Polynomial mixing for the complex Ginzburg–Landau equation perturbed by a random force at random times

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Abstract. In this paper we study the problem of ergodicity for the complex Ginzburg–Landau (CGL) equation perturbed by an unbounded random kick-force. Randomness is introduced both through the kicks and through the times between the kicks. We show that the Markov process associated with the equation in question possesses a unique stationary distribution and satisfies a property of polynomial mixing.

1 Introduction

We consider the CGL equation perturbed by a random kick-force on a domain $D \Subset \mathbb{R}^n$, $n \leq 4$ with $\partial D \in C^2$:

$$\dot{u} - \nu\Delta u + i\beta|u|^2 u = \eta(t,x), \quad x \in D, \tag{1.1}$$

$$u|_{\partial D} = 0, \tag{1.2}$$

$$u(0,x) = u_0(x), \tag{1.3}$$

where $u = u(t,x)$ and $\nu, \beta > 0$. We assume that $\eta(t,x)$ is a random process of the form

$$\eta(t,x) = \sum_{k=1}^{\infty} \eta_k(x)\delta(t-\tau_k), \tag{1.4}$$

where $\delta(t)$ is the Dirac measure, η_k are independent identically distributed (i.i.d.) random variables with range in the space $H := H_0^1(D)$, and the waiting times $t_k = \tau_k - \tau_{k-1}$, $k \geq 2$ and $t_1 = \tau_1$ are i.i.d. random variables exponentially distributed with parameter λ. Moreover, we assume that the sequences η_k, t_k are independent.

Suppose that $\{g_k\}_{k=1}^{\infty}$ is an orthonormal basis in H. The main result of the present paper is Theorem 4.2, which states that, if the low of η_k is non-degenerate on the space spanned by $\{g_k\}_{k=1}^{N}$ for sufficiently large N, then there is a unique stationary measure for the continuous time Markov process associated with (1.1), (1.2), (1.4). Moreover, any solution of the problem polynomially converges to the stationary measure in the dual Lipschitz norm.

Many authors have studied similar problems for various PDE's with different random perturbations (e.g., see [14, 1, 15, 16, 19, 13, 20, 12, 23] for discrete forcing and [5, 7, 2, 17, 6, 8, 22, 24, 3] for white noise). Several ideas of this article are taken from [16, 13, 23].

The problem of ergodicity for randomly forced Ginzburg–Landau equation was studied in the following articles. In [8], Hairer considered a real Ginzburg–Landau equation on multidimensional torus. Odasso [22] studied a class of CGL equations with strong nonlinear dissipation. In both of these works the property of exponential mixing is established. In [24], Shirikyan used a sufficient condition for ergodicity of Markov processes to show uniqueness and mixing for a class of CGL equations with linear dispersion. Finally, in [3], Debussche and Odasso proved the polynomial mixing property for a damped 1D Schrödinger equation.

The main novelty of the present paper is the condition over the waiting times. Note that the restriction of the solution at times τ_k looks like the random dynamical systems considered by Kuksin, Shirikyan [12],[14], [23] and Masmoudi, Young [19] :

$$u_{\tau_k} = S_{t_k}(u_{\tau_{k-1}}) + \eta_k, \tag{1.5}$$

but there are some essential differences. As the waiting times can be arbitrarily small, during any time interval the system can receive any number of kicks. This changes the dynamics of the associated process, for example:

- The distance between two trajectories having close initial data can be arbitrary large at any time $t > 0$.
- The phase space of the problem is not bounded even in the case of bounded kicks.

Let us give in a few words the ideas of the proof of Theorem 4.2. An important tool for the proof of the result is the Foiaş–Prodi type estimate. This kind of estimates are often used to prove ergodic properties of PDE's. Suppose that there are two sequences of kicks ζ_k and ζ'_k, having equal high Fourier modes for $k \geq l$, such that the solutions of corresponding problems have equal low Fourier modes at kicking times τ_k, $k \geq l$ (see Lemma 2.1 for the exact formulation). Let $\mathcal{N}_t$ be the number of kicks before time t, i.e. $\mathcal{N}_t = \max\{k : \tau_k \leq t\}$. Then, by Foiaş–Prodi Lemma, we have the following estimate for the distence between solutions at time t, if $t \geq \tau_l$:

$$\|u_t - u'_t\|_1 \leq e^{-C(\mathcal{N}_t - l)} \Big(\prod_{i=l+1}^{\mathcal{N}_t} t_i \Big)^{-\frac{1}{2}} e^{\varphi} \|u_{\tau_l} - u'_{\tau_l}\|_1, \tag{1.6}$$

where $\|\cdot\|_1$ stands for the norm in H, u_t and u'_t are solutions corresponding to the sequences ζ_k and ζ'_k respectively, φ is a polynomial function of $\{\|u_{\tau_i}\|_1\}_{i=l}^{\mathcal{N}_t}$

and $\{\|u'_{\tau_i}\|_1\}_{i=l}^{\mathcal{N}_t}$ and $C > 0$ is a large constant. Following the ideas from [23], we construct two sequences ζ_k and ζ'_k of i.i.d. random variables in H distributed as η_1 such that the conditions of Foiaş–Prodi Lemma are satisfied for a random integer $\ell \geq 1$. Moreover, using the low of large numbers and some martingale inequalities, we show that ℓ can be choosen in a such way that the following properties also hold:

(i) $\left(\prod_{i=\ell+1}^{\mathcal{N}_t} t_i\right)^{-\frac{1}{2}} e^{\varphi} \leq e^{(\mathcal{N}_t-\ell)}$, if $\mathcal{N}_t \geq \ell + 1$,

(ii) $\|u_{\tau_\ell}\|_1 + \|u'_{\tau_\ell}\|_1 \leq 1$,

(iii) $\mathbb{E}\ell^p \leq C_p$.

As we show in Section 4, properties (i)-(iii) and (1.6) imply the polynomial mixing property.

The random variables ζ_k, ζ'_k and ℓ are constructed in Proposition 4.3. In Section 4, we show properly how Theorem 4.2 is derived from Proposition 4.3. The proof of Proposition 4.3 is carried out in Sections 5 and 6.

Note that an exponential estimate for the random variable ℓ in (iii) implies immediately the exponential mixing property for the system. Finally, using (i)-(iii), one can show that the embedded Markov chain u_{τ_k} also satisfies a property of polynomial mixing. The stationary measure of the original process and that of embedded chain are connected with the Khasminskii relation (see Section 4).

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Notation

Let $D \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $\{g_j\}_{j\in\mathbb{N}}$ be an orthonormal basis in H. Let H_N be the vector span of $\{g_1, ..., g_N\}$ and $H_N^\perp$ be its orthogonal complement in H. We denote by P_N and Q_N the orthogonal projections onto H_N and $H_N^\perp$ in H. Denote by $\{e_j\}_{j\in\mathbb{N}}$ the set of normalized eigenfunctions of the Dirichlet Laplacian with eigenvalues $\{\alpha_j\}_{j\in\mathbb{N}}$ and denote by Q'_N the orthogonal projection onto the closure of the vector span of $\{e_N, e_{N+1}, ...\}$ in $L^2(D)$.

Let $H^s(D)$, $s \geq 0$ be the Sobolev space of order s. We denote by $\|u\|_1 = \|\nabla u\|, \|u\|_2 = \|\Delta u\|$ the norms in the spaces $H_0^1(D)$ and $H_0^1(D) \cap H^2(D)$ respectively, where $\|\cdot\|$ stands for the norm in $L^2(D)$. For a Banach space X, we shall use the following notation.

$\mathcal{B}(X)$ is the σ-algebra of Borel subsets of X.

$C(X)$ is the space of real-valued continuous functions on X.

$C_b(X)$ is the space of bounded functions $f \in C(X)$.

$\mathcal{L}(X)$ is the space of functions $f \in C_b(X)$ such that

$$\|f\|_{\mathcal{L}} := \|f\|_\infty + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|} < +\infty.$$

$\mathcal{P}(X)$ is the set of probability measures on $(X, \mathcal{B}(X))$. If $\mu \in \mathcal{P}(X)$ and $f \in C_b(X)$, we set

$$(f, \mu) = \int_X f(u)\mu(\mathrm{d}u).$$

If $\mu_1, \mu_2 \in \mathcal{P}(X)$, we set

$$\|\mu_1 - \mu_2\|^*_{\mathcal{L}} = \sup\{|(f, \mu_1) - (f, \mu_2)| : f \in \mathcal{L}(X), \|f\|_{\mathcal{L}} \leq 1\},$$

$$\|\mu_1 - \mu_2\|_{var} = \sup\{|\mu_1(\Gamma) - \mu_2(\Gamma)| : \Gamma \in \mathcal{B}(X)\}.$$

For any $\Gamma_1, \Gamma_2 \in \mathcal{B}(X)$, with $\mathbb{P}(\Gamma_2) \neq 0$, denote

$$\mathbb{P}(\Gamma_1|\Gamma_2) = \frac{\mathbb{P}(\Gamma_1\Gamma_2)}{\mathbb{P}(\Gamma_2)}.$$

The distribution of a random variable ξ is denoted by $\mathcal{D}(\xi)$. We denote by C, C_k unessential positive constants.

2 Preliminaries

It is well known that problem (1.1)-(1.3) with $\eta \equiv 0$ and $u_0 \in H$ has a unique solution in the space $C(\mathbb{R}_+, H) \cap L^2_{\text{loc}}(\mathbb{R}_+, H^2(D))$. Let $S_t : H \to H$ be the resolving semi-group for that problem. Let $\tau_0 \equiv 0$ and define u_t by the relation

$$u_t = \begin{cases} S_{t-\tau_k}(u_{\tau_k}), & \text{if } t \in [\tau_k, \tau_{k+1}), \ \ k \geq 0, \\ S_{t_{k+1}}(u_{\tau_k}) + \eta_{k+1}, & \text{if } t = \tau_{k+1}. \end{cases}$$

Then u_t is the unique solution of problem (1.1)-(1.4). Clearly, u_t exists for all $t > 0$ with probability 1, as $\mathbb{P}\{\sum t_k = \infty\} = 1$. Let us define a continuous functional on H:

$$\mathcal{H}(u) = \int_D \Big(\alpha|\nabla u(x)|^2 + \frac{\beta}{4}|u(x)|^4\Big)\mathrm{d}x, \tag{2.1}$$

where α is a positive constant. If α is sufficiently small, we have the estimate

$$\mathcal{H}(S_t(u)) \leq e^{-at}\mathcal{H}(u), \quad t \geq 0, \tag{2.2}$$

where a is a positive constant, and there is a constant C such that

$$\|S_t(u) - S_t(v)\|_1 \leq C\exp(C(\|u\|_1^6 + \|v\|_1^6))\|u - v\|_1, \ \ t \geq 0, \tag{2.3}$$

$$\|S_t(u) - S_t(v)\|_2 \leq Ct^{-\frac{1}{2}}\exp(C(\|u\|_1^6 + \|v\|_1^6))\|u - v\|_1, \ \ t > 0, \tag{2.4}$$

where $u, v \in H$. The proof (2.2), (2.3) and (2.4) is carried out by standard methods and is given in the Appendix.
For any sequence a_k, $m \le k \le n$, we set

$$\langle a_k \rangle_m^n = \frac{1}{n-m+1} \sum_{k=m}^{n} a_k.$$

Suppose $u_k, u_k' \in H$ and $t_k > 0$ are arbitrary sequences. Define ζ_k and ζ_k' by the relations

$$u_k = S_{t_k}(u_{k-1}) + \zeta_k, \quad u_k' = S_{t_k}(u_{k-1}') + \zeta_k'. \tag{2.5}$$

Lemma 2.1. *Suppose that*

$$P_N u_k = P_N u_k', \quad Q_N \zeta_k = Q_N \zeta_k', \qquad l+1 \le k \le n, \tag{2.6}$$

$$e_j \in H_N, \qquad j = 1, ..., N' \tag{2.7}$$

for some $N' \ge 1$ and $N \ge 1$. Then

$$\begin{aligned} \|u_k - u_k'\|_1 \le (C\alpha_{N'+1}^{-\frac{1}{2}})^{k-l} \Big(\prod_{i=l+1}^{k} t_i \Big)^{-\frac{1}{2}} \\ \times \exp\big(C(k-l)(\langle \|u_i\|_1^6 \rangle_l^{k-1} + \langle \|u_i'\|_1^6 \rangle_l^{k-1})\big) \|u_l - u_l'\|_1, \end{aligned} \tag{2.8}$$

for $l \le k \le n$, where C is a positive constant not depending on u_k, u_k', n, l, N and N'.

Proof. Using (2.4), (2.6) and (2.7), we see that

$$\begin{aligned} \|u_k - u_k'\|_1 &= \|Q_N(u_k - u_k')\|_1 = \|Q_N(S_{t_k}(u_{k-1}) - S_{t_k}(u_{k-1}'))\|_1 \\ &\le \|Q_{N'}'(S_{t_k}(u_{k-1}) - S_{t_k}(u_{k-1}'))\|_1 \\ &\le \alpha_{N'+1}^{-\frac{1}{2}} \|S_{t_k}(u_{k-1}) - S_{t_k}(u_{k-1}')\|_2 \\ &\le C\alpha_{N'+1}^{-\frac{1}{2}} t_k^{-\frac{1}{2}} \exp\big(C(\|u_{k-1}\|_1^6 + \|u_{k-1}'\|_1^6)\big) \|u_{k-1} - u_{k-1}'\|_1. \end{aligned}$$

Iteration of this inequality results in (2.8). □

3 Markov chains associated with CGL equation and existence of stationary measures

Let u_0 be an H-valued random variable, independent of $\{\eta_k\}$ and $\{t_k\}$, and let u_t be the solution of problem (1.1)-(1.4). Denote by $\mathcal{F}_t$, $t \ge 0$ the σ-algebra generated by u_0 and $\{\zeta(s), 0 \le s \le t\}$, where

$$\zeta(s) = \sum_{k=1}^{\infty} I_{\{\tau_k \le s\}} \eta_k. \tag{3.1}$$

Lemma 3.1. *Under the above conditions, u_t is a homogeneous Markov process with respect to $\mathcal{F}_t$.*

The proof of this lemma is given in the Appendix. For any $u \in H$ and $\Gamma \in \mathcal{B}(H)$, we set $P_t(u,\Gamma) = \mathbb{P}\{u_t(u) \in \Gamma\}$. The Markov operators corresponding to the process u_t have the form

$$\mathfrak{P}_t f(u) = \int_H P_t(u, \mathrm{d}v) f(v), \quad \mathfrak{P}_t^* \mu(\Gamma) = \int_H P_t(u,\Gamma)\mu(\mathrm{d}u),$$

where $f \in C_b(H)$ and $\mu \in \mathcal{P}(H)$.

The strong Markov property implies that u_{τ_k} is a homogeneous Markov chain with respect to σ-algebra $\mathcal{G}_k$ generated by $\{\eta_n, t_n, 1 \le n \le k\}$. In what follows, we shall write u_k instead of u_{τ_k}; this will not lead to confusion.

Lemma 3.2. *(i) For any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that*

$$\mathcal{H}(u_k) \le (1+\varepsilon)^k e^{-a\tau_k} \mathcal{H}(u_0) + C_\varepsilon \sum_{l=1}^{k} e^{-a(\tau_k - \tau_l)} (1+\varepsilon)^{k-l} \mathcal{H}(\eta_l). \quad (3.2)$$

(ii) Let $\mathbb{E}\mathcal{H}(\eta_k)^p < \infty$ for some $p \ge 1$. Then

$$\mathbb{E}\mathcal{H}(u_k)^p \le \gamma^k \mathbb{E}\mathcal{H}(u_0)^p + \frac{C_p}{1-\gamma} \mathbb{E}\mathcal{H}(\eta_k)^p, \quad (3.3)$$

where $0 < \gamma < 1$ and $C_p > 0$ are some constants not depending on k.

Proof. Using (2.2), we obtain

$$\mathcal{H}(u_k) \le (1+\varepsilon) e^{-t_k a} \mathcal{H}(u_{k-1}) + C_\varepsilon \mathcal{H}(\eta_k).$$

Iteration of this inequality results in (3.2).
To prove (3.3), note that for any $\varepsilon > 0$ there is a constant $C_{p,\varepsilon}$ such that

$$\mathcal{H}(u_k)^p \le (1+\varepsilon) e^{-t_k a p} \mathcal{H}(u_{k-1})^p + C_{p,\varepsilon} \mathcal{H}(\eta_k)^p. \quad (3.4)$$

Taking the expectation and using the independence of t_k and u_{k-1}, we obtain

$$\mathbb{E}\mathcal{H}(u_k)^p \le (1+\varepsilon) \frac{\lambda}{\lambda + ap} \mathbb{E}\mathcal{H}(u_{k-1})^p + C_{p,\varepsilon} \mathbb{E}\mathcal{H}(\eta_k)^p.$$

Choosing $\varepsilon > 0$ so small that $\gamma := (1+\varepsilon)\frac{\lambda}{\lambda+ap} < 1$ and iterating the resulting inequality, we arrive at (3.3). ☐

Lemma 3.3. *Let $\mathbb{E}\|\eta_k\|_1^p < \infty$ for all $p \ge 1$ and let $u_0 \in H$. Then*

(i) There is a constant $M > 0$ not depending on u_0 and a random integer $T = T(u_0) \ge 1$ such that

$$\langle \|u_k\|_1^6 \rangle_0^n \le M \quad \text{for } n \ge T, \quad (3.5)$$

$$\mathbb{E}T^p < \infty \quad \text{for all } p \ge 1. \quad (3.6)$$

(ii) For any $\delta \in (0,1)$ and $d > 0$, there is a constant $R = R(\delta, d) > 0$ such that

$$\mathbb{P}\{\langle \|u_k\|_1^6 \rangle_0^n \le R, \forall n \ge 0\} \ge \delta, \tag{3.7}$$

for any $u_0 \in B_d$, where $B_d = \{u \in H : \|u\|_1 \le d\}$.

Proof. Let us fix $\varepsilon > 0$. Using (3.4) with $p = 3$, we obtain

$$\begin{aligned}
&\mathcal{H}(u_k)^3 \le (1+\varepsilon)e^{-3at_k}\mathcal{H}(u_{k-1})^3 + C_\varepsilon \mathcal{H}(\eta_k)^3 \\
&= (1+\varepsilon)\Big(e^{-3at_k} - \frac{\lambda}{\lambda+3a}\Big)\mathcal{H}(u_{k-1})^3 + (1+\varepsilon)\frac{\lambda}{\lambda+3a}\mathcal{H}(u_{k-1})^3 + C_\varepsilon \mathcal{H}(\eta_k)^3.
\end{aligned} \tag{3.8}$$

Choosing $\varepsilon > 0$ so small that $q := (1+\varepsilon)\frac{\lambda}{\lambda+3a} < 1$ and summing up inequalities (3.8) for $1 \le k \le n$, we arrive at

$$\begin{aligned}
\sum_{k=1}^n \mathcal{H}(u_k)^3 \le (1+\varepsilon)\sum_{k=1}^n \Big(e^{-3at_k} - \frac{\lambda}{\lambda+3a}\Big)\mathcal{H}(u_{k-1})^3 + q\sum_{k=1}^n \mathcal{H}(u_{k-1})^3 \\
+ C_\varepsilon \sum_{k=1}^n \mathcal{H}(\eta_k)^3,
\end{aligned}$$

whence it follows that

$$\begin{aligned}
\alpha^3 \langle \|u_k\|_1^6 \rangle_0^n \le \langle \mathcal{H}(u_k)^3 \rangle_0^n \le \frac{1+\varepsilon}{1-q}\frac{1}{n+1}\sum_{k=1}^n \Big(e^{-3at_k} - \frac{\lambda}{\lambda+3a}\Big)\mathcal{H}(u_{k-1})^3 \\
+ \frac{1}{1-q}\frac{\mathcal{H}(u_0)^3}{n+1} + \frac{C_\varepsilon}{1-q}\mathfrak{m} + \frac{C_\varepsilon}{1-q}\frac{1}{n+1}\sum_{k=1}^n (\mathcal{H}(\eta_k)^3 - \mathfrak{m}),
\end{aligned} \tag{3.9}$$

where $\mathfrak{m} = \mathbb{E}\mathcal{H}(\eta_k)^3$. To complete the proof, we need the following lemma, whose proof is given in the Appendix.

Lemma 3.4. *Suppose that M_k is a sequence of random variables that satisfies the inequality*

$$\mathbb{E}|M_k|^{2p} \le C_p k^p \qquad \text{for all } p \ge 1. \tag{3.10}$$

Then the following assertions take place.

(i) There is a random integer $T \ge 1$ such that

$$\frac{1}{k}|M_k| \le 1 \quad \text{for } k \ge T, \tag{3.11}$$

$$\mathbb{E}T^p < \infty \quad \text{for all } p \ge 1. \tag{3.12}$$

(ii) For any $\delta \in (0,1)$, there is a constant $R > 0$ such that

$$\mathbb{P}\{\frac{|M_k|}{k} \le R, \forall k \ge 1\} \ge \delta. \tag{3.13}$$

Let us set

$$M'_k = \sum_{i=1}^{k}\Big(e^{-3at_i} - \frac{\lambda}{\lambda+3a}\Big)\mathcal{H}(u_{i-1})^3,$$
$$M''_k = \sum_{i=1}^{k}(\mathcal{H}(\eta_i)^3 - \mathfrak{m}), \quad M'_0 = M''_0 = 0.$$

Clearly M'_k and M''_k are martingales. For M''_k it is easy to verify that (3.10) holds, as $M''_i - M''_{i-1} = \mathcal{H}(\eta_i)^3 - \mathfrak{m}$, $i \geq 1$ are centered i.i.d. random variables. To prove (3.10) for M'_k, we need Burkholder's inequality for martingales ([9], Section 2.4):

$$C_1\mathbb{E}\Big|\sum_{i=1}^{k} X_i^2\Big|^p \leq \mathbb{E}|M'_k|^{2p} \leq C_2\mathbb{E}\Big|\sum_{i=1}^{k} X_i^2\Big|^p, \tag{3.14}$$

where $X_i = M'_i - M'_{i-1}$, $i \geq 1$, $p \geq 1$ and C_1, C_2 are positive constants depending only on p. Using (3.14) and (3.3), we obtain

$$\begin{aligned}\mathbb{E}|M'_k|^{2p} &\leq C_2\mathbb{E}\Big|\sum_{i=1}^{k}\Big(e^{-3at_i} - \frac{\lambda}{\lambda+3a}\Big)^2\mathcal{H}(u_{i-1})^6\Big|^p \\ &\leq C\sum_{i=1}^{k}\mathbb{E}\mathcal{H}(u_{i-1})^{6p}k^{p-1} \leq C'k^p,\end{aligned}$$

where C' depends on $\|u_0\|_1$. Applying Lemma 3.4, let T' and T'' be the random variables corresponding to martingales M'_k and M''_k. Setting

$$T = T_1 \vee T_2 \vee \Big(\mathcal{H}(u_0)\frac{1}{1-q}\Big), \quad M = \Big(\frac{1+\varepsilon}{1-q} + \frac{C_\varepsilon(\mathfrak{m}+1)}{1-q} + 1\Big)\alpha^{-3},$$

it is easy to verify that we have (3.5) and (3.6) for T and M.
To prove (3.7), we apply (3.13) to the sequence

$$M_k = \frac{1+\varepsilon}{1-q}M'_k + \frac{C_\varepsilon}{1-q}M''_k,$$

and using (3.9), we see that (3.7) holds with

$$R_1 = R + C_d\frac{1}{1-q} + \frac{C_\varepsilon}{1-q}\mathfrak{m},$$

where $C_d = \sup_{u\in B_d}\mathcal{H}(u)^3$. □

Let τ_R be the first hitting time of the ball B_R:

$$\tau_R = \min\{k \geq 0 : \|u_k\|_1 \leq R\}.$$

Lemma 3.5. *Let $\mathbb{E}\mathcal{H}(\eta_1) < +\infty$. Then there are positive constants δ, C and R not depending on u such that*

$$\mathbb{E}_u e^{\delta\tau_R} \le C(1 + \mathcal{H}(u)).$$

Proof. It suffices to show that u_k possesses a Lyapunov function (see [21]), i.e. there is a continuous functional F on H such that

(i) $F(u) \ge 1$ and $\lim_{\|u\|_1 \to \infty} F(u) = +\infty$.

(ii) There are positive constants n, R', C' and $a < 1$ that

$$\mathbb{E}_u F(u_n) \le aF(u) \quad \text{for } \|u\|_1 \ge R', \tag{3.15}$$
$$\mathbb{E}_u F(u_k) \le C' \quad \text{for } \|u\|_1 < R',\ k \ge 0. \tag{3.16}$$

Let

$$F(u) = \begin{cases} \mathcal{H}(u), & \text{if } \mathcal{H}(u) \ge A, \\ A, & \text{if } \mathcal{H}(u) < A, \end{cases}$$

where $A \ge 1$. Then (i) is satisfied. Let $\|u\|_1 \ge R'$. Note that

$$\begin{aligned} \mathbb{E}_u F(u_n) &= \mathbb{E}_u F(u_n) I_{\{\mathcal{H}(u_n) < A\}} + \mathbb{E}_u F(u_n) I_{\{\mathcal{H}(u_n) \ge A\}} \\ &\le A + \mathbb{E}_u \mathcal{H}(u_n) \le \gamma^n \mathcal{H}(u) + A + C\mathbb{E}\mathcal{H}(\eta_1), \end{aligned} \tag{3.17}$$

where we used (3.3). Choosing n and R' so large that $2\gamma^n < 1$ and $A + C\mathbb{E}\mathcal{H}(\eta_1) \le \gamma^n R'^2 \alpha$, where α is the constant in (2.1), we arrive at (3.15) with $a = 2\gamma^n$. It remains to note that (3.16) follows from (3.3). □

Definition 3.6. *A measure $\mu \in \mathcal{P}(H)$ is said to be stationary for problem (1.1), (1.2), (1.4), if $\mathfrak{P}_t^* \mu = \mu$ for any $t \ge 0$.*

Using the classical Bogolyubov-Krylov argument and Fatou's lemma, one can prove the following theorem. Its proof is outlined in the Appendix.

Theorem 3.7. *Let $\mathbb{E}\mathcal{H}(\eta_k) < \infty$, then problem (1.1), (1.2), (1.4) has at least one stationary measure. Moreover, if $\mathbb{E}\mathcal{H}(\eta_k)^p < \infty$ for some $p \ge 1$, then for any stationary measure μ we have:*

$$\mathcal{H}_p(\mu) := \int_H \mathcal{H}(u)^p \mu(\mathrm{d}u) < +\infty.$$

We denote by $\mathcal{P}_1(H)$ the set of measures $\mu \in \mathcal{P}(H)$ such that $\mathcal{H}(\mu) := \mathcal{H}_1(\mu) < +\infty$.

4 Main result

To show the uniqueness of stationary measure for (1.1), (1.2), (1.4), we shall need the following condition satisfied for η_k:

Condition 4.1. *The random variables* η_k *are i.i.d. and have the form*

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} g_j(x),$$

where $\{g_i\}_{i\in\mathbb{N}}$ *is an orthonormal basis in* H, $b_j \geq 0$ *are some constants with*

$$B := \sum_{j=1}^{\infty} b_j^2 < \infty,$$

and ξ_{jk} *are independent scalar random variables. Moreover, the distribution of* ξ_{jk} *possesses a density* $p_j(r)$ *(with respect to the Lebesgue measure), which is a function of bounded variation such that*

$$\int_{-\varepsilon}^{\varepsilon} p_j(r)\mathrm{d}r > 0, \quad \int_{-\infty}^{+\infty} |r|^p p_j(r)\mathrm{d}r \leq C_p < \infty, \tag{4.1}$$

for all $\varepsilon > 0$, $p \geq 1$, $j \geq 1$ *and for some constants* $C_p > 0$.

Clearly, if Condition 4.1 is satisfied, then

$$\mathbb{E}\|\eta_k\|_1^p < \infty \quad \text{for all } k \geq 1,\ p \geq 1. \tag{4.2}$$

Theorem 4.2. *Suppose that Condition 4.1 is satisfied. For any* $B > 0$ *there is an integer* $N' \geq 1$ *such that, if*

$$e_j \in H_N, \quad j = 1, ..., N' \tag{4.3}$$

for some $N \geq 1$, *and*

$$b_j \neq 0, \qquad j = 1, ..., N, \tag{4.4}$$

then there is a unique stationary measure $\mu \in \mathcal{P}(H)$. *Moreover, for any initial measure* $\mu' \in \mathcal{P}_1(H)$ *we have*

$$\|\mathfrak{P}_t^*\mu' - \mu\|_{\mathcal{L}}^* \leq C_p(1 + \mathcal{H}(\mu'))t^{-p}, \quad t > 0, \tag{4.5}$$

where C_p *is a constant not depending on* μ'.

Proof. **Step 1.** It suffices to show that for any $u, u' \in H$ we have

$$|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(u')| \leq C_p\|f\|_{\mathcal{L}}(1 + \mathcal{H}(u) + \mathcal{H}(u'))t^{-p}, \tag{4.6}$$

for any $p \geq 1$, $t > 0$ and some constant $C_p > 0$ not depending on (u, u') and t. Indeed, suppose that (4.6) is already proved. Then for any two initial measures $\mu', \mu'' \in \mathcal{P}_1(H)$ we derive from (4.6):

$$\|\mathfrak{P}_t^*\mu' - \mathfrak{P}_t^*\mu''\|_{\mathcal{L}}^* \leq C_p(1 + \mathcal{H}(\mu') + \mathcal{H}(\mu''))t^{-p}. \tag{4.7}$$

This inequality shows the uniqueness of stationary measure in $\mathcal{P}_1(H)$. It follows from Theorem 3.7 that any stationary measure μ is in $\mathcal{P}_1(H)$. Taking $\mu'' = \mu$ in (4.7), we arrive at (4.5).

Step 2. Inequality (4.6) is a direct consequence of the following proposition.

Proposition 4.3. *Under the conditions of Theorem 4.2, for any $B > 0$ there is an integer $N' \geq 1$ such that, if (4.3) and (4.4) hold for some integer $N \geq 1$, then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sequence of i.i.d. random variables $\{t_k\}$ that are exponentially distributed with parameter λ such that for any $u, u' \in H$ one can construct random sequences u_k, u'_k defined on Ω with the following properties:*

(i) The initial value of the trajectory (u_k, u'_k) is (u, u'):

$$u_0 = u, \;\; u'_0 = u'.$$

Furthermore, the random variables ζ_k and ζ'_k defined by (2.5) are i.i.d., and their distribution coincides with that of η_k:

$$\mathcal{D}(\zeta_k) = \mathcal{D}(\zeta'_k) = \mathcal{D}(\eta_k).$$

(ii) There is a random integer $\ell = \ell(u, u')$ and a constant M depending only on B such that

$$P_N u_k = P_N u'_k \qquad \text{for } k \geq \ell + 1, \tag{4.8}$$

$$Q_N \zeta_k = Q_N \zeta'_k \qquad \text{for } k \geq 1, \tag{4.9}$$

$$\langle \|u_i\|_1^6 + \|u'_i\|_1^6 \rangle_\ell^k \leq M \qquad \text{for } k \geq \ell + 1, \tag{4.10}$$

$$\frac{1}{2} \frac{1}{k - \ell} \Big| \sum_{i=\ell+1}^{k} \log t_i \Big| \leq M \qquad \text{for } k \geq \ell + 1. \tag{4.11}$$

(iii) There is a positive constant C_p not depending on (u, u') such that

$$\mathbb{E}\ell^p \leq C_p(1 + \mathcal{H}(u) + \mathcal{H}(u')) \qquad \text{for all } p \geq 1, \tag{4.12}$$

$$\|u_\ell\|_1 \vee \|u'_\ell\|_1 \leq 1. \tag{4.13}$$

To prove (4.6), let u_k and u'_k be the random sequences constructed in Proposition 4.3 and corresponding to the initial value (u, u'). Let $\tau_k = \sum_{n=1}^{k} t_n, n \geq 1$ and $\tau_0 = 0$. Define

$$u_t = \begin{cases} S_{t-\tau_k}(u_k), & \text{if } t \in [\tau_k, \tau_{k+1}), \;\; k \geq 0, \\ S_{t_{k+1}}(u_k) + \zeta_{k+1}, & \text{if } t = \tau_{k+1}, \end{cases}$$

and u'_t is defined in a similar way. Clearly, u_t and u'_t have the same distributions as the solutions of (1.1)-(1.4) corresponding to u and u', respectively. Thus

$$|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(u')| = |\mathbb{E}(f(u_t) - f(u'_t))|. \tag{4.14}$$

Let

$$\mathcal{N}_t = \max\{k \geq 0 : \tau_k \leq t\}, \tag{4.15}$$

then $\mathcal{N}_t$ is a Poisson random variable with parameter λt (e.g., see [11]). Define $G_t = \{\omega : 2\ell + 1 \le \mathcal{N}_t\} = \{\omega : \tau_{2\ell+1} \le t\}$. As τ_k is a Gamma random variable with parameters λ and k (e.g., see [4]), we have

$$\begin{aligned}\mathbb{E}\tau_{2\ell+1}^q &\le \sum_{n=1}^{\infty} \mathbb{E}[\tau_{2n+1}^q I_{\{\ell=n\}}] \le \sum_{n=1}^{\infty} (E\tau_{2n+1}^{2q})^{\frac{1}{2}} \mathbb{P}\{\ell = n\}^{\frac{1}{2}} \\ &\le C(1 + \mathcal{H}(u) + \mathcal{H}(u')) \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \qquad (4.16)\end{aligned}$$

for any $q \ge 1$, where we used the Cauchy–Schwarz inequality and (4.12) with $p = 2(2 + q)$. It follows that

$$\mathbb{P}(G_t^c) \le C_p'(1 + \mathcal{H}(u) + \mathcal{H}(u'))t^{-p} \quad \text{for any } p \ge 1. \qquad (4.17)$$

Using (2.3), we see that

$$\|u_t - u_t'\|_1 \le C \exp(C(\|u_{\tau_{\mathcal{N}_t}}\|_1^6 + \|u'_{\tau_{\mathcal{N}_t}}\|_1^6))\|u_{\tau_{\mathcal{N}_t}} - u'_{\tau_{\mathcal{N}_t}}\|_1,$$

whence, using (4.8)-(4.11), (4.13) and Lemma 2.1, we obtain

$$\mathbb{E}[I_{G_t}\|u_t - u_t'\|_1] \le \mathbb{E}[2(C\alpha_{N'+1}^{-\frac{1}{2}})^{\mathcal{N}_t - \ell} e^{2CM(\mathcal{N}_t - \ell)}].$$

Choosing N' so large that $\log \alpha_{N'+1} \ge 2(2CM + \log C + 2)$, we arrive at

$$\mathbb{E}[I_{G_t}\|u_t - u_t'\|_1] \le \mathbb{E}e^{-\mathcal{N}_t} = e^{-ct}, \qquad (4.18)$$

where $c = \lambda - \frac{\lambda}{e}$. Let $f \in \mathcal{L}(H)$. Then, using (4.14), (4.17) and (4.18), we derive

$$\begin{aligned}|\mathfrak{P}_t f(u) - \mathfrak{P}_t f(u')| &\le \mathbb{E}|f(u_t) - f(u_t')| \\ &\le \|f\|_{\mathcal{L}}\mathbb{E}[I_{G_t}\|u_t - u_t'\|_1] + \mathbb{E}I_{G_t^c}2\|f\|_\infty \\ &\le \|f\|_{\mathcal{L}}e^{-ct} + 2\|f\|_{\mathcal{L}}C_p'(1 + \mathcal{H}(u) + \mathcal{H}(u'))t^{-p} \\ &\le C_p\|f\|_{\mathcal{L}}(1 + \mathcal{H}(u) + \mathcal{H}(u'))t^{-p}. \qquad (4.19)\end{aligned}$$

This completes the proof of (4.6). □

Remark 4.4. The embedded Markov chain u_{τ_k} also satisfies a property of polynomial mixing. This follows from Proposition 4.3 and is proved using the same arguments as in the proof of Theorem 4.2. The stationary measures of the original process and that of embedded chain are connected with the Khasminskii relation:

$$(f, \mu) = \frac{1}{\mathbb{E}_\nu \tau_1}\mathbb{E}_\nu \int_0^{\tau_1} f(u_t)\mathrm{d}t,$$

where ν and μ are the stationary measures of u_{τ_k} and u_t respectively.

5 Coupling operators

Let η_k be a sequence of random variables with range in H and suppose that Condition 4.1 is satisfied for η_k. Clearly, if $b_j \neq 0$, $j = 1, ..., N$, then the distribution of the random variable $P_N(\eta_1)$ is absolutely continuous with respect to the Lebesgue measure, and its density has the form

$$p(x) := \prod_{j=1}^{N} q_j(x_j), \quad q_j(x_j) = b_j^{-1} p_j(x_j b_j^{-1}), \quad x = (x_1, ..., x_N) \in H_N.$$

Now we have the following lemma, which is a version of Lemma 3.2 in [16]:

Lemma 5.1. *Suppose that Condition 4.1 is satisfied and $b_j \neq 0$ for $j = 1, ..., N$, where $N \geq 1$ is an integer. Then there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $u, u' \in H$ there are H-valued random variables $\zeta = \zeta(u, u', \omega)$, $\zeta' = \zeta'(u, u', \omega)$ and a real-valued random variable $t = t(\omega)$ with the following properties:*

(i) The random variables ζ, ζ' and η_1 have the same distributions, and t is exponentially distributed with parameter λ.

(ii) The random variables $(P_N\zeta, P_N\zeta')$ and $(Q_N\zeta, Q_N\zeta')$ are independent, and ζ and ζ' are independent of t.

(iii) The random variables $Q_N\zeta$ and $Q_N\zeta'$ are equal for all $\omega \in \Omega$ and do not depend on (u, u').

(iv) The random variables ζ and ζ' are measurable functions of $(u, u', \omega) \in H \times H \times \Omega$.

Proof. Suppose that $t_1 = t_1(\omega_1)$ is a random variable that is exponentially distributed with parameter λ and is defined on the space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$. Let (v, v') be a maximal coupling for $(\nu_{u,\omega_1}, \nu_{u',\omega_1})$, where ν_{u,ω_1} is a measure on H_N given by the density $p(x - P_N S_{t_1(\omega_1)}(u))$ (see [18], Section I, 5). By Theorem 4.2 in [16], we can assume that the random variables v and v' are defined on the same probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ for all $u, u' \in H$, $\omega_1 \in \Omega_1$ and are measurable functions of $(u, u', \omega_1, \omega_2) \in H \times H \times \Omega_1 \times \Omega_2$. Suppose that η_1 is defined on the space $(\Omega_3, \mathcal{F}_3, \mathbb{P}_3)$. We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the direct product of $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $i = 1, 2, 3$, and define ζ, ζ' and t by the relations:

$$\begin{aligned} t(\omega) &= t_1(\omega_1), \\ P_N\zeta(\omega) &= v(u, u', \omega_1, \omega_2) - P_N S_{t(\omega_1)}(u), \\ P_N\zeta'(\omega) &= v'(u, u', \omega_1, \omega_2) - P_N S_{t(\omega_1)}(u'), \\ Q_N\zeta(\omega) &= Q_N\zeta'(\omega) = Q_N\eta_1(\omega_3), \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega$. Using the definition of ζ and Fubini's theorem, we see that

$$\begin{aligned}\mathbb{P}\{P_N\zeta \in \Gamma\} &= \mathbb{E}I_{\{P_N\zeta\in\Gamma\}} = \mathbb{E}_1\mathbb{P}_2\{P_N\zeta(\omega_1) \in \Gamma\} \\ &= \mathbb{E}_1\mathbb{P}_2\{v - P_N S_{t_1}(u) \in \Gamma\} = \int_\Gamma p(x)\mathrm{d}x = \mathbb{P}\{P_N\eta_1 \in \Gamma\}, \quad (5.1)\end{aligned}$$

for any $\Gamma \in \mathcal{B}(H_N)$, where $\mathbb{E}_1$ is the expectation corresponding to the measure $\mathbb{P}_1$. All assertions of lemma follow from the construction and relation (5.1). $\square$

Remark 5.2. Using inequality (3.8) in Lemma 3.2, [16] for the variational distance between ν_{u,ω_1} and ν_{u',ω_1}, we obtain the inequality:

$$\|\nu_{u,\omega_1} - \nu_{u',\omega_1}\|_{var} \le C_N\|S_t(u) - S_t(u')\|_1,$$

which holds $\mathbb{P}_1$-a.s.. Then the definition of maximal coupling gives

$$\mathbb{P}_2\{v \ne v'\} \le C_N\|S_t(u) - S_t(u')\|_1. \quad (5.2)$$

Remark 5.3. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be the direct product of $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $i = 2, 3$. For any $\omega_1 \in \Omega_1$, let $E_{\omega_1} = \{\omega' \in \Omega' : v(u, u', \omega_1, \omega') \ne v'(u, u', \omega_1, \omega')\}$. As (v, v') is a maximal coupling for $(\nu_{u,\omega_1}, \nu_{u',\omega_1})$, we have

$$\begin{aligned}&\mathbb{P}'\{v(u, u', \omega_1, \cdot) \in \Gamma, v'(u, u', \omega_1, \cdot) \in \Gamma'|E_{\omega_1}\} \\ &= \mathbb{P}'\{v(u, u', \omega_1, \cdot) \in \Gamma|E_{\omega_1}\}\mathbb{P}'\{v'(u, u', \omega_1, \cdot) \in \Gamma'|E_{\omega_1}\},\end{aligned}$$

if $\mathbb{P}'\{E_{\omega_1}\} > 0$ and $\Gamma, \Gamma' \in \mathcal{B}(H)$. Now it is easy to notice that

$$\mathbb{P}'\{v_{\omega_1} \in \Gamma, v'_{\omega_1} \in \Gamma', E_{\omega_1}\} \ge \mathbb{P}'\{v_{\omega_1} \in \Gamma, E_{\omega_1}\}\mathbb{P}'\{v'_{\omega_1} \in \Gamma', E_{\omega_1}\}. \quad (5.3)$$

Let us define coupling operators by the formulas

$$\mathcal{R}(u, u', \omega) = S_{t(\omega)}(u) + \zeta(u, u', \omega), \;\; \mathcal{R}'(u, u', \omega) = S_{t(\omega)}(u') + \zeta'(u, u', \omega),$$

where $u, u' \in H$ and $\omega \in \Omega$.

Lemma 5.4. *Under the conditions of Lemma 5.1, there exists a constant $\gamma \in (0, 1)$ such that for any $r > 0$ and an appropriate constant $\varepsilon := \varepsilon(r) > 0$ we have*

$$P_r^1 := \mathbb{P}\{\mathcal{H}(\mathcal{R}(u, u', \cdot)) + \mathcal{H}(\mathcal{R}'(u, u', \cdot)) \le \big(\gamma(\mathcal{H}(u) + \mathcal{H}(u'))\big) \vee r\} > \varepsilon, \quad (5.4)$$

for all $u, u' \in H$.

Proof. **Step 1.** It suffices to show that there is $C > 0$ such that for any $\delta > 0$ and an appropriate constant $\varepsilon_\delta > 0$ the following inequality holds $\mathbb{P}_1$-a.s.:

$$\begin{aligned}P_\delta^2(\omega_1) := \mathbb{P}'\{&\mathcal{H}(\mathcal{R}(u, u', \omega_1, \cdot)) + \mathcal{H}(\mathcal{R}'(u, u', \omega_1, \cdot)) \\ &\le C(\mathcal{H}(S_{t(\omega_1)}(u)) + \mathcal{H}(S_{t(\omega_1)}(u'))) + \delta\} \ge \varepsilon_\delta. \quad (5.5)\end{aligned}$$

Indeed, define the event

$$V = \{e^{-at(\omega_1)} \leq (2C)^{-1}\}.$$

Then $\mathbb{P}_1(V) > 0$, as t is exponentially distributed. We deduce from (2.2):

$$\begin{aligned} C(\mathcal{H}(S_{t(\omega_1)}(u)) + \mathcal{H}(S_{t(\omega_1)}(u'))) + \delta &\leq Ce^{-at(\omega_1)}(\mathcal{H}(u) + \mathcal{H}(u')) + \delta \\ &\leq \frac{1}{2}(\mathcal{H}(u) + \mathcal{H}(u')) + \delta, \end{aligned}$$

if $\omega_1 \in V$. Setting $\gamma = \frac{3}{4}$ and $\delta = \frac{r}{3}$, we see that

$$\frac{1}{2}(\mathcal{H}(u) + \mathcal{H}(u')) + \delta \leq \big(\gamma(\mathcal{H}(u) + \mathcal{H}(u'))\big) \vee r.$$

Combining this with (5.5), we obtain $P_r^1 \geq \mathbb{E}_1 P_\delta^2(\omega_1) I_V(\omega_1) \geq \varepsilon_\delta \mathbb{P}(V)$.
Step 2. Let us fix arbitrary $\delta > 0$ and $\omega_1 \in \Omega_1$. Suppose that

$$\mathcal{H}(P_N S_{t(\omega_1)}(u)) \leq \mathcal{H}(P_N S_{t(\omega_1)}(u')) \tag{5.6}$$

(the proof of the other case is similar). Define the events

$$\begin{aligned} A_\delta &= \{\omega' \in \Omega' : \mathcal{H}(P_N \mathcal{R}(\omega_1, \omega')) \leq 8\mathcal{H}(P_N S_{t(\omega_1)}(u')) + \frac{\delta}{32}\}, \\ F_\delta &= \{\omega' \in \Omega' : \mathcal{H}(Q_N \mathcal{R}(\omega_1, \omega')) \leq 8\mathcal{H}(Q_N S_{t(\omega_1)}(u)) + \frac{\delta}{32}\}, \\ G'_\delta &= \{\omega' \in \Omega' : \mathcal{H}(P_N \mathcal{R}'(\omega_1, \omega')) \leq 8\mathcal{H}(P_N S_{t(\omega_1)}(u')) + \frac{\delta}{32}\}, \\ F'_\delta &= \{\omega' \in \Omega' : \mathcal{H}(Q_N \mathcal{R}'(\omega_1, \omega')) \leq 8\mathcal{H}(Q_N S_{t(\omega_1)}(u')) + \frac{\delta}{32}\}, \end{aligned}$$

Clearly, if $\omega' \in A_\delta F_\delta$, then

$$\mathcal{H}(\mathcal{R}) \leq 8\mathcal{H}(P_N \mathcal{R}) + 8\mathcal{H}(Q_N \mathcal{R}) \leq 64(\mathcal{H}(P_N S_t(u')) + \mathcal{H}(Q_N S_t(u))) + \frac{\delta}{2}.$$

As $\dim H_N < \infty$, we obtain

$$\mathcal{H}(P_N S_t(u)) \leq C_1 \mathcal{H}(S_t(u)),$$

therefore

$$\mathcal{H}(Q_N S_t(u)) \leq C_2 \mathcal{H}(S_t(u)).$$

Finally, we have

$$\mathcal{H}(\mathcal{R}(u, u', \omega_1, \omega')) + \mathcal{H}(\mathcal{R}'(u, u', \omega_1, \omega')) \leq C(\mathcal{H}(S_t(u)) + \mathcal{H}(S_t(u'))) + \delta,$$

if $\omega' \in A_\delta G'_\delta F_\delta F'_\delta$. Using property (ii) of Lemma 5.1, we see that

$$P_\delta^2(\omega_1) \geq \mathbb{P}'(A_\delta F_\delta G'_\delta F'_\delta) = \mathbb{P}'(A_\delta G'_\delta)\mathbb{P}'(F_\delta F'_\delta).$$

Hence, it suffices to find a constant $k_\delta > 0$ not depending on $\omega_1 \in \Omega_1$ such that

$$\mathbb{P}'(A_\delta G'_\delta) \geq k_\delta, \;\; \mathbb{P}'(F_\delta F'_\delta) \geq k_\delta. \tag{5.7}$$

Step 3. It follows from (4.1) that for any $\tau > 0$ there is $q_\tau > 0$ such that

$$\mathbb{P}'\{\|\zeta\|_1 \leq \tau\} \geq q_\tau, \;\; \mathbb{P}\{\|\zeta'\|_1 \leq \tau\} \geq q_\tau. \tag{5.8}$$

In view of property (iii) of Lemma 5.1, we have

$$\mathbb{P}'\{8\mathcal{H}(Q_N\zeta) = 8\mathcal{H}(Q_N\zeta') \leq \frac{\delta}{32}\} \geq q'_\delta,$$

where $q'_\delta > 0$, therefore

$$\mathbb{P}'(F_\delta F'_\delta) \geq q'_\delta.$$

Step 4. We deduce from (5.8) and (5.6) that

$$\mathbb{P}'(A_\delta) \geq q'_\delta, \quad \mathbb{P}'(G'_\delta) \geq q'_\delta. \tag{5.9}$$

Let $E = \{P_N\mathcal{R} \neq P_N\mathcal{R}'\}$. Then $A_\delta E^c = G'_\delta E^c = A_\delta G'_\delta E^c$. If $\mathbb{P}'(E) = 0$, then

$$\mathbb{P}'(A_\delta G'_\delta) = \mathbb{P}'(A_\delta) \geq q'_\delta.$$

Suppose that $\mathbb{P}'(E) > 0$. Using Remark 5.3, we obtain

$$\mathbb{P}'(A_\delta G'_\delta) = \mathbb{P}'(A_\delta G'_\delta E^c) + \mathbb{P}'(A_\delta G'_\delta E) \geq \mathbb{P}'(A_\delta E^c) + \mathbb{P}'(A_\delta E)\mathbb{P}'(G'_\delta E). \tag{5.10}$$

If $\mathbb{P}'(A_\delta E^c) \geq \left(\frac{q'_\delta}{2}\right)^2 =: k_\delta$, then $\mathbb{P}'(A_\delta G'_\delta) \geq k_\delta$. If $\mathbb{P}'(A_\delta E^c) < k_\delta$, then

$$\mathbb{P}'(A_\delta G'_\delta) \geq (\mathbb{P}'(A_\delta) - \mathbb{P}'(A_\delta E^c))(\mathbb{P}'(G'_\delta) - \mathbb{P}'(A_\delta E^c)) \geq \Big(q'_\delta - \Big(\frac{q'_\delta}{2}\Big)^2\Big)^2 \geq k_\delta.$$

This completes the proof of the lemma. □

6 Proof of Proposition 4.3

Let $(\Omega^k, \mathcal{F}^k, \mathbb{P}^k)$, $k \geq 1$ be independent copies of the probability space constructed in Lemma 5.1, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be their direct product. Let $u_0 = u$ and $u'_0 = u'$, where $u, u' \in H$. We set

$$u_k(\omega) = \mathcal{R}(u_{k-1}(\omega), u'_{k-1}(\omega), \omega^k), \;\; u'_k(\omega) = \mathcal{R}'(u_{k-1}(\omega), u'_{k-1}(\omega), \omega^k),$$
$$\zeta_k(\omega) = \zeta(u_{k-1}(\omega), u'_{k-1}(\omega), \omega^k), \;\; \zeta'_k(\omega) = \zeta'(u_{k-1}(\omega), u'_{k-1}(\omega), \omega^k),$$
$$t_k(\omega) = t(\omega^k),$$

where $\omega = (\omega^1, \omega^2, ...) \in \Omega$. Clearly, for $U_k := (u_k, u'_k)$ assertion (i) of Proposition 4.3 is satisfied. Since ζ_k, ζ'_k and t_k are sequences of independent random variables and $\{\zeta_k\}_{k=1}^\infty$ and $\{\zeta'_k\}_{k=1}^\infty$ are independent of $\{t_k\}_{k=1}^\infty$, the sequence U_k is a Markov chain in the space $\mathbf{H} := H \times H$.
Let us introduce the stopping time

$$\tau_d = \min\{k \geq 0, \|u_k\|_1 \vee \|u'_k\|_1 \leq d\}.$$

Lemma 6.1. *For any $d > 0$ there are positive constants γ and C such that*

$$\mathbb{E}_U e^{\gamma\tau_d} \leq C(1 + \mathcal{H}(u) + \mathcal{H}(u')) \text{ for all } U := (u, u') \in \mathbf{H}. \tag{6.1}$$

Proof. It is well known (e.g., see [10] or Proposition 2.3 in [23]) that inequality (6.1) will follow from two statements below:

(i) There are positive constants δ, R and C such that

$$\mathbb{E}_U e^{\delta\tau_R} \leq C(1 + \mathcal{H}(u) + \mathcal{H}(u')) \text{ for all } U \in \mathbf{H}. \tag{6.2}$$

(ii) For any $R > 0$ and $d > 0$ there is an integer $l \geq 1$ and a constant $p > 0$ such that

$$\mathbb{P}_U\{U_l \in \boldsymbol{B}_d\} \geq p \text{ for any } U \in \boldsymbol{B}_R, \tag{6.3}$$

where $\boldsymbol{B}_d = \{(u, u') \in \mathbf{H} : \|u\|_1 \vee \|u'\|_1 \leq d\}$.

The proof of (i) is similar to that of Lemma 3.5. To prove (ii), we use the definition of $U_k = (u_k, u_k')$, Lemma 5.4 and the Markov property:

$$\mathbb{P}_U\{\mathcal{H}(u_l) + \mathcal{H}(u_l') \leq \big(\gamma^l(\mathcal{H}(u) + \mathcal{H}(u'))\big) \vee \big(d^2\alpha\big)\} \geq \varepsilon^l,$$

for all $l \geq 1$, where ε depends only on d. Choosing l so large that $\gamma^l C_R < d^2\alpha$, where $C_R = \sup_{U \in \boldsymbol{B}_R}(\mathcal{H}(u) + \mathcal{H}(u'))$, we obtain (6.3). □

The proof of the following lemma is similar to that of Lemma 3.3, and we shall not dwell on it.

Lemma 6.2. *(i) There is a constant $M > 0$ such that for any $U_0 = (u_0, u_0') \in \mathbf{H}$ and an appropriate random integer $T = T(u_0, u_0') \geq 1$ the following inequalities hold*

$$\langle \|u_k\|_1^6 + \|u_k'\|_1^6 \rangle_0^n \leq M \quad \text{for } n \geq T, \tag{6.4}$$

$$\mathbb{E}T^p < \infty \quad \text{for all } p \geq 1. \tag{6.5}$$

(ii) For any $\delta \in (0, 1)$ and $d > 0$, there is a constant $R = R(\delta, d) > 0$ such that

$$\mathbb{P}\{\langle \|u_k\|_1^6 + \|u_k'\|_1^6 \rangle_0^n \leq R, \forall n \geq 0\} \geq \delta, \tag{6.6}$$

for any $U_0 = (u_0, u_0') \in \boldsymbol{B}_d$.

For any $M > 0$, we introduce the stopping times

$$\begin{aligned}
T_1(M) &= \min\{k \geq 1 : \langle \|u_i\|_1^6 + \|u_i'\|_1^6 \rangle_0^k > M\}, \\
T_2(M) &= \min\{k \geq 1 : \frac{1}{2}\langle \log t_i \rangle_0^k > M\}, \\
T_3(M) &= \min\{k \geq 1 : P_N u_k \neq P_N u_k'\}, \\
\sigma(M) &= T_1(M) \wedge T_2(M) \wedge T_3(M).
\end{aligned}$$

Lemma 6.3. *For any $B \geq 0$, there is an integer $N' \geq 1$ and a constant $M > 0$ such that, if (4.3) and (4.4) hold for some integer $N \geq 1$, then*

$$\mathbb{P}_U\{\sigma(M) = \infty\} \geq \frac{1}{2}, \tag{6.7}$$

$$\mathbb{E}_U[I_{\{\sigma(M)<\infty\}}\sigma(M)^p] < +\infty \quad \text{for all } p \geq 1, \tag{6.8}$$

where $U \in \boldsymbol{B}_d$, $d = \frac{1}{2C_N}$ and $C_N \geq 1$ is the constant in (5.2).

Proof. Let $M > 0$ be sufficiently large and let $m \geq 1$. Then

$$\{\sigma(M) = m\} \subset \{T_1(M) = m\} \cup \{T_2(M) = m\} \cup A_m, \tag{6.9}$$

where $A_m = \{T_3(M) = m, T_1(M) \geq m, T_2(M) \geq m\}$. Note that

$$A_m = \{P_N u_m \neq P_N u'_m, \sigma(M) > m - 1\}.$$

It follows from Lemma 2.1 that for $\mathbb{P}_U$-a.e. $\omega \in \{\sigma(M) > m - 1\}$, we have

$$\|S_{t_m}(u_{m-1}) - S_{t_m}(u'_{m-1})\|_1 \leq 2d(C\alpha_{N'+1}^{-\frac{1}{2}})^m e^{2CMm}.$$

Choosing N' so large that $\log \alpha_{N'+1} \geq 2(2CM + \log C + 2)$, we see that

$$\|S_{t_m}(u_{m-1}) - S_{t_m}(u'_{m-1})\|_1 \leq 2de^{-2m}.$$

Using Remark 5.2, construction of the space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Markov property, we obtain

$$\mathbb{P}_U(A_m) \leq 2dC_N e^{-2m} = e^{-2m}. \tag{6.10}$$

Let T'_2 be the random integer constructed in Lemma 3.4 for the sequence $\frac{1}{2}\log t_i$, and T'_1 be the random integer in Lemma 6.2. Then, it follows from the definition of T_1 and T_2 that

$$T_1 I_{\{T_1<\infty\}} < T'_1,$$
$$T_2 I_{\{T_2<\infty\}} < T'_2.$$

To prove (6.8), note that

$$\begin{aligned}
\mathbb{E}_U[I_{\{\sigma<\infty\}}\sigma^p] &= \sum_{m=1}^{\infty} \mathbb{P}\{\sigma(M) = m\} m^p \\
&\leq \sum_{m=1}^{\infty} (\mathbb{P}\{T'_1 > m\} + \mathbb{P}\{T'_2 > m\} + \mathbb{P}\{A_m\}) m^p \\
&\leq C \sum_{m=1}^{\infty} (m^{-p-2} + e^{-2m}) m^p < \infty,
\end{aligned}$$

where we used (6.9), (6.10), (6.5) and (3.12).
To prove (6.7), we use (6.9) and (6.10):

$$\mathbb{P}\{\sigma < \infty\} \leq \mathbb{P}\{T_1 < \infty\} + \mathbb{P}\{T_2 < \infty\} + \frac{1}{e^2 - 1}. \tag{6.11}$$

It follows from (6.6) and (3.13) that for any $\delta \in (0,1)$ there is $M = M(\delta, d) > 0$ such that

$$\mathbb{P}\{T_1(M) < \infty\} + \mathbb{P}\{T_2(M) < \infty\} < \delta.$$

Choosing $\delta = \frac{1}{2} - \frac{1}{e^2-1}$, we arrive at (6.7). □

To construct the random integer ℓ in Proposition 4.3, we follow the ideas of [23]. Suppose that $N \geq 1$, M and $d \leq 1$ are the constants in Lemma 6.3. Let ρ_0 be the first hitting time of the set $\boldsymbol{B}_d$. If for some $\omega \in \Omega$ we have

$$P_N u_k = P_N u_k', \ \ \langle \|u_i\|_1^6 + \|u_i'\|_1^6 \rangle_{\rho_0}^k \leq M, \ \ \frac{1}{2}\langle \log t_i \rangle_{\rho_0}^k \leq M \ \ \text{for all } k \geq \rho_0 + 1. \tag{6.12}$$

we set $\ell(\omega) = \rho_0(\omega)$, otherwise, let ρ_1' be the first time when one of the conditions in (6.12) is not satisfied and let ρ_1 be the first hitting time of the ball $\boldsymbol{B}_d$ after ρ_1'. Suppose that $\rho_1 < \infty$ and (6.12) is verified for $\omega \in \Omega$, with ρ_0 replaced by ρ_1, then we set $\ell(\omega) = \rho_1(\omega)$. Continuing this process and using the same arguments as in [23], one can show that ℓ is well defined for a.e. $\omega \in \Omega$ and satisfies (4.12). The other assertions of Proposition 4.3 follow immediately from the construction.

7 Appendix

7.1 Proof of inequality (2.2)

Let $u_0 \in H$. Setting $u(t) = S_t(u_0)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u(t)) = (-2\alpha\Delta u + \beta|u|^2 u, \dot{u}), \tag{7.1}$$

where $(u,v) = \mathrm{Re}\int_D u\bar{v}\mathrm{d}x$. Since u is the solution of (1.1)-(1.3) with $\eta \equiv 0$, we deduce from (7.1) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u) &= (-2\alpha\Delta u + \beta|u|^2 u, \nu\Delta u - i\beta|u|^2 u) \\ &\leq -2\alpha\nu\|\Delta u\|^2 + (\beta|u|^2 u, \nu\Delta u) + (2\alpha\Delta u, i\beta|u|^2 u).\end{aligned} \tag{7.2}$$

It is clear that

$$(2\alpha\Delta u, i\beta|u|^2 u) \leq 2\alpha\beta(|\nabla u|^2, |u|^2), \tag{7.3}$$

$$(\beta|u|^2 u, \nu\Delta u) = -\beta\nu(|u|^2, |\nabla u|^2) - \beta\nu(u\nabla(|u|^2), \nabla u). \tag{7.4}$$

Substituting (7.3) and (7.4) into (7.2) and noting that

$$(u\nabla(|u|^2), \nabla u) = \mathrm{Re}\int_D u\nabla\bar{u}\nabla(|u|^2)\mathrm{d}x = \frac{1}{2}\,\mathrm{Re}\int_D (\nabla(|u|^2))^2\mathrm{d}x \geq 0,$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u) \leq -2\alpha\nu\|\Delta u\|^2 - \beta\nu(|u|^2, |\nabla u|^2) + 2\alpha\beta(|u|^2, |\nabla u|^2).$$

Choosing α sufficiently small and applying Poincaré's inequality to the function $|u|^2$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(u) + \alpha\nu\|\Delta u\|^2 \leq -a\mathcal{H}(u), \tag{7.5}$$

for some positive constant a. Application of Gronwall's inequality results in (2.2). Finally, note that the integration of (7.5) gives

$$\alpha\nu \int_0^t \|\Delta u\|^2 \mathrm{d}s \leq \mathcal{H}(u_0). \tag{7.6}$$

7.2 Proof of inequalities (2.3) and (2.4)

Step 1. Let $u(t) = S_t(u_0)$ and $u_0 \in H$. Then

$$u \cdot t^{\frac{1}{2}} \in C([0,\infty), H^2(D)). \tag{7.7}$$

Indeed, formally taking the scalar product of $-\Delta\dot{u}t$ and Equation (1.1) with $\eta \equiv 0$, we obtain

$$(\dot{u} - \nu\Delta u + i\beta|u|^2u, -\Delta\dot{u}t) = 0.$$

Integration of this equality in t results in

$$\frac{\nu}{2}t\|\Delta u\|^2 + \int_0^t s\|\nabla\dot{u}\|^2\mathrm{d}s \leq \frac{\nu}{2}\int_0^t \|\Delta u\|^2\mathrm{d}s + \beta\int_0^t s|(|u|^2u, \Delta\dot{u})|\mathrm{d}s. \tag{7.8}$$

Note that

$$\begin{aligned}\beta\int_0^t s|(|u|^2u, \Delta\dot{u})|\mathrm{d}s &\leq \frac{1}{2}\int_0^t s\|\nabla\dot{u}\|^2\mathrm{d}s + C\int_0^t s\|\nabla u\|_{L^4}^2\||u|^2\|_{L^4}^2\mathrm{d}s \\ &\leq \frac{1}{2}\int_0^t s\|\nabla\dot{u}\|^2\mathrm{d}s + C\int_0^t s\|\Delta u\|^2\|u\|_{L^8}^4\mathrm{d}s, \end{aligned} \tag{7.9}$$

where we used Sobolev embedding $H^1(D) \hookrightarrow L^4(D)$. Substituting this inequality into (7.8) and using Gronwall's inequality, we arrive at

$$t\|\Delta u\|^2 \leq \int_0^t \|\Delta u\|^2\mathrm{d}s \cdot \exp\big(C\int_0^t \|u\|_{L^8}^4\mathrm{d}s\big). \tag{7.10}$$

Using the Gagliardo–Nirenberg inequality

$$\|u\|_{L^8} \leq C\|u\|_{L^4}^{\frac{1}{2}}\|\Delta u\|^{\frac{1}{2}}, \tag{7.11}$$

and inequalities (2.2) and (7.6), we see that

$$\begin{aligned}\exp(C\int_0^t \|u\|_{L^8}^4\mathrm{d}s) &\leq \exp(C\int_0^t \|u\|_{L^4}^2\|\Delta u\|^2\mathrm{d}s) \\ &\leq \exp(C\mathcal{H}(u_0)^{\frac{3}{2}}) \leq C\exp(C\|u_0\|_1^6). \end{aligned} \tag{7.12}$$

Now substituting (7.12) into the right-hand side of (7.10), and using (7.6), we obtain

$$\sup_{\tau\in[0,t]} \tau\|\Delta u(\tau)\|^2 \le C\exp(C\|u_0\|_1^6). \tag{7.13}$$

To prove (7.7), we use Galerkin's method, choosing as a base in $L^2(D)$ the set of normalized eigenfunctions of the Dirichlet Laplacian. It is easy to verify that (7.13) holds for Galerkin approximations. Then passing to the limit, we arrive at (7.7) and (7.13).

Step 2. Let $u_0, v_0 \in H$ and $u = S_t(u_0)$, $v = S_t(v_0)$. Then we have the following estimate for $w = u - v$:

$$\|\nabla w\|^2 + \nu\int_0^t \|\Delta w\|^2 \mathrm{d}s \le C\|\nabla w_0\|^2 \exp(C(\|u_0\|_1^6 + \|v_0\|_1^6)). \tag{7.14}$$

where $w_0 = u_0 - v_0$ and C is a positive constant. Indeed, w is a solution of the following equation

$$\dot{w} - \nu\Delta w + i\beta(|u|^2 u - |v|^2 v) = 0. \tag{7.15}$$

Taking the scalar product of this equation with $-\Delta w$ and integrating the resulting equality in t, we see that

$$\begin{aligned}\frac{1}{2}\|\nabla w\|^2 + \nu\int_0^t \|\Delta w\|^2 \mathrm{d}s &\le \frac{1}{2}\|\nabla w_0\|^2 + \beta\int_0^t |(|u|^2 u - |v|^2 v, \Delta w)|\mathrm{d}s \\ &\le \frac{1}{2}\|\nabla w_0\|^2 + \frac{\nu}{2}\int_0^t \|\Delta w\|^2 \mathrm{d}s + C\int_0^t \||u|^2 + |v|^2\|_{L^4}^2 \|\nabla w\|^2 \mathrm{d}s. \end{aligned}\tag{7.16}$$

We deduce from Gronwall's inequality:

$$\|\nabla w\|^2 \le \|\nabla w_0\|^2 \exp(C\int_0^t \||u|^2 + |v|^2\|_{L^4}^2 \mathrm{d}s).$$

Now substituting this inequality into the right-hand side of (7.16) and using (7.12), we arrive at (7.14).

Step 3. Taking the scalar product of (7.15) with $-t\Delta\dot{w}$ and integrating the resulting equality, we obtain

$$\begin{aligned}\frac{\nu}{2}t\|\Delta w\|^2 + \int_0^t s\|\nabla\dot{w}\|^2 \mathrm{d}s &\le \frac{\nu}{2}\int_0^t \|\Delta w\|^2 \mathrm{d}s \\ &+ \frac{\beta^2}{2}\int_0^t s\|\nabla(|u|^2 u - |v|^2 v)\|^2 \mathrm{d}s + \frac{1}{2}\int_0^t s\|\nabla\dot{w}\|^2 \mathrm{d}s. \end{aligned}\tag{7.17}$$

Using Hölder's inequality, we see that

$$\begin{aligned}&\int_0^t s\|\nabla(|u|^2 u - |v|^2 v)\|^2 \mathrm{d}s \\ &\le C\int_0^t s(\||u|^2\nabla w\|^2 + \|wu\nabla v\|^2 + \|wv\nabla v\|^2)\mathrm{d}s \\ &\le C\int_0^t (s\|\Delta w\|^2\|u\|_{L^8}^4 + s\|w\|_{L^8}^2\|\nabla v\|_{L^4}^2(\|v\|_{L^8}^2 + \|u\|_{L^8}^2))\mathrm{d}s. \end{aligned}\tag{7.18}$$

Substituting (7.18) into (7.17) and using Gronwall's inequality, we arrive at

$$t\|\Delta w\|^2 \leq C\Big[\int_0^t \|\Delta w\|^2 \mathrm{d}s + \int_0^t s\|w\|_{L^8}^2\|\nabla v\|_{L^4}^2(\|v\|_{L^8}^2 + \|u\|_{L^8}^2)\mathrm{d}s\Big]$$
$$\times \exp(C\int_0^t \|u\|_{L^8}^4 \mathrm{d}s). \tag{7.19}$$

By the Cauchy–Schwarz inequality,

$$P := \int_0^t s\|w\|_{L^8}^2\|\nabla v\|_{L^4}^2(\|u\|_{L^8}^2 + \|v\|_{L^8}^2)\mathrm{d}s \leq C\sup_{[0,t]} s\|\Delta v\|^2$$
$$\times \Big(\int_0^t \|w\|_{L^8}^4 \mathrm{d}s\Big)^{\frac{1}{2}}\Big[\Big(\int_0^t \|u\|_{L^8}^4 \mathrm{d}s\Big)^{\frac{1}{2}} + \Big(\int_0^t \|v\|_{L^8}^4 \mathrm{d}s\Big)^{\frac{1}{2}}\Big]. \tag{7.20}$$

To estimate the right-hand side of this inequality, note that

$$\sup_{[0,t]} s\|\Delta v\|^2\Big[\Big(\int_0^t \|u\|_{L^8}^4 \mathrm{d}s\Big)^{\frac{1}{2}} + \Big(\int_0^t \|v\|_{L^8}^4 \mathrm{d}s\Big)^{\frac{1}{2}}\Big]$$
$$\leq C\exp(C(\|u_0\|_1^6 + \|v_0\|_1^6)), \tag{7.21}$$

where we used (7.12) and (7.13). Using (7.11) and (7.14), we see that

$$\Big(\int_0^t \|w\|_{L^8}^4 \mathrm{d}s\Big)^{\frac{1}{2}} \leq C\sup_{[0,t]}\|\nabla w\|\Big(\int_0^t \|\Delta w\|^2 \mathrm{d}s\Big)^{\frac{1}{2}}$$
$$\leq C\|\nabla w_0\|^2\exp(C(\|u_0\|_1^6 + \|v_0\|_1^6)). \tag{7.22}$$

We deduce from (7.21) and (7.22) that

$$P \leq C\|\nabla w_0\|^2\exp(C(\|u_0\|_1^6 + \|v_0\|_1^6)). \tag{7.23}$$

Finally, substituting (7.23) into (7.19), and using (7.14) and (7.12), we arrive at (2.4).

7.3 Proof of Lemma 3.1

Let $\mathcal{N}_t$ be defined by (4.15). Then, for $0 \leq s \leq t$, $\mathcal{N}_t - \mathcal{N}_s$ is a Poisson random variable with parameter $\lambda(t-s)$, independent of $\mathcal{F}_t$ (e.g., see [11]), where $\mathcal{F}_t$ is defined in Section 3. Let ζ be defined by (3.1), then we have

$$\zeta(t) - \zeta(s) = \sum_{k=\mathcal{N}_s+1}^{\mathcal{N}_t} \eta_k. \tag{7.24}$$

It follows from (7.24) that ζ has independent increments. Using (7.24) and the fact that the distribution of $\mathcal{N}_t - \mathcal{N}_s$ depends only on $t-s$, it is easy to see that the distributions of processes $\zeta(\cdot)$ and $\zeta(\cdot + s) - \zeta(s)$ coincide:

$$\mathcal{D}(\zeta(t), t \geq 0) = \mathcal{D}(\zeta(t+s) - \zeta(s), t \geq 0), \tag{7.25}$$

for any $s \geq 0$. Note that $u(t)$ is determined by $\{\zeta(\tau) : 0 \leq \tau \leq t\}$ and $u(t)$ is $\mathcal{F}_t$-measurable. We have

$$\begin{aligned}\mathbb{P}\{u(t,u_0,\{\zeta(\tau):0\leq\tau\leq t\})\in\Gamma|\mathcal{F}_s\}\\ &=\mathbb{P}\{u(t-s,u_s,\{\zeta(\tau)-\zeta(s):s\leq\tau\leq t\})\in\Gamma|\mathcal{F}_s\}\\ &=\mathbb{P}\{u(t-s,v,\{\zeta(\tau)-\zeta(s):s\leq\tau\leq t\})\in\Gamma\}\big|_{v=u_s},\end{aligned} \quad (7.26)$$

for any $\Gamma \in \mathcal{B}(H)$, where we used the independence of increments of ζ. Using (7.25), we arrive at

$$\mathbb{P}\{u(t,u_0)\in\Gamma|\mathcal{F}_s\}=\mathbb{P}\{u(t-s,v)\in\Gamma\}\big|_{v=u_s}, \quad (7.27)$$

which completes the proof of the lemma.

7.4 Proof of Lemma 3.3

Let us introduce the random variable

$$T=\min\{n\geq 1:\ \frac{1}{k}|M_k|\leq 1\ \text{ for all } k\geq n\},$$

where $\min\{\emptyset\}=+\infty$. It is easy to see that $\mathbb{P}\{T=\infty\}=0$, as

$$\mathbb{P}\{T=\infty\}\leq\sum_{k=m}^{\infty}\mathbb{P}\{\frac{1}{k}|M_k|>1\}\leq C\sum_{k=m}^{\infty}\frac{1}{k^2}\to 0,\ \ m\to\infty,$$

where we used (3.10) with $p=2$ and Chebyshev's inequality. To estimate the moments of T, we use (3.10) with $l=p+2$:

$$\begin{aligned}\mathbb{E}T^p=\sum_{n=1}^{\infty}\mathbb{P}\{T=n\}n^p\ &\leq\ 1+\sum_{k=1}^{\infty}\mathbb{P}\{\frac{1}{k}|M_k|>1\}(k+1)^p\\ &\leq\ 1+C\sum_{k=1}^{\infty}k^{-l}(k+1)^p<+\infty.\end{aligned}$$

To prove (3.13), we use (3.10) with $p=2$ and Chebyshev's inequality

$$\mathbb{P}\{\frac{|M_k|}{k}\leq R,\forall k\geq 1\}\geq 1-\sum_{k=1}^{\infty}\mathbb{P}\{\frac{|M_k|}{k}>R\}\geq 1-\frac{C_2}{R^4}\sum_{k=1}^{\infty}\frac{1}{k^2}.$$

Choosing R sufficiently large, we obtain (3.13).

7.5 Proof of Theorem 3.7

Let u_t be the trajectory of (1.1)-(1.4) with $u_0 \equiv 0$. It suffices to show that the family $\mathcal{D}(u_t)$ is tight in H. Let $\mathcal{N}_t$ be defined by (4.15). First we shall show that the family $\mathcal{D}(u_{\mathcal{N}_t})$ is tight. By Ulam's theorem, there is a compact K_ε^1 such that $\mathbb{P}\{\eta_1 \notin K_\varepsilon^1\} \leq \frac{\varepsilon}{2}$. Using the independence of $\{\eta_p\}$ and $\mathcal{N}_t$, we obtain

$$\begin{aligned}\mathbb{P}\{\eta_{\mathcal{N}_t} \notin K_\varepsilon^1, \mathcal{N}_t \neq 0\} &= \sum_{p=1}^{\infty} \mathbb{P}\{\eta_p \notin K_\varepsilon^1, \mathcal{N}_t = p\} \\ &= \sum_{p=1}^{\infty} \mathbb{P}\{\eta_p \notin K_\varepsilon^1\}\{\mathcal{N}_t = p\} \leq \frac{\varepsilon}{2}.\end{aligned}$$

Using (3.2), it is easy to show that there is a constant $M > 0$ such that

$$\mathbb{E}\big(\|u_{\tau_{\mathcal{N}_t - 1}}\|_1 I_{\{\mathcal{N}_t \neq 0\}}\big) \leq M, \qquad \text{for all } t \geq 0.$$

Let $R_\varepsilon \geq \frac{4M}{\varepsilon}$. By the Chebyshev inequality, we have

$$\mathbb{P}\{\|u_{\tau_{\mathcal{N}_t - 1}}\|_1 I_{\{\mathcal{N}_t \neq 0\}} \geq R_\varepsilon\} \leq \frac{M}{R_\varepsilon} \leq \frac{\varepsilon}{4}. \tag{7.28}$$

Define $B_\varepsilon = \{v \in H : \|v\|_1 \leq R_\varepsilon\}$ and $K_\varepsilon^2 = S_{[a,b]}(B_{R_\varepsilon})$, where $b > 0, a > 0$. Then K_ε^2 is compact in H. We deduce from (7.28) that

$$\begin{aligned}&\mathbb{P}\{S_{t_{\mathcal{N}_t}}(u_{\tau_{\mathcal{N}_t - 1}}) \notin K_\varepsilon^2, \mathcal{N}_t \neq 0\} \leq \mathbb{P}\{t_{\mathcal{N}_t} > b, \mathcal{N}_t \neq 0\} + \mathbb{P}\{t_{\mathcal{N}_t} < a, \mathcal{N}_t \neq 0\} \\ &+ \mathbb{P}\{u_{\tau_{\mathcal{N}_t - 1}} \notin B_\varepsilon, \mathcal{N}_t \neq 0\} \leq \sum_{p=1}^{\infty} (\mathbb{P}\{t_p > b, \mathcal{N}_t = p\} + \mathbb{P}\{t_p < a, \mathcal{N}_t = p\}) + \frac{\varepsilon}{4} \\ &\leq \mathbb{P}\{t_1 > b\} + \mathbb{P}\{t_1 < a\} + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2},\end{aligned}$$

if b is sufficiently large and a is sufficiently small. Let $K_\varepsilon = K_\varepsilon^1 + K_\varepsilon^2$. We can assume that $0 \in K_\varepsilon$. As $u_{\mathcal{N}_t} = 0$, if $\mathcal{N}_t = 0$, we have

$$\begin{aligned}\mathbb{P}\{u_{\tau_{\mathcal{N}_t}} \notin K_\varepsilon\} &= \mathbb{P}\{u_{\tau_{\mathcal{N}_t}} \notin K_\varepsilon, \mathcal{N}_t = 0\} + \mathbb{P}\{u_{\tau_{\mathcal{N}_t}} \notin K_\varepsilon, \mathcal{N}_t \neq 0\} \\ &\leq \mathbb{P}\{\eta_{\mathcal{N}_t} \notin K_\varepsilon^1, \mathcal{N}_t \neq 0\} + \mathbb{P}\{S_{t_{\mathcal{N}_t}}(u_{\tau_{\mathcal{N}_t - 1}}) \notin K_\varepsilon^2, \mathcal{N}_t \neq 0\} \leq \varepsilon.\end{aligned}$$

Thus $\mathcal{D}(u_{\tau_{\mathcal{N}_t}})$ is tight in H.
Define $T_\varepsilon = S_{[0,\infty)}(K_\varepsilon)$ and note that $u_t = S_{t - \tau_{\mathcal{N}_t}}(u_{\tau_{\mathcal{N}_t}})$, $t \geq 0$. Then T_ε is compact in H. Finally, we have

$$\mathbb{P}\{u_t \notin T_\varepsilon\} \leq \mathbb{P}\{u_{\tau_{\mathcal{N}_t}} \notin K\} \leq \varepsilon.$$

The proof of the other assertion of the theorem is standard.

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