Construction of the Ground State in Nonrelativistic QED by Continuous Flows

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Abstract

For a nonrelativistic hydrogen atom minimally coupled to the quantized radiation field we construct the ground state projection $P_{\rm gs}$ by a continuous approximation scheme as an alternative to the iteration scheme recently used by Fröhlich, Pizzo, and the first author [5]. That is, we construct $P_{\rm gs} = \lim_{t\to\infty} P_t$ as the limit of a continuously differentiable family $(P_t)_{t\geq 0}$ of ground state projections of infrared regularized Hamiltonians H_t. Using the ODE solved by this family of projections, we show that the norm $\|\dot{P}_t\|$ of their derivative is integrable in t which in turn yields the convergence of P_t by the fundamental theorem of calculus.

Key words: QED, Spectral Analysis, Renormalization Group, Ground State

I Introduction

The existence and uniqueness of ground state eigenvectors at the bottom of the spectrum of Hamiltonian operators that generate the dynamics of nonrelativistic particles coupled to the quantized radiation field have been intensively studied in the past decade [1–4,8,5]. Explicit constructions of these ground states were given in [2,3] for a class of models that derives from the Pauli-Fierz transformation and the dipole approximation, with the additional requirement of a cutoff of the interaction at large electron position or a regularization of the interaction for small photon

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momenta. Due to this additional requirement, the class of models treated in [2,3] does not include nonrelativistic matter minimally coupled to the quantized radiation field. The construction of a unique ground state for the latter model (with one electron only which, however, is an immaterial restriction in this context) has recently been achieved in [5], using a method that goes back to Pizzo [13]. Both [3] and [5] use an iteration scheme that produces the (rank-one) ground state projection $P_{\rm gs} = \lim_{n\to\infty} P_n$ as a limit of a convergent sequence $(P_n)_{n\in\mathbb{N}}$ of ground state projections of infrared regularized Hamiltonians H_n.

In the present paper, we consider the same model of a hydrogen atom minimally coupled to the quantized radiation field as in [5]. Our purpose is to give an alternative construction of the ground state projection $P_{\rm gs} = \lim_{t\to\infty} P_t$ as the limit of a continuously differentiable family $(P_t)_{t\geq 0}$ of ground state projections of infrared regularized Hamiltonians H_t. This family $(P_t)_{t\geq 0}$ solves a certain ordinary differential equation established in Theorem V.3. The analysis of this ordinary differential equation in Theorem VI.1 then shows that the norm $\|\dot{P}_t\|$ of the derivative of P_t decays exponentially, as $t \to \infty$. In particular, $\|\dot{P}_t\|$ is integrable in t. This insures the convergence of P_t, as $t \to \infty$, by the fundamental theorem of calculus. Namely, we obtain $P_{\rm gs} \equiv P_\infty = P_0 + \int_0^\infty \dot{P}_t \, dt$.

The present paper is hence a mathematically rigorous implementation of a continuous approximation scheme. It has the advantage over a (discrete) iteration that the equations are simpler in structure: In [5] the resolvent of H_{n+1} was expressed in terms of the resolvent of H_n and the interaction W_{n+1}^n by means of a Neumann series expansion. In contrast, those Neumann series expansions reduce to their leading order term in the limit $\Delta t \to 0$ in the present paper, when we express the resolvent of $H_{t+\Delta t}$ in terms of the resolvent of H_t and the interaction $W_{t,t+\Delta t}$. The price to pay, however, is that some of the estimates which trivially follow in the discrete setting in [5] from the tensor product structure now become considerably more involved on a technical level, because the smooth momentum cutoff induces a *memory* or *delay* on the right side of the ODE under consideration.

II Definitions, Notation, and Main Result

We begin this section by defining the model considered in this paper. The Hilbert space of electron and photons is

$$\mathcal{H} \;:=\; \mathcal{H}_{el} \otimes \mathcal{F} \,, \tag{II.1}$$

where $\mathcal{H}_{el} := L^2(\mathbb{R}^3)$ is the configuration space of an electron, and $\mathcal{F}$ denotes the bosonic Fock space

$$\mathcal{F} \;=\; \mathbb{C}\Omega \,\oplus\, \bigoplus_{n=1}^{\infty} \mathcal{S} \bigotimes_{m=1}^{n} \mathfrak{h}_{ph} \tag{II.2}$$

where $\mathfrak{h}_{ph} := L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ is the one-photon Hilbert space, and Ω the photon vacuum vector. Here, $\mathbb{R}^3$ accounts for the photon momentum and $\mathbb{Z}_2$ for the transverse polarization directions of the photon. The n-fold tensor product in (II.2) is projected by $\mathcal{S}$ onto its subspace of totally symmetric wave functions, in accordance with Bose-Einstein statistics.

The Hamiltonian studied in this paper is

$$H(\alpha) \;:=\; \Big(-\imath\vec{\nabla}_x - \alpha^{3/2}\vec{A}(\alpha\vec{x})\Big)^2 \;+\; V(x) \;+\; \check{H}\,, \tag{II.3}$$

where $\alpha \approx \frac{1}{137}$ is the feinstructure constant, and $V \in L^2(\mathbb{R}^3;\mathbb{R}^-_0) + L^\infty(\mathbb{R}^3;\mathbb{R}^-_0)$ is a nonpositive potential with $\lim_{|x|\to\infty} V(x) = 0$. This insures that $V(x)$ is an infinitesimal perturbation of $-\Delta_x$ in the sense of Kato and, hence, the self-adjointness and semiboundedness of $H_{el} := -\Delta_x + V(x)$. We further assume that its ground state energy $e_0 := \inf\sigma[H_{el}] < 0$ is an isolated negative eigenvalue of multiplicity one. These assumptions imply that $\sigma_{\rm ess}[H_{el}] = [0,\infty)$ and

$$\Delta e_{el} := \inf\Big(\sigma[H_{el} - e_0] \setminus \{0\}\Big) \;>\; 0\,, \tag{II.4}$$

$$-b := \inf\sigma[(-\Delta_x/4) + V(x)] \;>\; -\infty\,. \tag{II.5}$$

The operator

$$\check{H} \;=\; \sum_{\lambda=\pm}\int \omega(\vec{k},\lambda)\, a^*(\vec{k},\lambda)a(\vec{k},\lambda)\, d^3k\,, \tag{II.6}$$

with $\omega(\vec{k},\lambda) := |\vec{k}|$, is the Hamiltonian of the free quantized radiation field, representing its energy. $\vec{A}$ denotes the vector potential of the transverse modes of the electromagnetic field,

$$\vec{A}(\vec{x}) \;:= \tag{II.7}$$
$$\frac{1}{(2\pi)^{3/2}}\sum_{\lambda=\pm}\int \frac{d^3k}{\sqrt{2|\vec{k}|}}\Lambda(\vec{k})\Big\{\vec{\varepsilon}(\vec{k},\lambda)e^{-\imath\vec{k}\cdot\vec{x}}\, a^*(\vec{k},\lambda) \;+\; \vec{\varepsilon}^*(\vec{k},\lambda)e^{\imath\vec{k}\cdot\vec{x}}\, a(\vec{k},\lambda)\Big\}\,,$$

with $\Lambda(\vec{k}) := \mathbf{1}\Big[|\vec{k}| \le \kappa\Big]$, where $\kappa \ge \max\{1, \Delta e_{el}\}$ is the ultraviolet cutoff. This operator is expressed in terms of photon creation- and annihilation operators, which satisfy the canonical commutation relations

$$[a^*(\vec{k},\lambda), a^*(\vec{k}',\lambda')] \;=\; [a(\vec{k},\lambda), a(\vec{k}',\lambda')] \;=\; 0\,, \tag{II.8}$$

$$[a(\vec{k},\lambda), a^*(\vec{k}',\lambda')] \;=\; \delta_{\lambda,\lambda'}\,\delta(\vec{k}-\vec{k}')\,, \tag{II.9}$$

$$a(\vec{k},\lambda)\Omega \;=\; 0\,, \tag{II.10}$$

for all $(\vec{k},\lambda), (\vec{k}',\lambda') \in \mathbb{R}^3 \times \mathbb{Z}_2$, in the sense of operator-valued distributions. The

polarization vectors $\vec{\varepsilon}(k,\pm) \in \mathbb{C}\otimes\mathbb{R}^3$ satisfy

$$\vec{\varepsilon}(\vec{k},\lambda)^* \cdot \vec{\varepsilon}(\vec{k},\lambda') \;=\; \delta_{\lambda,\lambda'} \quad \text{and} \quad \vec{k}\cdot\vec{\varepsilon}(\vec{k},\lambda) \;=\; 0\,, \tag{II.11}$$

for almost all $\vec{k}\in\mathbb{R}^3$ and $\lambda,\lambda'\in\mathbb{Z}_2$. We remark, that $\vec{A}(x)$ is chosen in Coulomb gauge, $\vec{\nabla}_x\cdot\vec{A}(\vec{x})=0$. In this paper we prove the following theorem.

Theorem II.1 *Assume that* $\alpha^3 \le (2500\vartheta^2+b+6\kappa^2+e_0)^{-1}$*, where* $\vartheta := \Delta e_{el}/\kappa \le 1$*. Then the ground state energy* $E_{\rm gs} := \inf\sigma[H(\alpha)] > -\infty$ *is a simple eigenvalue of* $H(\alpha)$*.*

As said in the introduction, we are not interested in merely showing the existence, but rather in constructing the ground state eigenvector $\phi_{\rm gs}$ or, equivalently, the corresponding rank-one projection $P_{\rm gs}=|\phi_{\rm gs}\rangle\langle\phi_{\rm gs}|$ by a new method. In view of this goal, Theorem II.1 is *not* the main result of this paper, but rather Theorem II.2, below.

In the following, we simplify our notations by setting

$$k := (\vec{k},\lambda), \quad \omega(k)\equiv|k| := |\vec{k}|, \quad \text{and} \quad \int f(k)\,dk := \sum_{\lambda=\pm}\int f(\vec{k},\lambda)\,d^3k, \tag{II.12}$$

for any integrable (possibly vector-valued) function f. Given $k=(\vec{k},\lambda)\in\mathbb{R}^3\times\mathbb{Z}_2$ and $\vec{x}\in\mathbb{R}^3$, we define $\vec{G}(k;\vec{x})$ by

$$\vec{G}(k;\vec{x}) \;:=\; -\frac{\alpha^{3/2}}{(2\pi)^{3/2}}\frac{\Lambda(k)}{\sqrt{2|k|}}\,e^{-\imath\alpha\vec{k}\cdot\vec{x}}\,\vec{\varepsilon}(k)\,, \tag{II.13}$$

and then three corresponding multiplication operators on $\mathcal{H}_{el}$ by $[\vec{G}(k)\varphi](\vec{x}) := \vec{G}(k;\vec{x})\varphi(\vec{x})$. Introducing

$$a^*(f) \;:=\; \int\big\{f(k)\otimes a^*(k)\big\}dk\,, \quad a(f) \;:=\; \int\big\{f^*(k)\otimes a(k)\big\}dk\,, \tag{II.14}$$

$$a^*(\vec{f}) := \Big(a^*(f_1),\,a^*(f_2),\,a^*(f_3)\Big)\,, \quad a(\vec{f}) \;:=\; \Big(a(f_1),\,a(f_2),\,a(f_3)\Big)\,, \tag{II.15}$$

$$A(f) \;:=\; a^*(f)+a(f)\,, \quad A(\vec{f}) \;:=\; a^*(\vec{f})+a(\vec{f})\,, \tag{II.16}$$

$$v(\vec{f}) \;:=\; -i\vec{\nabla}_x \;+\; A(\vec{f})\,, \tag{II.17}$$

we can then rewrite $H(\alpha)$ as

$$H(\alpha) \;=\; v(\vec{G})^2 \;+\; V(x) \;+\; \check{H}\,. \tag{II.18}$$

Note that $H(\alpha)$ depends on α through $\vec{G}$ defined in (II.13).

We pick a smooth function $p\in C_0^\infty(\mathbb{R};\mathbb{R}_0^+)$ with $\mathrm{supp}\{p\}\subseteq(1,2)$, $\int_0^\infty p(s)\,ds = 1$, and $\|p\|_\infty \le \pi/2$ (note that necessarily $\|p\|_\infty\ge 1$ and that, for any $\varepsilon>0$ there

exists such a function p with $\|p\|_\infty \leq 1+\varepsilon$). For $\eta, t > 0$, we define

$$\chi(r) := \int_{-\infty}^{r} p(s)\, ds\,, \tag{II.19}$$

$$\chi_t(k) := \chi(|k|/\rho_t)\,, \quad \text{where} \quad \rho_t \;:=\; \kappa\, \exp[-\eta\, t]\,. \tag{II.20}$$

Note that χ_t is a smooth characteristic function of $\{|k| > \rho_t\}$. That is, $0 \leq \chi_t \leq 1$, χ_t identically vanishes on $\{|k| \leq \rho_t\}$, and χ_t is identically equal to 1 on $\{|k| \geq 2\rho_t\}$. The following properties of χ_t are easily verified,

$$\forall\, k \in \mathbb{R}^3 \times \mathbb{Z}_2 : \quad \Big(t \mapsto \chi_t(k)\Big) \;\in\; C^\infty(\mathbb{R})\,, \tag{II.21}$$

$$\mathrm{supp}(\chi_t) \subseteq \Big\{k \in \mathbb{R}^3 \times \mathbb{Z}_2 \;\Big|\; \rho_t \leq |k|\Big\}\,, \tag{II.22}$$

$$\mathrm{supp}(\partial_t \chi_t) \subseteq \Big\{k \in \mathbb{R}^3 \times \mathbb{Z}_2 \;\Big|\; \rho_t \leq |k| \leq 2\rho_t\Big\}\,, \tag{II.23}$$

$$\|\partial_t \chi_t\|_\infty \leq 2\,\eta\, \|p\|_\infty \;=\; 2\,\eta \sup_{s\in(1,2)} p(s)\,. \tag{II.24}$$

Here, (II.24) results from

$$\partial_t \chi_t(k) \;=\; \partial_t \chi\Big(\kappa^{-1}\, e^{\eta t}\, |k|\Big) \;=\; p\Big(|k|/\rho_t\Big)\, \frac{\eta\, |k|}{\rho_t} \;\leq\; 2\eta\, \|p\|_\infty\,. \tag{II.25}$$

For $s \geq t \geq 0$, we further set

$$\vec{G}_t \;:=\; \chi_t\, \vec{G} \quad \text{and} \quad \vec{G}_{t,s} \;:=\; \vec{G}_s - \vec{G}_t \;=\; \Big(\chi_s - \chi_t\Big)\, \vec{G}\,. \tag{II.26}$$

We also introduce

$$\check{H}_t \;=\; \int \omega_t(k)\, a^*(k) a(k)\, dk\,, \quad \text{where} \quad \omega_t(k) \;:=\; \max\{\omega(k)\,,\; \rho_t\} \tag{II.27}$$

and note that, for $s \geq t \geq 0$, we have $\rho_t \rho_s^{-1} \omega_s \geq \omega_t \geq \omega_s \geq \omega_\infty = \omega$ and thus

$$\frac{\rho_t}{\rho_s}\, \check{H}_s \;\geq\; \check{H}_t \;\geq\; \check{H}_s \;\geq\; \check{H}_\infty = \check{H}\,, \tag{II.28}$$

as quadratic forms.

For each $s \geq t \geq 0$, we define a Hamiltonian $H_{t,s}$ on $\mathcal{H}$ by replacing in $H(\alpha)$ the velocity operator $v(\vec{G})$ by $v(\vec{G}_t)$ and the field energy operator $\check{H}$ by $\check{H}_s$, i.e.,

$$H_{t,s} \;:=\; v(\vec{G}_t)^2 \;+\; V(x) \;+\; \check{H}_s\,. \tag{II.29}$$

Especially, we write $H_t := H_{t,t}$. Note that if $s > t \geq 0$ then

$$H_{t,s} \;=\; \widetilde{H}_t \otimes \mathbf{1}_\infty^t \;+\; \mathbf{1}_t \otimes \check{H}_{t,s} \tag{II.30}$$

where

$$\widetilde{H}_t := v(\vec{G}_t)^2 \,+\, V(x) \,+\, \int_{\omega(k)\geq\rho_t} \omega(k)\, a^*(k)a(k)\, dk\,, \tag{II.31}$$

$$\check{H}_{t,s} := \int_{\omega(k)<\rho_t} \omega_s(k)\, a^*(k)a(k)\, dk\,, \tag{II.32}$$

and we used the natural isomorphism

$$\mathcal{F}[\mathfrak{h}_{ph}] \;\cong\; \mathcal{F}[\mathfrak{h}_t] \otimes \mathcal{F}[\mathfrak{h}_\infty^t]\,, \quad \text{for} \;\; \mathfrak{h}_{ph} \;=\; \mathfrak{h}_t \oplus \mathfrak{h}_\infty^t\,, \tag{II.33}$$

with $\mathfrak{h}_t := L^2(\{\omega \geq \rho_t\} \times \mathbb{Z}_2)$ and $\mathfrak{h}_\infty^t := L^2(\{\omega < \rho_t\} \times \mathbb{Z}_2)$. It follows that the ground state projection $P_{t,s}$ of $H_{t,s}$ is given by

$$P_{t,s} \;\cong\; \widetilde{P}_t \otimes P_{\Omega_\infty^t}\,, \tag{II.34}$$

where $\widetilde{P}_t$ is the ground state projection of $\widetilde{H}_t$, and $P_{\Omega_\infty^t}$ is the projection onto the vacuum in $\mathcal{F}[\mathfrak{h}_\infty^t]$. In particular,

$$P_{t,s} \;=\; P_{t,t} \;=:\; P_t\,, \tag{II.35}$$

for all $s > t \geq 0$. Similarly, we define $E_{t,s} := \inf \sigma[H_{t,s}]$ to be the ground state energy of $H_{t,s}$ and observe that

$$E_{t,s} \;=\; E_{t,t} \;=:\; E_t\,, \tag{II.36}$$

for all $s > t \geq 0$. We are now in position to formulate our main result.

Theorem II.2 *Assume that $\alpha^3 \leq (2500\vartheta^2 + b + 6\kappa^2 + e_0)^{-1}$, where $\vartheta := \Delta e_{el}/\kappa \leq 1$. Then $P_{(\cdot)} \in C^1(\mathbb{R}^+; \mathcal{B}[\mathcal{H}])$ is continuously differentiable in operator norm, its derivative is given by*

$$\dot{P}_t \;=\; \frac{P_t^\perp}{H_t^+}\,(-\dot{W}_t)\,P_t \;+\; P_t\,(-\dot{W}_t)\,\frac{P_t^\perp}{H_t^+}\,, \tag{II.37}$$

where $H_t^+ := H_t - E_t$ and $\dot{W}_t := v(\vec{G}_t)\cdot A(\dot{\chi}_t\vec{G}) + A(\dot{\chi}_t\vec{G})\cdot v(\vec{G}_t)$, and there exists a constant $C_{\dot{P}} < \infty$ such that $\dot{P}_t$ obeys the following norm estimate,

$$\|\dot{P}_t\| \;\leq\; C_{\dot{P}}\, \alpha^{3/2}\, \rho_t^{1/2}\,, \tag{II.38}$$

for all $t > 0$. Moreover, the ground state energy $E_{\rm gs} := \inf \sigma[H(\alpha)] > -\infty$ is a simple eigenvalue of $H(\alpha)$ with corresponding eigenprojection $P_{\rm gs}$ given by

$$P_{\rm gs} \;=\; \lim_{t\to\infty} P_t\,. \tag{II.39}$$

Proof: We only prove that $E_{\rm gs}$ is an eigenvalue, not its simplicity, i.e., we only construct a ground state $\phi_{\rm gs} = \lim_{t\to\infty}\phi_t$, where $P_{\rm gs} = |\phi_{\rm gs}\rangle\langle\phi_{\rm gs}|$ and $P_t = |\phi_t\rangle\langle\phi_t|$,

but we do not show its uniqueness. The proof of uniqueness, however, can be found in [5]. Another fact we use is the boundedness of $\|\vec{x}\phi_t\|$ uniformly in $t \geq 0$. We refer the reader again to [5] for a proof of this bound.

Eqs. (II.37) and (II.38) are stated and proved separately as Theorems V.3 and VI.1, below. They imply that

$$P_\infty \;:=\; \lim_{t\to\infty} P_t \;=\; P_0 \;+\; \int_0^\infty \dot{P}_t \, dt \,, \tag{II.40}$$

exists and is a rank-one projection. (Note that due to the continuity of $\dot{P}_t$, we may define the integral in (II.40) as a Riemann integral.) We may choose normalized vectors $\phi_t, \phi_\infty \in \mathcal{H}$ such that $P_t = |\phi_t\rangle\langle\phi_t|$, $P_\infty = |\phi_\infty\rangle\langle\phi_\infty|$, and $\phi_\infty = \lim_{t\to\infty}\phi_t$. Lemma IV.1 implies that $E_\infty := \lim_{t\to\infty} E_t$ exists, and hence we also have the convergence of $H_t\phi_t = E_t\phi_t \to E_\infty\phi_\infty$. On the other hand, denoting $H_\infty := H(\alpha)$, it is not difficult to show (in a similar fashion as in the proof of Lemma III.3 below) that $W_{t,\infty} := v(\vec{G}_t)\cdot A(\vec{G}_{t,\infty}) + A(\vec{G}_{t,\infty})\cdot v(\vec{G}_t) + A(\vec{G}_{t,\infty})^2$ obeys a norm bound

$$\Big\| W_{t,\infty}\,(H_t^+ + \rho_t)^{-1} \Big\| \;\leq\; \mathcal{O}(\rho_t^{-1/2}) \,. \tag{II.41}$$

Since $H_\infty\phi_t = H_t\phi_t + W_{t,\infty}\phi_t$, this norm bound implies that $\phi_t \in \mathrm{dom}(H_\infty)$ and that

$$\Big\| (H_t - H_\infty)\,\phi_t \Big\| \;\leq\; \rho_t \,\Big\| W_{t,\infty}\,(H_t^+ + \rho_t)^{-1} \Big\| \;\leq\; \mathcal{O}(\rho_t^{1/2}) \;\to\; 0 \,, \tag{II.42}$$

as $t \to \infty$. Thus, the closedness of H_∞ implies that $\phi_\infty \in \mathrm{dom}(H_\infty)$, and we obtain

$$H_\infty\,\phi_\infty \;=\; \lim_{t\to\infty}\{H_t\,\phi_t\} \;=\; \lim_{t\to\infty}\{E_t\,\phi_t\} \;=\; E_\infty\,\phi_\infty \,. \tag{II.43}$$

Thus, ϕ_∞ is a normalized eigenvector of H_∞. Finally, a simple continuity argument using that $E_t = \inf\sigma[H_t]$ shows that $E_\infty = \inf\sigma[H_\infty]$. □

III Basic Estimates

In this section we collect some basic estimates and apply them to establish a relative quadratic form bound $\pm W_{t,s} \leq \mathcal{O}(\alpha^{3/2})(t-s)(H_t^+ + \rho_t)$, where

$$\begin{aligned} W_{t,s} :=\, & H_{t,s} - H_{t,t} \;=\; v(\vec{G}_s)^2 - v(\vec{G}_t)^2 \\ & = \Big(v(\vec{G}_t) + A(\vec{G}_{t,s})\Big)^2 \;-\; v(\vec{G}_t)^2 \\ & = v(\vec{G}_t)\cdot A(\vec{G}_{t,s}) \;+\; A(\vec{G}_{t,s})\cdot v(\vec{G}_t) \;+\; A(\vec{G}_{t,s})^2 \end{aligned} \tag{III.44}$$

and

$$H^+_{t,s} \;:=\; H_{t,s} - E_{t,s}\,, \quad H^+_t \;:=\; H_t - E_t\,, \tag{III.45}$$

for $s \geq t \geq 0$. We begin by recalling the following standard estimates.

Lemma III.1 *For all $\vec{f} \in \mathfrak{h}^3$, such that $\omega^{-1/2}\vec{f} \in \mathfrak{h}^3$, and all $r > 0$, we have*

$$\left\| a(\vec{f})\,(\check{H}(\vec{f}) + r)^{-1/2} \right\| \leq \left\| \omega^{-1/2}\vec{f} \right\|, \tag{III.46}$$

$$\left\| a^*(\vec{f})\,(\check{H}(\vec{f}) + r)^{-1/2} \right\| \leq \left\| (\omega^{-1} + r^{-1})^{1/2}\vec{f} \right\|, \tag{III.47}$$

and

$$A(\vec{f})^2 \leq 4 \left\| (\omega^{-1} + r^{-1})^{1/2}\vec{f} \right\|^2 (\check{H}(\vec{f}) + r)\,, \tag{III.48}$$

where

$$\check{H}(\vec{f}) \;:=\; \int_{\mathrm{supp}\,\vec{f}} \omega(k)\, a^*(k)a(k)\, dk\,. \tag{III.49}$$

If additionally $\|(\omega^{-1} + \kappa^{-1})^{1/2}\vec{f}\|^2 \leq 1/2$ holds true, then

$$v(\vec{f})^2 \;\leq\; 2\left(H^+(\vec{f}) + b + \kappa + \|\vec{f}\|^2\right), \tag{III.50}$$

where $b := -\inf\sigma[\frac{1}{4}(-\Delta_x) + V(x)] < \infty$, $H^+(\vec{f}) := H(\vec{f}) - E(\vec{f})$, $E(\vec{f}) := \inf\sigma[H(\vec{f})]$, and

$$H(\vec{f}) \;:=\; v(\vec{f})^2 \;+\; V(x) \;+\; \check{H}(\vec{f})\,. \tag{III.51}$$

Proof: The proof of (III.46)–(III.48) can be found, e.g., in [2,3]. We only give a proof of (III.50) which is similar to [5, Lemma II.2]. We first use (III.48) and $-\Delta_x = (-i\vec{\nabla}_x)^2$ to obtain

$$\begin{aligned} -\Delta_x &= \left(v(\vec{f}) - A(\vec{f})\right)^2 \\ &= v(\vec{f})^2 \;-\; v(\vec{f})\cdot A(\vec{f}) \;-\; A(\vec{f})\cdot v(\vec{f}) \;+\; A(\vec{f})^2 \\ &\leq 2\, v(\vec{f})^2 \;+\; 2\, A(\vec{f})^2 \\ &\leq 2\, v(\vec{f})^2 \;+\; 8 \left\| (\omega^{-1} + r^{-1})^{1/2}\vec{f} \right\|^2 (\check{H}(\vec{f}) + r)\,. \end{aligned} \tag{III.52}$$

Since $V \in L^2(\mathbb{R}^3;\mathbb{R}^-_0) + L^\infty(\mathbb{R}^3;\mathbb{R}^-_0)$, we have that $\frac{1}{4}(-\Delta_x) + V(x) \geq -b > -\infty$ and hence $0 \leq -V(x) \leq \frac{1}{4}(-\Delta_x) + b$. This and (III.52) imply

$$\begin{aligned} v(\vec{f})^2 &= H(\vec{f}) - V(x) - \check{H}(\vec{f}) \;\;\leq\;\; H(\vec{f}) + \frac{1}{4}(-\Delta_x) + b - \check{H}(\vec{f}) \\ &\leq H(\vec{f}) + b \;+\; \frac{1}{2}\, v(\vec{f})^2 \\ &\quad + \left(2\,\|(\omega^{-1} + r^{-1})^{1/2}\vec{f}\|^2 - 1\right)\check{H}(\vec{f}) \;+\; 2\,\|(\omega^{-1} + r^{-1})^{1/2}\vec{f}\|^2\, r\,. \end{aligned} \tag{III.53}$$

We choose $r := \kappa$ and use that $2\,\|(\omega^{-1}+r^{-1})^{1/2}\vec{f}\|^2 \leq 1$ to arrive at

$$v(\vec{f})^2 \;\leq\; 2\Big(H(\vec{f})+b+\kappa\Big) \;\leq\; 2\Big(H^+(\vec{f})+E(\vec{f})+b+\kappa\Big)\,. \tag{III.54}$$

Finally, we use the min-max principle with the trial state $\varphi_{el}\otimes\Omega$, where φ_{el} is the normalized ground state of $H_{el} = -\Delta_x + V(x)$, i.e., $H_{el}\varphi_{el} = e_0\varphi_{el}$, with $e_0 < 0$. This gives

$$E(\vec{f}) \;\leq\; \langle\varphi_{el}\otimes\Omega|\; H(\vec{f})\,\varphi_{el}\otimes\Omega\rangle \;=\; e_0 + \|\vec{f}\|^2 \;\leq\; \|\vec{f}\|^2\,. \tag{III.55}$$

□

In order to apply these estimates to the choices $\vec{G}_t$ and $\vec{G}_{t,s}$, for $\vec{f}$, we next derive bounds on the L^2-norms appearing on the right sides of (III.46)–(III.48) and (III.50). Suppose that $s > t \geq 0$. Using $0 \leq \chi_s - \chi_t \leq (s-t)\|\partial_t\chi_t\|_\infty$, we derive the following estimates on the $L^2(\mathbb{R}^3\times\mathbb{Z}_2;\mathbb{C}^3)$-norm, for $\beta \leq 1/2$,

$$\begin{aligned}
\|\omega^{-\beta}\,\vec{G}_{t,s}\|^2 &= \int \Big[\chi_s(k)-\chi_t(k)\Big]^2\,\omega(k)^{-2\beta}\,|\vec{G}(k)|^2\,dk \\
&\leq (s-t)^2\,\|\partial_t\chi_t\|_\infty^2 \int_{\{\rho_s\leq|k|\leq 2\rho_t\}} \omega(k)^{-2\beta}\,|\vec{G}(k)|^2\,dk \\
&\leq (s-t)^2\,\|\partial_t\chi_t\|_\infty^2\,\frac{8\pi\,\alpha^3}{2(2\pi)^3}\int_0^{2\rho_t} r^{1-2\beta}\,dr \\
&\leq (s-t)^2\,\|\partial_t\chi_t\|_\infty^2\,\frac{\alpha^3}{2\pi^2}\,2\,\rho_t^{2-2\beta}\,.
\end{aligned} \tag{III.56}$$

Hence,

$$\|\vec{G}_{t,s}\| \;\leq\; \eta\,\rho_t\,\alpha^{3/2}\,(s-t)\,, \quad \text{and} \quad \|\omega^{-1/2}\,\vec{G}_{t,s}\| \;\leq\; \eta\,\rho_t^{1/2}\,\alpha^{3/2}\,(s-t)\,, \tag{III.57}$$

using that $\|p\|_\infty \leq \pi/2$, and similarly

$$\|\vec{G}_t\| \;\leq\; \kappa\,\alpha^{3/2} \quad \text{and} \quad \|\omega^{-1/2}\vec{G}_t\| \;\leq\; \kappa\,\alpha^{3/2}\,. \tag{III.58}$$

For the formulation of the next lemma, we introduce $\tau := \frac{1}{\eta}\ln 2 > 0$ and observe that $e^{\eta\tau} = 2$ and hence $\rho_{t-\tau} = 2\rho_t$. We recall from (II.32) that

$$\check{H}_{t-\tau,s} \;=\; \int_{\omega(k)<\rho_{t-\tau}} \omega_s(k)\,a^*(k)a(k)\,dk \;=\; \int_{\omega(k)<2\rho_t} \omega_s(k)\,a^*(k)a(k)\,dk\,. \tag{III.59}$$

Lemma III.2 *Let $\tau := \frac{1}{\eta}\ln 2 > 0$ and $0 \leq t \leq s$, and suppose that $\alpha^3 \leq (b + 6\kappa^2 + e_0 + 1)^{-1}$. Then*

$$\check{H}_{t-\tau,s} + \rho_t \;\leq\; 2(H^+_{t,s} + \rho_t)\,. \tag{III.60}$$

Proof: We first set

$$\vec{F}_t \;:=\; \mathbf{1}[\omega \geq 2\rho_t]\vec{G} \quad \text{and} \quad \Delta\vec{G}_t \;:=\; (\omega^{-1}+\rho_t^{-1})^{-1/2}\Big(\vec{G}_t - \vec{F}_t\Big)\,, \tag{III.61}$$

and note that

$$\|\Delta\vec{G}_t\|^2 \;\leq\; \frac{2}{\rho_t}\int_{\rho_t\leq\omega(k)\leq 2\rho_t} |\vec{G}(k)|^2\,dk \;\leq\; \frac{\alpha^3}{2\pi\,\rho_t}\int_0^{2\rho_t} r\,dr \;=\; \frac{\alpha^3\,\rho_t}{\pi}\,. \tag{III.62}$$

Next, we observe that

$$H(\vec{F}_t)+\check{H}_{t-\tau,s} \;=\; v(\vec{F}_t)^2+V(x)+\check{H}(\vec{F}_t)+\check{H}_{t-\tau,s} \;\leq\; H_{t,s}-\Big\{v(\vec{G}_t)^2-v(\vec{F}_t)^2\Big\}, \tag{III.63}$$

which implies

$$\begin{aligned} &H(\vec{F}_t)+\check{H}_{t-\tau,s} \;\leq\; H_{t,s} \;-\; \Big\{v(\vec{G}_t)^2-v(\vec{F}_t)^2\Big\} \\ &= H_{t,s} \;+\; 2v(\vec{F}_t)\cdot a(\vec{G}_t-\vec{F}_t) \;+\; 2a^*(\vec{G}_t-\vec{F}_t)\cdot v(\vec{F}_t) \;+\; A(\vec{G}_t-\vec{F}_t)^2\,, \end{aligned} \tag{III.64}$$

since $v(\vec{F}_t)\cdot a(\vec{G}_t-\vec{F}_t) = a(\vec{G}_t-\vec{F}_t)\cdot v(\vec{F}_t)$ and $v(\vec{F}_t)\cdot a^*(\vec{G}_t-\vec{F}_t) = a^*(\vec{G}_t-\vec{F}_t)\cdot v(\vec{F}_t)$, due to the Coulomb gauge condition and the fact that $\operatorname{supp}\vec{F}_t \cap \operatorname{supp}\vec{G}_t = \emptyset$. Applying Eqs. (III.46), (III.48), and (III.50), we obtain that, for any $0<\varepsilon\leq 2$,

$$\begin{aligned} H(\vec{F}_t)+\check{H}_{t-\tau,s} &\leq H_{t,s} \;+\; 2\varepsilon v(\vec{F}_t)^2 \;+\; \frac{2}{\varepsilon}\|\Delta\vec{G}_t\|^2\,\check{H}_{t-\tau,s} \\ &\quad + \|\Delta\vec{G}_t\|^2\,(\check{H}_{t-\tau,s}+\rho_t) \\ &\leq H_{t,s} \;+\; 4\varepsilon\,H^+(\vec{F}_t) \;+\; \frac{4}{\varepsilon}\,\|\Delta\vec{G}_t\|^2\,\check{H}_{t-\tau,s} \\ &\quad + 4\varepsilon\Big(b+\kappa+e_0+\|\vec{G}\|^2\Big) \;+\; \|\Delta\vec{G}_t\|^2\,\rho_t\,. \end{aligned} \tag{III.65}$$

We choose $\varepsilon := \frac{1}{8}\|\Delta\vec{G}_t\|^2$ and observe that $4\varepsilon = \frac{1}{2}\|\Delta\vec{G}_t\|^2 \leq \frac{1}{2}$ and $\frac{4}{\varepsilon}\,\|\Delta\vec{G}_t\|^2 = \frac{1}{2}$. Inserting this choice into (III.65) yields

$$\begin{aligned} H^+(\vec{F}_t)+\check{H}_{t-\tau,s} &\leq H^+_{t,s} \;+\; \frac{1}{2}\Big(H^+(\vec{F}_t)+\check{H}_{t-\tau,s}\Big) \\ &\quad + \frac{1}{2}\|\Delta\vec{G}_t\|^2\Big(b+\kappa+e_0+\|\vec{G}\|^2+2\rho_t\Big) \;+\; E_t - E(\vec{F}_t)\,, \end{aligned} \tag{III.66}$$

which we can solve for $H^+(\vec{F}_t)+\check{H}_{t-\tau,s}$ with the result

$$H^+(\vec{F}_t)+\check{H}_{t-\tau,s} \;\leq\; 2\,H^+_{t,s} + \|\Delta\vec{G}_t\|^2\Big(b+\kappa+e_0+\|\vec{G}\|^2+2\rho_t\Big) + 2\Big(E_t-E(\vec{F}_t)\Big). \tag{III.67}$$

Denoting by ψ_ε an ε-approximate ground state of $H(\vec{F}_t)$, i.e., a normalized vector in $\operatorname{dom}(H(\vec{F}_t)) \supseteq \operatorname{dom}(H_{t,s})$, for which $\langle\psi_\varepsilon|\,H(\vec{F}_t)\psi_\varepsilon\rangle \leq E(\vec{F}_t)+\varepsilon$, we observe

that

$$E_t - E(\vec{F}_t) \;\leq\; \Big\langle \psi_\varepsilon \,\Big|\, \Big(H_{t,s} - H(\vec{F}_t)\Big)\,\phi_\varepsilon \Big\rangle \;=\; \Big\|\vec{G}_t - \vec{F}_t\Big\|^2 + \varepsilon \;\leq\; \rho_t\,\|\Delta\vec{G}_t\|^2 + \varepsilon\,. \tag{III.68}$$

Letting $\varepsilon \to 0$ and additionally using $\rho_t \leq \kappa$ and $\|\vec{G}\|^2 \leq \kappa^2$, we arrive at

$$H^+(\vec{F}_t) + \check{H}_{t-\tau,s} \;\leq\; 2\,H^+_{t,s} \;+\; \|\Delta\vec{G}_t\|^2\Big(b + 6\kappa^2 + e_0\Big) \;\leq\; 2(H^+_{t,s} + \rho_t)\,, \tag{III.69}$$

since $\alpha^3 \;\leq\; (b + 6\kappa^2 + e_0 + 1)^{-1}$. This establishes the claim because $\check{H}_{t-\tau,s} \;\leq\; H^+(\vec{F}_t) + \check{H}_{t-\tau,s}$. □

The desired estimate on $W_{t,s}$ is now a simple consequence of Lemmata III.1 and III.2

Lemma III.3 *For all* $0 \leq t < s$ *and* $\alpha^3 \leq (b + 6\kappa^2 + e_0 + 1)^{-1}$,

$$\Big\|(H^+_{t,s} + \rho_t)^{-1/2}\,W_{t,s}\,(H^+_{t,s} + \rho_t)^{-1/2}\Big\| \;\leq\; 8\,\eta\,\alpha^{3/2}\,(s-t)\,. \tag{III.70}$$

Proof: We first pick $0 < \varepsilon \leq 1$ and observe that, by Lemma III.1, the following quadratic form estimate holds true,

$$\begin{aligned}
\pm W_{t,s} &= \pm v(\vec{G}_t)\cdot A(\vec{G}_{t,s}) \;\pm\; A(\vec{G}_{t,s})\cdot v(\vec{G}_t) \;\pm\; A(\vec{G}_{t,s})^2 \qquad\qquad \text{(III.71)}\\
&\leq \varepsilon\, v(\vec{G}_t)^2 \;+\; 2\varepsilon^{-1}\,A(\vec{G}_{t,s})^2\\
&\leq 2\varepsilon\,\Big(H^+(\vec{G}_t) + 1\Big) \;+\; 8\varepsilon^{-1}\,\Big\|(\omega^{-1} + \rho_t^{-1})^{1/2}\vec{G}_{t,s}\Big\|^2\,\Big(\check{H}(\vec{G}_{t,s}) + \rho_t\Big)\,.
\end{aligned}$$

Using $H^+(\vec{G}_t) \leq H^+_{t,s}$, $\check{H}(\vec{G}_{t,s}) \leq \check{H}_{t,s}$, and (III.57), Eq. (III.71) and Lemma III.2 imply that

$$\pm W_{t,s} \;\leq\; \Big(\frac{2\varepsilon}{\rho_t} \;+\; \frac{8\eta^2\,\alpha^3\,\rho_t\,(s-t)^2}{\varepsilon}\Big)\,\Big(H^+_{t,s} + \rho_t\Big)\,. \tag{III.72}$$

Choosing $\varepsilon := 2\eta\alpha^{3/2}\rho_t(s-t)$, we obtain (III.70). □

IV Spectral Gap

The goal of this section is to prove that, for $t \geq 0$, the ground state energies $E_t = \inf\sigma[H_t]$ are isolated eigenvalues of multiplicity one and to derive a quantitative estimate on the spectral gap $\inf\Big(\sigma[H^+_{t,s}] \setminus \{0\}\Big)$ of $H_{t,s}$ above $E_{t,s} = E_t$, for $s \geq t$.

Lemma IV.1 *For all* $0 \leq t < s \leq t+\eta$ *and* $\alpha^3 \leq (64+b+6\kappa^2+e_0)^{-1}$*, the difference of the ground state energies obeys*

$$|E_t - E_s| \;\leq\; 8\eta\,\alpha^{3/2}\,(s-t)\,\rho_t\,. \tag{IV.73}$$

Proof: Due to Lemma III.3 we have the following lower bound for H_s

$$\begin{aligned} H_s &= E_t + H_{t,s}^+ + W_{t,s} \qquad\qquad (\text{IV.74})\\ &= E_t - \rho_t + (H_{t,s}^+ + \rho_t)^{1/2} \\ &\qquad\qquad \Big(\mathbf{1} \,+\, (H_{t,s}^+ + \rho_t)^{-1/2}\, W_{t,s}\, (H_{t,s}^+ + \rho_t)^{-1/2}\Big)\, (H_{t,s}^+ + \rho_s)^{1/2} \\ &\geq E_t - \rho_t + \Big(1 \,-\, 8\eta\alpha^{3/2}(s-t)\Big)\,(H_{t,s}^+ + \rho_t) \\ &= E_t \,-\, 8\eta\alpha^{3/2}(s-t)\rho_t \,+\, \Big(1 \,-\, 8\eta\alpha^{3/2}(s-t)\Big)\, H_{t,s}^+\,, \end{aligned}$$

using that $1 - 8\eta\alpha^{3/2}(s-t) \geq 1 - 8\alpha^{3/2} \geq 0$. Similarly, we derive the following upper bound

$$H_s \;\leq\; E_t \,+\, 8\eta\alpha^{3/2}(s-t)\rho_t \,+\, \Big(1 \,+\, 8\eta\alpha^{3/2}(s-t)\Big)\, H_{t,s}^+\,. \tag{IV.75}$$

By (IV.74), the ground state energy E_s is bound from below by

$$E_s \;\geq\; E_t \,-\, 8\eta\,\alpha^{3/2}\,(s-t)\,\rho_t\,. \tag{IV.76}$$

To establish an upper bound on E_s we fix $\varepsilon > 0$ and suppose that $\phi_\varepsilon \in \mathcal{H}$ is a normalized ε-approximate ground state of $H_{t,s}$, i.e., $\langle \phi_\varepsilon | H_{t,s}^+ \phi_\varepsilon\rangle \leq \varepsilon$. By the min-max principle and (IV.75), we have

$$E_s \;\leq\; \langle \phi_\varepsilon \,|\, H_s\,\phi_\varepsilon\rangle \;\leq\; E_t \,+\, 8\eta\alpha^{3/2}(s-t)\rho_t \,+\, \Big(1 \,+\, 8\eta\alpha^{3/2}(s-t)\Big)\,\varepsilon\,, \tag{IV.77}$$

and the desired bound follows from $\varepsilon \to 0$. □

For $0 \leq t < s$, we introduce three nonnegative numbers

$$\mathrm{gap}_{t,s} := \sup_{\psi_0 \in \mathcal{H}\setminus\{0\}} \inf_{\phi \perp \psi_0,\, \|\phi\|=1} \langle \phi \,|\, H_{t,s}^+\,\phi\rangle\,, \tag{IV.78}$$

$$\mathrm{gap}_{t} := \sup_{\psi_0 \in \mathcal{H}\setminus\{0\}} \inf_{\phi \perp \psi_0,\, \|\phi\|=1} \langle \phi \,|\, H_{t}^+\,\phi\rangle\,, \tag{IV.79}$$

$$\widetilde{\mathrm{gap}}_{t} := \sup_{\psi_0 \in \mathcal{H}_t\setminus\{0\}} \inf_{\phi \perp \psi_0,\, \|\phi\|=1} \langle \phi \,|\, \widetilde{H}_{t}^+\,\phi\rangle\,, \tag{IV.80}$$

which are later shown to be the spectral gaps above the ground state energy of $H_{t,s}$, H_t, and $\widetilde{H}_t$, respectively, where $\mathcal{H}_t := \mathcal{H}_{el} \otimes \mathcal{F}[\mathfrak{h}_t]$ (see Eqs. (II.31) and (II.33)).

Lemma IV.2 *Assume* $\alpha^3 \leq (64 + b + 6\kappa^2 + e_0)^{-1}$. *Then*

$$\mathrm{gap}_{t,s} \;=\; \min\{\widetilde{\mathrm{gap}}_t \,,\; \rho_s\} \;\geq\; \min\{\mathrm{gap}_t \,,\; \rho_s\}\,, \tag{IV.81}$$

for all $0 \leq t < s \leq t + \eta$.

Proof: We first remark that the first equality $\mathrm{gap}_{t,s} = \min\{\widetilde{\mathrm{gap}}_t, \rho_s\}$ in (IV.81) implies that $\mathrm{gap}_t = \mathrm{gap}_{t,t} = \min\{\widetilde{\mathrm{gap}}_t, \rho_t\} \leq \widetilde{\mathrm{gap}}_t$ and hence the second inequality in (IV.81). It therefore suffices to show that $\mathrm{gap}_{t,s} = \min\{\widetilde{\mathrm{gap}}_t, \rho_s\}$. To this end, we recall from (II.30) that $H_{t,s} = \widetilde{H}_t \otimes \mathbf{1}^t_\infty + \mathbf{1}_t \otimes \check{H}_{t,s}$ on $\mathcal{H} = \mathcal{H}_t \otimes \mathcal{F}[\mathfrak{h}^t_\infty]$. Denoting by $P_\Omega := \mathbf{1}_{el} \otimes \mathbf{1}_t \otimes |\Omega^t_\infty\rangle\langle\Omega^t_\infty|$ the orthogonal projection onto the vacuum vector $\Omega^t_\infty \in \mathcal{F}[\mathfrak{h}^t_\infty]$, we define $\chi : \mathcal{H} \to \mathcal{H}_t$ by $P_\Omega\phi =: \chi_\phi \otimes \Omega^t_\infty$. Note that $H_{t,s}$ and P_Ω commute. Hence, for any normalized ϕ with $P_\Omega\phi \neq 0$, i.e., $0 < \|\chi_\phi\| = \|P_\Omega\phi\| \leq 1$, we have

$$\begin{aligned}
\langle\phi \,|\, H^+_{t,s}\,\phi\rangle &= \Big\langle P_\Omega\phi \,\Big|\, (\widetilde{H}_t \otimes \mathbf{1}^t_\infty)\, P_\Omega\phi\Big\rangle \;+\; \Big\langle P^\perp_\Omega\phi \,\Big|\, (\mathbf{1}_t \otimes \check{H}_{t,s})\, P^\perp_\Omega\phi\rangle \\
&\geq \Big\langle P_\Omega\phi \,\Big|\, (\widetilde{H}_t \otimes \mathbf{1}^t_\infty)\, P_\Omega\phi\Big\rangle \;+\; \rho_s\, \|P^\perp_\Omega\phi\|^2 \\
&= \|\chi_\phi\|^2 \left\langle \frac{\chi_\phi}{\|\chi_\phi\|} \,\middle|\, \widetilde{H}_t\, \frac{\chi_\phi}{\|\chi_\phi\|}\right\rangle \;+\; \rho_s \left(1 - \|\chi_\phi\|^2\right),
\end{aligned} \tag{IV.82}$$

which we may extend to the case $\|\chi_\phi\| = 0$ by setting the first term of the sum in the last line equal to zero. It follows for any $\psi \in \mathcal{H} \setminus \{0\}$ with $\chi_\psi \neq 0$ that

$$\begin{aligned}
&\inf_{\phi\perp\psi,\, \|\phi\|=1} \langle\phi \,|\, H^+_{t,s}\,\phi\rangle \\
&\geq \inf_{\chi_\phi\perp\chi_\psi,\, 0\leq\|\chi_\phi\|\leq 1} \left\{ \|\chi_\phi\|^2 \left\langle \frac{\chi_\phi}{\|\chi_\phi\|} \,\middle|\, \widetilde{H}^+_t\, \frac{\chi_\phi}{\|\chi_\phi\|}\right\rangle \;+\; \rho_s \left(1 - \|\chi_\phi\|^2\right)\right\} \\
&= \min\left(\inf_{\chi\perp\chi_\psi,\, \|\chi\|=1} \left\{\langle\chi \,|\, \widetilde{H}^+_t\,\chi\rangle\right\},\; \rho_s\right).
\end{aligned} \tag{IV.83}$$

Therefore

$$\begin{aligned}
\mathrm{gap}_{t,s} &\geq \sup_{\psi\in\mathcal{H}:\, \chi_\psi\neq 0} \left\{ \min\left(\inf_{\chi\perp\chi_\psi,\, \|\chi\|=1} \left\{\langle\chi \,|\, \widetilde{H}^+_t\,\chi\rangle\right\},\; \rho_s\right)\right\} \\
&= \min\left(\sup_{\psi\in\mathcal{H}:\, \chi_\psi\neq 0} \left[\inf_{\chi\perp\chi_\psi,\, \|\chi\|=1} \left\{\langle\chi \,|\, \widetilde{H}^+_t\,\chi\rangle\right\}\right],\; \rho_s\right) \\
&= \min\left\{\widetilde{\mathrm{gap}}_t \,,\; \rho_s\right\},
\end{aligned} \tag{IV.84}$$

establishing the lower bound in (IV.81). For the upper bound, we fix $\varepsilon > 0$ and choose $\tilde{\chi} \in \mathcal{H}_t$ such that

$$\widetilde{\mathrm{gap}}_t \;\leq\; \varepsilon \;+\; \inf_{\chi\perp\tilde{\chi},\, \|\chi\|=1} \langle\chi \,|\, \widetilde{H}^+_t\,\chi\rangle\,. \tag{IV.85}$$

Then

$$\min\Big\{\widetilde{\mathrm{gap}}_t\,,\ \rho_s\Big\}\ \leq\ \varepsilon\ +\ \inf_{\phi\perp\tilde{\chi}\otimes\Omega^t_\infty,\,\|\phi\|=1}\Big\{\langle\phi\,|\,H^+_{t,s}\,\phi\rangle\Big\}\ \leq\ \varepsilon\ +\ \mathrm{gap}_{t,s}\,, \quad \text{(IV.86)}$$

and the upper bound in (IV.81) follows by taking $\varepsilon\to 0$. □

Theorem IV.3 *Assume $\alpha^3\leq(2500\vartheta^2+b+6\kappa^2+e_0)^{-1}$, where $\vartheta:=\Delta e_{el}/\kappa\leq 1$. Then E_t is an isolated, non-degenerate eigenvalue of H_t, and*

$$\inf\Big(\sigma[H^+_t]\setminus\{0\}\Big)\ =\ \mathrm{gap}_t\ \geq\ \vartheta\,\rho_t\,. \quad \text{(IV.87)}$$

Proof: We first remark that $\mathrm{gap}_t\geq\vartheta\,\rho_t$, for all $t\geq 0$, iff $J=\mathbb{R}^+_0$, where

$$J\ :=\ \Big\{s\in\mathbb{R}\,\Big|\,\forall 0\leq t\leq s:\ \mathrm{gap}_t\geq\vartheta\rho_t\Big\}\,. \quad \text{(IV.88)}$$

Obviously, J is an (possibly degenerate) interval. $J\ni 0$ is not empty since $\mathrm{gap}_0=\min\{\Delta e_{el},\kappa\}=\Delta e_{el}=\vartheta\kappa=\vartheta\rho_0$. We now demonstrate that the assumption

$$s_*\ :=\ \sup J\ <\ \infty \quad \text{(IV.89)}$$

leads to a contradiction. To this end, we fix $0<\delta\leq\min\{\eta,\ \eta^{-1}\ln(1/\vartheta)\}$ and choose $0\leq t\leq s_*<s$ such that $t\in J$, $s\notin J$, and $s-t\leq\delta$. Due to (IV.73), (IV.74), and Lemma IV.2, we have

$$H^+_s\ \geq\ E_t-E_s\ -\ 8\eta\alpha^{3/2}(s-t)\rho_t\ +\ \Big(1\ -\ 8\eta\alpha^{3/2}(s-t)\Big)\,H^+_{t,s}\,, \quad \text{(IV.90)}$$

which implies that

$$\begin{aligned}\mathrm{gap}_s\geq&\,E_t-E_s\ -\ 8\eta\alpha^{3/2}(s-t)\rho_t\ +\ \Big(1\ -\ 8\eta\alpha^{3/2}(s-t)\Big)\,\mathrm{gap}_{t,s} \quad \text{(IV.91)}\\ =&-16\eta\alpha^{3/2}(s-t)\rho_t\ +\ \Big(1\ -\ 8\eta\alpha^{3/2}(s-t)\Big)\ \min\Big\{\widetilde{\mathrm{gap}}_t\,,\ \rho_s\Big\}\\ \geq&-16\eta\alpha^{3/2}(s-t)\rho_t\ +\ \Big(1\ -\ 8\eta\alpha^{3/2}(s-t)\Big)\ \min\Big\{\mathrm{gap}_t\,,\ \rho_s\Big\}\,.\end{aligned}$$

Since $t\in J$ and $s-t\leq\delta\leq\eta^{-1}\ln(1/\vartheta)$,

$$\min\Big\{\mathrm{gap}_t\,,\ \rho_s\Big\}\ \geq\ \min\Big\{\vartheta\,\rho_t\,,\ \rho_s\Big\}\ =\ \vartheta\,\rho_t\,. \quad \text{(IV.92)}$$

On the other hand, $s\notin J$ implies $\mathrm{gap}_s\leq\vartheta\rho_s$ and hence

$$1\ \geq\ f[\eta(s-t)]\ :=\ e^{\eta(s-t)}\Big(1\ -\ 24\alpha^{3/2}\vartheta\eta(s-t)\Big)\,, \quad \text{(IV.93)}$$

additionally using $\vartheta\leq 1$. Note, however, that $f[0]=1$ and

$$f'[0]\ =\ 1-24\alpha^{3/2}\vartheta\ \geq\ \frac{1}{2}\,, \quad \text{(IV.94)}$$

since $\alpha^{3/2} \leq (50\vartheta)^{-1}$. So, $f[\varepsilon] > 1$, for sufficiently small values of $\varepsilon > 0$, yielding a contradiction to (IV.93). It follows that $s_* = \infty$ and $J = \mathbb{R}_0^+$, i.e., for all $t \geq 0$, we have $\mathrm{gap}_t \geq \vartheta\, \rho_t$. The min-max principle now immediately implies that E_t is an isolated, non-degenerate eigenvalue of H_t. □

V Ground State Projections

The present section is devoted to the construction of the spectral projection P_t onto the simple eigenvalue E_t of H_t, for $t > 0$. To this end, we choose $0 < \delta < \frac{\vartheta}{128\eta}$ and $s > 0$ such that $s - \delta < t < s$. Since $\eta\delta\, e^{\eta\delta} \leq \frac{\vartheta}{128}\, e^{\vartheta/128} \leq \frac{\vartheta}{64}$, we have the estimate

$$|E_t - E_s| \;\leq\; 8\,\eta\,\alpha^{3/2}\,(s-t)\,\rho_t \;\leq\; 16\,\alpha^{3/2}\,\eta\delta\, e^{\eta\delta}\,\rho_s \;\leq\; \frac{\alpha^{3/2}\,\vartheta}{8}\,\rho_s\,. \tag{V.95}$$

We define

$$\Gamma_t \;:=\; \left\{ z \in \mathbb{C} \;\middle|\; |z| = \frac{1}{4}\,\vartheta\,\rho_t \right\}, \tag{V.96}$$

and we observe that $\mathrm{gap}_{t,s} \geq \min\{\mathrm{gap}_t, \rho_s\} \geq \vartheta\rho_t \geq \vartheta\rho_s$ [see Eq. (IV.92)] and (V.95) imply

$$\begin{aligned} |\lambda - E_s - z_s| &\geq |\lambda - E_t| - |E_t - E_s| - |z_s| \\ &\geq \mathrm{gap}_{t,s} - \frac{\vartheta}{8}\,\rho_s - \frac{\vartheta}{4}\,\rho_s \;\;\geq\;\; \frac{5\vartheta}{8}\,\rho_s\,, \end{aligned} \tag{V.97}$$

for every $z_s \in \Gamma_s$ and $\lambda \in \sigma[H_{t,s}] \setminus \{E_t\}$. Therefore, E_t is the only point in $\sigma[H_{t,s}]$, which is encircled by $\{E_s\} - \Gamma_s$ and $\mathrm{dist}(\sigma[H_{t,s}],\, \Gamma_s) \geq \frac{\vartheta}{8}\rho_s$. The same statement holds for $\sigma[H_t]$ instead of $\sigma[H_{t,s}]$. The Riesz-formula yields

$$P_t = \frac{-1}{2\pi i} \int_{\Gamma_s} \frac{dz_s}{H_t - E_s + z_s} \tag{V.98}$$

$$= P_{t,s} = \frac{-1}{2\pi i} \int_{\Gamma_s} \frac{dz_s}{H_{t,s} - E_s + z_s}\,. \tag{V.99}$$

Lemma V.1 *Let $s > 0$ and $\beta \in \{0, 1\}$, and assume that $\alpha^3 \leq (2500\vartheta^2 + b + 6\kappa^2 + e_0)^{-1}$ and $0 < \delta < \min\{\frac{\vartheta}{128\eta}, s\}$. Then, for all $s - \delta < t < r \leq s$ and all $z_s \in \Gamma_s$, we have*

$$\Big\| \Big\{ (H_{r,s} - E_s + z_s)^{-1} - (H_{t,s} - E_s + z_s)^{-1} \Big\}\, (H_{t,s}^+ + \rho_t)^{\beta/2} \Big\| \;\leq\; \frac{96\,\eta}{\vartheta\,\rho_t^{1-\beta}}\,\alpha^{3/2}\,(r-t) \tag{V.100}$$

and

$$\left\|(P_s - P_t)\,(H^+_{t,s} + \rho_t)^{\beta/2}\right\| \;\leq\; 96\,\eta\,\vartheta^{-1}\,\alpha^{3/2}\,\rho_t^{\beta/2}\,(s-t)\,. \tag{V.101}$$

Proof: We first expand the difference of the resolvents in Eq. (V.100) in a Neumann series as

$$\begin{aligned}&(H_{r,s} - E_s + z_s)^{-1} - (H_{t,s} - E_s + z_s)^{-1} \\ &= \sum_{n=1}^{\infty} (H_{t,s} - E_s + z_s)^{-1}\Big[- W_{t,r}\,(H_{t,s} - E_s + z_s)^{-1}\Big]^n\,.\end{aligned} \tag{V.102}$$

We observe that, for $\beta = 0, 1$,

$$\begin{aligned}&\left\|(H_{t,s} - E_s + z_s)^{-1}\Big[- W_{t,r}\,(H_{t,s} - E_s + z_s)^{-1}\Big]^n\,(H_{t,s} + \rho_t)^{\beta/2}\right\| \\ &\leq \left\|(H^+_{t,r} + \rho_t)^{-1/2}\right\|^{2-\beta}\,\left\|\frac{H^+_{t,r} + \rho_t}{H^+_{t,s} - E_s + z_s}\right\|^n \\ &\quad\cdot\left\|(H^+_{t,r} + \rho_t)^{-1/2}\,W_{t,r}\,(H^+_{t,r} + \rho_t)^{-1/2}\right\|^n \\ &\leq \rho_t^{(\beta/2)-1}\left(4\vartheta^{-1}\cdot 8\eta\alpha^{3/2}(r-t)\right)^n\,,\end{aligned} \tag{V.103}$$

using Lemma III.3 and

$$\begin{aligned}&\left\|\frac{H^+_{t,r} + \rho_t}{H^+_{t,s} - E_s + z_s}\right\| \;\leq\; \left\|\frac{H^+_{t,r} + \rho_t}{H^+_{t,r} - E_s + z_s}\right\| \\ &\leq \max\left\{\frac{\rho_t}{|z_s - E_s + E_t|}\,,\;\sup_{\mu\geq \mathrm{gap}_{t,r}}\left|\frac{\mu + \rho_t}{\mu - z_s - E_s + E_t}\right|\right\} \;\leq\; \frac{4}{\vartheta}\,.\end{aligned} \tag{V.104}$$

Note that here $t > s - \delta$ insures that $\mu \geq \mathrm{gap}_{t,r} = \vartheta\rho_t$. Hence $\mu + \rho_t \leq (1 + \frac{1}{\vartheta})\mu \leq \frac{2}{\vartheta}\mu$ and $|\mu - z_s - E_s + E_t| \geq \mu - \frac{\vartheta}{4}\rho_t - \frac{\vartheta}{8}\rho_t \geq \frac{5}{8}\mu$.

For $s - t > 0$ sufficiently small, $32\eta\vartheta^{-1}\alpha^{3/2}(s-t) \leq \frac{1}{2}$. This implies the norm convergence of the Neumann series on the right side of (V.102) with $r := s$ and yields Estimate (V.100) by summing up a geometric series.

Eq. (V.101) is an immediate consequence of (V.100) and (V.99) because

$$P_s - P_t \;=\; P_{s,s} - P_{t,s} \;=\; \frac{-1}{2\pi i}\int_{\Gamma_s}\Big\{(H_s - E_s + z_s)^{-1} - (H_{t,s} - E_s + z_s)^{-1}\Big\}\,dz_s\,, \tag{V.105}$$

and hence

$$\left\|(P_s - P_t)\,(H_{t,s}^+ + \rho_t)^{\beta/2}\right\| \;\leq \tag{V.106}$$
$$\frac{\vartheta\,\rho_t}{3}\;\sup_{z_s\in\Gamma_s}\left\|\left\{(H_s - E_s + z_s)^{-1} - (H_{t,s} - E_s + z_s)^{-1}\right\}(H_{t,s}^+ + \rho_t)^{\beta/2}\right\|.$$

□

Next, we establish the continuous differentiability of the map $P_{(\cdot)} : \mathbb{R}^+ \to \mathcal{B}[\mathcal{H}]$, $t \mapsto P_t$, in operator norm. To this end, we introduce

$$\dot{W}_t \;:=\; v(\vec{G}_t)\cdot A(\dot{\chi}_t\vec{G}) \;+\; A(\dot{\chi}_t\vec{G})\cdot v(\vec{G}_t)\,, \tag{V.107}$$

with $\dot{\chi}_t := \partial_t\chi_t$. Our goal is to prove the following theorem.

Theorem V.2 *Assume that* $\alpha^3 \leq (2500\vartheta^2 + b + 6\kappa^2 + e_0)^{-1}$. *For* $0 < s < \infty$, $z_s \in \Gamma_s$, *and* $0 < \delta \leq \frac{\vartheta}{128\eta}$, *the map*

$$\mathbb{R}^+ \cap (s-\delta,\, s) \;\ni\; t \;\mapsto\; \Big(H_{t,s} - E_s + z_s\Big)^{-1} \;\in\; \mathcal{B}[\mathcal{H}] \tag{V.108}$$

is continuously differentiable, and its derivative is given by

$$\frac{\partial}{\partial t}\Big(H_{t,s} - E_s + z_s\Big)^{-1} \;=\; \Big(H_{t,s} - E_s + z_s\Big)^{-1}(-\dot{W}_t)\Big(H_{t,s} - E_s + z_s\Big)^{-1}. \tag{V.109}$$

Proof: We compute the differential quotient in $t \in \mathbb{R}^+ \cap (s-\delta,\, s)$ and compare it to

$$D_{t,s} \;:=\; (H_{t,s} - E_s + z_s)^{-1}(-\dot{W}_t)(H_{t,s} - E_s + z_s)^{-1}\,. \tag{V.110}$$

From the second resolvent equation, we get, for $r \in (t, s)$, that

$$\frac{1}{r-t}\Big\{(H_{r,s} - E_s + z_s)^{-1} - (H_{t,s} - E_s + z_s)^{-1}\Big\} \;-\; D_{t,s} \;=\; X_{t,r} \,+\, Y_{t,r}\,, \tag{V.111}$$

where the operators $X_{t,r}$ and $Y_{t,r}$ are given by

$$X_{t,r} := (H_{t,s} - E_s + z_s)^{-1}\,(-\Delta W_{t,r})(H_{r,s} - E_s + z_s)^{-1}\,, \tag{V.112}$$
$$\begin{aligned}\Delta W_{t,r} &:= (r-t)^{-1}W_{t,r} - \dot{W}_t \\ &= v(\vec{G}_t)\cdot A(\Delta\vec{G}_{t,r}) \;+\; A(\Delta\vec{G}_{t,r})\cdot v(\vec{G}_t) \;+\; A(\Delta\vec{G}_{t,r})^2\,,\end{aligned} \tag{V.113}$$

with $\Delta\vec{G}_{t,r} := (r-t)^{-1}\vec{G}_{t,r} - \dot{\chi}_t\vec{G}$, and

$$Y_{t,r} \;:=\; (H_{t,s} - E_s + z_s)^{-1}\,\dot{W}_t\left\{(H_{t,s} - E_s + z_s)^{-1} - (H_{r,s} - E_s + z_s)^{-1}\right\}. \tag{V.114}$$

An application of Lemma V.1 and (V.104) yields

$$\|Y_{t,r}\| \;\leq\; \frac{384\,\eta}{\vartheta^2\,\rho_t}\,\alpha^{3/2}\,(r-t)\left\|(H_{t,s}^+ + \rho_t)^{-1/2}\,(-\dot{W}_t)\,(H_{t,s}^+ + \rho_t)^{-1/2}\right\|, \tag{V.115}$$

where the norm on the right side is bounded uniformly in $t \in (s-\delta, s)$ by an estimate similar to (III.70).

For the estimate on $X_{t,r}$ we note that

$$Z_{t,r} \;:=\; \left\| (H^+_{t,r}+\rho_t)^{-1/2}\,(-\Delta W_{t,r})\,(H^+_{t,r}+\rho_t)^{-1/2} \right\| \;\leq\; \frac{32}{\rho_s^{3/2}} \left\| \Delta \vec{G}_{t,r} \right\|_2 . \quad \text{(V.116)}$$

A second-order Taylor expansion yields

$$\begin{aligned} |\Delta \vec{G}_{t,r}| &\leq (r-t)^{-1} \left| \chi_r - \chi_t - (r-t)\dot{\chi}_t \right| |\vec{G}| \;\leq\; (r-t) \left| \partial_t^2 \chi_t \right|_\infty |\vec{G}| \\ &\leq 6\,\eta^2\,\|p'\|_\infty\,(r-t)\,|\vec{G}|\,, \end{aligned} \quad \text{(V.117)}$$

pointwise a. e. in k. Since $|\vec{G}|$ is square-integrable, Lebesgue's dominated convergence theorem implies that $\lim_{r\to t} \|\Delta \vec{G}_{t,r}\|_2 = 0$ and hence also $\lim_{r\to t} \|Z_{t,r}\| = 0$. With this in mind, we apply Lemma V.1 and (V.104) again and obtain

$$\begin{aligned} \|X_{t,r}\| &\leq \frac{Z_{t,r}}{\rho_t^{1/2}} \left\| \frac{H^+_{t,r}+\rho_t}{H^+_{t,s}-E_s+z_s} \right\| \cdot \left\| \frac{(H^+_{t,s}+\rho_t)^{1/2}}{H^+_{r,s}-E_s+z_s} \right\| \qquad \text{(V.118)} \\ &\leq \frac{Z_{t,r}}{\rho_t^{1/2}} \left\| \frac{H^+_{t,r}+\rho_t}{H^+_{t,s}-E_s+z_s} \right\| \cdot \left\{ \left\| \frac{(H^+_{t,s}+\rho_t)^{1/2}}{H^+_{t,s}-E_s+z_s} \right\| \;+\; \frac{96\,\eta}{\vartheta\,\rho_t^{1/2}}\,\alpha^{3/2}\,(r-t) \right\} \\ &\leq \frac{16\,Z_{t,r}}{\vartheta^2\,\rho_t} \left\{ 1 \;+\; 24\eta\,\alpha^{3/2}\,\delta \right\} . \end{aligned}$$

Therefore, $\lim_{r\to t} \|X_{t,r}\| = \lim_{r\to t} \|Y_{t,r}\| = 0$ which establishes the differentiability of $t \mapsto (H_{t,s}-E_s-z_s)^{-1}$ and (V.109). The continuity of the derivative (and, in fact, even smoothness) is shown similarly. □

A direct consequence of Theorem V.2 is

Corollary V.3 *Assume that* $\alpha^3 \leq (2500\vartheta^2 + b + 6\kappa^2 + e_0)^{-1}$.
Then $P_{(\cdot)} \in C^1(\mathbb{R}^+; \mathcal{B}[\mathcal{H}])$ *is continuously differentiable in operator norm, and its derivative is given by*

$$\begin{aligned} \dot{P}_t &= \frac{-1}{2\pi i} \int_{\Gamma_t} \Big(H_t^+ + z_t \Big)^{-1} (-\dot{W}_t) \Big(H_t^+ + z_t \Big)^{-1} dz_t \qquad \text{(V.119)} \\ &= \frac{P_t^\perp}{H_t^+}\,(-\dot{W}_t)\,P_t \;+\; P_t\,(-\dot{W}_t)\,\frac{P_t^\perp}{H_t^+}\,, \qquad \text{(V.120)} \end{aligned}$$

for all $t > 0$.

Proof: Let $t > 0$ and $s > 0$ such that $t \in (s-\delta, s)$, for a suitable $\delta > 0$ as in Theorem V.2. We have

$$P_t \;=\; \frac{-1}{2\pi i}\int_{\Gamma_s} \frac{dz_s}{H_{t,s} - E_s - z_s}\,. \tag{V.121}$$

Its derivative is given by

$$\dot{P}_t \;=\; \frac{-1}{2\pi i}\int_{\Gamma_s} \Big(H_{t,s} - E_s + z_s\Big)^{-1}(-\dot{W}_t)\Big(H_{t,s} - E_s + z_s\Big)^{-1} dz_s\,. \tag{V.122}$$

We obtain (V.119) in the limit $s \to t$. Eq. (V.120) follows from Cauchy's integral formula, observing that $z_t \mapsto (H_t - E_t - z_t)^{-1}P_t^{\perp}$ is regular, for $z_t \in \Gamma_t$, and from the identity $\dot{P}_t = P_t\,\dot{P}_t\,P_t^{\perp} + P_t^{\perp}\dot{P}_t\,P_t$. □

VI Integrability of the Derivative of the Ground State Projection

In this last section we prove that the norm of the derivative $\dot{P}_t = \partial_t P_t$ of $P_{(\cdot)} \in C^1(\mathbb{R}^+; \mathcal{B}[\mathcal{H}])$ decays exponentially in t. This implies the integrability of $\dot{P}_t$ and the fundamental theorem of calculus then yields the existence of the limit $P_\infty = P_0 + \int_0^\infty \dot{P}_t\,dt$.

Theorem VI.1 *Assume that $\alpha^3 \leq (2500\vartheta^2 + b + 6\kappa^2 + e_0)^{-1}$. Then there exists a constant $C_{\dot{P}} < \infty$ such that the derivative $\dot{P}_t = \partial_t P_t$ of $P_{(\cdot)} \in C^1(\mathbb{R}^+; \mathcal{B}[\mathcal{H}])$ obeys the following norm estimate,*

$$\|\dot{P}_t\| \;\leq\; C_{\dot{P}}\,\alpha^{3/2}\,\rho_t^{1/2}\,, \tag{VI.123}$$

for all $t > 0$.

Proof: First, we observe that, for $\nu \in \{1,2,3\}$ and a sufficiently regular and decaying function $F : \mathbb{R}^3_{\vec{x}} \times (\mathbb{R}^3_{\vec{k}} \times \mathbb{Z}_2) \to \mathbb{C}$,

$$\begin{aligned}
&\Big[v_\nu(\vec{G}_t)\;,\; A(F)\Big] \qquad\qquad\qquad\

$$v(\vec{G}_t)\cdot A(\dot\chi_t\vec{G}) \;-\; A(\dot\chi_t\vec{G})\cdot v(\vec{G}_t) \;= \tag{VI.125}$$
$$-i\,A(\dot\chi_t\vec\nabla_x\cdot\vec{G}) \;+\; \mathrm{Im}\Big\{\int\Big|\vec{G}(k)\Big|^2\,\partial_t\Big(\chi_t^2(k)\Big)\,dk\Big\} \;=\; 0\,.$$

Thus $-\dot P_t = Q_t + Q_t^*$, where

$$Q_t \;:=\; 2\,\frac{P_t^\perp}{H_t^+}\,A(\dot\chi_t\vec{G})\cdot v(\vec{G}_t)\,P_t\,. \tag{VI.126}$$

Hence, it suffices to show the existence of a contant $C<\infty$ such that $\|Q_t\|\leq C\,\rho_t^{1/2}$. To this end, we use $2v(\vec{G}_t)=-i[H_t^+\,,\,\vec{x}]$ and $H_t^+P_t=0$ and obtain

$$\begin{aligned}Q_t &= -i\,\frac{P_t^\perp}{H_t^+}\,A(\dot\chi_t\vec{G})\cdot[H_t^+\,,\,\vec{x}]\,P_t \;\;=\;\; -i\sum_{\nu=1}^3\frac{P_t^\perp}{H_t^+}\,A(\dot\chi_t G_\nu)\,H_t^+\,x_\nu\,P_t\\ &= -i\sum_{\nu=1}^3\frac{P_t^\perp}{H_t^+}\,[A(\dot\chi_t G_\nu)\,,\,H_t]\,x_\nu\,P_t \;-\; i\,P_t^\perp\,A(\dot\chi_t\vec{G})\cdot\vec{x}\,P_t\,.\end{aligned} \tag{VI.127}$$

Note that

$$\begin{aligned}-i[A(\dot\chi_t G_\nu)\,,\,H_t] &= -i[A(\dot\chi_t G_\nu)\,,\,\check H_t] \;-\; i[A(\dot\chi_t G_\nu)\,,\,v(\vec{G}_t)^2]\\ &= A(i\omega\dot\chi_t G_\nu) \;-\; \sum_{\mu=1}^3 i[A(\dot\chi_t G_\nu)\,,\,v_\mu^2(\vec{G}_t)]\,.\end{aligned} \tag{VI.128}$$

A straightforward, but somewhat lengthy computation repeatedly using (VI.124) yields

$$\begin{aligned}&-i[A(\dot\chi_t G_\nu)\,,\,v_\mu^2(\vec{G}_t)]\\ &= -i[A(\dot\chi_t G_\nu)\,,\,v_\mu(\vec{G}_t)]\,v_\mu(\vec{G}_t) \;-\; v_\mu(\vec{G}_t)\,i[A(\dot\chi_t G_\nu)\,,\,v_\mu(\vec{G}_t)]\\ &= \Big\{A(i\dot\chi_t\partial_{x_\mu}G_\nu)+2i\,\mathrm{Im}\langle\chi_t G_\mu|\dot\chi_t G_\nu\rangle\Big\}\,v_\mu(\vec{G}_t)\\ &\quad + v_\mu(\vec{G}_t)\,\Big\{A(i\dot\chi_t\partial_{x_\mu}G_\nu)+2i\,\mathrm{Im}\langle\chi_t G_\mu|\dot\chi_t G_\nu\rangle\Big\}\\ &= -A(\dot\chi_t k^2 G_\nu)+2i\,\mathrm{Im}\langle k_\mu G_{t,\mu}|\dot\chi_t G_\nu\rangle\\ &\quad + 2\Big\{A(\dot\chi_t k_\mu G_\nu)+2i\,\mathrm{Im}\langle\chi_t G_\mu|\dot\chi_t G_\nu\rangle\Big\}\,v_\mu(\vec{G}_t)\,.\end{aligned} \tag{VI.129}$$

Summing over $\mu=1,2,3$ and taking the Coulomb gauge condition, which implies that $\vec{k}\cdot\vec{G}_t(k)=0$, we have that

$$\begin{aligned}-i[A(\dot\chi_t G_\nu)\,,\,H_t] &= A(i\omega\dot\chi_t G_\nu) \;-\; A(\dot\chi_t k^2 G_\nu)\\ &\quad + 2\sum_{\mu=1}^3\Big\{A(\dot\chi_t k_\mu G_\nu)+2i\,\mathrm{Im}\langle\chi_t G_\mu|\dot\chi_t G_\nu\rangle\Big\}\,v_\mu(\vec{G}_t)\,.\end{aligned} \tag{VI.130}$$

Inserting this into (VI.127), we obtain that $Q_t = Q_t^{(1)} + Q_t^{(2)} + Q_t^{(3)} + Q_t^{(4)}$, where

$$Q_t^{(1)} := -\frac{P_t^\perp}{H_t^+} A\Big((k^2 - i\omega)\dot\chi_t \vec G_t\Big) \cdot \vec x\, P_t\,, \tag{VI.131}$$

$$Q_t^{(2)} := -iP_t^\perp A(\dot\chi_t \vec G_t) \cdot \vec x\, P_t\,, \tag{VI.132}$$

$$Q_t^{(3)} := 2\sum_{\mu,\nu=1}^3 \frac{P_t^\perp}{H_t^+} A(\dot\chi_t k_\mu G_\nu)\, v_\mu(\vec G_t)\, x_\nu\, P_t\,, \tag{VI.133}$$

$$Q_t^{(4)} := 4i\sum_{\mu,\nu=1}^3 \frac{P_t^\perp}{H_t^+} \mathrm{Im}\langle \chi_t G_\mu | \dot\chi_t G_\nu\rangle\, v_\mu(\vec G_t)\, x_\nu\, P_t\,. \tag{VI.134}$$

We estimate each of these terms separately. Using the previous estimates, especially Lemma III.2, we have

$$\begin{aligned}
\|Q_t^{(1)}\| &\leq 8 \left\| \frac{P_t^\perp}{H_t^+} (\check H_{t-\tau,t} + \rho_t)^{1/2} \right\| \left\| \rho_t^{-1/2} (k^2 + i\omega) \dot\chi_t \vec G_t \right\| \|\vec x P_t\| \\
&\leq \frac{128\,\eta\,\|p\|_\infty\,\alpha^{3/2}}{\pi\,\vartheta}\,\rho_t\,\|\vec x \phi_t\|\,, \qquad\qquad \text{(VI.135)} \\
\|Q_t^{(2)}\| &\leq 16 \left\| \rho_t^{-1/2} \dot\chi_t \vec G_t \right\| \left\| (H_t^+ + \rho_t)^{1/2}\, \vec x \phi_t \right\| \\
&\leq \frac{32\,\eta\,\|p\|_\infty\,\alpha^{3/2}}{\pi}\,\rho_t^{1/2} \left\| (H_t^+ + \rho_t)^{1/2}\, \vec x \phi_t \right\|, \qquad\qquad \text{(VI.136)} \\
\|Q_t^{(3)}\| &\leq 8 \left\| \frac{P_t^\perp}{H_t^+} (\check H_{t-\tau,t} + \rho_t)^{1/2} \right\| \left\| \rho_t^{-1/2} |\vec k|\, \dot\chi_t \vec G_t \right\| \Big(\sum_{\mu,\nu=1}^3 \|v_\mu(\vec G_t)\, x_\nu\, \phi_t\|^2 \Big)^{1/2} \\
&\leq \frac{128\,\eta\,\|p\|_\infty\,\alpha^{3/2}}{\pi\,\vartheta}\,\rho_t \Big(\sum_{\mu,\nu=1}^3 \|v_\mu(\vec G_t)\, x_\nu\, \phi_t\|^2 \Big)^{1/2}, \qquad\qquad \text{(VI.137)} \\
\|Q_t^{(4)}\| &\leq \frac{6}{\vartheta\,\rho_t} \left\| \chi_t^{1/2} |\dot\chi_t|^{1/2} \vec G \right\|^2 \Big(\sum_{\mu,\nu=1}^3 \|v_\mu(\vec G_t)\, x_\nu\, \phi_t\|^2 \Big)^{1/2} \\
&\leq \frac{64\,\eta^2\,\|p\|_\infty^2\,\alpha^3}{\pi\,\vartheta}\,\rho_t \Big(\sum_{\mu,\nu=1}^3 \|v_\mu(\vec G_t)\, x_\nu\, \phi_t\|^2 \Big)^{1/2}, \qquad\qquad \text{(VI.138)}
\end{aligned}$$

We note that $v(\vec G_t)^2 \leq 2(H_t^+ + 1)$ and hence

$$\sum_{\mu,\nu=1}^3 \|v_\mu(\vec G_t)\, x_\nu\, \phi_t\|^2 \;=\; \sum_{\nu=1}^3 \|v(\vec G_t)\, x_\nu\, \phi_t\|^2 \;\leq\; 2 \left\| (H_t^+ + 1)^{1/2}\, \vec x\, \phi_t \right\|^2, \tag{VI.139}$$

which, together with (VI.135)–(VI.138), gives

$$\|Q_t\| \;\leq\; \frac{362\,\eta\,\|p\|_\infty\,\alpha^{3/2}}{\pi\,\vartheta}\,\rho_t^{1/2} \Big\{ \|(H_t^+)^{1/2}\, \vec x \phi_t\|^2 + \|\vec x \phi_t\|^2 \Big\}^{1/2}. \tag{VI.140}$$

Again using $[H_t^+, \vec x] = 2iv(\vec G_t)$ and $v(\vec G_t)^2 \leq 2(H_t^+ + 1)$, we obtain

$$\|(H_t^+)^{1/2}\,\vec{x}\phi_t\|^2 = \langle \vec{x}\phi_t \mid [H_t^+, \vec{x}]\phi_t\rangle \;\leq\; 2\,\|\vec{x}\phi_t\|\cdot\|v(\vec{G}_t)\phi_t\| \qquad \text{(VI.141)}$$
$$\leq 2\sqrt{2}\,\|\vec{x}\phi_t\|\cdot\|(H_t^+ + 1)^{1/2}\phi_t\| \;=\; 2\sqrt{2}\,\|\vec{x}\phi_t\|\,\|\phi_t\|\,.$$

We finally use the fact that $\sup_{t>0}\|\vec{x}\phi_t\| \leq C < \infty$ is uniformly bounded, which is the proved in [5, Theorem III.10]. Inserting this uniform bound on $\|\vec{x}\phi_t\|$ and (VI.141) into (VI.140), we arrive at the asserted claim. □

References

[1] A. Arai and M. Hirokawa. On the existence and uniqueness of ground states of the a generalized spin-boson model. *J. Funct. Anal.*, 151(2):455–503, 1997.

[2] V. Bach, J. Fröhlich, and I. M. Sigal. Quantum electrodynamics of confined non-relativistic particles. *Adv. in Math.* , 137:299–395, 1998.

[3] V. Bach, J. Fröhlich, and I. M. Sigal. Renormalization group analysis of spectral problems in quantum field theory. *Adv. in Math.* , 137:205–298, 1998.

[4] V. Bach, J. Fröhlich, and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Commun. Math. Phys.*, 207(2):249–290, 1999.

[5] V. Bach, J. Fröhlich, and A. Pizzo. Infrared-finite algorithms in QED: The groundstate of an atom interacting with the quantized radiation field. *Commun. Math. Phys.*, February 2006. DOI: 10.1007/s00220-005-1478-3.

[6] Ch. Gerard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, *Ann. H. Poincaré*,1(3): 443–459, 2000. DOI: 10.1007/s00023-005-000-2.

[7] Ch. Gerard, A remark on the paper: "On the existence of ground states for massless Pauli-Fierz Hamiltonians", preprint, mp-arc 06-146, 2006.

[8] M. Griesemer, E. Lieb, and M. Loss. Ground states in nonrelativistic quantum electrodynamics. *Invent. math.*, June 2001. DOI 10.1007/s00222-010-0159.

[9] F. Hiroshima. Functional Integral Representation of a Model in QED. *Rev. Math. Phys.*, 9(4):489–530, 1997.

[10] F. Hiroshima. Ground states of a model in nonrelativistic quantum electrodynamics II. *J. Math. Phys.* , 41(2): 611–674 , 2000.

[11] F. Hiroshima. Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary coupling constants. *Ann. Henri Poincaré*, 3(1): 171–201, 2002. DOI: 10.1007/s00023-002-8615-8.

[12] F. Hiroshima and H. Spohn. Groundstate Degeneracy of the Pauli-Fierz Hamiltonian with spin. *Adv. Theor. Math. Phys.*, 5(6): 1091–1104,2001.

[13] A. Pizzo. One-particle (improper) states in Nelson's massless model. *Ann. H. Poincaré*, 4(3):439–486, 2003.