Enhanced binding for N-particle system interacting with a scalar bose field I

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Abstract

An enhanced binding of an N-particle system interacting through a scalar bose field is investigated, where $N \geq 2$. It is not assumed that this system has a ground state for a zero coupling. It is shown, however, that there exists a ground state for a sufficiently large values of coupling constants. When the coupling constant is sufficiently large, N particles are bound to each other by the scalar bose field, and are trapped by external potentials. Basic ideas of the proofs in this paper are applications of a weak coupling limit and a modified HVZ theorem.

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1 Introduction

In this paper we are concerned with an enhanced binding of an N-particle system interacting with a scalar bose field. Here we assume that $N \geq 2$ and impose ultraviolet cutoffs on the scalar bose field. It may be expected that when N particles interact with each other through a scalar bose field, a strong coupling enhances the binding of this system if forces mediating between each two particles are attractive. We want to justify this heuristic consideration for a certain quantum field model, which is so-called the Nelson model [15].

1.1 The Nelson model

We begin with giving the definition of the Nelson model. In this paper we denote the scalar product and the norm on a Hilbert space $\mathcal{K}$ by $(f,g)_{\mathcal{K}}$ and $\|f\|_{\mathcal{K}}$, respectively. Here $(f,g)_{\mathcal{K}}$ is linear in g and antilinear in f. Unless confusions arise, we omit the suffix $\mathcal{K}$. Let $\mathcal{F}$ be the Boson Fock space over $L^2(\mathbb{R}^d)$ defined by $\mathcal{F} := \bigoplus_{n=0}^\infty [\otimes_s^n L^2(\mathbb{R}^d)]$, where $\otimes_s^n L^2(\mathbb{R}^d)$ denotes the n-fold symmetric tensor product of $L^2(\mathbb{R}^d)$ with $\otimes_s^0 L^2(\mathbb{R}^d) := \mathbb{C}$. Vector $\Psi \in \mathcal{F}$ is written as $\Psi = \{\Psi^{(n)}\}_{n=0}^\infty$ with $\Psi^{(n)} \in \otimes_s^n L^2(\mathbb{R}^d)$. The Fock vacuum $\Omega \in \mathcal{F}$ is defined by $\Omega := \{1,0,0,\ldots\}$. $a(f)$ and $a^*(f)$, $f \in L^2(\mathbb{R}^d)$, denote the annihilation operator and the creation operator in $\mathcal{F}$, respectively, which are defined by

$$(a(f)\Psi)^{(n)}(k_1,\ldots,k_n) := \sqrt{n+1}\int_{\mathbb{R}^d} f(k)\Psi^{(n+1)}(k,k_1,\ldots,k_n)dk, \quad n=0,1,2,\ldots$$

with

$$D(a(f)) := \left\{\Psi \in \mathcal{F} \middle| \sum_{n=0}^\infty \|(a(f)\Psi)^{(n)}\|^2_{\otimes_s^n L^2(\mathbb{R}^d)} < \infty\right\}$$

and $a^*(f) := (a(\bar{f}))^*$. They satisfy canonical commutation relations:

$$[a(f), a^*(g)] = (\bar{f}, g), \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)],$$

on $\mathcal{F}_0 := \{\Psi \in \mathcal{F} | \Psi^{(n)} = 0 \text{ for all } n \geq n_0 \text{ with some } n_0\}$. We informally write as

$$a(f) = \int_{\mathbb{R}^d} a(k)f(k)dk, \quad a^*(f) = \int_{\mathbb{R}^d} a^*(k)f(k)dk.$$

The Nelson Hamiltonian H is a self-adjoint operator acting on the Hilbert space

$$\mathcal{H} := L^2(\mathbb{R}^{dN}) \otimes \mathcal{F},$$

which is defined by

$$H := H_0 + H_{\rm I},$$
$$H_0 := H_{\rm p} \otimes 1 + 1 \otimes H_{\rm f}.$$

Here $H_{\rm p}$ is the N-particle Hamiltonian defined by

$$H_{\rm p} := \sum_{j=1}^{N} \left(-\frac{1}{2m_j}\Delta_j + V_j \right),$$

where m_j is the mass of the j-th particle. $H_{\rm f}$ is the free Hamiltonian of $\mathcal{F}$ given by

$$H_{\rm f} := \bigoplus_{n=0}^{\infty} \left(\sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes \overset{j\text{-th}}{\breve{\omega}} \otimes \cdots \otimes 1}_{n} \right)$$

with the dispersion relation $\omega(k) := |k|$, which is informally written as

$$H_{\rm f} = \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk.$$

Note that

$$\begin{aligned} (H_{\rm f}\Psi)^{(n)}(k_1, \ldots, k_n) &= (\omega(k_1) + \cdots + \omega(k_n))\Psi^{(n)}(k_1, \ldots, k_n), \quad n \geq 1, \\ H_{\rm f}\Omega &= 0. \end{aligned}$$

It is well known that $\sigma(H_{\rm f}) = [0, \infty)$, $\sigma_{\rm p}(H_{\rm f}) = \{0\}$, where $\sigma(K)$ (resp. $\sigma_{\rm p}(K)$) denotes the spectrum (resp. point spectrum) of K. Notation $\sigma_{\rm ess}(K)$ (resp. $\sigma_{\rm disc}(K)$) denotes the essential spectrum (resp. discrete spectrum) of K. We identify as

$$\mathcal{H} \cong \int_{\mathbb{R}^{dN}}^{\oplus} \mathcal{F} dx,$$

where $\int_{\mathbb{R}^{dN}}^{\oplus} \cdots dx$ denotes a constant fiber direct integral [18] and $x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$ the position of particles. Finally $H_{\rm I}$ denotes the interaction between N particles and the scalar field given by

$$H_{\rm I} := \sum_{j=1}^{N} \alpha_j \int_{\mathbb{R}^{dN}}^{\oplus} \phi_j(x_j) dx,$$

where α_j's are real coupling constants and the scalar field $\phi_j(x)$ is given by

$$\phi_j(x) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} (a^*(k)\hat{\lambda}_j(-k)e^{-ikx} + a(k)\hat{\lambda}_j(k)e^{ikx}) dk,$$

where $\hat{\lambda}_j$'s are ultraviolet cutoff functions. Note that

$$(H_{\rm I}\Psi)(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)\Psi(x), \quad a.e.x \in \mathbb{R}^{dN},$$

with

$$D(H_{\rm I}) := \Big\{ \Psi \in \mathcal{H} \Big| \Psi(x) \in \cap_{j=1}^N D(\phi_j(x)) \text{ and } \sum_{j=1}^N \int_{\mathbb{R}^{dN}} \|\phi_j(x)\Psi(x)\|_{\mathcal{F}}^2 dx < \infty \Big\}.$$

Ground states of H are defined by eigenvectors associated with eigenvalue $\inf\sigma(H)$. We want to show the existence of ground states of H. Generally $\inf\sigma(H)$ is the bottom of the essential spectrum of H. Although this makes troublesome to show the existence of ground states, it has been shown for various models in quantum field theory by many authors, e.g., [8], where one fundamental assumption is that $H_{\rm p}$ has a ground state. In this paper we do not assume the existence of ground states of $H_{\rm p}$, which implies the absence of ground state of H with $\alpha_1 = \cdots = \alpha_N = 0$, and show that H has a ground state for sufficiently large values of coupling constants. This phenomena, if it exists, is called the enhanced binding.

1.2 Weak coupling limits

In our model under consideration, it is seen that the enhanced binding is derived from the effective potential $V_{\rm eff}$ which is the sum of potentials between two particles. The effective potential can be derived from a *weak coupling limit* [5, 6, 11, 12], which is one of a key ingredient of this paper. Let us introduce a scaling. We define

$$H(\kappa) = H_{\rm p} \otimes 1 + \kappa^2 1 \otimes H_{\rm f} + \kappa H_{\rm I},$$

where $\kappa > 0$ is a scaling parameter. We shall outline a weak coupling limit in a heuristic level. Let $\mathcal{C} := C([0,\infty); \mathbb{R}^{dN})$. It can be seen that

$$(f \otimes \Omega, e^{-TH(\kappa)} g \otimes \Omega) = \int_{\mathcal{C}\times\mathbb{R}^{dN}} \overline{f(X_0)} g(X_t) e^{-\int_0^T V(X_s)ds} e^{W_\kappa} dP^x dx, \tag{1.1}$$

where $X_\cdot = (X_{1,\cdot}, ..., X_{N,\cdot}) \in \mathcal{C}$, dP^x, $x \in \mathbb{R}^{dN}$, denotes the Wiener measure on $\mathcal{C}$ with $P^x(X_0 = x) = 1$,

$$V(X_s) := \sum_{j=1}^N V_j(X_{j,s})$$

and

$$W_\kappa := \frac{1}{4} \sum_{i,j=1}^N \alpha_i \alpha_j \int_0^T ds \int_0^T dt \int_{\mathbb{R}^d} \hat{\lambda}_i(-k) \hat{\lambda}_j(k) \kappa^2 e^{-\kappa^2|s-t|\omega(k)} e^{-ik\cdot(X_{i,s}-X_{j,t})} dk. \tag{1.2}$$

Informally taking $\kappa \to \infty$ in (1.2), we see that the diagonal part of $\int_0^T ds \int_0^T dt$ survives and the off diagonal part is dumped by factor

$$\kappa^2 e^{-\kappa^2|s-t|\omega(k)} = \frac{\omega(k)\kappa^2 e^{-\kappa^2|s-t|\omega(k)}}{\omega(k)} \sim \delta(s-t)\frac{1}{\omega(k)}.$$

Thus we have

$$W_\kappa \sim \frac{1}{4}\sum_{i,j=1}^N \alpha_i\alpha_j \int_0^T ds \int_{\mathbb{R}^d} \frac{\hat{\lambda}_i(-k)\hat{\lambda}_j(k)}{\omega(k)} e^{-ik\cdot(X_{i,s}-X_{j,s})} dk \tag{1.3}$$

for a sufficiently large κ. Combining the right-hand side of (1.3) with $\int_0^T V(X_s)ds$ in (1.1), we can derive the Feynman-Kac type path integral:

$$\lim_{\kappa\to\infty}(1.1) = \int_{\mathcal{C}\times\mathbb{R}^{dN}} \overline{f(X_0)} g(X_t) e^{-\int_0^T [V(X_s)+V_{\rm eff}(X_s)+G]ds} dP^x dx, \tag{1.4}$$

where

$$V_{\rm eff}(x) := V_{\rm eff}(x_1,...,x_n) := -\frac{1}{4}\sum_{i\neq j}^N \alpha_i\alpha_j \int_{\mathbb{R}^d} \frac{\hat{\lambda}_i(-k)\hat{\lambda}_j(k)}{\omega(k)} e^{-ik(x_i-x_j)} dk \tag{1.5}$$

and

$$G := -\frac{1}{4}\sum_{j=1}^N \int_{\mathbb{R}^d} \frac{\hat{\lambda}_j(-k)\hat{\lambda}_j(k)}{\omega(k)} dk.$$

Note that when $\mathrm{supp}\hat{\lambda}_i \cap \mathrm{supp}\hat{\lambda}_j = \emptyset$, $i \neq j$, the effective potential $V_{\rm eff}$ vanishes. Heuristic arguments mentioned above can be operator theoretically established. Let

$$H_{\rm eff} := \sum_{j=1}^N \left(-\frac{1}{2m_j}\Delta_j + V_j\right) + V_{\rm eff}.$$

Proposition 1.1 *It follows that*

$$\mathrm{s\text{-}}\lim_{\kappa\to\infty} e^{-tH(\kappa)} = e^{-t(H_{\rm eff}+G)} \otimes P_\Omega,$$

where P_Ω denotes the projection onto the Fock vacuum.

See e.g., [11, 12] for details. Intuitively Proposition 1.1 suggests that $H(\kappa) \sim H_{\rm eff} + G$ for a sufficiently large κ. Then if $H_{\rm eff}$ has a ground state, $H(\kappa)$ also may have a ground state. This is actually proved by checking binding conditions introduced by [8] under the assumption that $H_{\rm eff}$ has a ground state. This is an idea in this paper.

Remark 1.2 *Probabilistically through a weak coupling limit, one can derive a Markov process from a non Markov process. The family of measures μ_κ, $\kappa > 0$, on $\mathcal{C}$ is given by*

$$\mu_\kappa(dX) = e^{-\int_0^t V(X_s)ds} e^{W_\kappa} dP^x. \tag{1.6}$$

The double integral W_κ in (1.6) breaks a Markov property of $(X_s)_{s>0}$ and

$$T_{\kappa,s} : f \longmapsto \int_{\mathcal{C}} f(X_s)\mu_\kappa(dX), \quad \kappa < \infty,$$

does not define a semigroup on $L^2(\mathbb{R}^{dN})$. The Markov property revives, however, as $\kappa \to \infty$, and we have $T_{\infty,s} = e^{-s(H_{\rm eff}+G)}$.

1.3 Effective Hamiltonians and enhanced bindings

Typical example of $V_{\rm eff}$ is a three dimensional N-body smeared Coulomb potential:

$$V_{\rm eff}(x_1, ..., x_N) = -\frac{1}{8\pi}\sum_{i\neq j}^N \frac{\alpha_i\alpha_j}{|x_i - x_j|}\varpi(|x_i - x_j|),$$

where $\varpi(|x|) > 0$ holds for a sufficiently small $|x|$. See (3.1). For this case it is determined by signs of $\alpha_1, ..., \alpha_N$ whether $V_{\rm eff}$ is attractive or repulsive for sufficiently small $|x_i - x_j|$. We can see from (1.5) that an identical sign of coupling constants and $\mathrm{supp}\hat{\lambda}_i \cap \mathrm{supp}\hat{\lambda}_j \neq \emptyset$, $i \neq j$, derive attractive effective potentials and enhances binding of the system. Notice that although in the case of $N = 1$ the enhanced binding in the Pauli-Fierz Hamiltonian occurs [3, 9, 13], the effective potential (1.5) disappears and then no enhanced binding in the Nelson model. This is a remarkable discrepancy between a nonrelativistic quantum electrodynamics and the Nelson model.

It is shown in e.g., [7, 10, 14] that the Nelson Hamiltonian with no infrared cutoff, $\hat{\lambda}/\omega \notin L^2(\mathbb{R}^d)$, has no ground state. So we do not discuss the infrared problem and assume that $\hat{\lambda}/\omega \in L^2(\mathbb{R}^d)$. Moreover since we take the Boltzmann statistics for N particles, it is established in [2] that the ground state is unique if it exists. Then we concentrate our discussion to showing the existence of a ground state of H. Systems including the Fermi statistics will be discussed somewhere. We unitarily transform $H(\kappa)$ to a self-adjoint operator of the form

$$H_{\rm eff} \otimes 1 + \kappa^2 1 \otimes H_{\rm f} + H'(\kappa). \tag{1.7}$$

See Proposition 2.4. It is checked that under some condition $H_{\rm eff}$ has a ground state for α_j's with $0 < \alpha_c < |\alpha_j|$, $j = 1, \ldots, N$, for some α_c, which suggests that for a sufficiently large κ, $H(\kappa)$ also has a ground state for α_j with $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, ..., N$, for some $\alpha_c(\kappa)$. Note that we do not assume the existence of ground states of $H_{\rm p}$, namely $H(\kappa)$ with $\alpha_1 = \cdots = \alpha_N = 0$ may have no ground state. We show the existence of a ground state by checking the binding condition [8] in Proposition 2.5 for (1.7).

If there is no interaction between particles, the j-th particle is influenced only by the potential V_j. In this case, a shallow external potential $\sum_{j=1}^N V_j$ can not trap these particles. But if these particles attractively interact through an effective potential derived from a scalar bose field, particles close up and behave just like as one particle with mass $\sum_{j=1}^N m_j$. This *one particle* may feel the force $-\sum_{j=1}^N \nabla_{x_j} V_j$. If N is large enough, this *one particle* feels $\sum_{j=1}^N V_j$ strongly, and finally it will be trapped. In Section 3, we will justify this intuition.

This paper is organized as follows. In Section 2 the Nelson model and its scaled one is introduced and show the main results. The proof of the main theorem is also given. Section 3 is devoted to giving examples of $V_{\rm eff}$ and V_j's. Finally in Appendix A we show some fundamental facts on approximation of the bottom of the essential spectrum of Schrödinger operators.

2 The main results and its proof

2.1 Statements and results

Throughout this paper we assume (L) below:

(L) For all $j = 1, ..., N$, (i),(ii),(iii) and (iv) are fulfilled.

(i) $\hat{\lambda}_j(-k) = \overline{\hat{\lambda}_j(k)}$ and $\hat{\lambda}_j \in L^2(\mathbb{R}^d)$, $\hat{\lambda}_j/\sqrt{\omega} \in L^2(\mathbb{R}^d)$.

(ii) There exists an open set $S \subset \mathbb{R}^d$ such that $\bar{S} = \operatorname{supp} \hat{\lambda}_j$ and $\hat{\lambda}_j \in C^1(S)$.

(iii) For all $R > 0$, $S_R := \{k \in S \| k| < R\}$ has a cone property.

(iv) For all $p \in [1, 2)$ and all $R > 0$, $|\nabla_k \lambda_j| \in L^p(S_R)$.

Remark 2.1 *(i) in (L) guarantees that $H_{\rm I}$ is a symmetric operator. In the proof of Proposition 2.5 below, (ii)-(iv) in (L) are used. In order to show the existence of a ground state, we applied a method invented in [8]. Precisely, we used the photon derivative bound and the Rellich-Kondrachov theorem. The conditions (ii)-(iv) are required to verify these procedures. See [19] for details. In [19] the dimension of the particle space equals three, but one can justify Proposition 2.5 in the dN-dimensional case.*

Let $D(K)$ denote the domain of K. It is well known and easily proved that H is self-adjoint on $D(H) := D(H_{\rm p} \otimes 1) \cap D(1 \otimes H_{\rm f})$ and bounded from below for an arbitrary $\alpha_j \in \mathbb{R}$, $j = 1, ..., N$, by the Kato-Rellich theorem with the inequality

$$\|H_{\rm I}\Psi\| \leq \epsilon\|H_0\Psi\| + b_\epsilon\|\Psi\|, \quad \Psi \in D(H_0),$$

for an arbitrary $\epsilon > 0$. It is also true that $H(\kappa)$ is self-adjoint on $D(H)$ for all $\kappa > 0$.

Assumptions (V1) and (V2) are introduced:

(V1) *There exists $\alpha_c > 0$ such that $\inf \sigma(H_{\rm eff}) \in \sigma_{\rm disc}(H_{\rm eff})$ for α_j with $|\alpha_j| > \alpha_c$, $j = 1, ..., N$.*

(V2) *$V_j(-\Delta + 1)^{-1}$, $j = 1, \ldots, N$, are compact.*

The main theorem is stated below.

Theorem 2.2 *Let $\hat{\lambda}_j/\omega \in L^2(\mathbb{R}^d)$, $j = 1, ..., N$, and assume (L),(V1) and (V2). Fix a sufficiently large $\kappa > 0$. Then for α_j with $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, ..., N$, $H(\kappa)$ has a ground state, where $\alpha_c(\kappa)$ is a constant but possibly infinity.*

The scaling parameter κ in Theorem 2.2 can be regarded as a dummy and absorbed into m_j's, V_j's and $\hat{\lambda}_j$'s. Let κ be sufficiently large. Define

$$\hat{H} := \sum_{j=1}^{N} \left(-\frac{1}{2\hat{m}_j} \Delta_j + \hat{V}_j \right) \otimes 1 + \sum_{j=1}^{N} \alpha_j \hat{\phi}_j + 1 \otimes H_{\rm f},$$

where $\hat{m}_j = m_j \kappa^2$, $\hat{V}_j = V_j/\kappa^2$ and $\hat{\phi}_j$ is defined by ϕ_j with $\hat{\lambda}_j$ replaced by $\hat{\lambda}_j/\kappa$.

Corollary 2.3 *Let $\hat{\lambda}_j/\omega \in L^2(\mathbb{R}^d)$, $j = 1, ..., N$, and assume (L),(V1) and (V2). Then $\hat{H}$ has a ground state for $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, ..., N$.*

Proof: We have $\kappa^{-2} H(\kappa) = \hat{H}$. Then by Theorem 2.2, $\hat{H}$ has a ground state. QED

2.2 Proof of Theorem 2.2

Let $\hat{\lambda}_j/\omega \in L^2(\mathbb{R}^d)$, $j = 1, ..., N$, and define the unitary operator T on $\mathcal{H}$ by

$$T := \exp\left(-i \sum_{j=1}^{N} \frac{\alpha_j}{\kappa} \pi_j \right),$$

where $\pi_j := \int_{\mathbb{R}^{dN}}^{\oplus} \pi_j(x_j) dx$ with

$$\pi_j(x) := \frac{i}{\sqrt{2}} \int_{\mathbb{R}^d} \left(a^*(k) e^{-ikx} \frac{\hat{\lambda}_j(-k)}{\omega(k)} - a(k) e^{ikx} \frac{\hat{\lambda}_j(k)}{\omega(k)} \right) dk.$$

Proposition 2.4 *T maps $D(H)$ onto itself and*

$$\begin{aligned} & T^{-1} H(\kappa) T \\ & = \sum_{j=1}^{N} \left\{ \frac{1}{2m_j} \left(-i\nabla_j \otimes 1 - \frac{\alpha_j}{\kappa} \tilde{\phi}_j \right)^2 + V_j \otimes 1 - \frac{\alpha_j^2}{2} \|\hat{\lambda}_j/\sqrt{\omega}\|^2 \right\} + \kappa^2 1 \otimes H_{\rm f} + V_{\rm eff} \otimes 1 \\ & = H_{\rm eff} \otimes 1 + \kappa^2 1 \otimes H_{\rm f} + H'(\kappa), \end{aligned}$$

where $\tilde{\phi}_j := \int_{\mathbb{R}^{dN}}^{\oplus} \tilde{\phi}_j(x_j) dx$ with

$$\tilde{\phi}_j(x) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} k \left(a^*(k) e^{-ikx} \frac{\hat{\lambda}_j(-k)}{\omega(k)} + a(k) e^{ikx} \frac{\hat{\lambda}_j(k)}{\omega(k)} \right) dk$$

and

$$H'(\kappa) = \sum_{j=1}^{N} \left\{ \frac{1}{\kappa} \frac{\alpha_j}{2m_j} ((-i\nabla_j \otimes 1)\tilde{\phi}_j + \phi_j(-i\nabla_j \otimes 1)) + \frac{1}{\kappa^2} \frac{\alpha_j^2}{2m_j} \tilde{\phi}_j^2 - \frac{\alpha_j^2}{2} \|\hat{\lambda}_j/\sqrt{\omega}\|^2 \right\}.$$

Proof: It is a fundamental identity. We omit the proof. QED

Let us set $C_N := \{1, ..., N\}$. For $\beta \subset C_N$, we define

$$H^0(\beta) = H^0(\beta, \kappa) := \sum_{j\in\beta} \frac{1}{2m_j} \left(-i\nabla_j \otimes 1 - \frac{\alpha_j}{\kappa}\tilde{\phi}_j\right)^2 + \kappa^2 1 \otimes H_f + V_{\rm eff}(\beta) \otimes 1,$$

$$V_{\rm eff}(\beta) := \begin{cases} -\dfrac{1}{4} \displaystyle\sum_{i,j\in\beta, i\neq j} \alpha_i\alpha_j \int_{\mathbb{R}^d} \frac{\hat{\lambda}_i(-k)\hat{\lambda}_j(k)}{\omega(k)} e^{-ik\cdot(x_i-x_j)} dk, & |\beta| \geq 2, \\ 0, & |\beta| = 0, 1, \end{cases}$$

$$H^V(\beta) = H^V(\beta, \kappa) := H^0(\beta) + \sum_{j\in\beta} V_j \otimes 1.$$

Simply we set $H^V := H^V(C_N)$. $H^V = H(\kappa) - \sum_{j=1}^N \alpha_j^2 \|\hat{\lambda}_j\|^2/4$ has ground states if and only if $H(\kappa)$ does, since $\sum_{j=1}^N \alpha_j^2 \|\hat{\lambda}_j\|^2/4$ is a fixed number. In what follows our investigation is focused on showing the existence of ground state of H^V. The operators $H^0(\beta)$ and $H^V(\beta)$ are self-adjoint operators acting on $L^2(\mathbb{R}^{d|\beta|}) \otimes \mathcal{F}$. We set

$$E^V(\kappa) := \inf\sigma(H^V), \qquad E^V(\kappa, \beta) := \inf\sigma(H^V(\beta)),$$
$$E^0(\kappa, \beta) := \inf\sigma(H^0(\beta)), \quad E^V(\kappa, \emptyset) := 0.$$

The lowest two cluster threshold $\Sigma^V(\kappa)$ is defined by

$$\Sigma^V(\kappa) := \min\{E^V(\kappa, \beta) + E^0(\kappa, \beta^{\rm c}) | \beta \subsetneqq C_N\}.$$

To establish the existence of ground state of $H(\kappa)$, we use the next proposition:

Proposition 2.5 ([8]) *Let $\Sigma^V(\kappa) - E^V(\kappa) > 0$. Then $H(\kappa)$ has a ground state.*

For $\beta \subset C_N$, we set

$$h^0(\beta) := -\sum_{j\in\beta} \frac{1}{2m_j}\Delta_j + V_{\rm eff}(\beta), \quad h^V(\beta) := h^0(\beta) + \textstyle\sum_{j\in\beta} V_j,$$
$$\mathcal{E}^0(\beta) := \inf\sigma(h^0(\beta)), \qquad \mathcal{E}^V(\beta) := \inf\sigma(h^V(\beta)),$$

where $h^0(\emptyset) := 0$ and $h^V(\emptyset) := 0$. Furthermore we simply put

$$h^V := h^V(C_N) = H_{\rm eff}, \quad \mathcal{E}^V := \inf\sigma(h^V). \tag{2.1}$$

We define the lowest two cluster threshold for h^V by

$$\Xi^V := \min\{\mathcal{E}^V(\beta) + \mathcal{E}^0(\beta^{\rm c}) | \beta \subsetneqq C_N\} \tag{2.2}$$

and we set

$$V_{{\rm eff}\,ij}(x) := -\frac{1}{4}\alpha_i\alpha_j \int_{\mathbb{R}^d} \frac{\hat{\lambda}_i(-k)\hat{\lambda}_j(k)}{\omega(k)} e^{-ik\cdot x} dk, \quad i \neq j.$$

Lemma 2.6 *Potentials $V_{\mathrm{eff}\,ij}$, $i,j=1,...,N$, are relatively compact with respect to the d-dimensional Laplacian.*

Proof: Since $\hat{\lambda}_i\hat{\lambda}_j/\omega \in L^1(\mathbb{R}^d)$, $i,j=1,...,N$, we can see that $V_{\mathrm{eff}\,ij}(x)$ is continuous in x and $\lim_{|x|\to\infty} V_{\mathrm{eff}\,ij}(x)=0$ by the Riemann-Lebesgue theorem. In particular $V_{\mathrm{eff}\,ij}$ is relatively compact with respect to the d-dimensional Laplacian. QED

We want to estimate $\inf\sigma_{\mathrm{ess}}(H_{\mathrm{eff}})$. For Hamiltonians with the center of mass motion removed, the bottom of the essential spectrum is estimated by HVZ theorem. By extending the IMS localization argument to a quantum field model, in [8] the lowest two cluster threshold of a Hamiltonian interacting with a quantized field (the Pauli-Fierz model) is shown. The following lemma is a simplified version of [8], since no interaction with a quantized radiation field exists. For a self consistency of this paper we give an outline of a proof.

Lemma 2.7 *Assume (V2). Then $\sigma_{\mathrm{ess}}(H_{\mathrm{eff}}) = [\Xi^V,\infty)$.*

Proof: We may assume that $V_i, V_{\mathrm{eff}\,ij} \in C_0^\infty(\mathbb{R}^d)$ by Proposition A.3. Then there exists a normalized sequence $\{g_n\}_n \subset C_0^\infty(\mathbb{R}^{dN})$ such that $\mathrm{supp} g_n \subset \{x\in\mathbb{R}^{dN} | V_i(x)=0, V_{\mathrm{eff}\,ij}(x_i-x_j)=0, i,j=1,...,N\}$ and $(g_n, h^V(\beta)g_n) = (g_n, \sum_{j\in\beta}(-\Delta/2m_j)g_n)\to 0$ as $n\to\infty$. Then we have

$$\mathcal{E}^V(\beta)+\mathcal{E}^0(\beta^c)\le 0. \tag{2.3}$$

Let $\tilde{j}_\beta \in C^\infty(\mathbb{R}^d)$, $\beta\in C_N$, be a Ruelle-Simon partition of unity [4, Definition 3.4], which satisfy (i)-(v) below:

(i) $\sum_{\beta\subseteq C_N}\tilde{j}_\beta(x)^2=1$,
(ii) $\tilde{j}_\beta(Cx)=\tilde{j}_\beta(x)$ for $|x|=1$, $C\ge 1$ and $\beta\ne C_N$,
(iii) $\mathrm{supp}\,\tilde{j}_\beta \subset \{x\in\mathbb{R}^d | \min_{i\in\beta, j\in\beta^c}\{|x_i-x_j|,|x_j|\}\ge c|x|\}$ for some $c>0$,
(iv) $\tilde{j}_\beta(x)=0$ for $|x|<\frac{1}{2}$ and $\beta\ne C_N$,
(v) $\tilde{j}_{C_N}$ has a compact support.

For a constant $R>0$ we put $j_\beta(x):=\tilde{j}_\beta(x/R)$. Note that for each $\beta\subset C_N$,

$$H_{\mathrm{eff}} = h^V(\beta)\otimes 1 + 1\otimes h^0(\beta^c) + \underbrace{\sum_{i\in\beta^c} 1\otimes V_i(x_i) + \sum_{\substack{i\in\beta, j\in\beta^c\\ i\in\beta^c, j\in\beta}} V_{\mathrm{eff}\,ij}(x_i-x_j)}_{=I_\beta}.$$

By the IMS localization formula [4, Theorem 3.2 and p. 34], we have

$$H_{\mathrm{eff}} = j_{C_N}H_{\mathrm{eff}}j_{C_N} + \sum_{\beta\subsetneqq C_N} j_\beta\left[h^V(\beta)\otimes 1 + 1\otimes h^0(\beta^c)\right]j_\beta + \sum_{\beta\subsetneqq C_N} j_\beta^2 I_\beta - \frac{1}{2}\sum_{\beta\subseteqq C_N}|\nabla j_\beta|^2.$$

Here we identify as $L^2(\mathbb{R}^{dN})\cong L^2(\mathbb{R}^{d|\beta|})\otimes L^2(\mathbb{R}^{d|\beta^c|})$. Since $j_{C_N}^2(\sum_{j=1}^N V_j + V_{\mathrm{eff}})$ and $\sum_{\beta\subsetneqq C_N} j_\beta^2 I_\beta$ are relatively compact with respect to the dN-dimensional Laplacian by the property (iii) and (v), it is seen that

$$\begin{aligned}
&\sigma_{\rm ess}(H_{\rm eff})\\
&=\sigma_{\rm ess}\left(j_{C_N}(-\frac{1}{2}\sum_{j=1}^N\Delta_j)j_{C_N}+\sum_{\beta\subsetneqq C_N}j_\beta\left[h^V(\beta)\otimes 1+1\otimes h^0(\beta^{\rm c})\right]j_\beta-\frac{1}{2}\sum_{\beta\subseteqq C_N}|\nabla j_\beta|^2\right).
\end{aligned}$$

We have

$$\sum_{\beta\subsetneqq C_N}j_\beta\left[h^V(\beta)\otimes 1+1\otimes h^0(\beta^{\rm c})\right]j_\beta\geq\sum_{\beta\subsetneqq C_N}(\mathcal{E}^V(\beta)+\mathcal{E}^0(\beta^{\rm c}))j_\beta^2.$$

By (ii) and (v),

$$\left\|\frac{1}{2}\sum_{\beta\subseteqq C_N}|\nabla j_\beta|^2\right\|\leq\frac{C}{R^2}$$

with some constant C independent of R. Hence we obtain that

$$\inf\sigma_{\rm ess}(H_{\rm eff})\geq\min_{x\in\mathbb{R}^d}\sum_{\beta\subsetneqq C_N}(\mathcal{E}^V(\beta)+\mathcal{E}^0(\beta^{\rm c}))j_\beta(x)^2-\frac{C}{R^2}\geq\Xi^V-\frac{C}{R^2}$$

for all $R>0$. Here we used (i) and (2.3). Thus $\sigma_{\rm ess}(H_{\rm eff})\subset[\Xi^V,\infty)$ follows. Next we shall prove the reverse inclusion $\sigma_{\rm ess}(H_{\rm eff})\supset[\Xi^V,\infty)$. Fix $\beta\subsetneqq C_N$. Let $\{\psi_n^V\}_{n=1}^\infty\subset C_0^\infty(\mathbb{R}^{d|\beta|})$ be a minimizing sequence of $h^V(\beta)$ so that

$$\lim_{n\to\infty}\|(h^V(\beta)-\mathcal{E}^V(\beta))\psi_n^V\|=0,\quad\|\psi_n^V\|=1.$$

and $\{\psi_n^0\}_{n=1}^\infty\subset C_0^\infty(\mathbb{R}^{d|\beta^{\rm c}|})$ a normalized sequence such that

$$\lim_{n\to\infty}\|(h^0(\beta^{\rm c})-\mathcal{E}^0(\beta^{\rm c})-K)\psi_n^0\|=0,\tag{2.4}$$

where $K\geq 0$ is a constant. Note that since $\sigma(h^0(\beta^c))=[\mathcal{E}^0(\beta^c),\infty)$, ψ_n^0 such as (2.4) exists. By the translation invariance of $h^0(\beta^{\rm c})$, for any function $\tau.:\mathbb{N}\to\mathbb{R}^d$ the translated sequence $\psi_n^0(x_{j_1}-\tau_n,\ldots,x_{j_{|\beta^{\rm c}|}}-\tau_n)$ also satisfies (2.4). Let $R_n>0$ be a constant satisfying

$$\operatorname{supp}\psi_n^V\subset\{x=(x_{j_1},\cdots,x_{j_{|\beta|}})\in\mathbb{R}^{d|\beta|}||x_{j_i}|<R_n,\,j_i\in\beta,i=1,...,|\beta|\}.$$

We take τ such that

$$\begin{aligned}
&\operatorname{supp}\psi_n^0(\cdot-\tau_n,\cdots,\cdot-\tau_n)\\
&\subset\{x=(x_{k_1},\cdots,x_{k_{|\beta^c|}})\in\mathbb{R}^{d|\beta^c|}||x_{k_i}|\geq R_n+n,\,k_i\in\beta^c,i=1,...,|\beta^c|\}.
\end{aligned}$$

We set $\Psi_n(x_1\cdots x_N)=\psi_n^V(x_{j_1}\cdots x_{j_{|\beta|}})\otimes\psi_n^0(x_{k_1}-\tau_n\cdots x_{k_{|\beta^c|}}-\tau_n)\in L^2(\mathbb{R}^{dN})$. Then, for all i,j with $i\in\beta$, $j\in\beta^{\rm c}$, we have

$$\|V_{{\rm eff}\,ij}(x_i-x_j)\Psi_n\|\le\sup_{x\in\mathbb{R}^d,|x|>n}|V_{{\rm eff}\,ij}(x)|\to 0,\quad (n\to\infty),$$
$$\|V_j(x_j)\Psi_n\|\le\sup_{x\in\mathbb{R}^d,|x|\ge R_n+n}|V_j(x)|\to 0,\quad (n\to\infty).$$

Hence, by a triangle inequality, we have that

$$\|(H_{\rm eff}-\mathcal{E}^V(\beta)-\mathcal{E}^0(\beta^{\rm c})-K)\Psi_n\|\to 0,\quad (n\to\infty).$$

Therefore $[\mathcal{E}^V(\beta)+\mathcal{E}^0(\beta^c)+K,\infty)\subset\sigma(H_{\rm eff})$. Since $\beta\subsetneqq C_N$ and $K>0$ are arbitrary, $[\Xi^V,\infty)\subset\sigma_{\rm ess}(H_{\rm eff})$ follows. Thus the proof is complete. QED

We define

$$\Delta_{\rm p}(\alpha_1,...,\alpha_N):=\Xi^V-\mathcal{E}^V.$$

Corollary 2.8 *Assume (V1) and (V.2). Then* $\Delta_{\rm p}(\alpha_1,...,\alpha_N)>0$ *follows for* α_j *with* $|\alpha_j|>\alpha_c$, $j=1,...,N$.

Proof: Since $\inf\sigma_{\rm ess}(H_{\rm eff})=\Xi^V$ by Lemma 2.7 and $\inf\sigma(H_{\rm eff})\in\sigma_{\rm disc}(H_{\rm eff})$ by (V1), the corollary follows from $\Delta_P(\alpha_1,...,\alpha_N)=\inf\sigma_{\rm ess}(H_{\rm eff})-\inf\sigma(H_{\rm eff})>0$. QED

Lemma 2.9 *For an arbitrary* $\kappa>0$, *it follows that* $\Sigma^V(\kappa)\ge\Xi^V$.

Proof: It is well known that $H^V(\beta)$ can be realized as a self-adjoint operator on a Hilbert space $\mathcal{H}_Q=L^2(\mathbb{R}^{|\beta|d})\otimes L^2(Q,d\mu)$ with some measure space (Q,μ), which is called a Schrödinger representation. It is established that

$$(\Psi,e^{-tH^V(\beta)}\Phi)_{\mathcal{H}_Q}\le(|\Psi|,e^{-t(h^V(\beta)\otimes 1+\kappa^2 1\otimes H_{\rm f})}|\Phi|)_{\mathcal{H}_Q}.$$

Hence for any $\beta\subset C_N$, it follows that $\inf\sigma(h^V(\beta)\otimes 1+\kappa^2 1\otimes H_{\rm f})\le\inf\sigma(H^V(\beta))$. Since $\inf\sigma(H_{\rm f})=0$ and $\inf\sigma(h^V(\beta)\otimes 1+\kappa^2 1\otimes H_{\rm f})=\inf\sigma(h^V(\beta))$, the lemma follows from the definition of lowest two cluster thresholds. QED

Lemma 2.10 *Assume (V1). Then* $E(\kappa)\le\mathcal{E}^V+\kappa^{-2}\sum_{j=1}^N\alpha_j^2\|\hat\lambda_j\|^2/(4m_j)$ *for* α_j *with* $|\alpha_j|>\alpha_c$, $j=1,...,N$.

Proof: By (V1), $H_{\rm eff}$ has a normalized ground state u for α_j with $|\alpha_j|>\alpha_c$, $j=1,...,N$. Set $\Psi:=u\otimes\Omega$. Then

$$\begin{aligned}E(\kappa)&\le(u,H_{\rm eff}u)+\sum_{j=1}^N\frac{\alpha_j}{2m_j\kappa}2\Re(i\nabla_j\Psi,\tilde\phi_j\Psi)+\sum_{j=1}^N\frac{\alpha_j^2}{2m_j\kappa^2}\|\tilde\phi_j\Psi\|^2\\&=\mathcal{E}^V+\sum_{j=1}^N\frac{\alpha_j^2}{4m_j\kappa^2}\|\lambda_j\|^2.\end{aligned}$$

Then the lemma follows. QED

Proof of Theorem 2.2
By Lemmas 2.9 and 2.10, we have

$$\Sigma^V(\kappa) - E(\kappa) \geq \Xi^V - \mathcal{E}^V - \sum_{j=1}^N \frac{\alpha_j^2}{4m_j\kappa^2}\|\lambda_j\|^2 = \Delta_{\rm p}(\alpha_1, ..., \alpha_N) - \sum_{j=1}^N \frac{\alpha_j^2}{4m_j\kappa^2}\|\lambda_j\|^2.$$

Note that $\Delta_p(\alpha_1, ..., \alpha_N) > 0$ is continuous in $\alpha_1, ..., \alpha_N$. Then for a sufficiently large κ, we can obtain that there exists $\alpha_c(\kappa) > \alpha_c$ such that for $\alpha_c < |\alpha_j| < \alpha_c(\kappa)$, $j = 1, ..., N$, $\Sigma^V(\kappa) - E(\kappa) > 0$. Thus $H(\kappa)$ has a ground state for such α_j's by Proposition 2.5. QED

3 Examples

3.1 Example of effective potentials

The typical example of ultraviolet cutoff function is of the form $\hat{\lambda}_j = \hat{\rho}_j/\sqrt{\omega}$, $j \in C_N$, with rotation invariant nonnegative functions $\hat{\rho}_j$. In this case $V_{\rm eff}(x_1, ..., x_N) = \sum_{i\neq j} \alpha_i\alpha_j V_{{\rm eff}\,ij}(x_i - x_j)$ satisfies that (1) $V_{{\rm eff}\,ij}$ is continuous, (2) $\lim_{|x|\to\infty} V_{ij}(x) = 0$ and (3) $V_{{\rm eff}\,ij}(0) < V_{{\rm eff}\,ij}(x)$ for all $x \in \mathbb{R}^d$ but $x \neq 0$. More explicitly effective potential $V_{\rm eff}$ is given by

$$\begin{aligned} V_{\rm eff}(x_1, \cdots, x_N) &= -\frac{1}{4}\sum_{i\neq j}^N \alpha_i\alpha_j \int_{\mathbb{R}^d} \frac{\hat{\rho}_i(-k)\hat{\rho}_j(k)}{\omega(k)^2} e^{-ik\cdot(x_i-x_j)}dk \\ &= -\frac{1}{4}\sum_{i\neq j}^N \alpha_i\alpha_j \frac{\sqrt{(2\pi)^d}}{|x_i - x_j|^{(d-1)/2}} \int_0^\infty \frac{r^{(d-1)/2}}{r^2}\hat{\rho}_i(r)\hat{\rho}_j(r)\sqrt{r|x_i - x_j|}J_{(d-2)/2}(r|x|)dr. \end{aligned}$$

Here J_ν is the Bessel function:

$$J_\nu(x) = (x/2)^\nu \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)}(x/2)^{2n}$$

where Γ denotes the Gamma function. In the case of $d = 3$, and

$$\hat{\rho}_j(k) = \begin{cases} 0 & |k| < \kappa, \\ 1/\sqrt{(2\pi)^3} & \kappa < |k| < \Lambda, \\ 0 & |k| \geq \Lambda, \end{cases}$$

we see that

$$V_{\rm eff}(x_1, \cdots, x_N) = -\frac{1}{8\pi^2}\sum_{i\neq j}^N \frac{\alpha_i\alpha_j}{|x_i - x_j|}\int_{\kappa|x_i-x_j|}^{\Lambda|x_i-x_j|} \frac{\sin r}{r}dr. \tag{3.1}$$

3.2 Example of V_j's

We give an example of $V_1, \cdots, V_N$ satisfying assumption (V1). Assume simply that $V_1 = \cdots = V_N = V$, $\alpha_1 = \cdots = \alpha_N = \alpha$, $\hat{\lambda}_1 = \cdots = \hat{\lambda}_N = \hat{\lambda}$ and $m_1 = \cdots = m_N = m$. Then $V_{\mathrm{eff}\,ij} = W$ for all $i \neq j$. Let

$$h^V(\alpha) := \sum_{j=1}^N \left(-\frac{1}{2m}\Delta_j + V(x_j) \right) + \alpha^2 \sum_{j\neq l}^N W(x_j - x_l),$$

which acts on $L^2(\mathbb{R}^{dN})$. We assume (W1)-(W3) below:

(W1) V is relatively compact with respect to the d-dimensional Laplacian Δ, and $\sigma(-(\Delta/2m) + V) = [0, \infty)$.

(W2) W satisfies that $-\infty < W(0) = \operatorname{ess.inf}_{|x|<\epsilon} W(x) < \operatorname{ess.inf}_{|x|>\epsilon} W(x)$ for all $\epsilon > 0$.

(W3) $\inf\sigma(-(\Delta/(2Nm) + NV) \in \sigma_{\mathrm{disc}}(-(\Delta/(2Nm) + NV)$.

Remark 3.1 *Note that examples of* V_{eff} *given in subsection 3.1 satisfies (W2). The condition (W1) means that the external potential* V *is shallow and the non-interacting Hamiltonian* $h^V(0)$ *has no negative energy bound state.*

When $W = 0$, (W1) implies that each particle independently behaves and is not trapped. When $W \neq 0$, W closes up N particles and they behave as *one particle* with mass Nm. The *one particle* may feel the force $-N\nabla V$ and be trapped by NV. The following theorem justifies this heuristic argument.

Theorem 3.2 *Assume (W1)-(W3). Then, there exists* $\alpha_c > 0$ *such that for all* α *with* $|\alpha| > \alpha_c$, $\inf\sigma(h^V(\alpha)) \in \sigma_{\mathrm{disc}}(h^V(\alpha))$.

To prove Theorem 3.2 we need several lemmas. For $\beta \subset C_N$, we define

$$h^0(\alpha,\beta) := -\frac{1}{2m}\sum_{j\in\beta}\Delta_j + \alpha^2 \sum_{\substack{j,l\in\beta \\ j\neq l}} W(x_j - x_l), \quad h^V(\alpha,\beta) := h^0(\alpha,\beta) + \sum_{j\in\beta} V(x_j),$$

$$\mathcal{E}^0(\alpha,\beta) := \inf\sigma(h^0(\alpha,\beta)), \quad \mathcal{E}^V(\alpha,\beta) := \inf\sigma(h^V(\alpha,\beta)),$$

where $\mathcal{E}^V(\alpha,\emptyset) := 0$ and $\mathcal{E}^0(\alpha,\emptyset) := 0$. Simply we set $\mathcal{E}^V(\alpha, C_N) = \mathcal{E}^V(\alpha)$ and $\mathcal{E}^0(\alpha, C_N) = \mathcal{E}^0(\alpha)$. Let $\Xi^V(\alpha)$ denote the lowest two cluster threshold of $h^V(\alpha)$ defined by (2.2). Then by (W1) and Lemma 2.7, we have

$$\sigma_{\mathrm{ess}}(h^V(\alpha)) = [\Xi^V(\alpha), \infty). \tag{3.2}$$

Lemma 3.3 *Let $\beta \subsetneqq C_N$ but $\beta \neq \emptyset$. Then there exists $\alpha' > 0$ such that, for all α with $|\alpha| > \alpha'$,*

$$\mathcal{E}^0(\alpha) < \mathcal{E}^V(\alpha,\beta) + \mathcal{E}^0(\alpha,\beta^c). \tag{3.3}$$

Proof: Since $h^0(\alpha,\beta)/\alpha^2$ and $h^V(\alpha,\beta)/\alpha^2$ converge to $\sum_{\substack{j,l\in\beta \\ j\neq l}} W(x_j - x_l)$ in the uniform resolvent sense, by (W2), one can show that

$$\lim_{\alpha\to\infty} \frac{\mathcal{E}^V(\alpha,\beta)}{\alpha^2} = \lim_{\alpha\to\infty} \frac{\mathcal{E}^0(\alpha,\beta)}{\alpha^2} = |\beta|(|\beta|-1)W(0).$$

Hence

$$\begin{aligned} \lim_{\alpha\to\infty} \frac{\mathcal{E}^0(\alpha)}{\alpha^2} &= N(N-1)W(0), \\ \lim_{\alpha\to\infty} \frac{\mathcal{E}^V(\alpha,\beta) + \mathcal{E}^0(\alpha,\beta^c)}{\alpha^2} &= \big\{(|\beta|(|\beta|-1) + |\beta^c|(|\beta^c|-1)\big\}W(0) \\ &= \big\{N(N-1) + 2|\beta|(|\beta|-N)\big\}W(0). \end{aligned}$$

Since $|\beta|(|\beta|-N) \leq -1$ and $W(0) < 0$ by (W2), we see that there exists $\alpha' > 0$ such that (3.3) holds for all α with $|\alpha| > \alpha'$. QED

Let $X = (x_1, ..., x_N)^t \in \mathbb{R}^{dN}$ and $Y := (x_c, y_1, \dots, y_{N-1})^t$ be its Jacobi coordinates:

$$x_c := \frac{1}{N}\sum_{j=1}^N x_j, \quad y_j := x_{j+1} - \frac{1}{j}\sum_{i=1}^j x_i, \quad j = 1, ..., N-1.$$

Let $T \in \mathrm{GL}(N,\mathbb{R})$ be such that $Y = TX$. Note that

$$T = \begin{bmatrix} \frac{1}{N} & \frac{1}{N} & \frac{1}{N} & \cdots & \cdots & \cdots & \frac{1}{N} \\ -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \ddots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \ddots & \cdots \\ -\frac{1}{N-1} & -\frac{1}{N-1} & -\frac{1}{N-1} & \cdots & \cdots & -\frac{1}{N-1} & 1 \end{bmatrix},$$

$$T^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} & \cdots & \cdots & -\frac{1}{N} \\ 1 & \frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{5} & \cdots & \cdots & -\frac{1}{N} \\ 1 & 0 & \frac{2}{3} & -\frac{1}{4} & -\frac{1}{5} & \cdots & \cdots & -\frac{1}{N} \\ 1 & 0 & 0 & \frac{3}{4} & -\frac{1}{5} & -\frac{1}{6} & \cdots & -\frac{1}{N} \\ \vdots & \vdots & \cdots & \cdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & \frac{N-2}{N-1} & -\frac{1}{N} \\ 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{N-1}{N} \end{bmatrix}.$$

T induces the unitary operator $U : L^2(\mathbb{R}^{dN}_X) \to L^2(\mathbb{R}^{dN}_Y)$ defined by $(U\psi)(Y) := \psi(T^{-1}Y)$. We have

$$Uh^0(\alpha)U^{-1} = -\frac{1}{2Nm}\Delta_{x_c} - \sum_{j=1}^{N}\frac{1}{2\mu_j}\Delta_{y_j} + \alpha^2\sum_{j\neq l}^{N} W(x_j(Y)-x_l(Y)),$$

$$Uh^V(\alpha)U^{-1} = Uh^0(\alpha)U^{-1} + \sum_{j=1}^{N} V(x_j(Y)),$$

where $\mu_j := jm/(j+1)$ is a reduced mass and $x_j(Y) := (T^{-1}Y)_j$. Let $k(\alpha)$ be $h^0(\alpha)$ with the center of mass motion removed:

$$k(\alpha) := -\sum_{j=1}^{N}\frac{1}{2\mu_j}\Delta_{y_j} + \alpha^2\sum_{j\neq l}^{N} W(x_j(Y)-x_l(Y)).$$

Set $\mathbb{R}^{dN} = \mathbb{R}^d_{x_c} \oplus \mathbb{R}^{d(N-1)}_{y_1,\dots,y_{N-1}} := \chi_c \oplus \chi_c^\perp$. Since $x_j(Y) - x_i(Y)$, $i, j = 1, ..., N-1$, depend only on $y_1, \dots, y_{N-1} \in \chi_c^\perp$, $k(\alpha)$ is a self-adjoint operator acting on $L^2(\chi_c^\perp)$.

Lemma 3.4 *There exists $\alpha'' > 0$ such that* $\inf\sigma(k(\alpha)) \in \sigma_{\rm disc}(k(\alpha))$ *for all α with $|\alpha| > \alpha''$.*

Proof: Assume that $\lim_{|x|\to\infty} W(x) = 0$. Let $\chi, \bar\chi \in C^\infty(\mathbb{R})$ be such that $\chi(x)^2 + \bar\chi(x)^2 = 1$ with $\chi(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 2. \end{cases}$ For a parameter R, we set

$$\chi_R(y_1) := \chi(|y_1|/R), \quad \bar\chi_R(y_1) := \bar\chi(|y_1|/R), \quad y_1 \in \mathbb{R}^d,$$
$$\theta_R(Y_1) := \chi(|Y_1|/2R), \quad \bar\theta_R(Y_1) := \bar\chi(|Y_1|/2R), \quad Y_1 := (y_2, \dots, y_{N-1}) \in \mathbb{R}^{d(N-2)}.$$

By the IMS localization formula, we have

$$\begin{aligned} k(\alpha) \;=\; & \chi_R\theta_R k(\alpha)\theta_R\chi_R + \chi_R\bar\theta_R k(\alpha)\bar\theta_R\chi_R + \bar\chi_R k(\alpha)\bar\chi_R \\ & \underbrace{-\frac{1}{2}\chi_R^2|\nabla\theta_R|^2 - \frac{1}{2}\chi_R^2|\nabla\bar\theta_R|^2 - \frac{1}{2}|\nabla\chi_R|^2 - \frac{1}{2}|\nabla\bar\chi_R|^2}_{=B(R)}. \end{aligned} \tag{3.4}$$

Here $B(R)$ is a bounded operator with

$$\|B(R)\| \le \frac{C}{R^2},$$

where C is a constant independent of R. Since $\chi_R^2\theta_R^2\alpha^2\sum_{j\neq l}^{N} W(x_j(Y) - x_l(Y))$ is relatively compact with respect to $-\sum_{j=1}^{N}(2\mu_j)^{-1}\Delta_{y_j}$, we have $\sigma_{\rm ess}(k(\alpha)) = \sigma_{\rm ess}(k'(\alpha))$, where

$$k'(\alpha) = \chi_R\theta_R\Big(-\sum_{j=1}^{N}\frac{1}{2\mu_j}\Delta_{y_j}\Big)\theta_R\chi_R + \chi_R\bar\theta_R k(\alpha)\bar\theta_R\chi_R + \bar\chi_R k(\alpha)\bar\chi_R + B(R).$$

We have

$$\begin{aligned}
k'(\alpha) \geq & \chi_R^2 \bar{\theta}_R^2 E\left(k(\alpha) - \alpha^2 W(x_2(Y) - x_3(Y)) - \alpha^2 W(x_3(Y) - x_2(Y))\right) && (3.5)\\
& + \chi_R^2 \bar{\theta}_R^2 \alpha^2 [W(x_2(Y) - x_3(Y)) + W(x_3(Y) - x_2(Y))] && (3.6)\\
& + \bar{\chi}_R^2 E\left(k(\alpha) - \alpha^2 W(x_1(Y) - x_2(Y)) - \alpha^2 W(x_2(Y) - x_1(Y))\right) && (3.7)\\
& + \bar{\chi}_R^2 \alpha^2 [W(x_1(Y) - x_2(Y)) + W(x_2(Y) - x_1(Y))] && (3.8)\\
& - C/R^2. && (3.9)
\end{aligned}$$

Note that $y_1 = x_2(Y) - x_1(Y)$ and $x_3(Y) - x_2(Y) = y_2 - y_1/2$. We have

$$\begin{aligned}
|(3.6)| &\leq 2 \sup_{\substack{y_1, y_2 \\ |y_1|<2R,\, |y_2|>4R}} \alpha^2 |W(y_2 - y_1/2)| \leq 2\alpha^2 \sup_{|y|>3R} |W(y)|,\\
|(3.8)| &\leq 2 \sup_{|y_1|>2R} \alpha^2 |W(y_1)|.
\end{aligned}$$

Since we assume that $\lim_{|x|\to\infty} W(x) = 0$, we obtain that $\lim_{R\to\infty} \|(3.6)\| = 0$ and $\lim_{R\to\infty} \|(3.8)\| = 0$. Thus, for all $R > 0$ we have

$$\begin{aligned}
\inf \sigma_{\rm ess}(k(\alpha)) &\geq \inf_{Y\in\mathbb{R}^{d(N-1)}} [(3.5) + (3.7)] - \|(3.6)\| - \|(3.8)\| - C/R^2\\
&\geq \min\{E(k(\alpha) - \alpha^2 W(x_1 - x_2) - \alpha^2 W(x_2 - x_1)),\\
&\qquad E(k(\alpha) - \alpha^2 W(x_2 - x_3) - \alpha^2 W(x_3 - x_2))\} + o(R), \quad (3.10)
\end{aligned}$$

where $\lim_{R\to\infty} o(R)/R = 0$. It is seen that

$$\lim_{\alpha\to\infty} \frac{E(k(\alpha) - \alpha^2 W(x_1 - x_2) - \alpha^2 W(x_2 - x_1))}{\alpha^2} = [N(N-1) - 2]W(0), \quad (3.11)$$

$$\lim_{\alpha\to\infty} \frac{E(k(\alpha) - \alpha^2 W(x_2 - x_3) - \alpha^2 W(x_3 - x_2))}{\alpha^2} = [N(N-1) - 2]W(0), \quad (3.12)$$

$$\lim_{\alpha\to\infty} \frac{E(k(\alpha))}{\alpha^2} = N(N-1)W(0). \quad (3.13)$$

By (W2), we have $W(0) < 0$. Therefore combining (3.10)-(3.13) we see that there exists $\alpha'' > 0$ such that $\inf \sigma_{\rm ess}(k(\alpha)) - \inf\sigma(k(\alpha)) > 0$ for $|\alpha| > \alpha''$. This implies the desired result. QED

Lemma 3.5 *Let u_α be a normalized ground state of $k(\alpha)$, where $|\alpha| > \alpha''$. Then $|u_\alpha(y_1, \ldots, y_{N-1})|^2 \to \delta(y_1) \cdots \delta(y_{N-1})$ as $\alpha \to \infty$ in the sense of distributions.*

Proof: It suffices to show that for all $\epsilon > 0$,

$$\lim_{\alpha\to\infty} \int_{|Y_0|>\epsilon} |u_\alpha(Y_0)|^2 dY_0 = 0, \quad Y_0 = (y_1, \ldots, y_{N-1}). \quad (3.14)$$

We prove (3.14) by a reductive absurdity. Assume that $\liminf_{\ell\to\infty}\int_{|Y_0|>\epsilon}|u_{\alpha_\ell}(Y_0)|^2dY_0>0$ for some constant $\epsilon>0$ and some sequence $\{\alpha_\ell\}_{\ell=1}^\infty\subset\mathbb{R}$ such that $\alpha_\ell\to\infty(\ell\to\infty)$. We can take a subsequence $\{\hat\alpha_\ell\}_{\ell=1}^\infty\subset\{\alpha_\ell\}_{\ell=1}^\infty$ so that

$$\gamma:=\lim_{\ell\to\infty}\int_{|Y_0|>\epsilon}|u_{\hat\alpha_\ell}(Y_0)|^2dY_0>0.$$

Since $k(\alpha)/\alpha^2\geq N(N-1)W(0)$ and $\lim_{\alpha\to\infty}E(k(\alpha)/\alpha^2)=N(N-1)W(0)$, we have

$$\begin{aligned}N(N-1)W(0)&=\lim_{\ell\to\infty}\frac{1}{\hat\alpha_\ell^2}(u_{\hat\alpha_\ell},k(\hat\alpha_\ell)u_{\hat\alpha_\ell})=\lim_{\ell\to\infty}(u_{\hat\alpha_\ell},\sum_{j\neq l}^N W(x_j(Y_0)-x_l(Y_0))u_{\hat\alpha_\ell})\\&\geq(1-\gamma)N(N-1)W(0)+\gamma\inf_{|Y_0|>\epsilon}\sum_{j\neq l}^N W(x_j(Y_0)-x_l(Y_0))\\&\geq N(N-1)W(0).\end{aligned}$$

Thus we have

$$\inf_{|Y_0|>\epsilon}\sum_{j\neq l}^N W(x_j(Y_0)-x_l(Y_0))=N(N-1)W(0). \tag{3.15}$$

By (W2) and (3.15) there exists a sequence $Z_n=(z_{1,n},\ldots,z_{(N-1),n})\in\mathbb{R}^{d(N-1)}$ such that $|Z_n|>\epsilon$ and $\lim_{n\to\infty}(x_j(Z_n)-x_l(Z_n))\to 0$ for $j\neq l$. By the definition of $x_j(Y)$, we have

$$\begin{aligned}&\lim_{n\to\infty}(x_2(Z_n)-x_1(Z_n))=\lim_{n\to\infty}z_{1,n}=0,\\&\lim_{n\to\infty}(x_3(Z_n)-x_2(Z_n))=\lim_{n\to\infty}(z_{2,n}-\frac{1}{2}z_{1,n})=\lim_{n\to\infty}z_{2,n}=0,\\&\qquad\cdots\\&\lim_{n\to\infty}(x_N(Z_n)-x_{N-1}(Z_n))=\lim_{n\to\infty}z_{N-1,n}=0.\end{aligned}$$

This is a contradiction to $|Z_n|>\epsilon>0$ for all n. QED

Proof of Theorem 3.2

Let u_α be a ground state of $k(\alpha)=Uh^0(\alpha)U^{-1}$. By Proposition A.3, we may assume that $V\in C_0^\infty(\mathbb{R}^d)$. Let $|\alpha|>\alpha''$. Let $v\in C_0^\infty(\mathbb{R}^d)$ be a normalized vector such that

$$(v,(-\frac{1}{2Nm}\Delta_{x_c}+NV(x_c))v)<0. \tag{3.16}$$

Such a vector exists by $(W3)$. We set $\Psi(Y)=\Psi(x_c,Y_0):=v(x_c)u_\alpha(Y_0)$ for $Y=(x_c,Y_0)=(x_c,y_1,\ldots,y_{N-1})\in\mathbb{R}^{dN}$. Then

$$(\Psi,Uh^V(\alpha)U^{-1}\Psi)=-\frac{1}{2mN}(v,\Delta_{x_c}v)+\mathcal{E}^0(\alpha)+(\Psi,\sum_{j=1}^N V(x_j(Y))\Psi). \tag{3.17}$$

We define

$$V^{\alpha}_{j,\mathrm{smeared}}(x_c) := \int_{\mathbb{R}^{d(N-1)}} dy_1 \cdots dy_{N-1} V(x_j(Y))|u_\alpha(y_1,\ldots,y_{N-1})|^2, \quad j=1,\ldots,N.$$

By Lemma 3.5, we have

$$\lim_{\alpha\to\infty}(\Psi, \sum_{j=1}^N V(x_j(Y))\Psi) = \lim_{\alpha\to\infty}\sum_{j=1}^N (v, V^{\alpha}_{j,\mathrm{smeared}} v) = (v, NV(x_c)v).$$

Therefore, by (3.16) and (3.17), $(\Psi, h^V(\alpha)\Psi) < \mathcal{E}^0(\alpha)$ for $|\alpha| > \alpha'''$ with some $\alpha''' > 0$. By this inequality, Lemma 3.3 and (3.2), we conclude that for α with $|\alpha| > \alpha_c := \max\{\alpha', \alpha'''\}$, $\Xi^V(\alpha) - \mathcal{E}^V(\alpha) \geq \mathcal{E}^0(\alpha) - \mathcal{E}^V(\alpha) > 0$. Then the theorem follows. QED

A The bottom of an essential spectrum

We give a general lemma.

Lemma A.1 *Let K_ϵ, $\epsilon > 0$, and K be self-adjoint operators on a Hilbert space $\mathcal{K}$ and $\sigma_{\mathrm{ess}}(K_\epsilon) = [\xi_\epsilon, \infty)$. Suppose that $\lim_{\epsilon\to 0} K_\epsilon = K$ in the uniform resolvent sense, and $\lim_{\epsilon\to 0}\xi_\epsilon = \xi$. Then $\sigma_{\mathrm{ess}}(K) = [\xi, \infty)$. In particular $\lim_{\epsilon\to 0}\inf\sigma_{\mathrm{ess}}(K_\epsilon) = \inf\sigma_{\mathrm{ess}}(K)$.*

Proof: Let $a > \xi$. Then there exists ϵ_0 such that for all ϵ with $\epsilon < \epsilon_0$, $\xi_\epsilon < a$, from which we have $a \in \sigma(K_\epsilon)$ for all $\epsilon < \epsilon_0$. Since K_ϵ uniformly converges to K in the resolvent sense, $a \in \sigma(K)$ follows from [16, Theorem VIII.23 and p.291]. Since a is arbitrary, $(\xi, \infty) \subset \sigma(K)$ follows and then $[\xi, \infty) \subset \sigma_{\mathrm{ess}}(K)$. It is enough to show $\inf\sigma_{\mathrm{ess}}(K) = \xi$. Let $\lambda \in [\inf\sigma_{\mathrm{ess}}(K), \xi)$ but $\lambda \notin \sigma(K)$. Note that for all sufficiently small ϵ, $\lambda \notin \sigma(K_\epsilon)$ by [16, Theorem VIII.24]. Since $\mathbb{R}\setminus\sigma(K)$ is an open set, there exists $\delta > 0$ such that $(\lambda-\delta, \lambda+\delta) \not\subset \sigma(K)$. Let $P_A(T)$ denote the spectral projection of a self-adjoint operator T on a Borel set $A \subset \mathbb{R}$. We have $\lim_{\epsilon\to 0} P_{(\inf\sigma_{\mathrm{ess}}(K)-\delta',\lambda)}(K_\epsilon) = P_{(\inf\sigma_{\mathrm{ess}}(K)-\delta',\lambda)}(K)$ uniformly by [16, Theorem VIII.23 (b)]. In particular, for some $\delta' > 0$,

$$\|P_{(\inf\sigma_{\mathrm{ess}}(K)-\delta',\lambda)}(K_\epsilon) - P_{(\inf\sigma_{\mathrm{ess}}(K)-\delta',\lambda)}(K)\| < 1,$$

which implies that $P_{(\inf\sigma_{\mathrm{ess}}(K)-\delta',\lambda)}(K_\epsilon)\mathcal{K}$ is isomorphic to $P_{(\inf\sigma_{\mathrm{ess}}(K)-\delta',\lambda)}(K)\mathcal{K}$, and then $P_{(\inf\sigma_{\mathrm{ess}}(K)-\delta',\lambda)}(K)\mathcal{K}$ is a finite dimensional space, since that of $P_{(\inf\sigma_{\mathrm{ess}}(K_\epsilon)-\delta',\lambda)}(K)\mathcal{K}$ is finite. Thus $(\inf\sigma_{\mathrm{ess}}(K) - \delta', \lambda) \cap \sigma(K) \subset \sigma_{\mathrm{disc}}(K)$. This is a contradiction. Hence we have $[\inf\sigma_{\mathrm{ess}}(K), \xi) \subset \sigma(K)$. Suppose that $\inf\sigma_{\mathrm{ess}}(K) < \xi$. Let $\tau > 0$ be sufficiently small. Note that $(\inf\sigma_{\mathrm{ess}}(K) - \tau, \inf\sigma_{\mathrm{ess}}(K) + \tau) \subset \sigma_{\mathrm{disc}}(K_\epsilon)$ for all sufficiently small ϵ. Let $\theta \in C_0^\infty(\mathbb{R})$ satisfy that

$$\theta(z) = \begin{cases} 1, & |z - \inf\sigma_{\mathrm{ess}}(K)| < \tau, \\ 0, & |z - \inf\sigma_{\mathrm{ess}}(K)| > 2\tau. \end{cases}$$

Then we have $\lim_{\epsilon\to 0}\theta(K_\epsilon) = \theta(K)$ uniformly by [16, Theorem VIII.20]. Since $\theta(K_\epsilon)$ is a finite rank operator for all sufficiently small ϵ, $\theta(K)$ has to be a compact operator. It contradicts with the fact, however, that the spectrum of $\theta(K)$ is continuous. Then we can conclude that $\inf\sigma_{\rm ess}(K) = \xi$ and the proof is complete. QED

Let $V : \mathbb{R}^d \to \mathbb{R}^d$ be a real measurable function.

Lemma A.2 *Let Δ be the d-dimensional Laplacian. Assume that $V(-\Delta+1)^{-1}$ is compact. Then there exists a sequence $\{V(\epsilon)\}_{\epsilon>0}$ such that $V(\epsilon) \in C_0^\infty(\mathbb{R}^d)$ and $\lim_{\epsilon\to 0} V(\epsilon)(-\triangle+1)^{-1} = V(-\Delta+1)$ uniformly.*

Proof: Generally, let A be a compact operator and $\{B_n\}_n$ bounded operators such that $\text{s-lim}_{n\to\infty} B_n = 0$, then $B_nA \to 0$ as $n \to \infty$ in the operator norm. Since $V(-\Delta+1)^{-1}$ is a compact operator, we obtain that for a sufficiently large $R > 0$,

$$\|(1-\chi_R)V(-\triangle+1)^{-1}\| < \epsilon/3, \tag{1.1}$$

where χ_R denotes the characteristic function of $\{x \in \mathbb{R}^d | |x| < R\}$. Let $\chi^{(n)}$ denote the characteristic function of $\{x \in \mathbb{R}^d | |V(x)| < n\}$. Since $(1-\chi^{(n)}) \to 0$ strongly as $n \to \infty$,

$$\|(1-\chi^{(n)})\chi_R V(-\triangle+1)^{-1}\| < \epsilon/3 \tag{1.2}$$

for a sufficiently large n. Since $C_0^\infty(\text{supp}(\chi_R\chi^{(n)}))$ is dense in $L^2(\text{supp}(\chi_R\chi^{(n)}))$, there exists a sequence $\{V_m\}_m \subset C_0^\infty(\text{supp}(\chi_R\chi^{(n)}))$ such that $\|V_m - \chi_R\chi^{(n)}V\|_{L^2(\mathbb{R}^d)} \to 0$ as $m \to \infty$. Since $\chi_R\chi^{(n)}V$ has a compact support and is bounded, we obtain that $\text{s-lim}_{m\to\infty} V_m = \chi_R\chi^{(n)}V$ as an operator. Thus for a sufficiently large m,

$$\|(V_m - \chi_R\chi^{(n)}V)(-\triangle+1)^{-1}\| < \epsilon/3. \tag{1.3}$$

By (1.1)-(1.3) we can obtain that for an arbitrary $\epsilon > 0$, $\|(V - V_m)(-\triangle+1)^{-1}\| < \epsilon$ for a sufficiently large m. Thus the lemma follows by setting $V_m = V(\epsilon)$. QED

Let $\beta \subset C_N$. Set

$$k^0(\beta) := -\sum_{j\in\beta}\frac{1}{2m_j}\Delta_j + \sum_{i,j\in\beta}V_{ij}, \quad k^V(\beta) := h^0(\beta) + \sum_{j\in\beta}V_j$$

with $V_i \in L^2_{\rm loc}(\mathbb{R}^d)$ and $V_{ij} \in L^2_{\rm loc}(\mathbb{R}^d)$ such that $V_i(-\triangle+1)^{-1}$ and $V_{ij}(-\triangle+1)^{-1}$ are compact operators. We define $K := k^V(C_N)$. Let

$$\Xi^V := \min_{\beta\subsetneqq C_N}\{\inf\sigma(k^0(\beta)) + \inf\sigma(k^V(\beta))\} \tag{1.4}$$

be the lowest two cluster threshold of K.

Proposition A.3 *There exist sequences* $\{V_i^\epsilon\}_\epsilon, \{V_{ij}^\epsilon\}_\epsilon \subset C_0^\infty(\mathbb{R}^d)$, $i, j = 1, ..., N$, *such that*

$$(1)\ \lim_{\epsilon\to 0} \Xi^V(\epsilon) = \Xi^V, \quad (2)\ \lim_{\epsilon\to 0} \inf\sigma_{\rm ess}(K(\epsilon)) = \inf\sigma_{\rm ess}(K),$$

where $\Xi^V(\epsilon)$ *(resp.* $K(\epsilon)$ *) is* Ξ^V *(resp.* K*) with* V_i *and* V_{ij} *replaced by* V_i^ϵ *and* V_{ij}^ϵ*, respectively.*

Proof: By Lemma A.2, there exist sequences $\{V_i^\epsilon\}_{\epsilon>0}, \{V_{ij}^\epsilon\}_{\epsilon>0} \subset C_0^\infty(\mathbb{R}^d)$, such that

$$V_i^\epsilon(x_i)(-\Delta_i + 1)^{-1} \to V_i(x_i)(-\Delta_i + 1)^{-1}$$

and

$$V_{ij}^\epsilon(x_i - x_j)(-\Delta_i - \Delta_j + 1)^{-1} \to V_{ij}(x_i - x_j)(-\Delta_i - \Delta_j + 1)^{-1}$$

uniformly as $\epsilon \to 0$ for $i, j = 1, ..., N$. Hence $\inf\sigma(k^V(\epsilon))$ and $\inf\sigma(k^0(\epsilon))$ converge to $\inf\sigma(k^V)$ and $\inf\sigma(k^0)$ as $\epsilon \to 0$, respectively. Then (1) follows from the definition (1.4). By this and the uniform convergence of $K(\epsilon)$ to K in the resolvent sense, Lemma A.1 yields (2). QED

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