by $\mathcal{T}$ and that $\rho^\star = \lim_{n\to\infty} \mathcal{T}^n \rho^{(0)}$. □

A direct consequence of this result is that for any $\hat\psi_0$, and sufficiently small ε, there is a unique $\rho^\star = \rho^\star(\hat\psi_0)$ (and thus a unique $\chi^\star(\hat\psi_0)$) satisfying the equations $\partial\widetilde{G}_N/\partial\chi_q = 0$ for all $q \in \mathcal{Q}$. Indeed, we take R_0 sufficiently large that our a priori estimates on the χ_q imply that $\rho \in \mathcal{B}_0(0,R_0)$. Then it follows that ρ is unique. Furthermore, for any $\lambda < \lambda_0$, making ε sufficiently small we obtain an estimate on $\|\rho^\star\|_\lambda$.

3.7 Stationary Values of $\hat\psi_0$

We now consider the condition $\partial\widetilde{G}_N/\partial\hat\psi_0 = 0$. As pointed out at the end of Section 3.5, $\widetilde{G}_N(\hat\psi_0,\chi)$ is a $\pi K/N$-periodic function of $\hat\psi_0$. For the same reasons, $\chi^\star(\hat\psi_0)$ is also $\pi K/N$-periodic. Hence it follows that the function $\hat\psi_0 \mapsto \widetilde{G}_N(\hat\psi_0,\chi^\star(\hat\psi_0))$ has the same period as well, and thus admits a Fourier series of the form

$$\widetilde{G}_N(\hat\psi_0,\chi^\star(\hat\psi_0)) = \sum_{k=-\infty}^{\infty} \hat g_k \,\mathrm{e}^{2\,\mathrm{i}\,k\hat\psi_0 N/K} , \tag{3.98}$$

with Fourier coefficients

$$\hat g_k = \frac{1}{2\pi}\int_0^{2\pi} \sum_{p=-\infty}^{\infty} \mathrm{e}^{2\,\mathrm{i}(p-kN/K)\hat\psi_0}\, \tilde g_p(\hat\psi_0,\chi^\star(\hat\psi_0))\,\mathrm{d}\hat\psi_0 \tag{3.99}$$

(we have chosen $[0,2\pi]$ as interval of integration for later convenience). Using the change of variables $\hat\psi_0 \mapsto -\hat\psi_0$ in the integral, and the various symmetries of the coefficients (in particular (3.57)), one checks that $\hat g_{-k} = \hat g_k$. Therefore (3.98) can be rewritten in real form as

$$\widetilde{G}_N(\hat\psi_0,\chi^\star(\hat\psi_0)) = \hat g_0 + 2\sum_{k=1}^{\infty} \hat g_k \cos\bigl(2k\hat\psi_0 N/K\bigr) . \tag{3.100}$$

Now $\partial\widetilde{G}_N/\partial\hat\psi_0$ vanishes if and only if the total derivative of $\widetilde{G}_N(\hat\psi_0,\chi^\star(\hat\psi_0))$ with respect to $\hat\psi_0$ is equal to zero. This function obviously vanishes for $\hat\psi_0 = \ell\pi K/2N$, $\ell = 1,\dots,4N/K$, and we have to show that these are the only roots.

We first observe that the Fourier coefficients $\hat g_k$ can be expressed directly in terms of the generating function (3.38), written in the form

$$\widetilde{G}(u,v,\varepsilon) = \varepsilon G_0(u,\varepsilon) + \varepsilon^3 \sum_{p\neq 0} \widehat{G}_p(u,\varepsilon)\,\mathrm{e}^{2\,\mathrm{i}\,pv} . \tag{3.101}$$

In the sequel, $\alpha^\star_n(\hat\psi_0)$ and $\beta^\star_n(\hat\psi_0)$ denote the quantities introduced in (3.54), evaluated at $\hat\psi_q = \mathrm{i}\,\omega^{q\hat\psi_0/2\pi\nu}\chi^\star_q(\hat\psi_0)$.

Lemma 3.10. *The Fourier coefficients $\hat{g}_k$ are given in terms of the generating function by*

$$\hat{g}_k = \frac{N}{2\pi} \int_0^{2\pi} \mathrm{e}^{-2\,\mathrm{i}\,k\hat{\psi}_0 N/K}\, \Lambda_0(\hat{\psi}_0)\, \mathrm{d}\hat{\psi}_0 \;, \tag{3.102}$$

where

$$\Lambda_0(\hat{\psi}_0) = \widetilde{G}\Bigl(\Delta + \varepsilon^2 \alpha_0^\star(\hat{\psi}_0), \hat{\psi}_0 + \pi\nu + \frac{1}{2}\varepsilon^2 \beta_0^\star(\hat{\psi}_0), \varepsilon\Bigr) \;. \tag{3.103}$$

Proof: The coefficient $\hat{g}_k$ can be rewritten as

$$\hat{g}_k = \frac{1}{2\pi} \sum_{n=1}^{N} \int_0^{2\pi} \mathrm{e}^{-2\,\mathrm{i}\,k\hat{\psi}_0 N/K}\, \Lambda_n(\hat{\psi}_0)\, \mathrm{d}\hat{\psi}_0 \;, \tag{3.104}$$

where

$$\begin{aligned}\Lambda_n(\hat{\psi}_0) &= \varepsilon G_0(\Delta + \varepsilon^2 \alpha_n^\star(\hat{\psi}_0)) \\ &\quad + \varepsilon^3 \sum_{p\neq 0} p\, \mathrm{e}^{2\,\mathrm{i}\,p\hat{\psi}_0}\, \widehat{G}_p(\Delta + \varepsilon^2\alpha_n^\star(\hat{\psi}_0))\omega^{pM(2n+1)}\, \mathrm{e}^{\mathrm{i}\,\varepsilon^2 p \beta_n^\star(\hat{\psi}_0)} \;. \end{aligned} \tag{3.105}$$

Using the periodicity of $\chi^\star$, one finds that $\alpha_n^\star(\hat{\psi}_0 + 2\pi\nu) = \alpha_{n+1}^\star(\hat{\psi}_0)$ and similarly for $\beta_n^\star$, which implies $\Lambda_n(\hat{\psi}_0) = \Lambda_0(\hat{\psi}_0 + 2\pi\nu n)$. Inserting this into (3.104) and using the change of variables $\hat{\psi}_0 \mapsto \hat{\psi}_0 - 2\pi\nu$ in the nth summand allows to express $\hat{g}_k$ as the $(2kN/K)$th Fourier coefficient of Λ_0. Finally, $\Lambda_0(\hat{\psi}_0)$ can also be written in the form (3.103). □

Relation (3.102) implies that the $\hat{g}_k$ decrease exponentially fast with k, like $\mathrm{e}^{-2\lambda_0 kN/K}$. Hence the Fourier series (3.100) is dominated by the first two terms, provided N is large enough. In order to obtain the existence of exactly $4N/K$ stationary points, it is thus sufficient to prove that $\hat{g}_1$ is also bounded below by a quantity of order $\mathrm{e}^{-2\lambda_0 N/K}$.

Proposition 3.11. *For any $\Delta > 0$, there exists $\varepsilon_1(\Delta) > 0$ such that whenever $\varepsilon < \varepsilon_1(\Delta)$,*

$$\operatorname{sign}(\hat{g}_1) = (-1)^{1+2M/K} \;. \tag{3.106}$$

Furthermore,

$$\frac{|\hat{g}_k|}{|\hat{g}_1|} \leqslant \exp\biggl\{-\frac{3k-5}{4}\lambda_0(\Delta)\frac{N}{K}\biggr\} \qquad \forall k \geqslant 2 \;, \tag{3.107}$$

where $\lambda_0(\Delta)$ is a monotonously increasing function of Δ, satisfying $\lambda_0(\Delta) = \sqrt{2}\pi\Delta + \mathcal{O}(\Delta^2)$ as $\Delta \searrow 0$, and diverging logarithmically as $\Delta \nearrow 1$.

Proof: First recall that $\Delta = 2\pi\nu/\varepsilon = 2\pi M/N\varepsilon$, where M is fixed. Thus taking ε small for given Δ automatically yields a large N. Combining the expression (3.22) for the twist map and the defining property (3.37) of the generating function with the relations $u = (\psi_{n+1} - \psi_n)/\varepsilon$ and $v = \psi_n + \psi_{n+1}$, one obtains the relation

$$\partial_v \widetilde{G}(u, v, \varepsilon) = \frac{\varepsilon^3}{2}\Bigl[g\Bigl(\tfrac{1}{2}(v - \varepsilon u), \overline{\Omega}^{-1}(u), \varepsilon\Bigr) + \mathcal{O}(\varepsilon^2)\Bigr] \;. \tag{3.108}$$

It follows from (3.24) and the definition (3.20) of $h(C)$ that

$$\begin{aligned} g(\psi, I, 0) &= \frac{1}{\overline{\Omega}(I)}\frac{xw}{4}[1 + 2w^2 - 4x^2 + 3x^4] \\ &= \frac{1}{\overline{\Omega}(I)}\frac{xw}{4}[1 + 4C - 6x^2 + 4x^4] \;, \end{aligned} \tag{3.109}$$

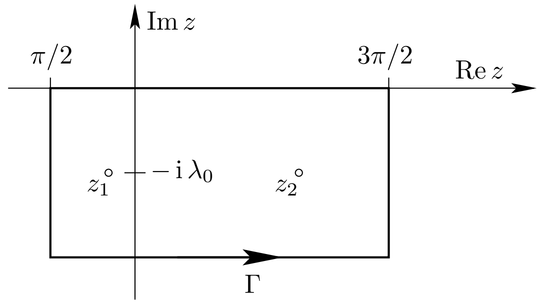

FIGURE 6. The integration contour Γ used in the integral (3.113).

where x and w have to be expressed as functions of ψ and I via (3.17) and (3.19). In particular, we note that

$$w = \frac{\sqrt{2C}}{a}\frac{\pi}{2\,\mathrm{K}(\kappa)}\frac{\mathrm{d}x}{\mathrm{d}\psi} = \overline{\Omega}(I)\frac{\mathrm{d}x}{\mathrm{d}\psi}\ , \tag{3.110}$$

where we used (3.20) again. This allows us to write

$$g(\psi, I, 0) = \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}\psi}\left[(1+4C)x^2 - 3x^4 + \frac{4}{3}x^6\right]\ . \tag{3.111}$$

A similar argument would also allow to express the first-order term in ε of $g(\psi, I, \varepsilon)$ as a function of $x = x(\psi, I)$. Also note the equality

$$\Delta = \overline{\Omega}(I) + \mathcal{O}(\varepsilon) = \frac{\pi b(C)}{2\sqrt{2}\,\mathrm{K}(\kappa)} + \mathcal{O}(\varepsilon) = \frac{1}{\sqrt{1+\kappa^2}\,\mathrm{K}(\kappa)} + \mathcal{O}(\varepsilon)\ , \tag{3.112}$$

which follows from the relation (3.30) between ν and $\overline{\Omega}(I)$.

The properties of elliptic functions imply that for fixed I, $\psi \mapsto x(\psi, I)$ is periodic in the imaginary direction, with period $2\lambda_0 = \pi\,\mathrm{K}(\sqrt{1-\kappa^2})/\,\mathrm{K}(\kappa)$, and has poles located in $\psi = n\pi + (2m+1)\,\mathrm{i}\,\lambda_0$, $n, m \in \mathbb{Z}$. As a consequence, the definition of the map $T = \Phi \circ T_2 \circ \Phi^{-1}$ implies in particular that $g(\psi, I, \varepsilon)$ is a meromorphic function of ψ, with poles at the same location, and satisfying $g(\psi + 2\,\mathrm{i}\,\lambda_0, I, \varepsilon) = g(\psi, I, \varepsilon)$. These properties yield informations on periodicity and location of poles for $\Lambda_0(\hat{\psi}_0)$, in particular $\Lambda_0(\hat{\psi}_0 + 2\,\mathrm{i}\,\lambda_0) = \Lambda_0(\hat{\psi}_0) + \mathcal{O}(\varepsilon^2)$.

Let Γ be a rectangular contour with vertices in $-\pi/2$, $-\pi/2 - 2\,\mathrm{i}\,\lambda_0$, $3\pi/2 - 2\,\mathrm{i}\,\lambda_0$, and $3\pi/2$, followed in the anticlockwise direction (Figure 6), and consider the contour integral

$$J = \frac{1}{2\pi}\oint_\Gamma \mathrm{e}^{-2\,\mathrm{i}\,kzN/K}\,\Lambda_0(z)\ \mathrm{d}z\ . \tag{3.113}$$

The contributions of the integrals along the vertical sides of the rectangle cancel by periodicity. Therefore, by Lemma 3.10 and the approximate periodicity of Λ_0 in the imaginary direction,

$$J = -\frac{1}{N}\left[\hat{g}_k - \mathrm{e}^{-2k\lambda_0 N/K}\Big(\hat{g}_k + \mathcal{O}(N\varepsilon^5)\Big)\right]\ . \tag{3.114}$$

On the other hand, the residue theorem yields

$$J = 2\pi\,\mathrm{i}\sum_{z_j}\mathrm{e}^{-2\,\mathrm{i}\,kz_jN/K}\,\mathrm{Res}(\Lambda_0(z), z_j)\ , \tag{3.115}$$

where the z_j denote the poles of the function $\Lambda_0(z)$, lying inside Γ. There are two such poles, located in $z_1 = -\,\mathrm{i}\,\lambda_0 - \pi\nu + \varepsilon\Delta + \mathcal{O}(\varepsilon^2)$, and $z_2 = z_1 + \pi$, and they both yield the same contribution, of order $\mathrm{e}^{-\lambda_0 k N(1+\mathcal{O}(\varepsilon^2))/K}$, to the sum. Comparing (3.114) and (3.115) shows that $\hat{g}_k/N$ is of the same order. Finally, the leading term of $\hat{g}_1$ can be determined explicitly using (3.111) and Jacobi's expression (A.11) for the Fourier coefficients of powers of elliptic functions, and is found to have sign $(-1)^{1+2M/K}$ for sufficiently large N. Choosing ε small enough (for fixed εN) guarantees that $\hat{g}_1$ dominates all $\hat{g}_k$ for $k \geqslant 2$. □

Corollary 3.12. *For $\varepsilon < \varepsilon_1(\Delta)$, the N-point generating function $\widetilde{G}_N$ admits exactly $4N/K$ stationary points, given by $\hat{\psi}_0 = \ell\pi K/2N$, $\ell = 1, \dots, 4N/K$, and $\chi = \chi^\star(\hat{\psi}_0)$.*

PROOF: In the points $\hat{\psi}_0 = \ell\pi K/2N$, the derivative of the function $\hat{\psi}_0 \mapsto \widetilde{G}_N(\hat{\psi}_0, \chi^\star(\hat{\psi}_0))$ vanishes, while its second derivative is bounded away from zero, as a consequence of Estimate (3.107). Thus these points are simple roots of the first derivative, which is bounded away from zero everywhere else. □

3.8 Index of the Stationary Points

We finally examine the stability type of the various stationary points, by first determining their index as stationary points of the N-point generating function $\overline{G}_N$, and then examining how this translates into their index as stationary points of the potential V_γ.

Proposition 3.13. *Let $(\hat{\psi}_0, \chi^\star(\hat{\psi}_0))$ be a stationary point of $\widetilde{G}_N$ with rotation number $\nu = N/M$. Let $x^\star = x^\star(\hat{\psi}_0)$ be the corresponding stationary point of the potential V_γ, and let $K = \gcd(N, 2M)$.*

- *If $2M/K$ is odd, then the points $x^\star(0)$, $x^\star(K\pi/N)$, ... are saddles of even index of V_γ, while the points $x^\star(K\pi/2N)$, $x^\star(3K\pi/2N)$, ... are saddles of odd index of V_γ.*
- *If $2M/K$ is even, then the points $x^\star(0)$, $x^\star(K\pi/N)$, ... are saddles of odd index of V_γ, while the points $x^\star(K\pi/2N)$, $x^\star(3K\pi/2N)$, ... are saddles of even index of V_γ.*

PROOF: We first determine the index of $(\hat{\psi}_0, \chi^\star(\hat{\psi}_0))$ as stationary point of $\widetilde{G}_N$. Using the fact that $G_0''(\Delta)$ is negative ($\overline{\Omega}^{-1}(\Delta)$ being decreasing), one sees that the Hessian matrix of $\widetilde{G}_N$ is a small perturbation of a diagonal matrix with $N-1$ negative eigenvalues. The Nth eigenvalue, which corresponds to translations of $\hat{\psi}_0$, has the same sign as the second derivative of $\hat{\psi}_0 \mapsto \widetilde{G}_N(\hat{\psi}_0, \chi^\star(\hat{\psi}_0))$, which is equal to $(-1)^{2M/K} \operatorname{sign}\cos(2\hat{\psi}_0 N/K)$. Thus $(\hat{\psi}_0, \chi^\star(\hat{\psi}_0))$ is an N-saddle of $\widetilde{G}_N$ if this sign is negative, and an $(N-1)$-saddle otherwise. The same is true for the index of $\psi = (\psi_1, \dots, \psi_N)$ as a stationary point of $\overline{G}_N$.

Let R be the so-called *residue* of the periodic orbit of T associated with the stationary point. This residue is equal to $(2 - \mathrm{Tr}(DT^N))/4$, where DT^N is the Jacobian of T^N at the orbit, and indicates the stability type of the periodic orbit: The orbit is hyperbolic if $R < 0$, elliptic if $0 < R < 1$, and inverse hyperbolic if $R > 1$. It is known [MM83] that the residue R is related to the index of ψ be the identity

$$R = -\frac{1}{4} \frac{\det(\operatorname{Hess} \overline{G}_N(\psi))}{\prod_{j=1}^N (-\partial_{12} G(\psi_j, \psi_{j+1}))} \,. \tag{3.116}$$

In our case, $-\partial_{12} G(\psi_j, \psi_{j+1})$ is always negative, so that R is positive if ψ is an $(N-1)$-saddle, and negative if ψ is an N-saddle.

Now $x^\star(\hat{\psi}_0)$ also corresponds to a periodic orbit of the map (3.2), whose generating function is $H(x_n, x_{n+1}) = \frac{1}{2}(x_n - x_{n+1})^2 + \frac{2}{\gamma} U(x_n)$. The corresponding N-point generating function is precisely $(2/\gamma)V_\gamma$. Since the residue is invariant under area-preserving changes of variables, we also have

$$R = -\frac{1}{2\gamma}\frac{\det(\operatorname{Hess} V_\gamma(x^\star))}{\prod_{j=1}^N(-\partial_{12}H(x_j^\star, x_{j+1}^\star))} . \tag{3.117}$$

In this case, the denominator is positive. Therefore, $\operatorname{Hess} V_\gamma(x^\star)$ has an even number of positive eigenvalues if ψ is an N-saddle, and an odd number of positive eigenvalues if ψ is an $(N-1)$-saddle. □

We can now complete the proofs of Theorem 2.1 and Theorem 2.2. We first recall the following facts, established in [BFG06a]. Whenever $\widetilde\gamma$ crosses a bifurcation value $\widetilde\gamma_M$, say from larger to smaller values, the index of the origin changes from $2M-1$ to $2M+1$. Thus the bifurcation involves a centre manifold of dimension 2, with $2M-1$ unstable and $N-2M-1$ stable directions transversal to the manifold. Within the centre manifold, the origin repels nearby trajectories, and attracts trajectories starting sufficiently far away. Therefore, all stationary points lying in the centre manifold, except the origin, are either sinks or saddles for the reduced two-dimensional dynamics. For the full dynamics, they are thus saddles of index $(2M-1)$ or $2M$ (c.f. [BFG06a, Section 4.3]), at least for $\widetilde\gamma$ close to $\widetilde\gamma_M$.

We now return to the twist map in action-angle variables (3.22). The frequency $\overline\Omega(I)$ being maximal for $I=0$, as ε increases, new orbits appear on the line $I=0$, which corresponds to the origin in x-coordinates. Orbits of rotation number $\nu = M/N$ can only exist if $\varepsilon\overline\Omega(0) = \varepsilon \geqslant 2\pi\nu + \mathcal{O}(\varepsilon^2)$, which is compatible with the condition $\widetilde\gamma < \widetilde\gamma_M$.

Consider now the case of a winding number $M=1$, that is, of orbits with rotation number $\nu = 1/N$, which are the only orbits existing for $\widetilde\gamma_2 < \widetilde\gamma < \widetilde\gamma_1$. Note that $\widetilde\gamma > \widetilde\gamma_2$ implies $\Delta = 2\pi/N\varepsilon > 1/2 - \mathcal{O}(\varepsilon)$, and thus there exists $N_1 < \infty$ such that the condition $N \geqslant N_1$ automatically implies that ε is small enough for Corollary 3.12 to hold. Now, by Proposition 3.13,

- If N is even, then $K = \gcd(N,2) = 2$, and there are $2N$ stationary points. The points $x^\star(0), x^\star(2\pi/N), \dots$ must be 2-saddles, while the points $x^\star(\pi/N), x^\star(3\pi/N), \dots$ are 1-saddles;
- If N is odd, then $K = \gcd(N,2) = 1$, and there are $4N$ stationary points. The points $x^\star(0), x^\star(\pi/N), \dots$ must be 1-saddles, while the points $x^\star(\pi/2N), x^\star(3\pi/2N), \dots$ are 2-saddles.

Going back to original variables, we obtain the expressions (2.14) and (2.15) for the coordinates of these stationary points. The fact that they keep the same index as $\widetilde\gamma$ moves away from $\widetilde\gamma_M$ is a consequence of Relation (3.117) and the fact that the corresponding stationary points of $\widetilde G_N$ also keep the same index. Finally, Relation (2.16) on the potential difference is a consequence of Proposition 3.2. This proves Theorem 2.1.

For larger winding number M, one can proceed in an analogous way, provided N is sufficiently large, as a function of M, for the conditions on ε to hold. This proves Theorem 2.2.

Finally, Theorem 2.3 is proved in an analogous way as Theorems 2.7 and 2.8 in [BFG06a], using results from [FW98] (see also [Kif81, Sug96]).

A Jacobi's Elliptic Integrals and Functions

Fix some $\kappa \in [0,1]$. The *incomplete elliptic integrals of the first and second kind* are defined, respectively, by[2]

$$\mathrm{F}(\phi,\kappa) = \int_0^\phi \frac{\mathrm{d}t}{\sqrt{1-\kappa^2\sin^2 t}} , \qquad \mathrm{E}(\phi,\kappa) = \int_0^\phi \sqrt{1-\kappa^2\sin^2 t}\,\mathrm{d}t . \tag{A.1}$$

The *complete elliptic integrals of the first and second kind* are given by

$$\mathrm{K}(\kappa) = \mathrm{F}(\pi/2,\kappa) , \qquad \mathrm{E}(\kappa) = \mathrm{E}(\pi/2,\kappa) . \tag{A.2}$$

Special values include $\mathrm{K}(0) = \mathrm{E}(0) = \pi/2$ and $\mathrm{E}(1) = 1$. The integral of the first kind $\mathrm{K}(\kappa)$ diverges logarithmically as $\kappa \nearrow 1$.

The *Jacobi amplitude* $\mathrm{am}(u,\kappa)$ is the inverse function of $\mathrm{F}(\cdot,\kappa)$, i.e.,

$$\phi = \mathrm{am}(u,\kappa) \quad \Leftrightarrow \quad u = \mathrm{F}(\phi,\kappa) . \tag{A.3}$$

The three standard Jacobi elliptic functions are then defined as

$$\begin{aligned} \mathrm{sn}(u,\kappa) &= \sin(\mathrm{am}(u,\kappa)) , \\ \mathrm{cn}(u,\kappa) &= \cos(\mathrm{am}(u,\kappa)) , \\ \mathrm{dn}(u,\kappa) &= \sqrt{1-\kappa^2\sin^2(\mathrm{am}(u,\kappa))} . \end{aligned} \tag{A.4}$$

Their derivatives are given by

$$\begin{aligned} \mathrm{sn}'(u,\kappa) &= \mathrm{cn}(u,\kappa)\,\mathrm{dn}(u,\kappa) , \\ \mathrm{cn}'(u,\kappa) &= -\,\mathrm{sn}(u,\kappa)\,\mathrm{dn}(u,\kappa) , \\ \mathrm{dn}'(u,\kappa) &= -\kappa^2\,\mathrm{sn}(u,\kappa)\,\mathrm{cn}(u,\kappa) . \end{aligned} \tag{A.5}$$

The function sn satisfies the periodicity relations

$$\begin{aligned} \mathrm{sn}(u+4\,\mathrm{K}(\kappa),\kappa) &= \mathrm{sn}(u,\kappa) , \\ \mathrm{sn}(u+2\,\mathrm{i}\,\mathrm{K}(\sqrt{1-\kappa^2}),\kappa) &= \mathrm{sn}(u,\kappa) , \end{aligned} \tag{A.6}$$

and has simple poles in $u = 2n\,\mathrm{K}(\kappa) + (2m+1)\,\mathrm{i}\,\mathrm{K}(\sqrt{1-\kappa^2})$, $n,m \in \mathbb{Z}$, with residue $(-1)^m/\kappa$. The functions cn and dn satisfy similar relations. Since $\mathrm{am}(u,0) = u$, one has $\mathrm{sn}(u,0) = \sin u$, $\mathrm{cn}(u,0) = \cos u$ and $\mathrm{dn}(u,0) = 1$. As κ grows from 0 to 1, the elliptic functions become more and more squarish. This is also apparent from their Fourier series, given by

$$\begin{aligned} \frac{2\,\mathrm{K}(\kappa)}{\pi}\,\mathrm{sn}\Big(\frac{2\,\mathrm{K}(\kappa)}{\pi}\psi,\kappa\Big) &= \frac{4}{\kappa}\sum_{p=0}^{\infty}\frac{\mathrm{q}^{(2p+1)/2}}{1-\mathrm{q}^{2p+1}}\sin\big((2p+1)\psi\big) , \\ \frac{2\,\mathrm{K}(\kappa)}{\pi}\,\mathrm{cn}\Big(\frac{2\,\mathrm{K}(\kappa)}{\pi}\psi,\kappa\Big) &= \frac{4}{\kappa}\sum_{p=0}^{\infty}\frac{\mathrm{q}^{(2p+1)/2}}{1+\mathrm{q}^{2p+1}}\cos\big((2p+1)\psi\big) , \\ \frac{2\,\mathrm{K}(\kappa)}{\pi}\,\mathrm{dn}\Big(\frac{2\,\mathrm{K}(\kappa)}{\pi}\psi,\kappa\Big) &= 1 + 4\sum_{p=0}^{\infty}\frac{\mathrm{q}^{p}}{1+\mathrm{q}^{2p}}\cos\big(2p\psi\big) , \end{aligned} \tag{A.7}$$

[2]One should beware of the fact that some sources use $m = \kappa^2$ as parameter.

where $\mathrm{q} = \mathrm{q}(\kappa)$ is the *elliptic nome* defined by

$$\mathrm{q} = \exp\left\{ -\pi \frac{\mathrm{K}(\sqrt{1-\kappa^2})}{\mathrm{K}(\kappa)} \right\} . \tag{A.8}$$

The elliptic nome has the asymptotic behaviour

$$\mathrm{q}(\kappa) = \begin{cases} \dfrac{\kappa^2}{16} + \dfrac{\kappa^4}{32} + \mathcal{O}\big(\kappa^6\big) & \text{for } \kappa \searrow 0 , \\ \exp\left\{ \dfrac{\pi^2}{\log[(1-\kappa^2)/16]} \right\} \left[1 + \mathcal{O}\left(\dfrac{1-\kappa^2}{\log^2[(1-\kappa^2)/16]} \right) \right] & \text{for } \kappa \nearrow 1 . \end{cases} \tag{A.9}$$

We also use the following identities, derived in [Jac69, p. 175]. For $k \geqslant 1$,

$$\left(\frac{2\,\mathrm{K}(\kappa)}{\pi} \right)^{2k} \mathrm{sn}^{2k}\left(\frac{2\,\mathrm{K}(\kappa)}{\pi} \psi, \kappa \right) = \hat{c}_{2k,0} + \sum_{p=1}^{\infty} \hat{c}_{2k,p} \frac{\mathrm{q}^p}{1-\mathrm{q}^{2p}} \cos\big(2p\psi\big) , \tag{A.10}$$

where the $\hat{c}_{2k,0}$ are positive constants (independent of ψ), and the other Fourier coefficients are given for the first few k by

$$\begin{aligned} \hat{c}_{2,p} &= -\frac{4}{\kappa^2}(2p) , \\ \hat{c}_{4,p} &= \frac{4}{3!\kappa^4} \left[(2p)^3 - 4(2p)(1+\kappa^2)\left(\frac{2\,\mathrm{K}(\kappa)}{\pi} \right)^2 \right] , \\ \hat{c}_{6,p} &= -\frac{4}{5!\kappa^6} \left[(2p)^5 - 20(2p)^3(1+\kappa^2)\left(\frac{2\,\mathrm{K}(\kappa)}{\pi} \right)^2 + 8(2p)(8+7\kappa^2+8\kappa^4)\left(\frac{2\,\mathrm{K}(\kappa)}{\pi} \right)^4 \right] . \end{aligned} \tag{A.11}$$

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