On the Bernoulli property of planar hyperbolic billiards

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Abstract

We consider billiards in bounded non-polygonal domains of $\mathbb{R}^2$ with piecewise smooth boundary. More precisely, we assume that the curves forming the boundary are straight lines or strictly convex inward curves (dispersing) or strictly convex outward curves of a special type (absolutely focusing). It follows from the work of Sinai, Bunimovich, Wojtkowski, Markarian and Donnay [S, Bu2, W2, M1, Do, Bu4] that these billiards are hyperbolic (non-vanishing Lyapunov exponents) provided that proper conditions are satisfied. In this paper, we show that if some additional mild conditions are satisfied, then not only these billiards are hyperbolic but are also Bernoulli (and therefore ergodic). Our result generalizes previous works [Bu4, Sz, M3, LW, CT] and applies to a very large class of planar hyperbolic billiards. This class includes, among the others, the convex billiards bounded by straight lines and absolutely focusing curves studied by Donnay [Do].

Contents

1 Introduction

Let Q be a bounded connected open subset of the Euclidean plane $\mathbb{R}^2$ whose boundary Γ is an union of finitely many compact curves of class C^3. The billiard in Q is the dynamical system generated by the motion of a point particle that, inside Q, moves along straight lines with unit speed, and it is reflected by Γ so that the angle of reflection equals the angle of incidence when there is a collision with Γ. The billiard in Q is endowed with a natural invariant measure that is the Liouville measure on the unit tangent bundle of Q.

In this paper, we consider a large class of planar hyperbolic billiards, and prove that they are Bernoulli with respect to the Liouville measure. A billiard is called hyperbolic if all its Lyapunov exponents are non-zero almost everywhere. It is well known from Pesin's theory for smooth systems [P] and its extension to systems with singularities due to Katok and Strelcyn [KS] that a hyperbolic system has positive entropy and countably many ergodic components of positive measure. In addition, each ergodic component is an union of finitely many disjoint sets of positive measure with the properties that they are cyclically permuted by the dynamics, and the first return map on each of them is K-mixing, and by general results of [CH, OW], Bernoulli. These sets are called Bernoulli components of the system. We say that a system is Bernoulli if it has only one Bernoulli component. We recall that a Bernoulli system is also ergodic, since ergodicity, mixing, K-mixing and the Bernoulli property form a hierarchy of increasingly strong statistical properties [CFS].

The proof that a hyperbolic system is Bernoulli can be achieved by demonstrating that its ergodic components are open modulo a set of zero measure. A system having this property is called locally ergodic. Local ergodicity can be proved by using a simple and yet powerful argument devised by Hopf back in the 30's to show that geodesic flows on surfaces of negative curvature are ergodic with respect to the volume [Ho].

The billiards considered here are not smooth systems, and some serious complications arise when one tries to apply Hopf's argument to them. In fact, on the one hand Hopf's argument relies on the existence of uniformly "long" stable and unstable manifolds, on the other, for billiards, these manifolds are arbitrarily "short" as the singularities prevent them from growing in size. The way of overcoming this major obstacle was found by Sinai at the end of the 60's. In his seminal paper [S], he considered a special class of hyperbolic billiards consisting of two-dimensional tori with strictly convex inwards scatterers, and managed to show that they have "enough" stable and unstable manifolds sufficiently "long" to carry over Hopf's argument. Sinai then went on to prove that these billiards are locally ergodic and K-mixing. The Bernoulli property of Sinai's billiards was proved later in [GO].

Since the publication of Sinai's paper, his method of proving local ergodicity has been improved and extended to larger and larger classes of hyperbolic systems. The papers [SC, KSS, C, M2, LW] contain extensions of Sinai's argument valid for some classes of multidimensional billiards and systems with singularities (non only billiards). We will refer to results of this type as local ergodic theorems, LET's for short. Our proof of the Bernoulli property for the billiards considered in this paper relies on a modified version of the LET proved in [LW]. We will come back on this later.

Besides Sinai's billiards, another important class of hyperbolic billiards is represented by semifocusing billiards. A billiard is called semifocusing if its boundary is formed by straight lines and strictly convex outwards curves, which we will call focusing. In his famous papers [Bu1, Bu2], Bunimovich gave several examples of semifocusing billiards that are hyperbolic; the most celebrated one is certainly the so called stadium, i.e., a billiard with the shape of a stadium. The geometry of Bunimovich's billiards is somewhat rigid, because their focusing curves can only be arcs of circles. The mechanism generating hyperbolicity in semifocusing billiards was further clarified by Wojtkowski using invariant cones. He also discovered many other focusing curves besides arcs of circles that can be used to design hyperbolic billiards [W2]. Markarian further enlarged this class of curves and also elaborated a technique to prove hyperbolicity based on quadratic forms [M1]. Another class of focusing curves that can be used to construct hyperbolic billiards was introduced by Bunimovich and independently by Donnay [Do, Bu4]. These curves are called absolutely focusing, and indeed form a large class, as it turned out that Wojtkowski's and Markarian's curves of class C^6 and any sufficiently short piece of a C^6 focusing curve are of this type.

The billiards considered in this paper are characterized by the following properties:

- the boundary of the billiard table is formed by finitely many curves chosen among straight lines, dispersing curves and absolutely focusing curves,
- each pair of non-flat boundary components are sufficiently apart,

- the subset of billiard trajectories that eventually hit only straight lines has measure zero.

The main result of this paper is the proof that the billiards just described are locally ergodic, and moreover Bernoulli if additional conditions on the smallness of the set of the non-hyperbolic billiard trajectories are verified. Previous results of this type were obtained for dispersing and certain semidispersing billiards [S, SC], and for semifocusing billiards like Bunimovich's billiards and some generalizations of these [Bu2, Bu3, Sz, LW, CT, De, DM]. We stress the fact that, in all the results just mentioned, there is a limitation on the generality of the focusing curves admitted in the billiard boundary, which is dropped in the results presented in this paper.

As already mentioned, to obtain our results, we use a modified version of the LET proved by Liverani and Wojtkowski [LW]. The application of this theorem in its original form or any other LET found in literature [SC, KSS, Bu3, M2, C, LW] is not allowed in our situation, since not all the hypotheses of these theorems are verified by all the billiards that we consider. More precisely, the hypotheses of the LET in [LW] not being true for all our billiards are two: the existence of a continuous invariant cone field on the tangent bundle and a nondegenerate noncontracting semimetric defined on stable and unstable spaces. This is so because general absolutely focusing boundary components have invariant cone fields that are only piecewise continuous [Do], and the only noncontraction semimetric we found for them is degenerate (see Subsection 8.1). For similar reasons, we cannot use the other LET's. To solve this problem, we prove a slight generalization of the LET of [LW] (only for two dimensional systems) that applies to billiards with general absolutely focusing curves. In this version of the LET, the original conditions on the continuity of the cone field and the nondegeneracy of the noncontraction metric are replaced by weaker ones. We have to warn the reader that the proof that the hypotheses of this LET are verified by our billiards requires lengthy and sometimes involved computations. And this is due, to the generality of the billiards considered.

To exemplify our results, we apply them to several hyperbolic billiards: dispersing and focusing billiards, stadia and semidispersing billiards and billiards in polygons with pockets and bumps. We prove that all these systems are Bernoulli. Some of our conclusions, like those concerning dispersing and Bunimovich's billiards, are not new. The others concerning hyperbolic billiards with general absolutely focusing curves in their boundaries are new instead. One of these new results is the proof that the convex billiards bounded by straight lines and absolutely focusing curves studied by Donnay in [Do] are Bernoulli.

The paper is organized as follows. In Section 2, we provide some background information on planar billiards. Section 3 is devoted to the description of the billiards studied and the formulation of the the main results of the paper. In Section 4, we introduce the first return map induced by the billiard map on a suitable region of the billiard phase space. We do so, because the LET that we prove applies only to the induced system rather and not to the original billiard map. More generalities concerning billiards, geometric optics and invariant cone fields are given in Section 5. In particular, a detailed description of Donnay's construction of invariant cone fields for absolutely focusing curves is provided in Subsection 5.3. In this subsection, we also introduce the invariant cone field for billiards used throughout this paper. One of the conditions characterizing our billiards is quite technical, and its precise formulation is given in Section 6. In this section, we also prove several results, like the hyperbolicity of our billiards and some properties concerning their stable and unstable manifolds used in subsequent proofs. In Section 7, we prove a LET building on the LET demonstrated in [LW]. Then in Section 8, we prove that the induced system of the billiards considered verify the hypotheses of the new LET, and derive our main result: billiards with absolutely focusing curves in the boundary are locally ergodic, and moreover Bernoulli if they satisfy a mild additional condition on the topology of the non-hyperbolic billiard trajectories. In Section 9, we consider several examples of billiards which our result apply to. We also study a class of billiards in polygons with pockets and bumps that do not satisfy the additional condition mentioned above. Despite this, we prove that they are Bernoulli. Donnay's billiard are included in this class. Finally the appendixes contain several technical results: in Appendixes A and B, we study the regularity of the singular sets of the billiard map T and its induced system, whereas in Appendix C, we prove a lemma, certainly known but which we did not find in the literature, concerning the relation between the measure of a compact subset of a smooth curve and the volume of tubular neighborhoods of this subset.

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2 Generalities

Let Q be a bounded connected open subset of the Euclidean plane $\mathbb{R}^2$. Let $k \geq 2$. We assume that its boundary Γ of is an union of n compact C^{k+1} curves $\Gamma_1, \ldots, \Gamma_n$ that will be called *components* of Γ. We also assume that Γ is a finite union of disjoint Jordan curves (homeomorphic images of the unit circle) intersecting only at their endpoints. Note that, under these hypotheses, Q is not necessarily simply connected as it may contain a finite number of two-dimensional scatterers. Given a C^2 curve, let κ denote its curvature. A component of Γ is called *dispersing* (*focusing*) if it is strictly convex inward (outward) and its curvature is everywhere non-zero. We adopt the convention that $\kappa(\cdot) < 0$ ($\kappa(\cdot) > 0$) on a dispersing curve (focusing curve). A point $q \in \Gamma$ is called a *corner* of Γ if it belongs to several components of Γ, otherwise it is called a *regular* point of Γ. For any regular point q of Γ, the symbol $n(q)$ denotes the unit normal of Γ at q pointing inside Q.

The *billiard in Q* is the dynamical system generated by the motion of a point particle obeying the rule: inside Q, the particle moves along straight lines at unit speed, and when the particle hits Γ, it gets reflected so that the angle of reflection equals the angle of incidence. The billiard in Q can be described either by a flow (billiard flow) or a map (billiard map). In this paper, we will focus our attention on the billiard map that is introduced below. For a definition and detailed discussion of the properties of the billiard flow, we refer the reader to the book [CFS]. Finally a word about the terminology used throughout the paper: we will use the word smooth as synonym of C^1.

Billiard phase space. Consider a smooth component Γ_i of Γ, and assign an orientation to it. For a given unit tangent vector $(q,v) \in \mathcal{T}_1\mathbb{R}^2$, let $\theta(q,v)$ denote the angle that the oriented tangent of Γ_i at q forms with (q,v). If Γ_i is not a closed curve, and q is an endpoint of Γ_i, then the one-sided tangent has to be taken in this definition. We assume that the orientation of Γ_i is chosen so that $\theta(q,v) \in [0,\pi]$ when (q,v) points inside Q. Then the set $M_i := \{(q,v) \in \mathcal{T}_1\mathbb{R}^2 : \theta(q,v) \in [0,\pi]\}$ collects the unit vectors attached to Γ_i and pointing inside Q. If s is the arclength-parameter of Γ_i, then the pair (s,θ) forms a a coordinate system for M_i. We see immediately that M_i is diffeomorphic to a cylinder or a rectangle whether or not Γ_i is closed.

The billiard phase space is the set M that coincides with $\cup_{i=1}M_i$ after having identified elements of any two sets M_i and M_j with $i \neq j$ corresponding to the same vector of $\mathcal{T}_1\mathbb{R}^2$. Although, in many significant cases, M is a manifolds (with boundary and corners), in general, it turns out to be a less regular object, namely, a subset of a finite union of smooth manifolds with boundary and corners identified along subsets of their boundaries.

We close this subsection by giving few more definitions. The map $\pi : M \to \Gamma$ given by $\pi(q,v) = q$ for all $(q,v) \in M$ is the canonical projection map of M onto Γ. Next let $\Gamma_-, \Gamma_+, \Gamma_0$ be the union of the dispersing and focusing curves and the straight lines forming Γ, respectively. Then the corresponding subsets of billiard phase space are $M_+ := \pi^{-1}(\Gamma_+), M_- := \pi^{-1}(\Gamma_-), M_0 := \pi^{-1}(\Gamma_0)$.

Billiard map. In the definition of the billiard map and, above all in the study of its ergodic properties, four subsets of the billiard phase space play a key role. The first set S_1 consists of unit vectors attached at the endpoints of non-closed components of Γ, and the second set S_2 consists of the unit tangent vectors of the components of Γ. Formally $S_1 = \cup_{i=1}\partial\Gamma_i \times [0,\pi]$ where the union is taken over all non-closed components of Γ, and $S_2 = \cup_{i=1}\Gamma_i \times \{0,\pi\}$. Let $\partial M = S_1 \cup S_2$ and $\operatorname{int} M = M \setminus \partial M$. For any $z = (q,v) \in \operatorname{int} M$, let $L(z) = \{q + tv : t \geq 0\}$ that is the ray starting at q and parallel to v. Then define $q_1 = q_1(z)$ to be the point in the set $L(z) \cap \Gamma$ having the smallest distance from q such that $q_1 \neq q$ and the segment with endpoints q and q_1 is contained in the closure of Q. Clearly $q_1(z)$ is the point where z hits first Γ. In order to characterize completely the collision at $q_1(z)$, we need to compute the velocity of the particle after this collision. However such a velocity is well defined if and only if q_1 is a regular point. This fact leads us to introduce the other two subsets of M mentioned at the beginning of this subsection. Let $S_3 = \{z \in \operatorname{int} M : q_1(z) \text{ is a corner of } \Gamma\}$ and $S_4 = \{z \in \operatorname{int} M : L(z) \text{ is tangent to } \Gamma \text{ at } q_1(z)\}$. In [KS, Part V, Theorem 6.1], it is proved that S_3 and S_4 are both unions of finitely many points and finitely many C^{k+1} curves of finite length. We can now define the billiard map T. Let $D_T = \operatorname{int} M \setminus S_3$. Then $T : D_T \to M$ is given by

$$T(q,v) = (q_1, v_1), \quad (q,v) \in D_T$$

where

$$v_1 = v + 2\langle v, n(q_1)\rangle n(q_1)$$

is the velocity of the particle after the collision at q_1. This map is discontinuous at points of S_4 and a C^k diffeomorphism on $\operatorname{int} M \setminus (S_3 \cup S_4)$ onto its image.

There is a natural probability measure on M that is T-invariant. In coordinates (s,θ), such a measure is given by $d\mu = (2\,\text{length}(\Gamma))^{-1} \sin\theta ds d\theta$. Throughout this paper, we will always have in mind this measure, unless we specify otherwise. We see immediately that $\mu(S_i) = 0$ for $1 \leq i \leq 4$ so that $\mu(D_T) = \mu(M)$. We finally point out another important feature of the billiard map T: the time-reversing property. More precisely, if $R : M \to M$ that reflects vectors of M about Γ given by $R(s, \theta) = (s, \pi - \theta)$ for $(s, \theta) \in M$, then $T^{-1} = RTR$ on $\operatorname{int} M \setminus (S_3 \cup S_4)$. Thanks to this symmetry, any property that is proved for T has a counterpart holding for T^{-1}.

Singular sets. We call $S_1^+ := S_3 \cup S_4$ the *singular set* of T. On this is the subset of M, T is not defined or C^1. Similarly there is a singular set S_1^- for T^{-1} that, by the time-reversing symmetry, coincides with RS_1^+. By previous observations on S_3 and S_4, it follows that S_1^+ and S_1^- are unions of finitely many points and C^{k+1} curves of finite length. In Theorem A.1, we show that the closures of S_1^+, S_1^- are finite union of smooth compact curves intersecting at most their endpoints. Since we are interesting in the dynamics of T, we need to understand how these singular sets evolve under T. In other words, we have to compute the T-iterates of S_1^+ and S_1^-. For any $n > 1$, let $S_n^+ = S_1^+ \cup T^{-1}S_1^+ \cup \cdots \cup T^{-n+1}S_1^+$. This is the singular set of T^n, i.e., the subset of M where T^n is not defined or C^1. Similarly $S_n^- = S_1^- \cup TS_1^- \cup \cdots \cup T^{n-1}S_1^-$ is the singular set of T^{-n}, and we have $S_n^- = RS_n^+$. We collect the singularities of all the positive and negative powers of T into $S_\infty^+ = \cup_{n \geq 1} S_n^+$ and $S_\infty^- = \cup_{n \geq 1} S_n^-$, and denote by S_∞ their intersection. The last set consists of elements of M for which both the positive semi-trajectory and the negative semi-trajectory hits a corner of Γ or hits tangentially a dispersing component of Γ. Since all the sets $S_1, \ldots, S_4$ has zero measure, it follows immediately that all the sets introduced in this subsection have zero measure as well.

Metrics on M. In this subsection, we introduce two Riemannian metrics g and g' for the billiard phase space M. Selecting a suitable Riemannian metric for a specific class of billiards is quite a delicate matter. For a discussion on this issue, see, for example, Section 2.1 of [BCST].

The metric g is just the Euclidean metric $ds^2 + d\theta^2$ in coordinates (s, θ). The metric g' is defined in terms of transversal Jacobi fields J and their derivatives J'. Given a $z \in M$ and a vector $u \in \mathcal{T}_z M$, let $(ds, d\theta)$ be the components of u with respect to the basis $\{\partial/\partial s, \partial/\partial \theta\}$. The transversal Jacobi field and its derivate associated to u are given by

$$\begin{cases} J = \sin\theta ds, \\ J' = -\kappa(s) ds - d\theta. \end{cases} \tag{1}$$

J is a Jacobi field along the billiard trajectory of z and is called transversal, because it is orthogonal to that trajectory. For further readings on Jacobi fields and billiards, see [W3, Do, M4]. The metric g' is then given by $J^2 + J'^2$. This metric[1] appeared before in several papers (for instance, [BCST]; see also the references therein) where J and g' are called the p-norm of u and the invariant metric, respectively. The relevance of g' for the study of our billiards is due to the fact that it has the Noncontraction property which is crucial for the proof of the LET (Theorem 7.5). This property is proved in Subsection 8.1. The norms corresponding to g and g' will be denoted by $\|\cdot\|$ and $\|\cdot\|'$, respectively. We also introduce two distances on M in the following way: let $w, z \in M$, if $w, z \in M_i$, then $d(w, z)(d'(w, z))$ is equal to the distance generated by $g(g')$ on M_i, otherwise $d(w, z) = 1(d'(w, z) = 1)$.

[1]Actually, g' is a semi-metric, because it degenerates on S_2.

The differential of T. It is convenient to refer to element of M in terms of their local coordinates (s,θ). Therefore, throughout the paper, we will use freely both notations (q,v) and (s,θ) to denote the same element of M. Let $(q,v) \in D_T$ and $(q_1,v_1) = T(q,v)$. The functions s_1, θ_1 are defined by $s_1(s,\theta) = s(q_1(s,\theta))$ and $\theta_1(s,\theta) = \theta(v_1(s,\theta))$. We denote by

- $l = l(q,v)$ the Euclidean distance in $\mathbb{R}^2$ between q and q_1,
- $d(q,v)$ the Euclidean length of the segment of $L(q,v)$ contained in the disk tangent to Γ at q and having radius which is half of the radius of curvature of Γ at q,
- r_0 and r_1 the radius of curvature of Γ at q and q_1,
- $d_0 = d(q,v)$, i.e., $d_0 = r_0 \sin\theta(q,v)$,
- $d_1 = d_1(q,v) := d(q_1,v_1)$, i.e., $d_1(q,v) = r_1 \sin\theta_1(q,v)$.

After choosing proper parameterizations for the boundary components of Γ, the matrix of the differential of T at $z = (s,\theta)$ computed with respect to the coordinates $(ds, d\theta)$ and coordinates (J, J') takes, respectively, the forms [KS, LW]

$$D_zT = \begin{pmatrix} \frac{l-d_0}{r_0 \sin\theta_1} & \frac{l}{\sin\theta_1} \\ \frac{l-d_0-d_1}{r_0 d_1} & \frac{l-d_1}{d_1} \end{pmatrix} \qquad \text{and} \qquad D_zT = \begin{pmatrix} -1 & -l \\ \frac{2}{d_1} & \frac{2l}{d_1} - 1 \end{pmatrix}. \tag{2}$$

Note that, in the last form, D_zT factorizes as follows

$$D_zT = \begin{pmatrix} -1 & 0 \\ \frac{2}{d_1} & -1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix},$$

where the upper triangular matrix evolves vectors between consecutive reflections, whereas the lower triangular matrix "reflects" the vectors when a collision occurs.

Absolutely focusing curves. The billiard tables considered in this paper are bounded by straight lines, dispersing curves and special focusing curves called absolutely focusing.

Consider a C^3 focusing curve γ of length $\mathcal{L}$. Let s be its arclength parameter, and we denote by $r(s)$ and $\kappa(s)$ the radius of curvature and the curvature of γ, respectively. An incoming ray to γ is said to be focused by γ if the infinitesimal family of rays parallel to the incoming ray focuses (in linear approximation) between any two consecutive reflections and after leaving γ provided that the ray leaves eventually γ.

Definition 2.1. *A focusing curve γ is called absolutely focusing if*

a) *all the incoming rays are focused by γ,*

b) $\int_0^{\mathcal{L}} \kappa(s) ds \le \pi$.

By a result due to Halpern [Ha], Condition (b) implies that no billiard trajectory can undergo an infinite sequence of consecutive reflections at γ. Absolutely focusing curves were studied independently by Bunimovich [Bu4] and Donnay [Do]; the term absolutely focusing was introduced in [Bu4]. Two important classes of curves are known to satisfy Condition (a).

The first class is formed by the so called *convex scatterers* introduced by Wojtkowski [W2]. A convex scatterer γ is a focusing curve verifying the condition

$$d(z) + d(z_1) \leq l(z)$$

for any two consecutive collisions $z = (q, v)$ and $z_1 = (q_1, v_1)$ at γ. If γ is C^4, then this condition is equivalent to $d^2r/ds^2 \leq 0$. The second class of curves satisfying Condition (a) was introduced by Markarian [M1] (see also [CM1]); these curves have the property that

$$d(z)(l(z) + l(z')) \leq l(z)l(z')$$

for any two consecutive reflections z and z' on them. Examples of Wojtkowski's and Markarian's curves are arcs of circles, cardioids, logarithmic spirals and the arcs of the ellipse $x^2/a^2+y^2/b^2 = 1, |x| \leq a/\sqrt{2}$ and $x^2/a^2 + y^2/b^2 = 1, |x| \geq b^4/(a^2 + b^2)$ where $a, b > 0$. An example of an absolutely focusing curve which is not a Wojtkowski's or Markarian's curve is the half-ellipse $x^2/a^2 + y^2/b^2 = 1, x \geq 0$ with $a/b < \sqrt{2}$ [Do].

Remark 2.2. *In this paper, we will only consider absolutely focusing curves of class C^6. We should however note that our results holds as well for Wojtkowski's and Markarian's curves which are of class C^4. The C^6 regularity is only required for the construction of an invariant cone fields for general absolutely focusing curves (see [Do] and Section 5.3 of this paper). It is well known that C^4 Wojtkowski's and Markarian's curves have a continuous or a piecewise continuous invariant cone field with finite focusing times [W2, M1]; these two properties make it possible to apply the results of this paper to Wojtkowski's and Markarian's curves.*

Some important properties of C^6 absolutely focusing curves are summarized here

1. Sufficiently short arcs of a focusing curve are absolutely focusing (Theorem 1 of [Do]).
2. Consider the space of focusing curves of class C^6 having the same length $\mathcal{L}$ and satisfying the condition $\int_0^{\mathcal{L}} \kappa(s)ds \leq \pi$. If we endow this space with the C^6 topology, then the subset of absolutely focusing curves is open (Theorem 4 of [Do]).

3 Main results

The billiards considered in this paper satisfy the following conditions besides the general ones described at the beginning of Section 2.

Let N_∞^+ be the set consisting of elements of $\operatorname{int} M \setminus S_\infty^+$ whose positive semi-trajectories hit only Γ_0 eventually. Similarly we define N_∞^- by replacing T with T^{-1}. In fact, we could simply define $N_\infty^- := RN_\infty^+$. Both semi-trajectories of elements of $N_\infty := N_\infty^- \cap N_\infty^+$ are infinite and hits only Γ_0 eventually so that element of N_∞ are not hyperbolic. Finally let ℓ be the one-dimensional volume on $S_1^- \cup S_1^+$ generated by the Euclidean metric.

B1. Each component of the boundary of the billiard table Q is either a straight line or a dispersing curve or an absolutely focusing curve of class C^6 (C^4 Wojtkowski's and Markarian's focusing curves are allowed as well). Furthermore Q is not a polygon, i.e, $\Gamma_+ \cup \Gamma_- \neq \emptyset$.

B2. The infimum of the length of all trajectories starting at any focusing component of Γ and ending at any other non-flat component of Γ is uniformly bounded below by a positive constant depending on the focusing components of Γ.

B3. $\mu(N_\infty^+) = 0$ and $\ell(N_\infty^+ \cap S_\infty^-) = 0$.

The description of Condition B2 given here is somewhat vague, and we postpone a more precise formulation to Section 6. By now we only observe that B2 imposes some restrictions on the distance and the angle between components of Γ (see Remark 6.3). When these are Wojtkowski's and Markarian's curves, B2 has a simple geometrical formulation in terms of the relative position of the circles of semi-curvature of distinct focusing curves [Bu2, W2], or in terms of the distance of the circles of curvature of focusing curves from the other curves of Γ [M1]. As we deal with a larger class of focusing components our Condition B2 has to be more more general, and, consequently, more involved.

Condition B3 concerns the measure of certain subsets of M supporting non-hyperbolic trajectories. The first part of Condition B3 means that the subset of trajectories which hit only straight lines eventually is irrelevant from the point of view of the invariant measure. This is a necessary condition to obtain hyperbolicity over the entire phase space M, and it is generically true because polygonal billiards are generically ergodic [KMS]. The second part of B3 is a technical condition, and allows to prove that the Sinai-Chernov Ansatz, one of the hypotheses of the LET (Condition E4 of Theorem 7.5), is verified. We do not know whether the second equality of B3 is implied by the first maybe together with some simple geometric conditions on Q. Note that the time-reversing simmetry implies the symmetric equalities $\mu(N_\infty^-) = \ell(N_\infty^- \cap S_\infty^+) = 0$.

Let $NS = (N_\infty^- \cap S_\infty^+) \cup (N_\infty^+ \cap S_\infty^-)$. This set, which consists of elements of M for which one semi-trajectory is finite, and the other hits only Γ_0 eventually, also supports non-hyperbolic trajectories. The set

$$H := \operatorname{int} M \setminus (S_\infty \cup N_\infty \cup NS)$$

contains the hyperbolic set of M. We will see that the LET applies not only to hyperbolic points of M but to every point of H.

A billiard (M, μ, T) satisfying B1-B3 is hyperbolic (see Section 6). By general results on hyperbolic systems [P, KS, CH, OW], any ergodic component E of T has a finite partition (mod 0) $\{A_1, \ldots, A_N\}$ with N depending on E such that $TA_k = A_{k+1}, TA_N = A_1$ and $T^N|_{A_k}$ is Bernoulli with respect to the probability measure $\mu/\mu(E)$. Any set A_k will be called a *Bernoulli component of* T.

The main result of this paper is the following theorem.

Theorem 3.1. *For any billiard satisfying B1-B3, we have*

1. *the billiard map* T *is hyperbolic,*
2. *every point of* H *has a neighborhood belonging (mod 0) to one Bernoulli component of* T.

To ensure that T is globally ergodic and Bernoulli, we need a further condition which guarantees that the complement of H is topologically small.

B4. The set $S_\infty \cup N_\infty \cup NS$ does not disconnect $\operatorname{int} M_i$ for every $1 \le i \le n$.

Theorem 3.2. *Any billiard satisfying B1-B4 is Bernoulli.*

Theorems 3.1 and 3.2 are proved in Subsection 8.2. We will see that Conditions B1-B3 imply that S_∞ is countable so that the validity of B4 depends only on the topological properties of the sets N_∞ and NS. Condition B4 can be weaken, and one can still obtain the Bernoulli property. In Section 9, we will consider several classes of billiards which do not satisfy B4, and even though Theorem 3.2 cannot be applied directly to them, we will show that these billiards are Bernoulli.

4 The induced system (Ω, ν, Φ)

In all the analysis carried out in this paper, an extremely important role is played by an invariant cone field for the billiard dynamics. On this topic, we refer the reader to Section 5 and the references therein contained. While certain billiards, like Bunimovich's billiards and Wojtkowski's billiards, admit everywhere continuous invariant cone fields on the phase phase, it seems that general hyperbolic planar billiards can only admit a piecewise invariant continuous cone field (see [Do]). The discontinuity set of this field can be particularly complicated on the subset M_0 of the billiard phase space M; this fact makes very difficult the verification of the hypotheses (in particular E4) of the LET (Theorem 7.5) which is the central result in the proof of the Bernoulli property of hyperbolic billiards. To overcome this problem, we will work with a new system (Ω, ν, Φ) which is the first return map induced by T on a suitable subset Ω of M. It turns out that it is much easier to deal with the discontinuity set of the invariant cone field of Φ (see Subsection 5.3) than with that of T. In Section 8, we will show that Φ satisfies the hypotheses of Theorem 7.5, and then that the conclusion of this theorem is valid for T as well. Throughout this section, we will use several results proved in Appendix.

Given a smooth curve $\gamma \in M$, we denote by $\operatorname{int}\gamma, \partial\gamma, \overline{\gamma}$, respectively, the interior, the boundary and the closure of γ in the relative topology of γ. Given two points $q_1, q_2 \in \mathbb{R}^2$, let (q_1, q_2) be the open segment joining q_1, q_2.

Definition 4.1. *We say that a set $\Lambda \subset M$ is regular if its closure is an union of $k > 0$ smooth compact curves $\gamma_1, \ldots, \gamma_k$ such that $\gamma_i \cap \gamma_j \subset \partial\gamma_i \cap \partial\gamma_j$ for $i \neq j$.*

Definition 4.2. *A compact and connected set $B \subset M$ is called a* box *if B coincides with the closure of* int *and ∂B is a regular set.*

In the following, the symbol B will always denote a finite union of boxes of M intersecting at most at their boundaries.

Definition 4.3. *Let $\Lambda \subset B$ be a regular set, and $\gamma_1, \ldots, \gamma_k$ be smooth compact curves as in Definition 4.1. We say that Λ is* neat *in B if $\partial\gamma_i \subset \partial B \cup (\cup_{j \neq i} \gamma_j)$ for any $1 \leq i \leq k$. The word* neat *alone is used in place of* neat in M.

Remark 4.4. *If Λ is neat in B, then it is easy to show that Λ* partitions *B, meaning that there exist $k > 0$ boxes $B_1, \ldots, B_k$ contained in B such that $\operatorname{int} B_1, \ldots, \operatorname{int} B_k$ are the connected components of $\operatorname{int} B \setminus \Lambda$.*

Definition 4.5. *Let $V_1^+ = S_1^+ \cap M_0$, and for $k > 1$, define inductively*

$$V_k^+ = (T^{-1}V_{k-1}^+ \cap M_0) \cup V_1^+.$$

We define similarly V_n^- by replacing S_1^+, T^{-1} by S_1^-, T. The sets $V_k^\pm$ consist of elements of M_0 having at most $k-1$ consecutive collisions with Γ_0 before hitting a corner of Γ or having a tangential collision at Γ_-. Let $V_k = V_k^- \cup V_k^+$, and $\partial M_0 = \partial M \cap M_0$.

Lemma 4.6. *V_n partitions M_0 into $N = N(n) > 0$ boxes $B_1, \dots, B_N$ such that for every $1 \le i \le N$ only one of the following possibilities can occur:*

1. *$T^j \operatorname{int} B_i \subset M_0$ for any $0 \le j \le n-1$,*
2. *$\exists\, 0 \le k < n-1$ such that $T^j \operatorname{int} B_i \subset M_0$ for any $0 \le j \le k$ and $T^{k+1} \operatorname{int} B_i \subset M_- \cup M_+$.*

The same is true if T is replaced by T^{-1}.

Proof. To prove that V_n partitions M_0, it is enough to show that V_n is neat. This is a consequence of Corollary A.5 and the finiteness of $\overline{V_n^+} \cap \overline{V_n^-}$. The last fact can be proved as follows. By Corollary A.5, the sets V_n^+, V_n^- are finite unions of proper sets of type $C_{\boldsymbol{i}}^m$ (see Definition A.2). Consider two of such sets $C_{\boldsymbol{i}}^m, C_{\boldsymbol{i}'}^{m'}$ such that $C_{\boldsymbol{i}}^m \in V_n^+$ and $C_{\boldsymbol{i}'}^{m'} \in V_n^-$. Let γ_1 be the closure of a connected component of $C_{\boldsymbol{i}}^m$, and γ_2 be the closure of a connected component of $C_{\boldsymbol{i}'}^{m'}$. We are done, if we show that $\gamma_1 \cap \gamma_2$ is finite. For $i = 1, 2$, let $I_i \ni t \mapsto \gamma_i(t)$ be a regular parametrization of γ_i where I_i is a closed interval. We have $\gamma_i(t) = (s_i(t), \theta_i(t))$ in coordinates (s, θ). It is easy to check that $s_1'\theta_1' > 0$ for any $t \in \operatorname{int} I_1$, and $s_2'\theta_2' < 0$ for any $t \in \operatorname{int} I_2$. Therefore, in coordinates (s, θ), the curves γ_1 and γ_2 are strictly increasing and strictly decreasing, respectively. It follows that $\gamma_1 \cap \gamma_2$ can contain at most one element.

We only prove the second statement of the lemma for T, because the proof for T^{-1} is the same. Let

$$k = \max\{0 \le l \le n-1 : T^j \operatorname{int} B_i \subset M_0 \text{ for any } 0 \le j \le l\}.$$

If $k = n-1$, then (1) is verified, and there is nothing to prove. Thus assume that $k < n-1$. Since $V_n \cap \operatorname{int} B_i = \emptyset$, the definition of k implies that $T^k \operatorname{int} B_i \cap S_1^+ = \emptyset$. It follows that either $T^{k+1} \operatorname{int} B_i \subset M_- \cup M_+$ or $T^{k+1} \operatorname{int} B_i \subset M_0$. The latter possibility is ruled out by the definition of k. □

Definition 4.7. *Let Δ_n be the union of boxes B_i for which the conclusion (1) of Lemma 4.6 holds, i.e.,*

$$\Delta_n = \bigcup_{i=1}^{N} \{B_i : T^{-k} \operatorname{int} B_i \subset M_0 \textit{ for all } |k| < n\}.$$

Let Λ_n be the infimum of the length of the trajectories connecting Δ_n and $M_- \cup M_+$. There are two possibilities: either there is a $n > 0$ such that $\Delta_n = \emptyset$ or not, i.e., $\Delta_n \neq \emptyset$ for every $n > 0$. In the second case, we claim that $\lim_{n \to +\infty} \Lambda_n = +\infty$: this fact is only not immediately evident for trajectories entering one or more wedges formed by straight lines of Γ, which, one could think, may have lots of reflections in a short amount of time. However it is easy to show that this is not the case: such trajectories, in fact, leave a wedge after a finite number of collisions which only depends on the angle of wedge and not on the trajectory [CM2].

An example of the dichotomy pertaining Δ_n is the following: if Q is a stadium-like billiard with parallel straight lines, then we have $\Delta_n \neq \emptyset$ for any $n > 0$, but if the two straight lines are not parallel, then $\Delta_n = \emptyset$ for any n sufficiently large. Note that $N_\infty \subset \cap_n \Delta_n$.

We choose $\bar{n} > 0$ sufficiently large so that $\Lambda_{\bar{n}} > \max_i \tau_i + \bar{\tau}$ where the maximum is taken over all the focusing components of Γ. The constants τ_i and $\bar{\tau}$ will be introduced in Section 6; they depend on the geometry of the billiard table Q and its invariant cone field (see Subsection 5.3).

We can now define the induced system (Ω, ν, Φ). Let $\Delta = \Delta_{\bar{n}}$.

Definition 4.8. *Let $\Omega = M_- \cup M_+ \cup \Delta$ and $\Phi : \Omega \to \Omega$ be the first return map on Ω induced by T, i.e., if $t(z) = \inf\{k > 0 : z \notin S_k^+ \text{ and } T^k z \in \Omega\}$ is the first return time of z to Ω, then*

$$\Phi z = T^{t(z)} z, \quad z \in \Omega.$$

Also let $\nu = (\mu(\Omega))^{-1}\mu$ be a probability measure on Ω. Since Φ preserves the measure μ, it also preserves ν.

We stress again that the reason for introducing this new system is that for a general billiard satisfying B1-B3, the invariant cone field (see Section 5.3) could vary on M_0 in a quite complicated way. For the induced systems, instead, the cone field turns out to be much nicer.

As a consequence of the definition of Δ, the space Ω is a finite union of boxes with boundaries lying on $\partial M \cup V_{\bar{n}}$. This is an important property because it allows us to apply the LET to (Ω, ν, Φ). Let $\partial\Delta$ be the boundary of Δ and $\operatorname{int}\Delta = \Delta \setminus \partial\Delta$. Furthermore let $\partial\Omega$ be the union of $\partial\Delta$ and every ∂M_i such that $M_i \subset M_- \cup M_+$, and let $\operatorname{int}\Omega = \Omega \setminus \partial\Omega$. It is easy to check that $1 \leq t(z) \leq 2\bar{n} - 1$ for every $z \in \Omega$. Like T, the induced map Φ has singularities. In fact, during the process of inducing, in addition to the singularities produced by taking powers of T, new singularities are created as the return time t is not constant. We denote by $\mathcal{S}_1^+$ the singularity set of Φ, i.e., the set of points of Ω where Φ is not defined or is not C^1. The singular set for Φ^{-1} is defined similarly. For $k > 1$, the singular set of Φ^k is given by $\mathcal{S}_k^+ = \mathcal{S}_1^+ \cup \Phi^{-1}\mathcal{S}_1^+ \cup \cdots \cup \Phi^{-k+1}\mathcal{S}_1^+$, and the singular set of Φ^{-k} is given by $\mathcal{S}_k^- = \mathcal{S}_1^- \cup \Phi\mathcal{S}_1^- \cup \cdots \cup \Phi^{k-1}\mathcal{S}_1^-$. The union of all $\mathcal{S}_k^+$ is denoted by $\mathcal{S}_\infty^+$, and the union of all $\mathcal{S}_k^-$ by $\mathcal{S}_\infty^-$. Note that the map Φ inherits from T the time-reversing property, i.e., $\Phi^{-1} = R\Phi R$, so that the singular sets of Φ^{-1} are just the image under R of the singular sets of Φ and vice-versa. We also define $\mathcal{S}_\infty = \mathcal{S}_\infty^- \cap \mathcal{S}_\infty^+$. By previous observations, $\mathcal{S}_k^\pm$ are union of finitely many smooth curves contained in $S_{k(3\bar{n}-1)}^\pm$. It follows that all the sets just defined have zero ν-measure. In Proposition 8.2, we will show that $\mathcal{S}_k^+$ and $\mathcal{S}_k^-$ are neat (as we deal now with Ω, ∂M has to be replaced by $\partial\Omega$ in the definition on neatness).

5 Geometric optics and cone fields

Invariant cone fields are used to prove hyperbolicity and statistical properties of dynamical systems like ergodicity, decay of correlations, etc. [W1, LW, Li]. In this section, we give the necessary definitions, and recall some general concepts from geometrical optics. In Subsection 5.3, we will introduce an invariant cone field for billiards satisfying Conditions B1 and B2.

5.1 Geometric optics

Variations. A *variation* $\{\eta(\alpha)\}_{\alpha\in I}$ is an one-parameter smooth family of lines in $\mathbb{R}^2$

$$\eta(\alpha) = \{q(\alpha) + tv(\alpha), t \in \mathbb{R}\}$$

where $I = (-\epsilon, \epsilon), \epsilon > 0$, $q, v : I \to \mathbb{R}^2$ are smooth, and $v(\alpha)$ is a unit vector for every $\alpha \in I$. Let $\eta(\alpha, t) = q(\alpha) + tv(\alpha)$, and let $v^\perp$ be a vector orthogonal to $v(0)$. We say that the variation $\{\eta(\alpha)\}$ focuses along $\eta(0)$ if

$$\langle \frac{\partial \eta}{\partial \alpha}(0,t), v^\perp\rangle = 0 \quad \text{for some } t \in \mathbb{R}. \tag{3}$$

If $v'(0) \neq 0$, then

$$t = -\frac{\langle q'(0), v'(0)\rangle}{\langle v'(0), v'(0)\rangle}$$

is the unique solution of (3), and we call it the *focusing time* of $\{\eta(\alpha)\}$. When $v'(0) = 0$, the variation consists, in linear approximation, of parallel lines. If $\langle q'(0), v^\perp\rangle \neq 0$, then (3) does not have a solution, and we set $t = \infty$. Finally if $\langle q'(0), v^\perp\rangle = 0$, then we set $t = 0$. We say that a variation is *convergent*, *divergent* or *flat* if its focusing time is positive, non-positive and infinite, respectively.

Focusing times. For every vector $u \in \mathcal{T}_z M, z \in M$, there is a family of curves in M associated to it. Each curve $\zeta = (q, v) : (-\epsilon, \epsilon) \to M$ of this family has the property that $\zeta(0) = (q(0), v(0)) = 0$ and $\zeta'(0) = (q'(0), v'(0)) = u$. One can associate to such a curve ζ, the variation $\eta(\alpha) = \{q(\alpha) + tv(\alpha), t \in \mathbb{R}\}$. The focusing time is the same for every curve ζ in the same family so that it only depends on u, and it makes sense to call it the focusing time of u. We say that u is convergent or divergent or flat if a variation $\eta(\alpha)$ associated to u has the corresponding property.

For a vector $u \in \mathcal{T}_z M$, let (u_s, u_θ) and (J, J') be its components with respect to the two bases described in Section 2. A straightforward computation gives (see for instance [W2])

$$t = \frac{\sin\theta}{\kappa(s) + \frac{u_\theta}{u_s}} = -\frac{J}{J'}.$$

the focusing time is a local coordinate of the projectivization of $\mathcal{T}_z M$ (the space of lines in $\mathcal{T}_z M$) and it will be uses to describe cone fields on M. Recall that the $R(s,\theta) = (s, \pi - \theta)$ for any $(s,\theta) \in M$. The vector $-u = D_z R u \in \mathcal{T}_{Rz} M$ is obtained by reflecting u about Γ at $\pi(z)$. Its focusing time is given by

$$t = \frac{\sin\theta}{\kappa(s) - \frac{u_\theta}{u_s}}.$$

For any $u \in \mathcal{T}_z M, z \in M$, let $\tau^+(z,u)$ be the the focusing time of u, and let $\tau^-(z,u)$ be the focusing time of $-u$. Hence

$$\tau^\pm(z,u) = \frac{\sin\theta}{\kappa(s) \pm \frac{u_\theta}{u_s}}. \tag{4}$$

The proof of the next statements can be found in [W2, Do].

Reflection Law. Let $z = (s, \theta) \in D_T$, and $Tz = (s_1, \theta_1)$. For any $u \in \mathcal{T}_z M$, let $\tau_0 = \tau^+(z, u)$, and let $\tau_1 = \tau^+(Tz, D_z T u)$. The relation between τ_0 and τ_1 is given by

$$\frac{1}{\tau_1} + \frac{1}{l(z) - \tau_0} = \frac{2\kappa(s_1)}{\sin\theta_1}. \tag{5}$$

Ordering property. Let $z \in D_T$ such that $Tz \in M_+$, and let $u, w \in \mathcal{T}_z M$. Assume that $0 < \tau^+(z, w) < l(z)$ and $0 < \tau^+(Tz, D_z T w)$. Then, as a direct consequence of (5), we obtain

$$\tau^+(z, u) \le \tau^+(z, w) \Rightarrow 0 < \tau^+(Tz, D_z T u) \le \tau^+(Tz, D_z T w).$$

The implication is also true if we replace the inequalities with strict inequalities.

5.2 Cone fields

Let V be a two dimensional vector space. A cone in V is the subset $C(X_1, X_2) = \{aX_1 + bX_2 : ab \ge 0\}$ where X_1 and X_2 are linearly independent vectors of V and a, b real numbers. If $\overline{0}$ denotes the zero element of V, then $\operatorname{int} C(X_1, X_2) = \{aX_1 + bX_2 : ab > 0\} \cup \{\overline{0}\}$ is the interior of C. The cone $C'(X_1, X_2) := C(X_1, X_2)$ is called the *complementary cone* of $C(X_1, X_2)$.

Definition 5.1. *A measurable cone field C on the billiard phase space M is a family of cones $C(z) = C(X_1(z), X_2(z)) \subset \mathcal{T}_z M$ defined for μ-almost $z \in M$ such that the vectors $X_1(z)$ and $X_2(z)$ vary measurably with z. We say that the cone field C is invariant (strictly) if $D_z T C(z) \subset C(Tz)$ $(\operatorname{int} C(Tz))$ for μ-almost every $z \in M$. We say that C is eventually strictly invariant if it is invariant, and for μ-almost every $z \in M$ there exists a positive integer $m(z)$ such that $D_z T^{m(z)} C(z) \subset \operatorname{int} C(T^{m(z)} z)$. Let C' denote the complementary cone field of C.*

For every $z \in M$, define

$$\tau^+(z) = \sup_{u \in C(z)} \tau^+(z, u),$$
$$\tau^-(z) = \sup_{u \in C'(z)} \tau^-(z, u).$$

The next lemma provides a simple criterion for checking whether a cone field on M_+ is invariant. For its proof, see [Do].

Lemma 5.2. *Let $z \in M_+ \cap D_T$ such that $Tz \in M_+$. Suppose that $0 \le \tau^+(z), \tau^-(Tz) \le l(z)$. Then*

$$0 < \tau^+(z) + \tau^-(Tz) \le (<)\, l(z) \Rightarrow D_z T C(z) \subseteq C(Tz)\, (\operatorname{int} C(Tz)).$$

5.3 Cone fields for billiards

Following [S, W1, Do], we introduce a cone field C on the restricted phase space Ω for billiards satisfying Condition B1. In Section 6, we will show that if Condition B2 and B3 are also verified, then C is eventually strictly invariant. This cone field is not everywhere continuous, and so, in the last part of this section, we study its discontinuity set.

Consider a focusing component Γ_i of Γ. We adopt the convention that if a vector $z \in M_i$ hits along its orbit an endpoint of Γ_i, then it gets reflected by Γ_i in the usual way. This assumption makes the map T a diffeomorphism from $M_i \cap T^{-1}M_i$ to $M_i \cap TM_i$ (see [KS]).

Let

$$R_1^+ = \{z \in \operatorname{int} M_i : q_1(z) \in \partial\Gamma_i\},$$

and for $n > 1$, define inductively

$$R_n^+ = (T^{-1}R_{n-1}^+ \cap M_i) \cup R_1^+.$$

The set R_n^+ consists of elements of M_i having at most $n-1$ consecutive collisions with Γ_i before hitting $\partial\Gamma_i$; it is somewhat the singular set of T^n for trajectories moving only along Γ_i. Also let $R_\infty^+ = \cup_{n>0} R_n^+$. The sets R_n^-, R_∞^- are defined similarly by considering trajectories backward in time. It follows from that the time-reversing symmetry that $R_n^- = RR_n^+$ and $R_\infty^- = RR_\infty^+$.

The function $\varrho : M_i \to \mathbb{N} \cup \{+\infty\}$ is defined as follows

$$\varrho(z) = \begin{cases} +\infty & \text{if } T^k z \in M_i, k = 1, 2, \ldots, \\ \min\{k \geq 0 : q_1(T^k z) \notin \Gamma_i\} & \text{otherwise.} \end{cases}$$

This function gives the number of consecutive reflections of z along Γ_i before leaving it. Thus

$$E := R\varrho^{-1}(0) \setminus S_2$$

is the subset of M_i consisting of vectors entering Γ_i, i.e., vectors leaving Γ_i in the past. For every $m \geq 0$, the set $E_m := E \cap \varrho^{-1}(m)$ is made of entering vectors of M_i which leave Γ_i exactly after m consecutive collisions. Clearly $E = \cup_{m\geq 0} E_m$, and we will see at the end of the next subsection that the closure of each set E and E_m is a box (see Fig. 1).

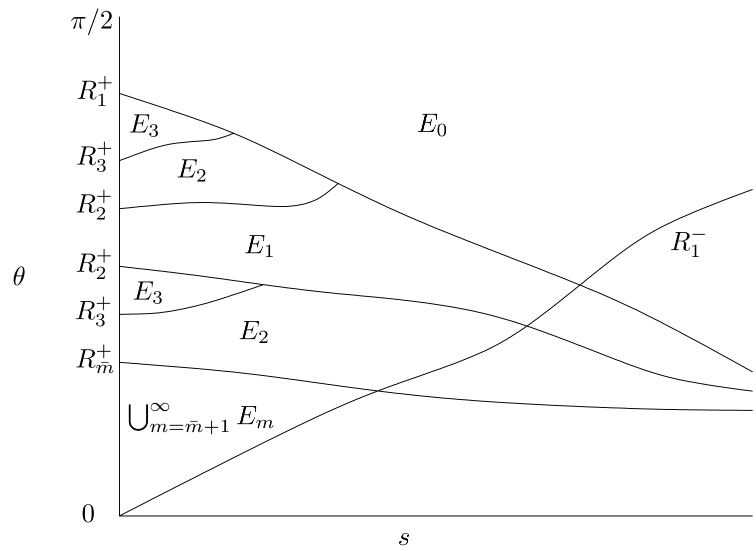

Figure 1: The sets $R_n^\pm$ and E_m in a neighborhood of $s = \theta = 0$

A cone field for absolutely focusing curves. We construct now the cone field C on M_+. This construction is due to Donnay [Do] and is done for a single focusing component of Γ_i. By repeating it for every focusing component of Γ, we obtain C on the entire M_+.

For every $z \in E$, let $X^-(z)$ and $X^+(z)$ be vectors of $\mathcal{T}_z M_i$ such that their s-component is non-negative and

$$\tau^-(z, X^-(z)) = \tau^+(T^{\varrho(z)}z, D_z T^{\varrho(z)} X^+(z)) = +\infty.$$

The second condition means that the evolution of $X^-(z)$ backward in time gives raise to a parallel variation, and that the $\varrho(z)$ iteration of $X^+(z)$ forward in time, i.e., when it leaves Γ_i, is a parallel variation. Given a tangent vector $u = (u_s, u_\theta)$, let $m(u) := u_\theta/u_s$ be its slope. Using the fact that Γ_i is absolutely focusing, one can easily check that

- $-\kappa(z) < m(X^+(z)) < m(X^-(z)) = \kappa(z)$ for any $z \in E$,
- X^- is continuous on E, and X^+ is continuous on every E_m.

Suppose now that there exists a unit vector field X_l on E such that

1. $m(X^+(z)) < m(X_l(z)) < m(X^-(z))$ for any $z \in E$,
2. $\sup_{z\in E_i} \tau^-(z, X_l(z))$ and $\sup_{z\in E_i} \tau^+(\Phi^{\varrho(z)}z, D_z\Phi^{\varrho(z)}X_l(z))$ are both finite.

These conditions say that the vector $X_l(z)$ leaves Γ_i after finitely many collisions, that it focuses between any two of such collisions and after the last one, and that the focusing times before the first and the last collisions are uniformly bounded on $z \in E$. If we define

$$C_E(z) := C\left(\frac{\partial}{\partial\theta}(z), X_l(z)\right), \quad z \in E \setminus S_2,$$

or, equivalently, $C_E(z) := \{u \in \mathcal{T}_z E : m(X_l(z)) \le m(u)\}$, then Donnay's cone field on $M_i \setminus S_2$ is given by

$$C = \{D_z T^k C_E(z) : z \in E \text{ and } 0 \le k \le \varrho(z)\}.$$

By the properties above, and using the ordering property (see the previous subsection), we see that every vector of C has forward and backward focusing times which are uniformly bounded.

To show the existence of a vector field X_l, we first observe that for every fixed $\bar{m} > 0$, any linear combination $aX^- + bX^+$ such that $ab > 0$ verifies Properties (1) and (2) on $\cup_{0\le m\le\bar{m}} E_m$ (not on the entire E). We introduce now Lazutkin's coordinates (x, y) on M_i; these are given by

$$x = C_1 \int_0^s r^{-\frac{2}{3}}(t)dt, \qquad y = C_2 r^{\frac{1}{3}}(s) \sin\frac{\theta}{2} \tag{6}$$

where C_1 and C_2 are constants depending on Γ_i (see [La, Do] for more details). Lemma 5.9 of [Do] implies that the vector $\partial/\partial x$ verifies Properties (1) and (2) on $\cup_{m=\bar{m}+1} E_m$ for $\bar{m} > 0$ sufficiently large depending on Γ_i. Therefore a choice for the vector field X_l is given by

$$X_l(z) = \begin{cases} aX^-(z) + bX^+(z) & \text{if } z \in \bigcup_{m=0}^{\bar{m}} E_m, \\ \frac{\partial}{\partial x}(z) & \text{if } z \in \bigcup_{m=\bar{m}+1}^{+\infty} E_m, \end{cases}$$

for any positive a, b and $\bar{m} > 0$ sufficiently large.

In this way, we obtain a family of cone fields on M_i depending on the parameters $a, b, \bar{m}$. Note that $\bar{m}$ is the lower bound for the number of consecutive reflections of vectors in $E \setminus \cup_{m=0}^{\bar{m}} E_m$. We select now a specific cone field, which will be used through all this paper, by choosing $a, b, \bar{m}$ as follows: a, b are any two positive reals, whereas $\bar{m}$ is any sufficiently large positive integer such that the results of Section 5 of [Do] [2] apply to $\partial/\partial x(z)$ on $E \setminus \cup_{m=0}^{\bar{m}} E_m$, and the right-hand side of (10) in the proof of Lemma 8.15 is greater than $e^{-2(b_1'+b_1'')b_3}/2$ (which is a constant depending only on Γ_+). We observe that the smallness of the Lazutkin's coordinate y in (8.15) is controlled by $\bar{m}$, and that the numbers b_1', b_1'', b_3 depend on Γ_i. This choice of $\bar{m}$ is technical and serves to simplify the proof of Lemma 8.15.

We introduce now a quantity τ_i which measure how far the focusing component Γ_i has to be placed from other non-flat components of Γ in order to obtain hyperbolicity. This quantity plays an important role in Condition B2 (see Section 6). Here the notation is as at the end the previous section. Let

$$\tau_i^{\pm} = \sup_{z \in M_i \setminus S_2} \tau^{\pm}(z) \qquad \text{and} \qquad \tau_i = \max\{\tau_i^+, \tau_i^-\}. \tag{7}$$

It follows from Property (2) that τ_i is finite.

Remark 5.3. *We observe that the choice of $a, b, \bar{m}$ effects the value of τ_i. We do not know how these parameters in order to obtain the optimal, i.e., the smallest τ_i. For a discussion on this point, see also Section 4 of [Do].*

By repeating the construction just described for every focusing component of Γ, we obtain a cone field on $M_+ \setminus S_2$. If $\tilde{m}$ is the maximum of $\bar{m}$ over all focusing components of Γ, then we can assume that all values $\bar{m}$ are equal to $\tilde{m}$. Finally denote by τ the maximum of τ_i where over all the focusing components of Γ.

A cone field for dispersing curves and straight lines (Δ). To complete the construction of C on M, it remains to define it on $M_- \cup \Delta$. Using coordinates (s, θ), we set

$$C(z) = \{(u_s, u_\theta) \in \mathcal{T}_z M : u_s u_\theta \leq 0\}, \quad z \in M_- \cup \Delta.$$

This means that the cone $C(z)$ consists of divergent vectors of $\mathcal{T}_z M$ which focus inside the half-osculating disk of Γ at $\pi(z)$ for $z \in M_-$, whereas $C(z)$ consists of all divergent vectors of $\mathcal{T}_z M$ for $z \in \Delta$. We could have equivalently defined C on $\operatorname{int} M_- \cup \operatorname{int} \Delta$ as follows

$$C(z) = \begin{cases} \{u \in \mathcal{T}_z M : -|d(z)| \leq \tau^+(z, u) \leq 0\} & \text{if } z \in \operatorname{int} M_- \\ \{u \in \mathcal{T}_z M : \tau^+(z, u) \leq 0\} & \text{if } z \in \operatorname{int} \Delta. \end{cases}$$

This finishes the construction of the cone field C for billiards satisfying B1.

[2] Donnay's results are formulated in terms of the closeness of the angle θ to 0 or π. Large m's correspond to θ's close to 0 or π.

5.4 Discontinuities of cone fields

We investigate now the discontinuity set C and its iterates under the map Φ.

Let $\mathcal{R}_n^{\pm}$ and $\mathcal{E}$ be, respectively, the union of the sets $R_n^{\pm}$ and E corresponding to every focusing component of Γ. The sets $\mathcal{R}_\infty^{\pm}, \mathcal{E}_m$ are defined similarly. We will show, in Proposition 8.2, that $\mathcal{S}_m^+, \mathcal{S}_n^-$ are neat in Ω, and that their intersection consists of finitely many points. The sets $\mathcal{R}_m^+, \mathcal{R}_n^-$ have the same properties, because they are union of smooth compact curves contained in $\mathcal{S}_m^+$ and $\mathcal{S}_n^-$, respectively. It follows immediately that any set of the form $\mathcal{R}_m^+ \cup \mathcal{R}_n^-$ partitions M_+ into finitely many boxes. Results contained in Section 5 and Appendix A2 of [Do] imply that $\lim_{m\to+\infty} \operatorname{dist}(\mathcal{R}_m^{\pm}, S_2) = 0$, where dist is the distance generated by the metric $ds^2 + d\theta^2$.

Using the previous observations and keeping in mind the construction of C, one can easily prove the next proposition that collects several useful facts concerning the continuity of C.

Proposition 5.4. *The cone field C has the following properties:*

1. *C is continuous on $(\cup_{0\le k\le m} T^k\mathcal{E}_m) \cup M_- \cup \Delta$ for any $m > 0$,*
2. *the restriction of $C|_{T^k\mathcal{E}_m}$ has a continuous extension to $\overline{T^k\mathcal{E}_m}$,*
3. *if D_0 is the set of the discontinuities of C, then $D_0 \subset \mathcal{R}_{\tilde m+1}^+ \cup \mathcal{R}_\infty^-$,*
4. *for $k > 0$, let $D_k = D_0 \cup \Phi^{-k} D_0$, and we have $D_k \subset \mathcal{S}_{\tilde m+1+k}^+ \cup \mathcal{R}_\infty^-$,*
5. *the closure of $\mathcal{S}_{\tilde m+k+1}^+ \cup \mathcal{R}_\infty^-$ is a countable union of smooth compact curves intersecting at their endpoints such that only finitely many curves can intersect at points of* $\operatorname{int}\Omega$.

6 Condition B2 and hyperbolicity

In this section, we give a precise formulation of Condition B2 and give a proof of the hyperbolicity of billiards satisfying Conditions B1-B3. The cone field C considered in this section (and the remaining sections of this paper) is the one defined in the previous section.

6.1 Condition B2

Definition 6.1. *Consider two distinct non-flat components Γ_i, Γ_j of Γ. Let $d_{i,j}$ be the infimum of the length of the trajectories starting at Γ_i and ending at Γ_j possibly after hitting Γ_0 finitely many times. Also define*

$$\tau_{i,j} = \begin{cases} \tau_i + \tau_j & \text{if } \Gamma_i, \Gamma_j \subset \Gamma_+, \\ \tau_i & \text{if } \Gamma_i \subset \Gamma_+, \Gamma_j \subset \Gamma_-, \\ 0 & \text{if } \Gamma_i, \Gamma_j \subset \Gamma_-, \end{cases}$$

where τ_i is defined in (7).

Definition 6.2. *We say that a corner of Γ is* polygonal *if it does not belong to any focusing or dispersing component of Γ. Given a focusing component Γ_i, let p_i be the distance with respect to the Euclidean metric in $\mathbb{R}^2$ between Γ_i and the set of polygonal corners of Γ.*

Condition B2: A billiard in a domain Q satisfies Condition B2 if there exists $\bar{\tau} > 0$ such that

1. $d_{i,j} > \tau_{i,j} + \bar{\tau}$ for any $\Gamma_i, \Gamma_j \subset \Gamma_+ \cup \Gamma_-$,

2. $p_i > \tau_i + \bar{\tau}$ for any $\Gamma_i \subset \Gamma_+$.

The hyperbolicity of Φ (and T) is just a consequence of (1). The second condition is required for the local ergodicity of Φ. This condition, in fact, rules out billiard tables for which the singular sets $\mathcal{S}_m^+$ and $\mathcal{S}_n^-$ may have smooth components which coincide, and when this occurs, the singular sets may separate ergodic components of Φ. Consider, for example, a mushroom-like billiard where the cap of the mushroom is a semi-ellipse (which can be chosen to absolutely focusing, see Subsection 2), and the stem is a rectangle such that one side coincides with the segment joining the foci of the semi-ellipse and the other side is sufficiently long (say larger than twice the major axis of the semi-ellipse). This billiard is not locally ergodic, and, in fact, it is an example of coexistence of regular and ergodic behavior. Its phase space consists of three invariant sets of positive measure: the billiard map is regular on two of them, and is ergodic on the third. The problem here is that two polygonal corners of this billiard lie at the foci of the semi-ellipse. This geometry - we leave the computations to the reader - violates Condition (2) and creates components of $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$ which coincide. To be more specific, we observe that (2) allows us to prove the Neatness and the Proper Alignment of the singular sets $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$, Properties E2 and E3 of the hypotheses of Theorem 7.5. One may wonder whether the first condition of B2 indeed implies the second. The first condition imposes some restrictions on the distance between focusing curves and polygonal corners, but it is not difficult to construct examples of billiard tables for which only (1) (and B1) is satisfied.

Remark 6.3. *Condition B2 imposes several constrains on the geometry of a billiard table Q. Here are listed some of these constrains to which we will refer several times in the rest of this paper:*

i) *the distance between any two non-adjacent non-flat components of Γ such that one is focusing sufficiently large,*

ii) *the internal angles between two adjacent focusing curves is greater than π,*

iii) *the internal angle between a focusing curve and a dispersing curve is greater than π,*

iv) *the internal angle between a focusing curve and a straight line is greater than $\pi/2$.*

6.2 Hyperbolicity

We prove now the billiards verifying B1-B3 are hyperbolic, and the we discuss some properties of the stable and unstable manifolds of T and Φ. For billiards in convex domains bounded by absolutely focusing curves and straight lines, the hyperbolicity was first proved in [Do].

Lemma 6.4. *The maps Φ and T of billiards satisfying Conditions B1-B3 are hyperbolic.*

Proof. First we show that the hyperbolicity of T is a consequence of the hyperbolicity of Φ. Denote by $\lambda_\Phi(z)$ and $\lambda_T(z)$ the positive Lyapunov exponents of T and Φ at the $z \in \Omega$ provided they exist. By Birkhoff's Theorem

$$\bar{t}(z) := \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} t(T^k z)$$

exists for ν-a.e. $z \in \Omega$ where $t(z)$ is the first return time to Ω (see Section 4). The relation between λ_Φ and λ_T reads as follows

$$\lambda_\Phi(z) = \bar{t}(z)\lambda_T(z), \quad \nu\text{-a.e. } z \in \Omega.$$

We have $0 \le \bar{t}(z) \le 2\bar{n} + 1$ if $\bar{t}(z)$ exists (see Section 4). Thus if $\lambda_\Phi(z) > 0$ for ν-a.e. $z \in \Omega$, then $\lambda_T(z) > 0$ for μ-a.e. $z \in \Omega$. For μ-a.e. $z \in M \setminus \Omega$, there exists an integer $1 \le k \le 2\bar{n} + 1$ such that $T^k z \in \Omega$. The invariance of the Lyapunov exponents implies that $\lambda_T(z) > 0$ for μ-a.e. $z \in M$.

To prove that Φ is hyperbolic, it suffices to show that the cone field C is eventually strictly invariant on Ω. By Theorem 1 of [W2], then Φ has non-zero Lyapunov exponents. The proof that C is eventually strictly invariant is standard, and we only sketch it. We have to consider three cases: (1) $z, \Phi z \in M_+$, (2) $z \in M_+$ and $\Phi z \in M_- \cup \Delta$, and (3) $z \in M_- \cup \Delta$. Using Lemma 5.2, one can easily show that $D_z\Phi C(z) \subset C(\Phi z)$ in all cases. This is a straightforward consequence of Condition B2. In fact, $C(z)$ is pushed strictly inside $C(\Phi z)$ in all cases except when $\pi(z), \pi(\Phi z)$ belong to the same focusing arc or to Δ. To finish, we observe that the absolutely focusing property implies that every vector leaves a focusing curve after a finite number of reflections and, by B3, μ-a.e. vector with base point on Δ eventually hits a focusing or dispersing curve of Γ. □

Local manifolds. The fact that (Ω, Φ, ν) has non-zero Lyapunov exponents ν-a.e. does not automatically imply the existence of local stable manifolds and local unstable manifolds ν-a.e.. According to general results on systems with singularities, this happens if (Ω, Φ, ν) satisfies Conditions 1.1-1.3 of [KS, Part I]. It turns out that we do not need to check whether (Ω, Φ, ν) satisfies these conditions, because we know that the billiard map T does, and we can use this fact to show that the local stable manifolds and local unstable manifolds of T are also local stable manifolds and local unstable manifolds of Φ. This is done in the next lemma.

Lemma 6.5. *The map Φ of billiards satisfying B1-B3 has local stable manifolds and local unstable manifolds ν-a.e. on Ω. These manifolds are the intersection of the stable and unstable manifolds of T with Ω and are absolute continuous.*

Proof. In [KS, Part V], it is proved that T satisfies Conditions 1.1-1.3 of [KS, Part I]. We can then apply Pesin's theory to T. Since T is hyperbolic, it has local stable manifolds and local unstable manifolds μ-a.e. on M which are absolutely continuous. We claim that the connected components of these manifolds contained in Ω are local stable manifolds and local unstable manifolds of Φ. This implies automatically that the local stable and unstable manifolds of Φ are absolute continuous.

We prove the claim only for local unstable manifolds. The proof for the local stable manifolds is similar. Let $z \in \Omega$, and let $W^u_{loc}(z)$ be the local unstable manifold of T at z. The connected component of $W^u_{loc}(z) \cap \Omega$ containing z is a local stable manifold of Φ if the set $\mathcal{S}^-_k \cap W^u_{loc}(z)$ does not accumulate at z as $k \to +\infty$. In fact, this intersection is empty for any $k > 0$, because $W^u_{loc}(z)$ is a local stable manifold of T and, by Proposition 8.2, $\mathcal{S}^-_k \subset S^-_{k(2\bar{n}-1)}$. □

7 A local ergodic theorem

In this section, we prove a LET valid for two-dimensional hyperbolic systems with singularities. As explained in the introduction, the LET's found in literature [SC, KSS, Bu3, M2, C, LW] do not apply to the generality of the billiards satisfying B1-B3 and their induced systems. In fact, many of these billiards and their induced systems do not have a continuous invariant cone field and a special type of noncontracting metrics on their tangent stable and unstable spaces, which are among the hypotheses of the mention LET's. Thus the need for a LET with less restrictive assumptions. The LET presented here builds on the LET proved in [LW], and applies to systems having a piecewise continuous invariant cone field and satisfying a weak form of the Noncontraction Property (explained later in this section). We point out that even this new LET do not apply directly to all the billiards satisfying B1-B3, and we will use it with the induced systems of these billiards rather than with the billiard maps themselves.

We start by describing the hypotheses of this theorem and introducing the new mathematical objects involved in its formulation. Let $(\mathcal{M}, m, f)$ be a smooth system with singularities having the following properties.

The phase space $\mathcal{M}$. The set $\mathcal{M}$ is an union of $n > 0$ boxes $\mathcal{M}_1, \ldots, \mathcal{M}_n$ of $\mathbb{R}^2$ (see Definition 4.2) which can only intersect along their boundaries. The union of the interior and the union of the boundary of the boxes of $\mathcal{M}$ are denoted by $\operatorname{int}\mathcal{M}$ and $\partial\mathcal{M}$ respectively. The space $\mathcal{M}$ has a natural Riemannian metric which is the restriction of the Euclidean metric of $\mathbb{R}^2$ to $\mathcal{M}$. We assume that $\mathcal{M}$ is endowed with another Riemannian metric $\tilde{g}$ which can be degenerate on $\partial\mathcal{M}$ and have to satisfies certain conditions described later (see E5 and E6, later in this subsection). We denote by $\|\cdot\|$ and $|\cdot|$ the norms generated by the Euclidean metric and $\tilde{g}$, respectively. We assume that m is absolutely continuous with respect to the volume of the Euclidean metric, and has bounded density which is positive on $\operatorname{int}\mathcal{M}$.

Singular sets. Let A^+_1 and A^-_1 be neat subsets of $\mathcal{M}$ such that $f : \operatorname{int}\mathcal{M} \setminus A^+_1 \to \operatorname{int}\mathcal{M} \setminus A^-_1$ is a diffeomorphism. A^+_1 and A^-_1 are called the singular sets of f and f^{-1}, respectively. For any $k > 0$, let $A^\pm_k = A^\pm_1 \cup f^{\mp 1} A^\pm_1 \cup \ldots \cup f^{\mp(k-1)} A^\pm_1$.

Cone field. We assume that there exists an eventually strict invariant cone field C on $\operatorname{int}\mathcal{M}$. Let D_0 denote the set of the discontinuity points of C, and for any $k \in \mathbb{Z} \setminus \{0\}$, let $D_k = D_0 \cup f^{-k} D_0$. We assume that for any $k \in \mathbb{Z} \setminus \{0\}$, there exists a set $\mathcal{D}_k \subset \mathcal{M}$ consisting of at most countably many smooth compact curves intersecting at most at their endpoints such that $D_k \subset \mathcal{D}_k$ and $\mathcal{D}_k$ partitions $\mathcal{M}$ into countably many boxes $\{B_i\}_{i \in \mathbb{N}}$. We also assume that $C|_{\operatorname{int} B_i}$ has a continuous extension from $\operatorname{int} B_i$ up to B_i. We recall that C was assumed to be everywhere continuous in [LW].

The previous assumptions imply that the system $(\mathcal{M}, m, f)$ has non-zero Lyapunov exponents m-a.e.. We also assume that $(\mathcal{M}, m, f)$ satisfies the Conditions 1.1-1.3 of [KS]. This implies that there exit local stable and unstable manifolds at m-a.e. point of $\mathcal{M}$, and that they are absolutely continuous.

We introduce now two quantities that measure the expansion generated by f on the vectors in the cone field C. These are the analogues of σ, σ_* of [LW]. Let $z \in \operatorname{int} \mathcal{M}$ and $u \in \mathcal{T}_z\mathcal{M}$. If $X_1(z)$ and $X_2(z)$ are the vectors belonging to the edges of $C(z)$, then $u = u_1 X_1(z) + u_2 X_2(z)$ for suitable $u_1, u_2 \in \mathbb{R}$. Following [LW, W2], we define a quadratic form $Q_z : \mathcal{T}_z\mathcal{M} \to \mathbb{R}$ as follows

$$Q_z(u) := A(X_1(z), X_2(z)) u_1 u_2, \quad u \in \mathcal{T}_z\mathcal{M},$$

where $A(\cdot,\cdot)$ is the area form of $\tilde{g}$.

Definition 7.1. *Let $\{\sigma_k\}_{k>0}, \{\sigma_k^*\}_{k>0}$ be two families of functions $\sigma_k, \sigma_k^* : \operatorname{int} \mathcal{M} \setminus A_k^+ \to [1, +\infty)$ given by*

$$\sigma_k(z) = \liminf_{\substack{y \to z \\ y \notin D_k}} \inf_{u \in \operatorname{int} C(y)} \sqrt{\frac{Q_{f^k y}(D_y f^k u)}{Q_y(u)}}, \quad z \in \operatorname{int} \mathcal{M} \setminus A_k^+,$$

and

$$\sigma_k^*(z) = \liminf_{\substack{y \to z \\ y \notin D_k}} \inf_{u \in \operatorname{int} C(y)} \frac{\sqrt{Q_{f^k y}(D_y f^k u)}}{\|u\|}, \quad z \in \operatorname{int} \mathcal{M} \setminus A_k^+.$$

For $k < 0$, the functions $\sigma_k, \sigma_k^* : \operatorname{int} \mathcal{M} \setminus A_k^- \to [1, +\infty)$ are defined similarly after replacing C by its complementary cone.

Note that $\sigma_k(z), \sigma_k^*(z)$ coincide with the quantities $\sigma(D_z f^k), \sigma_*(D_z f^k)$ defined in [LW] when $z \in \operatorname{int} \mathcal{M} \setminus D_k$.

Lemma 7.2. *For $k > 0$, the functions σ_k, σ_k^* have the following properties:*

1. *σ_k, σ_k^* are lower semicontinuous on $\operatorname{int} \mathcal{M} \setminus A_k^+$,*
2. *σ_k, σ_k^* are continuous on $\operatorname{int} \mathcal{M} \setminus (A_k^+ \cup \mathcal{D}_k)$,*
3. *for any smooth compact curve γ forming $\mathcal{D}_k$ (see the definition of $\mathcal{D}_k$), the restrictions $\sigma_k|_{\operatorname{int} \gamma}$ is continuous on $\operatorname{int} \gamma \setminus A_k^+$.*

The same properties holds for $k < 0$ with A_k^+ replaced by A_k^-.

Proof. It is enough to prove the lemma for σ_k and when $k > 0$. Define

$$\bar{\sigma}_k(y) = \inf_{u \in \operatorname{int} C(y)} \sqrt{\frac{Q_{f^k y}(D_y f^k u)}{Q_y(u)}}, \quad y \in \operatorname{int} \mathcal{M} \setminus (A_k^+ \cup \mathcal{D}_k).$$

We recall that $\mathcal{D}_k$ partitions $\mathcal{M}$ into at most countably many boxes $\{B_i\}_{i \in \mathbb{Z}}$. The assumptions made on C guarantee that $\bar{\sigma}_k$ can be continuously extended from $\operatorname{int} B_i \setminus A_k^+$ to $B_i \setminus A_k^+$. For every $i > 0$, we define the function $\bar{\sigma}_k^{(i)} : \operatorname{int} \mathcal{M} \setminus A_k^+ \to \mathbb{R} \cup \{+\infty\}$ to be the continuous extension of $\bar{\sigma}_k$ to $B_i \setminus S_k^+$, and $+\infty$ on $\operatorname{int}\mathcal{M} \setminus (A_k^+ \cup B_i)$. Each $\bar{\sigma}_k^{(i)}$ is lower semicontinuous. We can express σ_k in terms of the quantities $\bar{\sigma}_k^{(i)}$ as follows

$$\sigma_k(z) = \min\{\bar{\sigma}_k^{(i)} : z \in B_i\} \tag{8}$$

for every $z \in \operatorname{int}\mathcal{M}\setminus A_k^+$. Note that the minimum here makes sense, because we assume that only finitely many boxes B_i's can share the same vertex. It follows that σ_k is lower semicontinuous. This proves the first statement of the lemma.

To prove the second statement, it is enough to observe that if $z \in \operatorname{int}\mathcal{M} \setminus (A_k^+ \cup \mathcal{D}_k)$, then $\sigma_k(z) = \bar{\sigma}_k(z)$, and that $\bar{\sigma}_k$ is continuous on $\mathcal{M} \setminus (A_k^+ \cup \mathcal{D}_k)$.

It remains to prove the third statement of the lemma. Let γ be one of the curves forming $\mathcal{D}_k$; clearly γ belongs to the boundary of two boxes B_i's. As $\bar{\sigma}_k^{(i)}$ is continuous on $B_i \setminus A_k^+$, it follows that $\bar{\sigma}_k^{(j)}|_{\operatorname{int}\gamma\setminus A_k^+}$ is continuous for the (two) j's for which $\gamma \subset \partial B_j$. For $z \in \operatorname{int}\gamma$, we have

$$\sigma_k|_{\operatorname{int}\gamma\setminus A_k^+}(z) = \min\{\bar{\sigma}_k^{(i)}|_{\operatorname{int}\gamma\setminus A_k^+}(z) : z \in \operatorname{int}\gamma \setminus A_k^+\}.$$

We then see that $\sigma_k|_{\operatorname{int}\gamma\setminus A_k^+}$ is continuous. □

Lemma 7.3. *The function σ_k has the following properties:*

1. $\sigma_k \geq 1$,
2. *σ_k is supermultiplicative, i.e., $\sigma_{k_1+k_2}(z) \geq \sigma_{k_2}(f^{k_1}z)\sigma_{k_1}(z)$ for any positive integers k_1, k_2,*
3. $\lim_{k\to\pm\infty} \sigma_k(z) = +\infty$ *if and only if* $\lim_{k\to\pm\infty} \sigma_k^*(z) = +\infty$.

Proof. Statements 1 and 2 are valid for $\bar{\sigma}_k, \bar{\sigma}_k^{(i)}$ (see [LW, W2]). Hence they are also valid for σ_k in virtue of (8). The proof of Statement 4 is as the proof of Theorem 6.8 of [LW]. □

The rest of the hypotheses of the LET are listed below.

E1. (Regularity) The singular sets $A_k^\pm$ are neat for any $k > 0$.

E2. (Discontinuities of $\sigma_{\mp k}$) For $k > 0$, let $\Sigma_k^\pm$ be the subset of $A_1^\pm$ where the restriction $\sigma_{\mp k}|_{A_1^\pm}$ is not defined or discontinuous. We assume that $\Sigma_k^\pm$ is finite.

E3. (Proper Alignment) For every $z \in A_1^-$, the tangent spaces of every smooth component of A_1^- at z is contained in $C(z)$. A similar condition holds for points of A_1^+ with $C(z)$ replaced by its complementary cone.

E4. (Sinai-Chernov Ansatz) Let m_1 be the one-dimensional measure on $A_1^+ \cup A_1^-$ generated by the Euclidean metric on $\mathbb{R}^2$. We have

$$\lim_{k\to\pm\infty} \sigma_k(z) = +\infty, \quad m_1\text{-a.e. } z \in A_1^\mp.$$

E5. (Noncontraction) There exists a real number $a > 0$ such that for every $k > 0$

$$|D_z f^k u| \geq a|u|$$

for any $z \in \operatorname{int}\mathcal{M}$ and any $u \in C(z)$. Furthermore a similar condition holds for $k < 0$ and points of $\operatorname{int}\mathcal{M} \setminus A_k^-$ with C replaced by its complementary cone.

E6. (Volume estimates) Denote by $\mathcal{L}$ the length generated by $\tilde{g}$. Let A be a subset of A_1^-. Given $\delta > 0$, let

$$A_u^\delta = \{z \in \operatorname{int}\mathcal{M} : \exists W_{loc}^u(z) \text{ and a curve } \gamma \subset W_{loc}^u(z) \text{ such that } \partial\gamma \cap A \neq \emptyset \text{ and } \mathcal{L}(\gamma) < \delta\}.$$

We assume that for any A closed in the relative topology of A_1^-, there exists a positive number $c = c(A)$ such that

$$\limsup_{\delta \to 0^+} \frac{m(A_u^\delta)}{\delta} \leq c m_1(A).$$

We also assume that the same condition is verified by A_s^δ for any $A \subset A_1^+$ closed in the relative topology of A_1^+ where A_s^δ is defined as A_u^δ after replacing W_{loc}^u by W_{loc}^s.

Definition 7.4. *Let*

$$\mathcal{H} = \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \sigma_k^{-1}((3, +\infty)).$$

The points of $\mathcal{H}$ *will be called* sufficient.

Theorem 7.5. *Let* $(\mathcal{M}, m, f)$ *be a smooth system with singularities with a cone field* C *verifying all the hypotheses described earlier in this section. Then every sufficient point of* $\mathcal{M}$ *has a neighborhood contained (mod 0) in one Bernoulli component of* $(\mathcal{M}, m, f)$.

Proof. This proof is adapted from the proof of the Main Theorem of [LW]. We assume the reader to be familiar with this proof, because, while we will describe in detail the modifications needed by our more general setting, we will only mention the steps of the original proof which are still valid.

The proof of the Main Theorem occupies Sections 8-13 of [LW]. The first step of this proof is Proposition 8.4 (in Section 8). To extend the validity of this proposition to our setting, we have to modify the part of its proof where a suitable "reference" neighborhood $\mathcal{U} \subset \operatorname{int}\mathcal{M}$ of a sufficient point z is constructed. The original construction of $\mathcal{U}$ has to be replaced by the following. We can assume without loss of generality that there exists $k > 0$ such that $z \notin A_k^-$ and $\sigma_{-k}(z) > 3$. Set $\bar{z} = f^{-k}z$ and $\tilde{\rho} = 1/\sigma_{-k}(z)$. Since $z \notin A_k^-$, we can find a neighborhood U of $\bar{z}$ such that f^k is a diffeomorphism from U to $f^k U$. Let $B_1, \dots, B_m$ be the boxes generated by $\mathcal{D}_k$ and containing $\bar{z}$. If $\bar{z} \notin \mathcal{D}_k$, then there is only one box as above, and $\bar{z}$ is contained in its interior: in this case, the original construction in [LW] gives $\mathcal{U}$. We continue with the general case: let C_i be the continuous extension of $C|_{(U \cap \operatorname{int} B_i) \cup f^k(U \cap \operatorname{int} B_i)}$, and Q_i be the corresponding quadratic form. Then define

$$\sigma^{(i)} = \inf_{u \in \operatorname{int} C_i(\bar{z})} \sqrt{\frac{Q_i(D_{\bar{z}} f^k u)}{Q_i(u)}}.$$

It is easy to see that $\sigma_{-k}(z) = \min\{\sigma^{(1)}, \ldots, \sigma^{(m)}\}$. For every map $f^k : U \cap B_i \to f^k(U \cap B_i)$ and the cone field C_i, we construct a neighborhood $\mathcal{U}_i$ of z in the relative topology of $f^k(U \cap B_i)$ using the original argument of [LW]: we obtain a cone field C_ρ on $\mathcal{U}_i$ where ρ satisfies the relation $\tilde{\rho} < \rho < 1$, and a priori depends on i. Since $\sigma^{(i)} \geq \sigma_{-k}(z)$, it is easy to see that we can choose the same ρ for every i. Let $\mathcal{U} = \cup_i \mathcal{U}_i$. It is easy to check that Proposition 8.4 of [LW] is also valid in our setting with the cone field C_ρ on $\mathcal{U}$ just constructed.

All the results contained in Sections 9 and 10 of [LW] rely on Proposition 8.4 and general properties of hyperbolic systems, thus they are also valid for $(\mathcal{M}, m, f)$.

Section 11 of [LW] contains an improved version of the Hopf's argument based on Sinai's approach [S] and valid for hyperbolic systems with singularities. This argument is the last step of the Main Theorem of [LW]. Since it is general and relies on the results of Sections 8-10 and 12-13 of [LW], it carries over to $(\mathcal{M}, m, f)$ if we show that the results of Sections 12 and 13 extend to $(\mathcal{M}, m, f)$.

Sections 12 and 13 form the so called Sinai's Theorem. This theorem provides an estimate of the m-measure of the set of the points of $\mathcal{U}$ which have "short" stable manifolds or unstable manifolds (see [LW]). It consists of two parts: the first part is Proposition 12.2, the only result contained in Section 12, and the second part is the Tail Bound Lemma in Section 13. Proposition 12.2 tuns out to be valid for $(\mathcal{M}, m, f)$ because it is a consequence of the results contained in Sections 8-10 [LW], and these results, as we have seen, extend to our setting. Note that in the proof of this proposition, it is used the fact that the transformation f is a symplectic, and, since our setting is two-dimensional, the volume form generating m is a symplectic form (preserved by f).

The Tail Bound Lemma of [LW] is not valid for $(\mathcal{M}, m, f)$ because, among its hypotheses, there are a stronger form of our Property E5 and the continuity of C. Hence we explain now how the proof of this lemma has to be modified in order to work in our setting. As in [LW], we will only deal with the unstable version of this lemma, since the stable version can be proved exactly in the same way.

Property E4 and part (4) of Lemma 7.3 imply that for m_1-a.e $z \in A_1^-$, we have

$$\lim_{k \to +\infty} \sigma_k^*(z) = +\infty. \tag{9}$$

Given two positive numbers M, t, let

$$\tilde{E}_t = \{z \in A_1^- : \sigma_M^*(z) \leq t + 1\}.$$

It follows from (9) that for any pair $h, t > 0$ there exists an integer $M = M(h, t) > 0$ such that $m_1(\tilde{E}_t) \leq h$. Let

$$B_M = \Sigma_M^- \cup \left(\overline{A_1^-} \cap \partial \mathcal{M} \right).$$

If B_M^ζ denotes the ζ-neighborhood of B_M in A_1^- with respect to the Euclidean metric, i.e.,

$$B_M^\zeta := \{z \in A_1^- : d(z, B_M) < \zeta\},$$

then E1 and E2 implies that there exists a $\zeta > 0$ such that the m_1-measure of $\overline{B_M^\zeta}$ is less than h. Another consequence of $B2$ is that $\sigma_M^*|_{A_1^-}$ is continuous on $A_1^- \setminus B_M^\zeta$ which in turns implies that the sets

$$E_t = \{z \in A_1^- \setminus B_M^\zeta : \sigma_M^*(z) \leq t + 1\}$$

and

$$S_t = \{z \in A_1^- \setminus B_M^\zeta : \sigma_M^*(z) \geq t + 1\}$$

are compact. Furthermore since $E_t \subset \tilde{E}_t$, we have $m_1(E_t \cup \overline{B_M^\zeta}) \leq 2h$. Finally the semicontinuity of σ_M^* (part (1) of Lemma 7.2) implies that there exists a $r > 0$ such that $\sigma_M^*(z) > t$ for any $z \in S_t^r$ where S_t^r is the r-neighborhood of S_t with respect to the Euclidean metric.

The proof now continues as in Section 13 of [LW] up to the point where CASE 1 and CASE 2 are discussed. In order for the analysis of CASE 1 done in [LW] to work in our setting, we need to make the following change: use the metric $\tilde{g}$ instead of the Euclidean metric in order to evaluate the length of $f^{-m}\gamma$ (see [LW]; note that f corresponds to T in that paper). We can always choose $\mathcal{U}$ such that its closure does not intersect $\partial\mathcal{M}$. Since $\tilde{g}$ can only degenerate on $\partial\mathcal{M}$, the Euclidean metric and $\tilde{g}$ are equivalent on $\mathcal{U}$. Thus using Property E5, we obtain that the length of $f^{-m}\gamma$ with respect to $\tilde{g}$ differs from the value obtained in [LW] by a positive factor accounting for the equivalence of the two metrics on $\mathcal{U}$. To conclude the analysis of CASE 1, we have to use Property E6, as the radius of the neighborhood of $E_t \subset \tilde{E}_t$ is now computed with respect to $\tilde{g}$. The analysis of CASE 2 in [LW] can be repeated word by word in our setting. However the final estimate is obtained for the Lebesgue measure, whereas we need a similar estimate for m. This follows from the latter, because, by assumption, the density of m is bounded.

This concludes the proof of the Tail Bound Lemma and so the proof of Sinai's Theorem for $(\mathcal{M}, m, f)$. To finish our proof, we need a final remark. In the formulation of the theorem of Liverani and Wojtkowski, there is no reference to Bernoulli components of f, only to ergodic components of f. Nevertheless since local stable and unstable manifolds of f are also local stable and unstable manifolds of any positive power of f, we see that as the property that $\mathcal{U}$ contains a full m-measure set of points such that any pair of them is connected by a Hopf's chain of stable and unstable manifolds implies that $\mathcal{U}$ is contained (mod 0) in one ergodic component of f so it implies that $\mathcal{U}$ is contained (mod 0) in one ergodic component of an arbitrary positive power of f (in fact, any power). Thus $\mathcal{U}$ is contained (mod 0) in one Bernoulli component of f. □

Remark 7.6. *Using the same argument at the end of Section 7 in [LW], one can show that if a system $(\mathcal{M}, m, f)$ verifies E1-E6, then m-a.e. points of $\mathcal{M}$ satisfies the hypothesis of Theorem 7.5 which, in turn, implies that Bernoulli components of f are open (mod 0).*

8 Bernoulli property of planar billiards

We go back to billiards satisfying Conditions B1-B3. In this subsection, we prove that the hypotheses of Theorem 7.5 are verified by the induced system (Ω, ν, Φ) of these billiards endowed with the invariant cone field C introduced in Subsection 5.3. Of course, the space Ω, the singular sets $\mathcal{S}_k^+$ of Φ and C verify the basic hypotheses of the theorem, and we are going to check only that Conditions E1-E6 are verified. Once this is done, in the next subsection, will use Theorem 7.5 and Condition B4 to prove that the billiard map T is Bernoulli. As we are now dealing with billiards, the notation used through this and the following subsections is as in Sections 2-5.

8.1 Verification of Conditions E1-E6

Before directing our attention the proof of Conditions E1-E6, we make a remark which will simplify our task. By the time-reverse symmetry of the billiard dynamics, we see that every result valid for any object defined in terms of T, is also valid for the symmetric object defined in terms of T^{-1}. Examples of such objects are the local stable and unstable manifolds W^s, W^u and the singular sets $\mathcal{S}_k^-, \mathcal{S}_k^+$. This symmetry is somewhat incorporated in Conditions E1-E6, as each of these conditions consists of two symmetric parts. We then see that to prove E1-E6 for the billiards considered in this paper, it is enough only to check one part of each condition. This is what we will do in this section.

Consider a billiard satisfying B1-B3. Let (Ω, ν, Φ) be the corresponding induced system (see Section 4), C the invariant cone field of (Ω, ν, Φ) (see Section 5.3), and set $\tilde{g} = g'$ where g' is the semimetric introduced in Section 2.

Theorem 8.1. *The system (Ω, ν, Φ) verifies Conditions E1-E6.*

Proof. We will show that (Ω, ν, Φ) satisfies the unstable part of E1-E6. The proof is subdivided into Propositions 8.2, 8.5, 8.7, 8.19 and 8.22. □

E1. (Regularity)

Proposition 8.2. *The singular sets $\mathcal{S}_k^\pm$ of Φ have the following properties:*

1. *$\mathcal{S}_k^\pm$ are neat in Ω, and $\mathcal{S}_k^\pm \subset S_{k(2\bar{n}-1)}^\pm$. In particular, (Ω, ν, Φ) verifies Condition E1.*
2. *$\mathcal{S}_j^- \cap \mathcal{S}_k^+$ is finite for any $j, k > 0$,*
3. *$\mathcal{S}_\infty$ is at most countable.*

Proof. Part (1). We will only prove this part of the proposition for sets $\mathcal{S}_k^+$; in virtue of the relation $\mathcal{S}_k^- = R\mathcal{S}_k^+$, the result extends automatically to the sets $\mathcal{S}_k^-$ as well. We say that $\mathcal{S}_k^+$ verifies the property (P) if it has finitely many smooth components, and it is the union of these components. It follows from Lemma B.2 that if $\mathcal{S}_k^+$ verifies (P), then the closure of $\mathcal{S}_k^+$ is regular. Accordingly, to prove the regularity of the closure of $\mathcal{S}_k^+$, we will show that $\mathcal{S}_k^+$ verifies (P). Once this has been done, it is simple to show that $\mathcal{S}_k^+$ is neat in Ω: it is enough to note that if the closure of a smooth component of $\overline{\mathcal{S}_k^+}$ does not intersect any other smooth component of $\overline{\mathcal{S}_k^+}$, then it can be extended up to $\partial\Omega$.

We use an induction argument. Let us start by analyzing the case $k = 1$. It is clear that it suffices to prove that $\mathcal{S}_1^+ \cap \Delta$ and $\mathcal{S}_1^+ \cap (M_- \cup M_+)$ both verify (P). We first consider the set $\mathcal{S}_1^+ \cap \Delta$. The initial step is to show that $(\operatorname{int}\Delta \setminus V_{\bar{n}}^+) \cap S_{\bar{n}}^+ = \emptyset$ (see Section 4 for the definition of the quantities involved here). Suppose that this is not true, i.e., there exists an integer $0 \le j \le \bar{n} - 1$ such that $(\operatorname{int}\Delta \setminus V_{\bar{n}}^+) \cap T^{-j} S_1^+ \neq \emptyset$. As a consequence of the definition Δ, then one obtains that $T^j(\operatorname{int}\Delta \setminus V_{\bar{n}}^+) \cap V_1^+ \neq \emptyset$ and $T^j(\operatorname{int}\Delta \setminus V_{\bar{n}}^+) \cap V_1^+ = \emptyset$ are verified simultaneously. Thus $(\operatorname{int}\Delta \setminus V_{\bar{n}}^+) \cap S_{\bar{n}}^+ = \emptyset$. We recall that $\Phi z = T^{t(z)} z$ (see Definition 4.8). Since t on Δ (where it is defined) has only value 1 or $\bar{n}$, the restriction of Φ to every connected component of $\operatorname{int}\Delta \setminus V_{\bar{n}}^+$ is smooth, and therefore $\mathcal{S}_1^+ \cap \Delta \subset V_{\bar{n}}^+ \cap \operatorname{int}\Delta$. It follows from the definition of Φ that $V_{\bar{n}}^+ \cap \operatorname{int}\Delta \subset \mathcal{S}_1^+ \cap \Delta$. Hence $\mathcal{S}_1^+ \cap \Delta = V_{\bar{n}}^+ \cap \operatorname{int}\Delta$. We have

$V_{\bar n}^+ = \cup_{1\le m\le \bar n} \cup_{\boldsymbol{i}\in I_m} C_{\boldsymbol{i}}^m$ where the union over the $\boldsymbol{i}$'s is restricted as explained in Corollary A.5. Every $\overline{C_{\boldsymbol{i}}^m}$ has finitely many connected components whose properties are described in Lemma A.3. These properties are inherited by the sets $\overline{C_{\boldsymbol{i}}^m \cap \operatorname{int}\Delta}$, as one can easily check after recalling the definition of Δ, and that $\overline{V_{\bar n}^+} \cap \overline{V_{\bar n}^-}$ is finite. Then it is not difficult to see that the connected components of $\overline{C_{\boldsymbol{i}}^m \cap \operatorname{int}\Delta}$ are smooth components of $\overline{\mathcal{S}_1^+ \cap \Delta}$, and that $\mathcal{S}_1^+ \cap \Delta$ verifies (P).

We consider now $\mathcal{S}_1^+ \cap M'$ where $M' := M_- \cup M_+$. Let $W_{\bar n} = S_1^+ \cup T^{-1}V_{2\bar n-2}^+$. To prove (1) for $\mathcal{S}_1^+ \cap M'$, we use an argument similar to the one used above for $\mathcal{S}_1^+ \cap \Delta$. We start by showing that $\mathcal{S}_1^+ \cap (M' \setminus W_{\bar n}) = \emptyset$. Let B be any connected component of $\operatorname{int} M' \setminus W_{\bar n}$. Since $B \cap S_1^+ = \emptyset$, it follows that $T|_B$ is smooth and either $TB \subset M'$ or $TB \subset M_0$. In the first case, $\Phi|_B = T|_B$ and $\Phi|_B$ is smooth so that $\mathcal{S}_1^+ \cap B = \emptyset$. In the second case, we have $TB \cap S_1^+ = \emptyset$ because $B \cap W_{\bar n} = \emptyset$, and there are two possibilities: the first is that $T^2B \subset M'$ which implies that $\Phi|_B = T^2|_B$ and $\mathcal{S}_1^+ \cap B = \emptyset$, and the second is that $T^2B \subset M_0$ which implies $T^2B \cap S_1^+ = \emptyset$ since $B \cap W_{\bar n} = \emptyset$. Then we apply the same argument to $T^2B \subset M_0$ and then to the higher images of B under T. We conclude that there are three possibilities for B: i) $TB \subset M'$, ii) there is a $1 < j \le 2\bar n - 1$ such that $T^iB \subset M_0$ for every $1 \le i < j$ and $T^jB \subset M'$, and iii) $T^iB \subset M_0$ for every $1 \le i \le 2\bar n - 1$. Accordingly, we have $\Phi|_B = T^j|_B$ for $1 \le j \le 2\bar n - 1$ or $\Phi|_B = T^{\bar n}|_B$. In all cases, $\Phi|_B$ is smooth so that $\mathcal{S}_1^+ \cap B = \emptyset$. This allows us to obtain the wanted equality $(M' \setminus W_{\bar n}) = \emptyset$. It is easy to check that $W_{\bar n} \cap M' \subset \mathcal{S}_1^+ \cap M'$, and hence $\mathcal{S}_1^+ \cap M' = W_{\bar n} \cap M'$. Corollary A.5 implies that $W_{\bar n} = \cup_{1\le m\le 2\bar n-2} \cup_{\boldsymbol{i}\in I_m} C_{\boldsymbol{i}}^m$ where the union over the $\boldsymbol{i}'s$ is restricted as explained in Lemma A.3. Every connected component of $\overline{C_{\boldsymbol{i}}^m} \subset M'$ is a smooth component of $\mathcal{S}_1^+ \cap M'$. By recalling the properties of the sets $C_{\boldsymbol{i}}^m$ and their connected components (Lemma A.3 and A.4), we see that $\mathcal{S}_1^+ \cap M'$ verifies (P).

We complete now the induction argument. We assume that $\mathcal{S}_k^+$ verifies (P), and show that $\mathcal{S}_{k+1}^+$ verifies (P). We have $\overline{\mathcal{S}_{k+1}^+} = \overline{\Phi^{-1}\mathcal{S}_k^+} \cup \overline{\mathcal{S}_1^+}$. The set $\overline{\mathcal{S}_k^+}$ is the union of its smooth components, i.e., $\overline{\mathcal{S}_k^+} = \cup_{i=1}^m \gamma_i$. Hence $\overline{\Phi^{-1}\mathcal{S}_k^+} = \cup_{i=1}^m \overline{\Phi^{-1}\operatorname{int}\gamma_i}$. As $\mathcal{S}_1^-$ verifies (P), $\overline{\mathcal{S}_1^-}$ is the union of its smooth components, i.e., $\overline{\mathcal{S}_1^-} = \cup_{j=1}^n \gamma_j'$. The map Φ^{-1} is not defined at $\operatorname{int}\gamma_i \cap \mathcal{S}_1^- \subset \gamma_i \cap \overline{\mathcal{S}_1^-} \subset \cup_{j=1}^n \gamma_i \cap \gamma_j'$. Using Lemma B.3, we then see that $\operatorname{int}\gamma_i \cap \mathcal{S}_1^-$ is finite, and so is the number of disjoint smooth open curves $\xi_1^{(i)}, \dots, \xi_{l(i)}^{(i)}$ forming $\operatorname{int}\gamma_i \setminus \mathcal{S}_1^-$. This, of course, implies that $\overline{\Phi^{-1}\operatorname{int}\gamma_i} = \cup_j^{l(i)} \overline{\Phi^{-1}\xi_j^{(i)}}$. Each set $\overline{\Phi^{-1}\xi_j^{(i)}}$ is smooth by Theorem B.4, and, in fact, it is easy to see that it is contained in a smooth component of $\overline{\mathcal{S}_{k+1}^+}$. Since $\overline{\Phi^{-1}\mathcal{S}_k^+} = \cup_i^m \cup_j^{l(i)} \overline{\Phi^{-1}\xi_j^{(i)}}$, it follows that $\overline{\Phi^{-1}\mathcal{S}_k^+}$ is a finite union of smooth compact curves contained in smooth components of $\overline{\mathcal{S}_{k+1}^+}$. The set $\overline{\mathcal{S}_1^+}$ is a finite union of smooth components of $\overline{\mathcal{S}_{k+1}^+}$ because $\mathcal{S}_1^+$ verifies (P). We have just proved that $\overline{\mathcal{S}_{k+1}^+}$ consists of a finite union of its smooth components, and so $\mathcal{S}_{k+1}^+$ verifies (P).

To finish the proof, we observe that Part (2) follows from Part (1) and Lemma B.3, and that Part (3) is a consequence of Parts (1) and (2). □

Lemma 8.3. *The singular set S_∞ of T is at most countable.*

Proof. Given a $z \in S_\infty$, there are two integers $j, k \ge 0$ such that $z \in T^kS_1^- \cap T^{-j}S_1^+$. If $T^iz \notin \Omega$ for any $-k \le i \le j$, then $z \in V_\infty$, otherwise $T^iz \in \mathcal{S}_\infty$ for some $-k \le i \le j$. Hence

$S_\infty \subset V_\infty \cup (\cup_{i\in\mathbb{Z}} T^{-i}\mathcal{S}_\infty)$, and all we need to do is to show that V_∞ and $\cup_{i\in\mathbb{Z}} T^{-i}\mathcal{S}_\infty$ are both at most countable. The neatness (Lemma A.5) and transversality of $\overline{V_\infty^-}, \overline{V_\infty^+}$ (see the proof of Lemma 4.6) imply that $\overline{V_\infty^-} \cap \overline{V_\infty^+}$ is at most countable which, it turn, implies the same for V_∞ since $V_\infty \subset \overline{V_\infty^-} \cap \overline{V_\infty^+}$. Proposition 8.2 implies immediately that $\cup_{i\in\mathbb{Z}} T^{-i}\mathcal{S}_\infty$ is at most countable. □

E2. (Discontinuities of C)

Proposition 8.4. (Ω, ν, Φ) *verifies Condition E2.*

Proof. According to the analysis carried out in Subsection 5.4, we can set $\mathcal{D}_k = \mathcal{S}^+_{\tilde{m}+k+1} \cup \mathcal{R}^-_\infty$. To estimate the cardinality of Σ_k^-, we argue as follows. The set of points where $\sigma_k|_{\mathcal{S}_1^-}$ is not defined is given by $\mathcal{S}_1^- \cap \mathcal{S}_k^+$. Let B be a box in the partition of Ω generated by $\mathcal{D}_k$, and γ be the intersection of $\operatorname{int}\Omega$ and a smooth component of $\overline{\mathcal{S}_1^-}$. Using Statements 2 and 3 of Lemma 7.2, we see that the points where $\sigma_k|_{\gamma\cap B\setminus\mathcal{S}_k^+}$ is discontinuous are contained in

$$(\partial\left(\gamma \cap \operatorname{int} B\right) \cap \partial B) \cup \left(\gamma \cap \partial\partial B\right),$$

where $\partial\partial B$ is the union of the vertexes of B. Thus it is not difficult to see that

$$\Sigma_k^- \subset \left(\mathcal{S}_1^- \cap \overline{\mathcal{S}^+_{\tilde{m}+k+1}}\right) \cup \left(\bigcup_{i>1} \mathcal{S}_1^- \cap \overline{\mathcal{R}_i^-}\right) \cup \left(\bigcup_{i=1}^n \partial\gamma_i\right),$$

where $\gamma_1, \ldots, \gamma_n$ are the smooth components of $\overline{\mathcal{S}_1^-}$. The first set from the right in the previous expression contains the branching points of $\overline{\mathcal{S}_1^-}$, and therefore those of $\overline{\mathcal{R}_1^-}$. The regularity of $\overline{\mathcal{S}_1^-}, \overline{\mathcal{S}^+_{\tilde{m}+k+1}}$ and Lemma B.3 imply that $\mathcal{S}_1^- \cap \overline{\mathcal{S}^+_{\tilde{m}+k+1}}$ and $\cup_{i=1}^n \partial\gamma_i$ are finite. We have $\overline{\mathcal{S}_1^-} \cap \overline{\mathcal{R}_i^-} = \emptyset$ for any i sufficiently large (see Case (IV) in Appendix A2 of [Do]). Using this fact and Lemma B.2, we obtain that the set $\cup_{i>1} \mathcal{S}_1^- \cap \overline{\mathcal{R}_i^-}$ is finite. This concludes the proof. □

E3. (Proper Alignment)

Proposition 8.5. (Ω, ν, Φ) *verifies Condition E3.*

Proof. That (Ω, ν, Φ) verifies E3 is a byproduct of the proof of Lemma B.3. □

E4. (Sinai-Chernov Ansatz) Let $z \in \operatorname{int}\Omega$ and $u \in \mathcal{T}_z\Omega$. If $X_1(z)$ and $X_2(z)$ are non-zero vectors belonging to the edges of $C(z)$, then we can write $u = u_1 X_1(z) + u_2 X_2(z)$ for proper $u_1, u_2 \in \mathbb{R}$. We denote by $|X_1(z), X_2(z)|$ the Euclidean area of the parallelogram with sides $X_1(z), X_2(z)$. We associate to $C(z)$ the quadratic form

$$Q_z(u) = \sin\theta |X_1(z), X_2(z)| u_1 u_2, \quad u \in \mathcal{T}_z\Omega.$$

See Section 3, for the definition of S^+_∞ and N^+_∞.

Lemma 8.6. $\lim_{k\to+\infty} \sigma_k(z) = +\infty, \quad \forall z \in \operatorname{int}\Omega \setminus (S^+_\infty \cup N^+_\infty)$.

Proof. It is enough to prove the following property: there exists a positive number $\bar{\sigma}$ such that for any $z \in \operatorname{int}\Omega \setminus (S^+_\infty \cup N^+_\infty)$, one can find an increasing sequence $\{m_i\}_{i\in\mathbb{N}}$ for which $\sigma_{m_{i+1}-m_i}(\Phi^{m_i} z) \geq \bar{\sigma}$ for every $i \geq 1$. In fact, let us assume that this property is verified, and let $z \in \operatorname{int}\Omega \setminus (S^+_\infty \cup N^+_\infty)$ and $\{m_i\}_{i\in\mathbb{N}}$ be the sequence associated to z. If $i(n) := \max\{i > 0 : m_i \leq n\}$, then, for large n, we would obtain (using the supermultiplicativity property of σ)

$$\sigma_n(z) \geq \prod_{i=1}^{i(n)-1} \sigma_{m_{i+1}-m_i}(\Phi^{m_i} z) \geq \bar{\sigma}$$

which implies $\lim_{n\to+\infty} \sigma_n(z) = +\infty$ because $\lim_{n\to+\infty} i(n) = +\infty$ as a consequence of the fact that $\{m_i\}$ is increasing.

We explain now how to construct the sequence $\{m_i\}$ for a given $z \in \operatorname{int}\Omega \setminus (S^+_\infty \cup N^+_\infty)$. We claim that the positive semi-trajectory of z hits non-flat components of Γ infinitely many times, and its ω-limit set cannot be equal to a corner of Γ. Suppose, in fact, that z does not have this property. Since $z \notin N^+_\infty$, the positive semi-trajectory of z hits non-flat components of Γ infinitely many times, and so, in order to violate the property we want to prove, the ω-limit set of z must be a corner of Γ. This corner cannot belong to a focusing component of Γ for two reasons: the first is that focusing components of Γ are absolutely focusing, and so every trajectory can only have a finite number of consecutive collisions with a focusing component of Γ; the second reason is that to satisfy B2, the internal angle of the corner must be greater than either π or $\pi/2$ according to the fact that the other component of Γ forming the corner is either a non-flat component or a flat component. It is easy to see that these two facts make it impossible for the positive semi-trajectory of z to accumulate at a corner contained in a focusing component of Γ. Thus the corner must be formed by two dispersing components or by a dispersing component and a straight line. In both cases, however, any trajectory entering such a corner leaves it after a finite number of reflections (for example, see Appendix A1.3 of [BSC]). But then the ω-limit set of z cannot be a single corner which contradicts the assumption made on z. Let U_{cusp} be a small neighborhood of the corners of Γ formed by adjacent dispersing components. For the chosen z, the sequence $\{m_i\}$ is defined recursively as follows: we set $m_1 = 1$, and given m_i for $i > 1$, we choose $m_{i+1} > m_i$ in such a way that $\Phi^{m_i+1} z \in M_- \cup R\mathcal{E}$ and $\pi(\Phi^{m_i+1} z) \notin U_{cusp}$. This can be done because of the property verified by points in $\Omega \setminus (S^+_\infty \cup N^+_\infty)$ proved before. Next we show that there exists a positive $\bar{\sigma}$ independent of $z \in \Omega \setminus (S^+_\infty \cup N^+_\infty)$ such that $\sigma_{m_{i+1}-m_i}(\Phi^{m_i} z) \geq \bar{\sigma}$ for every $i \geq 1$. It is easy to show that for any $i > 1$, we can find an integer $k_i > 0$ such that $T^{-k_i}\Phi^{m_i} z \in \mathcal{E}$ and $T^{-j}\Phi^{m_i} z \in M_0$ for $1 < j < k_i$. Using the supermultiplicativity property of σ, then it is enough to show that there is $\bar{\sigma} > 0$ independent of $z \in \Omega \setminus (S^+_\infty \cup N^+_\infty)$ such that $\sigma_{k_i}(T^{-k_i}\Phi^{m_i} z) \geq \bar{\sigma}$ for any $i > 1$. This can be proved as follows. For any $i > 1$, let $z_i = T^{-k_i}\Phi^{m_i} z$ and $w_i = \Phi^{m_i} z$. Of course $\sigma_{k_i}(T^{-k_i}\Phi^{m_i} z) = \sigma_{m_i}(z_i)$. we denote by l_i be the length of the trajectory connecting z_i, w_i. To compute $\sigma_{k_i}(z_i)$, we use the Cross Ratio Formula (see Section 1 of [W2]). This is justified by the fact that although our definition of $\sigma_k(z)$ differs from that one of $\sigma(D_z T^k)$ in [W2], it turns out that $\sigma_k(z)$ coincides with a certain object $\bar{\sigma}^{(j)}_k(z)$ (see (8) in the proof of Lemma 7.2) for which the Cross Ratio Formula is valid. Hence this formula is also valid for $\sigma_k(z)$, and the cones at the points z and $T^k z$ that we have to consider for its computation are those corresponding to $\sigma^{(j)}_k(z)$. A simple computation gives

$$\sigma_{k_i}(z_i) = \sqrt{1+\omega_i} + \sqrt{\omega_i}$$

where

$$\omega_i = \begin{cases} \frac{l_i(l_i-\tau^+(z_i)-\tau^-(w_i))}{\tau^+(z_i)\tau^-(w_i)} & \text{if } z_i, w_i \in M_+ \\ \frac{l_i(l_i+|d(z_i)|+|d(w_i)|)}{|d(z_i)|+|d(w_i)|)} & \text{if } z_i, w_i \in M_- \\ \frac{(l_i-\tau^+(z_i))(l+|d(w_i)|)}{\tau^+(z_i)|d(w_i)|} & \text{if } z_i \in M_+, w_i \in M_- \\ \frac{(l_i+|d(w_i))(l_i-\tau^-(w_i))}{|d(z_i)|\tau^-(w_i)} & \text{if } z_i \in M_-, w_i \in M_+, \end{cases}$$

$\tau^\pm(z)$ are the forward and backward focusing times of C at z (see Section 5), and $d(z)$ is the (Euclidean) length of the segment of the trajectory of z contained in the disk tangent to Γ at q and having radius which is half of the radius of curvature of Γ at q (see Section 2). By B2 and the definition of z_i, w_i, we have

$$l_i \geq l_i - \tau^+(z_i), l_i - \tau^-(w_i), l_i - \tau^+(z_i) - \tau^-(w_i) \geq \bar{\tau}, \quad \forall i > 1.$$

If we set $\bar{r} := \max_{s\in\Gamma_-} |r(s)|$, $\bar{\omega} := \bar{\tau}/(\max \tau, \bar{r})^2$ and $\bar{\sigma} := \sqrt{1+\bar{\omega}} + \sqrt{1+\bar{\omega}}$, then we obtain the wanted conclusion

$$\sigma_{m_i}(z_i) \geq \bar{\sigma}, \quad \forall i > 1.$$

□

Proposition 8.7. (Ω, Φ, ν) *verifies Condition E4.*

Proof. It follows from Lemma 8.6 that it is enough to show that $\ell(\mathcal{S}_1^- \cap (S_\infty^+ \cup N_\infty^+)) = 0$. Since $\mathcal{S}_1^- \subset S_{2\bar{n}-1}^-$ (see Proposition 8.2) and $S_\infty^- \cap \Omega = \mathcal{S}_\infty^+$, we obtain $\mathcal{S}_1^- \cap (S_\infty^+ \cup N_\infty^+) \subset (\mathcal{S}_1^- \cap \mathcal{S}_\infty^+) \cup (S_{2\bar{n}-1}^- \cap N_\infty^+)$. The set $\mathcal{S}_1^- \cap \mathcal{S}_\infty^+$ is countable by Proposition 8.2, and $\ell(S_{2\bar{n}-1}^- \cap N_\infty^+) = 0$ by B3. Hence $\ell(\mathcal{S}_1^- \cap (S_\infty^+ \cup N_\infty^+)) \leq \ell(\mathcal{S}_1^- \cap \mathcal{S}_\infty^+) + \ell(S_{2\bar{n}-1}^- \cap N_\infty^+) = 0$. □

E5. (Noncontraction) We recall $\|\cdot\|'$ denotes the norm generated by the semimetric g' (see Section 2).

Definition 8.8. *Let $k > 0$, and $z \in \operatorname{int}\Omega \setminus \mathcal{S}_k^+$. We say that the orbit $\alpha := \{z, Tz, \ldots, T^k z\}$ verifies the* noncontraction property *if there is a constant $a > 0$ such that $\|D_z\Phi^k u\|' \geq a\|u\|'$ for any $u \in C(z)$.*

The following families of orbits called *blocks* play an important role in the proof of E5. We assume that $z \in \operatorname{int}\Omega \setminus \mathcal{S}_k^+$.

1. $\mathcal{F}_1$ consists of orbits $\{z, \Phi z\}$ such that $z \in M_i \subset M_+$ and $\Phi z \in M_j \cap \Omega$ for some $i \neq j$,
2. $\mathcal{F}_2$ consists of orbits $\{z, \Phi z, \ldots, \Phi^k z\} \subset \Delta \cup M_-$ for some $k > 0$,
3. $\mathcal{F}_3$ consists of orbits $\{z, \Phi z\}$ such that $z \in \Delta \cup M_-$ and $\Phi z \in M_+$,
4. $\mathcal{F}_4$ consists of orbits $\{z, \Phi z, \ldots, \Phi^k z\} \subset M_i \subset M_+$ for some $k > 0$ and i,
5. $\mathcal{F}_5$ consists of orbits $\{z, \Phi z, \ldots, \Phi^k z\}$ such that $\{z, \Phi^k z\} \in R\mathcal{E}$ for some $k > 0$.

Remark 8.9. *We observe that for the billiards considered in this paper, the noncontraction is not always verified along every block of type 1-5 if we replace the semimetric g' by the more natural metric g. This is the reason why we considered g' instead of g. It follows from the results of this subsection and Remark 8.21 that the noncontraction is always (for any billiard considered here) satisfied with respect to g along blocks of type 1,4,5. It is not satisfied, instead, along all blocks of type 2 and 3. In fact, it is not difficult to construct billiard tables satisfying B1-B3 such that there is a one-parameter family of vectors $\{z(c)\}$ contained in $\mathcal{F}_2(\mathcal{F}_3)$ for $c \in (0, \epsilon), \epsilon > 0$ such that*

$$\lim_{\epsilon \to 0} \theta(z(\epsilon)) = 0 \quad \textit{and} \quad \lim_{\epsilon \to 0} \theta(\Phi z(\epsilon)) = \tilde{\theta}$$

where $0 < \tilde{\theta} < \pi$. It is now easy to see that if $u = (1, 0) \in C(z(\epsilon))$, then

$$\lim_{\epsilon \to 0} \frac{\|D_{z(\epsilon)} \Phi u\|}{\|u\|} = 0.$$

Clearly this implies that the noncontraction property is not verified with respect to g along $\{z, \Phi z\}$.

Given a block α, we say that α is *of type* $i(\alpha)$ if $\alpha \in \mathcal{F}_{i(\alpha)}$. Every finite orbit can be represented as a finite sequence of blocks $\alpha_1, \ldots, \alpha_n$ of type 1-4 satisfying the following rule

$$i(\alpha_{k+1}) = \begin{cases} i(\alpha_k) + 1 \mod 4 & \text{if } i(\alpha_k) \in \{2, 3, 4\}, \\ 2, 4 & \text{if } i(\alpha_k) = 1, \end{cases}$$

for every $k = 1, \ldots, n-1$. In other words, given any block representation of an orbit, we can always reduce it to a block representation described above by grouping together adjacent blocks of the same type. We when talk about the block representation of an orbit, we will always have in mind this representation. Note that by definition a block of type 4 is allowed to contain just one element (this is important to derive the representation above), and that for an orbit of type 5, its block representation $\alpha_1, \ldots, \alpha_n$ has the property that $i(\alpha_1) = 1$ and $i(\alpha_4) = 4$.

We prove now that any finite orbit can be decomposed in at most eight blocks.

Lemma 8.10. *Let $k > 0$, and $z \in \operatorname{int} \Omega \setminus \mathcal{S}_k^+$. There exist an integer $1 \le m \le 8$ and a finite sequence of integers $0 = k_0 < k_1 < \cdots < k_m = k$ such that $\{z, \Phi z, \ldots, \Phi^k z\} = \bigcup_{i=1}^m \{\Phi^{k_{i-1}} z, \ldots, \Phi^{k_i} z\}$ where $\{\Phi^{k_{i-1}} z, \ldots, \Phi^{k_i} z\} \in \mathcal{F}_j$ for some $1 \le j \le 5$.*

Proof. Consider a finite orbit α with block representation $\alpha_1, \ldots, \alpha_n$. After having removed the largest block $\tilde{\alpha}$ of type 5 from α, we are left with at most two orbits, one coming before and another coming after $\tilde{\alpha}$. It is immediate to see that if the first of these orbits exist, then it can contain at most three blocks; when it contains exactly three blocks, these must be a block of type 2, one of type 3 and one of type 4. Similarly if the second orbit exists, then it can contain at most four blocks; when it contains exactly four blocks, these must be a block of type 1, one of type 2, one of type 3 and one of type 4. The orbit α admits therefore a decomposition with a maximum of 8 blocks. ☐

To prove that (Ω, ν, Φ) has Property E5, we show, in the following series of lemmas, that every block of type 1-5 verifies the noncontraction property. We recall that the values $\tau, \bar{\tau}$ are defined in Section 6.

Lemma 8.11. *If $z \in M_+ \setminus S_2$, then $\|u\|'/\sqrt{1+\tau^2} \le |J'| \le \|u\|'$ for any $u = (J, J') \in C(z)$.*

Proof. The lemma follows trivially from the fact that $|J/J'| \le \tau^+(z) \le \tau$ for any $u = (J, J') \in C(z)$ and $z \in M_+ \setminus S_2$. □

Lemma 8.12. *Any block of type 1 verifies the noncontraction with constant $a_1 = \min\{1/(1+\tau^2)^{1/2}, \bar{\tau}/\tau\}$.*

Proof. Let $u = (J_0, J_0') \in C(z)$, and $D_z\Phi u = (J_1, J_1')$. We assume first that $\Phi z \in M_- \cup \Delta$. By (2), we have

$$\begin{pmatrix} J_1 \\ J_1' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -\frac{2}{|d(\Phi z)|} & -1 \end{pmatrix} \begin{pmatrix} 1 & l(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} J_0 \\ J_0' \end{pmatrix}$$

where $l(z)$ is the length of the trajectory connecting z and Φz, and the factor $|d(\Phi z)|$ has to be replaced by ∞ if $\Phi z \in \Delta$. It follows from the construction of C on M_+ that $\tau(z,u)J_0 + J_0' = 0$. A straightforward computation then gives

$$\begin{pmatrix} J_1 \\ J_1' \end{pmatrix} = -\begin{pmatrix} l(z) - \tau^+(z,u) \\ 2\frac{l(z)-\tau^+(z,u)}{|d(\Phi z)|} + 1 \end{pmatrix} J_0'.$$

Condition B2 implies that $l(z) - \tau^+(z,u) \ge \bar{\tau} > 0$ so that $|J_1'| \ge |J_0'|$. Finally

$$\|D_z\Phi u\|' \ge |J_1'| \ge |J_0'| \ge \frac{\|u\|'}{\sqrt{1+\tau^2}}$$

where the last inequality follows from Lemma 8.11.

We assume now that $\Phi z \in M_+$. In this case, we have

$$\left|\frac{J_1'}{J_0'}\right| = \left|\frac{J_1'}{J_1}\right|\left|\frac{J_1}{J_0'}\right| = \frac{\tau^-(\Phi z, D_z\Phi u)}{\tau^+(\Phi z, D_z\Phi u)} \ge \frac{\bar{\tau}}{\tau},$$

because $\tau^-(\Phi z, D_z\Phi u) = l(z) - \tau^+(z,u) \ge l(z) - \tau^+(z) \ge \bar{\tau}$ by B2, and $\tau^+(\Phi z, D_z\Phi u) \le \tau$ by construction of C on M_+. Using Lemma 8.11 like before, we conclude that

$$\|D_z\Phi u\|' \ge \frac{\bar{\tau}}{\tau}\|u\|'.$$

□

Lemma 8.13. *Any block of type 2 verifies the noncontraction with constant $a_2 = 1$.*

Proof. Let $u = (J_0, J_0') \in C(z)$, and $D_z\Phi u = (J_1, J_1')$. We prove the case when $k = 1$, and then explain why it implies the general case. By (2), we have

$$\begin{pmatrix} J_1 \\ J_1' \end{pmatrix} = -\begin{pmatrix} 1 & l(z) \\ \frac{2}{|d(\Phi z)|} & \frac{2l(z)}{|d(\Phi z)|} + 1 \end{pmatrix} \begin{pmatrix} J_0 \\ J_0' \end{pmatrix}.$$

where $l(z)$ is the length of the trajectory connecting z and Φz, and the factor $|d(\Phi z)|$ has to be replaced by ∞ if $\Phi z \in \Delta$. Since $J_0J_0' \ge 0$ and $l(z) > 0$, it follows that $|J_0| \le |J_1|$ and $|J_0'| \le |J_1'|$, and therefore $\|D_z\Phi u\|' \ge \|u\|'$. The invariance of C implies $J_iJ_i' \ge 0$ for any $0 \le i \le k$. Hence the result obtained for $k = 1$ is valid for every segment of orbit $\{\Phi^i z, \Phi^{i+1} z\}$. This proves the lemma in the general case. □

Lemma 8.14. *Any block of type 3 verifies the noncontraction with constant* $a_3 = \min\{1, \bar{\tau}/\tau\}$.

Proof. Arguing exactly like in the second part of the proof of Lemma 8.12, we obtain $|J_1'/J_0'| \geq \bar{\tau}/\tau$. Since $J_0 J_0' \geq 0$, it follows that $|J_1| \geq |J_0|$. Hence $\|D_z \Phi u\|' \geq \|u\|' \bar{\tau}/\tau$. □

Lemma 8.15. *Any block of type 4 verifies the noncontraction with a certain constant* $a_4 > 0$.

Proof. To prove the lemma, it is enough to verify the noncontraction property with respect to the seminorm $|J'|$. This is justified by Lemma 8.11, and by the fact that blocks of type 4 are contained in M_+.

Let $\alpha := \{z, \Phi z, \ldots, \Phi^k z\}$ be a block of type 4. We assume that $k > 0$, otherwise there is nothing to prove. Note that $\Phi^i z = T^i z$ for $i = 1, \ldots, k$, and so, through this proof, we are allowed to use T instead of Φ. Let $0 \neq u = (J_0, J_0') \in C(z)$, and denote $(J_i, J_i') = D_z T^i u$ for $0 \leq i \leq k$. We have

$$\left|\frac{J_k'}{J_0'}\right| = \prod_{i=1}^{k} \left|\frac{J_i'}{J_i}\right| \left|\frac{J_i}{J_{i-1}'}\right| = \prod_{i=1}^{k} \frac{\tau^-(D_z T^i u, T^i z)}{\tau^+(D_z T^i u, T^i z)} \geq \prod_{i=1}^{k} \frac{\tau^-(T^i z)}{\tau^+(T^i z)}.$$

Let us define

$$\Xi(\alpha) = \prod_{i=1}^{k} \frac{\tau^-(T^i z)}{\tau^+(T^i z)}.$$

The lemma is proved if we show that $\Xi(\alpha)$ is uniformly bounded away from zero over all blocks of type 4. We will do this in two steps: first for blocks of type 4 contained in $\mathcal{A} := \cup_{m=0}^{\tilde{m}} \cup_{i=0}^{m} T^i \mathcal{E}_m$, and then for blocks of type 4 contained in $\mathcal{B} := \cup_{m=\tilde{m}+1}^{\infty} \cup_{i=0}^{m} T^i \mathcal{E}_m$. Note that these sets are invariant and disjoint.

Let α be a block of type 4 contained in $\mathcal{A}$. We recall that $0 < \tau^{\pm}(z) \leq \tau$ for any $z \in M_+ \setminus S_2$ by construction of C. Since $\mathcal{A}$ is bounded away from S_2, there exists a constant $C = C(\tilde{m}) > 0$ such that $d(z) \geq C$ for any $z \in \mathcal{A}$. Using (5), we obtain $0 < 1/\tau^+(z) + 1/\tau^-(z) \leq 2/C$ which implies $\tau^{\pm}(z) \geq C/2$ for any $z \in \mathcal{A}$. In conclusion, we have

$$\Xi(\alpha) \geq \left(\frac{C}{2\tau}\right)^{\tilde{m}}$$

for every block α of type 4 contained in $\mathcal{A}$.

Consider now a block α of type 4 contained in $\mathcal{B}$. Let (x, y) be the Lazutkin's coordinates of z. For $i = 1, \ldots, k$, denote by X_i be the lower edge of $C(T^i z)$ which, by construction, is equal to $\partial_{\tilde{z}} T^m \partial/\partial x(\tilde{z})$ for some $\tilde{z} \in \mathcal{E}$ and $m > 0$. We have

$$\tau^{\pm}(T^i z) = \frac{\sin\theta(T^i z)}{\kappa(T^i z) \pm m(X_i)}$$

where $m(X_i)$ is the slope of X_i in coordinates (s', θ') (see Section 5.3). It follows from the results of Section 5 of [Do] that there are constants b_1, b_2, b_3 depending only on Γ_+ such that

$$|m(X_i)| \leq b_1 y + b_2 y^3, \quad i = 1, \ldots, k,$$

and

$$k \leq \frac{b_3}{y}.$$

Hence

$$\Xi(\alpha) \geq \left(\frac{1 - b'_1 y - b'_2 y^3}{1 + b''_1 y + b''_2 y^3} \right)^{\frac{b_3}{y}} \tag{10}$$

where if $\kappa_{\max} = \max_{\Gamma_+} \kappa$ and $\kappa_{\min} = \min_{\Gamma_+} \kappa$, then $b'_i = b_i/\kappa_{\max}$ and $b''_i = b_i/\kappa_{\min}$ for $i = 1, 2$. The choice of $\tilde{m}$ in Section 5.3 was made to conclude that the right hand-side of this inequality is greater than $e^{-2(b'_1 + b''_1)b_3}/2$ which is a constant depending only on Γ_+. Thus

$$\Xi(\alpha) \geq \frac{e^{-2(b'_1 + b''_1)b_3}}{2}$$

for any block α of type 4 contained in $\mathcal{B}$. □

Definition 8.16. *A block α of type 5 is called* minimal *if it does not contain any smaller block of type 5. The block representation of a minimal block of type 5 is given by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ or α_1, α_4 where, in both case, it must be $i(\alpha_k) = k$.*

It is clear that every block of type 5 decomposes into finitely many minimal blocks of type 5.

Lemma 8.17. *For any block $\{z, \dots, \Phi^k z\}$ of type 5 containing $m > 0$ minimal blocks, the operator $D_z\Phi^k$ can be decomposed as follows*

$$D_z\Phi^k = F_2^{-1} \circ L_{2m} \circ L_{2m-1} \circ \dots \circ L_1 \circ F_1.$$

In coordinates (J, J'), the operators $F_1(F_2), L_{2i-1}, L_{2i}$ have, respectively, the matrix form

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad -\begin{pmatrix} 1 + \delta_1^{(i)} & \bar{\tau} + \delta_3^{(i)} \\ \delta_2^{(i)} & 1 + \delta_4^{(i)} \end{pmatrix}, \quad \begin{pmatrix} \epsilon_1^{(i)} & 0 \\ \epsilon_2^{(i)} & \frac{1}{\epsilon_1^{(i)}} \end{pmatrix}$$

where $\delta_1^{(i)}, \dots, \delta_4^{(i)} \geq 0$ and $\epsilon_1^{(i)}, \epsilon_2^{(i)} > 0$.

Proof. It is enough to prove the lemma for a minimal block $\alpha = \{z, \dots, \Phi^k z\}$ of type 5. Let $0 < \tilde{k} \leq k$ be the integer such that $\Phi^{\tilde{k}} z \in \alpha$ which is the first element in the sequence of collisions with the second focusing component of Γ visited by z. Set $z_0 = z, z_1 = \Phi^{\tilde{k}} z, z_2 = \Phi^k z$. Using coordinates (q, v), we write $z_i = (q_i, v_i)$, and define $p_i = q_i + \tau^+(z_i) v_i$ for $i = 0, 1, 2$. Next set $\tilde{z}_0 = (p_0, v_0)$ and $\tilde{z}_1 = (p_1, w_1)$ where w_1 is obtained by reflecting v_1 about Γ. Finally let t_0, t_3 be the length of the pieces of orbit α joining q_0, p_0 and q_2, p_2, respectively, and t_i be the length of the piece joining p_{i-1}, p_i for $i = 1, 2$. If $\{\phi_t\}_{t \in \mathbb{R}}$ denotes the billiard flow in Q, then we can write $D_z\Phi^k$ as follows

$$D_z\Phi^k = (D_{z_2}\phi_{t_3})^{-1} \circ D_{\tilde{z}_1}\phi_{t_2} \circ D_{\tilde{z}_0}\phi_{t_1} \circ D_{z_0}\phi_{t_0}.$$

Let $F_1 = D_{z_0}\phi_{t_0}, F_2 = D_{z_2}\phi_{t_3}, L_1 = D_{\tilde{z}_0}\phi_{t_1}, F_1 = D_{\tilde{z}_1}\phi_{t_2}$. It is easy to check that, in coordinates (J, J'), we have

$$F_1 = \begin{pmatrix} 1 & t_0 \\ 0 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & t_3 \\ 0 & 1 \end{pmatrix}.$$

To compute the matrix form of L_1, L_2, we argue as follows. In coordinates (J, J'), the vector $(1, 0)$ represents a parallel variation, whereas $(0, 1)$ represents a variation of lines emerging from the same point. Since along the orbit between p_0, p_1, there are only reflections at $\Gamma_- \cup \Gamma_0$, it is not difficult to see

$$L_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = - \begin{pmatrix} 1 + \delta_1 \\ \delta_2 \end{pmatrix}, \quad L_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} t_1 + \delta_3 \\ 1 + \delta_4 \end{pmatrix}$$

where $\delta_1, \ldots, \delta_4 \geq 0$, and the sign of the two vectors coincides. Condition B2 implies that $t_1 \geq \bar{\tau}$ so that

$$L_1 = - \begin{pmatrix} 1 + \delta_1 & \bar{\tau} + \delta_3 \\ \delta_2 & 1 + \delta_4 \end{pmatrix}.$$

Note that every variation focusing at p_1 must also focus at p_2. So it is easy to see that

$$L_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad L_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \epsilon_3 \end{pmatrix}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, and the sign of the two vectors coincides. The billiard flow in coordinates (J, J') preserves the standard area form. Hence $\det L_2 = \pm 1$, and so

$$L_2 = \begin{pmatrix} \epsilon_1 & 0 \\ \epsilon_2 & \epsilon_1^{-1} \end{pmatrix}.$$

□

Lemma 8.18. *Any block of type 5 verifies the noncontraction with constant* $a_5 = \sqrt{\frac{\bar{\tau}}{\tau(1+\tau^2)}}$.

Proof. Consider a block of type 5 $\{z, \Phi z, \ldots, \Phi^k z\}$ containing $m > 0$ minimal blocks of type 5. Let $F_1, F_2, L_{2i}, L_{2i-1}$ for $i = 1, \ldots, m$ be the matrices associated this block as in Lemma 8.17. Given $u = (J_0, J_0') \in C(z)$, let $(J_2, J_2') = D_z \Phi^k u$. Also define $(\tilde{J}_0, \tilde{J}_0') = F_1 u, (J_1, J_1') = L_1(\tilde{J}_0, \tilde{J}_0')$ and $(\tilde{J}_2, \tilde{J}_2') = F_2(J_2, J_2')$. We will use the symbol $|u|_j$ to denote the absolute value of the jth component of u.

By Lemma 8.11, it is enough to prove that there is a positive constant a_5 such that $|J_2'| \geq a_5 |J_0'|$ for any $(J_0, J_0') \in C(z)$. It follows from Lemma 8.17 that there exist two 2×2 matrices $R, \tilde{R}$ with non-negative entries such that

$$L_{2m} \circ L_{2m-1} \circ \ldots \circ L_3 \circ L_2 = \pm (L_{2m} \circ L_{2m-2} \circ \ldots L_4 \circ L_2 + R), \tag{11}$$

and

$$L_{2m} \circ L_{2m-2} \circ \ldots L_4 \circ L_2 = \tilde{L}_{2m} \circ \tilde{L}_{2m-2} \circ \ldots \tilde{L}_4 \circ \tilde{L}_2 + \tilde{R}, \tag{12}$$

where

$$\tilde{L}_{2i} = \begin{pmatrix} \epsilon_1^i & 0 \\ 0 & \frac{1}{\epsilon_1^i} \end{pmatrix}. \tag{13}$$

If $\tilde{L} = \tilde{L}_{2m} \circ \tilde{L}_{2m-2} \circ \ldots \tilde{L}_4 \circ \tilde{L}_2$ and $\epsilon = \prod_{i=1}^m \epsilon_1^i$, then

$$\tilde{L} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}. \tag{14}$$

Let $(J_0, J_0') \in C(z)$, and $v = (J_1, J_1')$. Condition B2 implies that $J_1/J_1' \geq \bar{\tau}$ so that, by using (12) and (13), we obtain

$$|L_{2m} \circ L_{2m-1} \circ \ldots \circ L_3 \circ L_2 v|_j \geq |\tilde{L}v|_j.$$

This gives

$$|\tilde{J}_2| \geq \epsilon |J_1| \geq \epsilon \bar{\tau} |J_1'|, \tag{15}$$

and

$$|\tilde{J}_2'| \geq \epsilon^{-1} |J_1'|. \tag{16}$$

Another consequence of B2 is that $J_2/J_2' \leq \tau$. Using (15), we obtain

$$|\tilde{J}_2'| \geq \frac{\epsilon \bar{\tau}}{\tau} |J_1'|. \tag{17}$$

The average of (16) and (17) gives

$$|\tilde{J}_2'| \geq \frac{1}{2} \left(\epsilon^{-1} + \frac{\epsilon \bar{\tau}}{\tau} \right) |J_1'|.$$

A simple computation shows that $\epsilon^{-1} + \epsilon\bar{\tau}/\tau \geq 2(\bar{\tau}/\tau)^{1/2}$ for $\epsilon > 0$, and so

$$|\tilde{J}_2'| \geq \left(\frac{\bar{\tau}}{\tau} \right)^{1/2} |J_1'|.$$

To conclude the proof, we only need to observe that $|J_1'| \geq |\tilde{J}_0'| = |J_0'|$ and $|J_2'| = |\tilde{J}_2'|$. □

Proposition 8.19. (Ω, ν, Φ) *verifies Condition E5.*

Proof. In Lemmas 8.14-8.15, it is proved that the noncontraction is verified by any block of type i with constant $a_i > 0$ for any $1 \leq i \leq 5$. Then (Ω, ν, Φ) satisfies the noncontraction, because if $a = (\min_{1 \leq i \leq 5} a_i)^8$, then for every finite orbit $\{z, \Phi z, \ldots, \Phi^k z\}$, we have

$$\|D_z \Phi^k u\|' \geq a \|u\|', \quad \forall u \in C(z).$$

□

E6. (Volume estimates) We now prove that (Ω, Φ, μ) has Property E6. The key points is an inequality for the semi-norms $\| \cdot \|$ and $\| \cdot \|'$ valid in a neighborhood of $\mathcal{S}_1^-$ which is proved in Lemma 8.20. The notation here is as in the statement of Property E6.

Lemma 8.20. *There exists a* $\alpha > 0$ *such that for every* $Y \subset \mathcal{S}_1^-$*, we have* $Y_u^\delta \subset Y^{\alpha\delta}$ *for sufficiently small* $\delta > 0$.

Proof. It is enough to prove that there exists a positive number $\alpha > 0$ such that

$$\sup_{0 \neq u \in C(z)} \frac{\|u\|^2}{\|u\|'^2} \leq \alpha, \quad z \in V \cup M_- \cup (M_+ \setminus S_2), \tag{18}$$

where V is a proper neighborhood of $\mathcal{S}_1^- \cap \Delta$ in Δ. We will prove (18) separately for points of $V, M_-, M_+ \setminus S_2$ with proper constants $\alpha_0, \alpha_-, \alpha_+$.

Note that if $z = (s, \theta)$ and $u = (ds, d\theta) \in \mathcal{T}_z M$, then it follows from (1) that the ratio $\|u\|^2/\|u\|'^2$ has the following expression

$$\frac{\|u\|^2}{\|u\|'^2} = \frac{1 + m(u)^2}{\sin^2\theta + (\kappa(s) + m(u))^2},$$

where $m(u) = d\theta/ds$.

We explain now how to find the set V, and then show that (18) is verified on V. We claim that $d(\overline{\mathcal{S}_1^-} \cap \Delta, S_2) > 0$. Suppose that this is not true, then we have $d(\overline{\mathcal{S}_1^-} \cap \Delta, S_2) = 0$, and so there is a smooth component γ of $\overline{\mathcal{S}_1^-} \cap \Delta$ such that $\gamma \cap S_2 \neq \emptyset$. This implies that the continuation of the segment $\Gamma_i \supset \pi(\gamma)$ either contains the corner of Γ generating γ (if γ is generated by a corner), or is tangent to the dispersing component of Γ generating γ (if γ is generated by a dispersing curve). But then we must have $\gamma \subset S_2$, contradicting $\emptyset \neq \operatorname{int}\gamma \subset \operatorname{int}\Omega$, which follows from the definition of smooth component (see Definition B.1). Since $d(\overline{\mathcal{S}_1^-} \cap \Delta, S_2) > 0$, we can find a neighborhood V of $\overline{\mathcal{S}_1^-} \cap \Delta$ in Δ such that $d(V, S_2) > 0$. On such a V, the function $\sin\theta$ is uniformly bounded away from zero. Since $\kappa \equiv 0$ on V, we see that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent on V. This proves (18) for $z \in V$.

By the general assumption on the dispersing components of Γ, there exists a negative number κ_- such that $\kappa(s) \leq \kappa_- < 0$ for every $s \in \Gamma_-$. Also, by construction of C, we have $-\infty \leq m(u) \leq 0$ for every $u \in C(z)$ and $z \in M_-$. Thus

$$\sup_{0 \neq u \in C(z)} \frac{\|u\|^2}{\|u\|'^2} \leq \sup_{0 \neq u \in C(z)} \frac{1 + m(u)^2}{(\kappa(s) + m(u))^2} \leq 1 + \frac{1}{\kappa_-^2} =: \alpha_-, \quad z \in M_-,$$

which proves (18) for $z \in M_-$.

By the general assumption on the focusing curves of Γ, there are two positive constants $\kappa_{\min} < \kappa_{\max}$ such that $0 < \kappa_{\min} \leq \kappa(s) \leq \kappa_{\max} < +\infty$ for every $s \in \Gamma_+$. By construction of C, we can find a small $a > 0$ such that $-\kappa_{\min}/2 \leq m(u) \leq +\infty$ for every $u \in C(z)$ and $z \in M_+ \setminus S_2$ provided that $0 \leq \sin\theta < a$. For such u and z, we then have

$$\frac{\|u\|^2}{\|u\|'^2} \leq \frac{1 + m(u)^2}{(\kappa(s) + m(u))^2} \leq 1 + \frac{4}{\kappa_{\min}^2}.$$

Let now $u \in C(z)$ and $z \in M_+$ such that $\sin\theta \geq a$. In this case,

$$\frac{1 + m(u)^2}{a^2 + (\kappa(s) + m(u))^2} \leq \frac{1}{a^2 + (\kappa(s) + m(u))^2} + \frac{m(u)^2}{a^2 + (\kappa(s) + m(u))^2},$$

and we obtain

$$\frac{\|u\|^2}{\|u\|'^2} \leq 1 + \frac{1}{a^2} + \frac{4}{\kappa_{\min}^2}.$$

Hence

$$\sup_{0 \neq u \in C(z)} \frac{\|u\|^2}{\|u\|'^2} \leq 1 + \frac{1}{a^2} + \frac{4}{\kappa_{\min}^2} =: \alpha_+, \quad z \in M_+ \setminus S_2.$$

This finishes the proof. $\square$

Remark 8.21. *The previous proof can be strengthen, and it can be proved that* $\|\cdot\|, \|\cdot\|'$ *are, in fact, equivalent on* $V \cup M_- \cup M_+$. *This stronger result is, however, more than we need to use Theorem 7.5. We also stress that* $\|\cdot\|, \|\cdot\|'$ *are not equivalent on* Δ.

Proposition 8.22. (Ω, ν, Φ) *verifies Condition E6.*

Proof. By Lemmas 8.20 and C.1, we have

$$\nu(Y_u^\delta) \le \nu(Y^{\alpha\delta}) \le \alpha C \ell(Y)\delta.$$

□

8.2 Bernoulli Property

In this subsection, we prove that all billiards satisfying B1-B3 are locally ergodic, and are also Bernoulli provided that they satisfy the additional Condition B4. The notation here is as in Sections 2 and 3.

We recall that $\mathcal{H} = \{z \in \operatorname{int}\Omega : \exists k > 0 \text{ s.t. } z \notin \mathcal{S}_k^+ \text{ and } \sigma_k(z) > 3 \text{ or } z \notin \mathcal{S}_k^- \text{ and } \sigma_{-k}(z) > 3\}$ and $H = \operatorname{int} M \setminus (S_\infty \cup NS \cup N_\infty)$. In the previous section, we have proved that Theorem 7.5 applies to the induced system (Ω, ν, Φ) of a billiard satisfying B1-B3. In the first result contained in this section, we show that the conclusion of Theorem 7.5 is also valid for the billiard system (M, μ, T) at every point of H.

Theorem 8.23. *Consider a billiard satisfying B1-B3. Then for every* $z \in H$ *there is a neighborhood of* z *contained (mod 0) in one Bernoulli component of* T.

Proof. Let $z \in H$, and assume without of loss of generality that $z \notin S_\infty^+ \cup N_\infty^+$. In this case, there is an integer $m > 0$ for which $T^m z \in \operatorname{int}\Omega \setminus (S_\infty^+ \cup N_\infty^+)$. Using Lemma 8.6, we can then find an integer $j > 0$ such that $\sigma_j(T^m z) > 3$. Hence $T^m z \in \mathcal{H}$, and so Theorem 7.5 tells us that there exists a neighborhood U be of $T^m z$ contained (mod 0) in a Bernoulli component of Φ.

The key point in the proof of Theorem 7.5 is to show that the neighborhood $\mathcal{U}$ contains a full m-measure set of points such that any pair of them is connected by a Hopf's chain of stable and unstable manifolds of Φ. The Hopf's argument then implies that $\mathcal{U}$ is contained (mod 0) in one Bernoulli component of Φ. Since local stable and unstable manifolds of Φ and T coincide, it follows that $\mathcal{U}$ is also contained (mod 0) in one Bernoulli component of T.

To conclude, we observe that T^m is continuous at z, and that the image under an arbitrarily power of T of a Bernoulli component of T is also a Bernoulli component of T.

□

Theorem 8.24. *For billiards satisfying B1-B4, the map* T *is Bernoulli.*

Proof. We recall that by assumption Γ is an union of $m > 0$ disjoint Jordan curves $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_m$, each of them being a finite union of components of Γ. The geometry of Q allows us to order $\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_m$ so that there is a finite sequence $1 = i_1 < i_2 < \cdots < i_l = m$ for some $l > 1$ for which $\tilde{\Gamma}_1$ contains in its interior all the other curves, and each curve $\tilde{\Gamma}_{i_j+1}, \ldots, \tilde{\Gamma}_{i_{j+1}}$ is connected by a segment contained in the interior of Q to at least one of the curves $\tilde{\Gamma}_{i_{j-1}+1}, \ldots, \tilde{\Gamma}_{i_j}$ for $2 \le j \le l-1$. Denote by $\tilde{M}_i$ the set $\pi^{-1}(\tilde{\Gamma}_i)$. It follows immediately from Theorem 8.23 and

Figure 2: If U is a neighborhood of z, then $\mu(TU \cap M_i)$ and $\mu(TU \cap M_j)$ are positive in both cases

Condition B4 that the each $M_1, \ldots, M_n$ (recall that $M_i = \pi^{-1}(\Gamma_i)$) is contained (mod 0) in one Bernoulli component of T. To prove the theorem, we have to show that all these components coincide. Consider two boundary components Γ_i, Γ_j as in Fig. 2: they are adjacent in the first case, and non-adjacent in the second case. The ℓ-measure of $S_1^+ \cap S_\infty^-$ and $S_1^+ \cap N_\infty^-$ are zero, because S_∞ is countable (Lemma 8.3), and $\ell(S_1^+ \cup N_\infty^-) = 0$ by B3. Hence $S_1^+ \cap H$ is dense in S_1^+. We can then find a point $z \in S_1^+ \cap H$ not in $M_i \cup M_j$ such that $q_1(z) \in \Gamma_j$, and the continuation of the segment $[\pi(z), q_1(z)]$ intersects the interior of Γ_i. Furthermore, since $z \in H$, there exists a neighborhood $U \subset \operatorname{int} M$ of z contained (mod 0) in one Bernoulli component of T. The same is true for TU. Note that TU contains two open sets: one contained in M_i, and the other contained in M_j (see Fig. 2). Thus M_i, M_j belong (mod 0) to the same Bernoulli component of T. Using this argument, we can immediately prove that each set $\tilde{M}_1, \ldots, \tilde{M}_m$ is contained (mod 0) in one Bernoulli component of T. Moreover using the same argument over and over, we obtain first that all the sets $\tilde{M}_1, \tilde{M}_2, \ldots, \tilde{M}_{i_2}$ are contained (mod 0) in the same Bernoulli component of T containing $\tilde{M}_1$, then that all the sets $\tilde{M}_1, \ldots, \tilde{M}_{i_3}$ are contains (mod 0) in the same Bernoulli component of T, and so on, until we obtain that all the sets $\tilde{M}_1, \ldots, \tilde{M}_m$ are contained (mod 0) in the same Bernoulli component of T. □

As a consequence of the previous theorem, also the billiard flow (of billiards satisfying B1-B4) is ergodic. This is a simple fact which we do not prove. By general results [CH, OW], it follows then that the billiard flow is either Bernoulli or Bernoulli times a rotation. In the second case, the flow would have a rotation as a factor, and therefore, a non-constant eigenfunction. But this is impossible, because a billiard flow is a contact flow, and as such, it has only constant eigenfunctions (Theorem 3.6 of [KB]). Hence, for billiards satisfying B1-B4, only the first possibility occurs. Note that although Theorem 3.6 of [KB] is formulated for $C^{1+\epsilon}$ flows, its proof is also valid for billiard flows which are not smooth. This proves the following theorem.

Theorem 8.25. *For billiards satisfying B1-B4, the billiard flow is Bernoulli.*

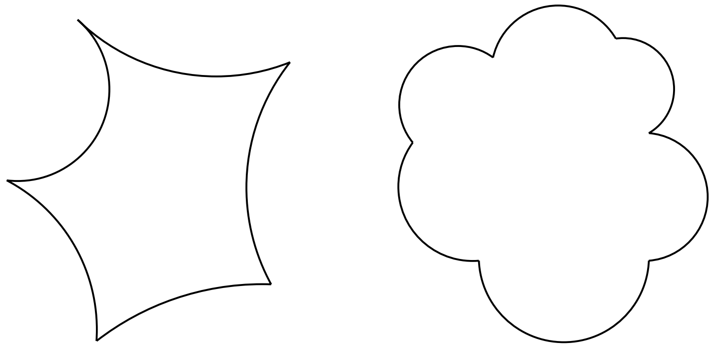

Figure 3: Dispersing and focusing billiards

9 Examples

In this section, we use Theorems 8.23 and 8.24 to reprove and generalize several results on the ergodicity of hyperbolic planar billiards previously obtained. We will implicitly assume that all billiards considered in this section satisfy Conditions B1 and B2. Also note that, by Theorem 8.25, if a certain billiard map is Bernoulli, then the corresponding billiard flow is also Bernoulli. We mentioned this fact once and for all, thus avoiding to repeat it every time in this section, we prove that a billiard map is Bernoulli.

9.1 Dispersing and focusing billiards

A billiard is called dispersing (focusing) if $\Gamma = \Gamma_-$ ($\Gamma = \Gamma_+$). See Fig. 3. Dispersing billiards were introduced by Sinai [S]. In his seminal paper, he proved that dispersing billiards without cusps are hyperbolic, and have the K-property. Later Gallavotti and Ornstein proved that these billiards are also Bernoulli [GO]. Focusing billiards were introduced by Bunimovich. Following Sinai's approach, Bunimovich showed that billiards bounded by arcs of circles are Bernoulli [Bu2]. Several authors [Bu3, M3, LW, Sz] extended this result by replacing the arcs of circles by a variety of curves forming certain classes (including arcs of circles). However the union of these classes does not contain the whole family of absolutely focusing curves. We prove now that focusing billiards bounded by general absolute focusing curves are Bernoulli. At the same time, we reprove that dispersing billiards are Bernoulli.

For dispersing billiards and focusing billiards, Condition B3 is trivially satisfied, because $N^-_\infty = N^+_\infty = \emptyset$. Thus the set of non-sufficient points is contained in S_∞ which is at most countable by Lemma 8.3. It follows immediately that Condition B4 is satisfied. Using Theorem 8.24, we can then conclude that dispersing and focusing billiards (the last satisfying the condition on the distance and position of the focusing components prescribed by B2) with absolutely focusing curves are Bernoulli.

The same argument proves the Bernoulli property for other two classes of billiards: billiards with boundary containing both dispersing curves and absolutely focusing curves, and billiards in tables obtained by removing a finite number of strictly convex obstacles (the curvature of their

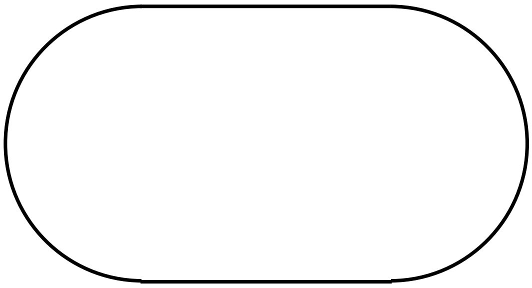

Figure 4: Stadium

boundaries is strictly negative) from a flat two-torus. While our argument applies directly to the first class, its application to the second class requires few observations. First, we stress that the boundary of each obstacles has to be formed by a finite number of strictly convex inward curves. Next by adding a "transparent" boundary to the "border" of the torus, we obtain a semi-dispersing billiard. Although the set N^-_∞, N^+_∞ may be not empty for these billiards, it is easy to show that Conditions B1-B3 are satisfied and $N^-_\infty, N^+_\infty \subset M_0$. The proof of Theorem 8.24 shows then that every obstacle belongs (mod 0) to one Bernoulli component of T. This, of course, implies the Bernoulli property of the billiard in the torus.

9.2 Stadia and some semidispersing billiards

A billiard table is called a stadium if its boundary consists of two absolutely focusing curves joined together by two straight segments not necessarily parallel. See Fig. 4. In the famous Bunimovich's stadium, the focusing curves are semicircles, and the segments are parallel. The Bernoulli property of Bunimovich's stadium was showed in [Bu2, Bu3].

We stress that the stadia considered here are bounded by general absolutely focusing curve and not only by semicircles. This generality however comes with a price: the distance between the focusing curves cannot be arbitrary (as required by B2) as in the case of Bunimovich's stadium, and in general, it will be bounded from below. This is the case, for instance, for stadia bounded by semiellipses with small eccentricity [CMOP, MOP]. Accordingly, we assume that the focusing curves in the stadia considered are sufficiently far apart (so that B2 is satisfied).

If the segments of the stadium are not parallel, then it easy to check that $N^-_\infty = N^+_\infty = \emptyset$. In this case, Conditions B3 and B4 are satisfied and the stadium has the Bernoulli property by Theorem 8.24. When the segments of the stadium are parallel, N^-_∞, N^+_∞, N_∞ coincide and consist of two horizontal segments in M corresponding to vectors of M attached to the segments and perpendicular to them (i.e., two curves $\theta = \{\pi/2\}$ in M_0). It is not difficult to see that B3 is verified in this case. However Condition B4 is not, because N_∞ disconnects M_0. Hence Theorem 8.24 cannot be used now. Nevertheless, in the next section, we will prove that stadia with parallel segments, as part of a large class of hyperbolic billiards, are Bernoulli. This will be done by improving a little the proof of Theorem 8.24.

We mention that the Bernoulli property of stadia with semiellipses was first proved in [DM], and that the ergodicity of some truncated elliptical billiards was proved in [De].

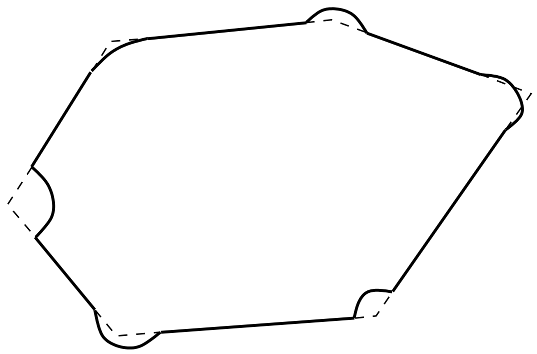

Figure 5: A Polygon with pockets and bumps

9.3 Polygons with pockets and bumps

A polygon with pockets and bumps is a simply connected domain Q of $\mathbb{R}^2$ obtained from a polygon P by "replacing" a neighborhood of the vertices of P by absolutely focusing curves (pockets) and dispersing curve (bumps). In Fig. 5, P is a convex polygon bounded by dashed lines, and Q is bounded by solid curves. Note that the pockets and straight lines in ∂Q can join to form C^1 curves but not C^2. We assume that the polygon P, and the curves are chosen so that the billiard in Q satisfies Condition B2. This can achieved, for example, by using sufficiently short focusing curves as pockets. Stadia (with parallel and non parallel segments) can be thought of as degenerate convex polygons with pockets. So we include them in the family of polygons with pockets and bumps. Accordingly, the theorem proved below applies also to stadia.

In [CT], it was proved that convex polygons with pockets have the Bernoulli property when the pockets are arcs of circles contained in Q. We consider here the more general when the pockets are absolutely focusing curves. As in the mentioned papers, we also allow Γ to have bumps. The hyperbolicity of convex polygons with absolutely focusing pockets was first proved [Do].

Theorem 9.1. *Convex polygons with bumps and absolutely focusing pockets are Bernoulli.*

Proof. A general result for polygonal billiards states that every semiorbit of a polygon billiard is either periodic or accumulates at least at one vertex of the polygon [GKT]. This fact and the convexity of P imply that, for billiards with pockets and bumps, the sets N_∞^-, N_∞^+, N_∞ coincide and consist of periodic orbits bouncing off straight lines (as we have seen earlier, the same phenomenon occurs in stadia with parallel segments). In a polygon, every periodic orbit is contained in a family of "parallel" orbits having the same period. This family is called *strip.* Thus a strip is a finite union of horizontal segments (θ=const) contained in M_0 with endpoints either belonging to ∂M or $S_\infty^- \cup S_\infty^+$. The number of distinct strips in a polygonal billiard is finite [CT]. Hence, for the billiards studied here, we have $N_\infty \subset M_0$ and $\mu(N_\infty) = 0$. Consider now the set $S_\infty^- \cap N_\infty^+$. Since any element of this set is contained in a strip, and has a singular orbit, it is easy to see that $S_\infty^- \cap N_\infty^+$ must be the set of the endpoints of all the horizontal

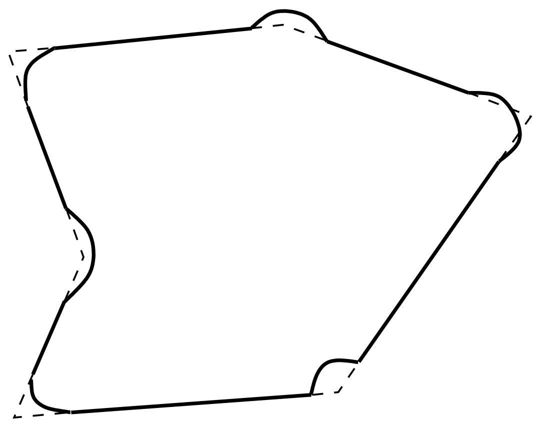

Figure 6: A non-convex polygon with pockets and bumps

segments forming all the strips. This set is finite, and so Condition B3 is satisfied. We do not know, however, whether B4 is satisfied, because N_∞ might disconnect sets M_i contained in M_0 (this is indeed the case for the stadium with parallel segments).

Thus we cannot use Theorem 8.24. We argue, instead, as follows. Since $N_\infty \subset M_0$, every set $M_i \cap H$ is connected provided that $M_i \subset M' := M_- \cup M_+$. Using Theorem 8.23 and a standard argument involving the connectedness of $M_i \cap H$, we can show that every $M_i \subset M'$ belongs (mod 0) to one Bernoulli component of T. Consider now three distinct sets $M_i, M_j, M_l \subset M'$. It is trivial to check, using the convexity of P, that $\mu(TM_i \cap M_j), \mu(TM_i \cap M_l) > 0$ so that M_j, M_l are contained (mod 0) in the same Bernoulli component of T. Thus the whole set M' is contained (mod 0) in one Bernoulli component of T. Note that this argument does not work for stadia, because, in this case, Γ has only two focusing components. To prove that same result for stadia, we have to use instead the fact that there are orbits connecting the two focusing components consisting of an arbitrarily number of collisions.

Let $N > 0$ be the number of Bernoulli components of T. If $N > 1$, then we have $\mu(T^k M' \cap M') = 0$ for any $1 \leq k \leq N - 1$. However the geometry of Q is such that $\mu(TM' \cap M') > 0$. Hence $N = 1$, i.e., T is Bernoulli. □

It is clear from this proof, that Theorem 9.1 is still valid if the condition on the convexity of the polygon P is replaced by the the following condition:

(A): vertices of P with an internal angle greater than π are replaced by bumps in such a way that these vertices lie outside the table Q. See Fig. 6.

Theorem 9.2. *Polygons with bumps and absolutely focusing pockets satisfying Condition (A) are Bernoulli.*

Appendices

A Singular sets: $S_1^\pm$ and $V_k^\pm$

In this appendix, we will not use the fact that focusing boundary components are absolutely focusing. This means that the billiard table considered here satisfy only the conditions described in Section 2. For such billiards, it follows from Theorem 6.1 of [KS, Part V] that S_1^-, S_1^+ are unions of finitely many points and finitely many smooth curves of finite length. In the next theorem, we study the closure of $S_1^\pm$, and prove that $S_1^\pm$ are neat. Note that, whereas the mentioned results of [KS] are valid for billiard table with boundary C^3 piecewise, we assume that such a boundary is C^4 piecewise.

Theorem A.1. *For a billiard with boundary formed by straight lines and C^4 focusing and dispersing curves, the sets S_1^+, S_1^- are neat.*

Proof. The billiards considered in this theorem satisfy the hypotheses of Theorem 6.1 of [KS, Part V] (we warn the reader that the notation for singular sets used in that paper differs from ours). A byproduct of the proof of this theorem is that the closure of the sets S_3 and $\tilde{S}_4$ are regular, where $\tilde{S}_4$ is the union of curves γ such that the trajectories emerging from γ are tangent to dispersing components of Γ that do not intersect $\pi(\gamma)$.

In this proof, we will show that also the closure of $S_4 \setminus \tilde{S}_4$ is regular. Since any two smooth curves contained in the closure of S_1^+ can only intersect at the endpoint of at least one of them, we then obtain that the closure of S_1^+ is regular. It follows immediately that S_1^- is regular as well, because $S_1^- = RS_1^+$.

To show that the closure of $S_4 \setminus \tilde{S}_4$ is regular, we have only to consider the case of two boundary components Γ_i, Γ_j having a common endpoint q. One of these components, let us say Γ_j, has to be dispersing, whereas the other can be dispersing or focusing or a segment. Let $\gamma \subset (S_4 \setminus \tilde{S}_4)$ be a curve consisting of vectors (q, v) such that $q \in \Gamma_i$ and the ray $L(q, v)$ is tangent Γ_j. We consider an orthogonal system of coordinates (x, y) such that the point q coincides with the origin $(0, 0)$. Then the curve Γ_j is the graph of a C^4 function $g : [0, a) \to \mathbb{R}$ for some $a > 0$ such that $g(0) = g'(0) = 0$ and $g''(0) < 0$. The curve Γ_i is the graph of a C^4 function $f : (b, c) \to \mathbb{R}$ with b or c equal to 0. Moreover f is such that either $f(0) = f'(0) = 0$ and $f''(0) > 0$ or $f \equiv 0$. The former case corresponds to a Γ_i dispersing or focusing, whereas the last to a Γ_i flat. We will discuss here only the case when $c = 0$ and $f''(0) > 0$, which is the case of a focusing Γ_i (see Fig. 7). Our argument however covers also all the other cases.

Let $I = (b, 0] \times [0, a)$. We start by noting that there exists a small neighborhood (in M_i) of γ admitting the following C^1 parameterization

$$I \ni (t, u) \mapsto (q = (t, f(t)), v = -(\cos\beta(u), \sin\beta(u)))\,,$$

where $\beta(u) = \tan^{-1} g'(u)$ is a C^1 function as $g''(u) < 0$ for $u \in [0, a)$. Now, let

$$F(t, u) = f(t) - g(u) - g'(u)(t - u), \quad (t, u) \in I.$$

The function F is C^3 because f, g are C^4 functions. The lemma is proved if we prove that $F^{-1}(0)$ is a C^1 curve (see Fig. 7). Since

$$D_{(t,u)}F = \left(f'(t) - g'(u), -g''(u)(t - u)\right)\,,$$

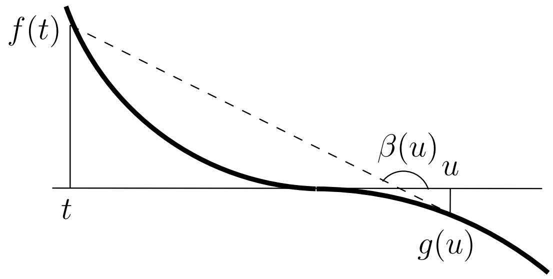

Figure 7: A billiard orbit between the curves Γ_i and Γ_j generating a component of S_1^+

and $g''(u) < 0$ for $u \in [0, a)$, it follows that F is a submersion on $\operatorname{int} I$, and so $F^{-1}(0) \cap \operatorname{int} I$ is a C^1 curve. Note that $F(0,0) = 0$, but F is not a submersion on the entire domain I, because $(0,0)$ is a critical point of F. To conclude that $F^{-1}(0)$ is a C^1 curve on I, we have to analyze the behavior of $F^{-1}(0)$ in a neighborhood of $(0,0)$. To do this, we observe that F has a smooth extension up to a neighborhood of $(0,0)$ (as f, g extend smoothly up to a neighborhood of 0). Next, since

$$\operatorname{Hess}_{(t,u)} F = \begin{pmatrix} f''(t) & -g''(u) \\ -g''(u) & g''(u) - g'''(u)(t-u) \end{pmatrix},$$

we have $\operatorname{Det} \operatorname{Hess}_{(0,0)} = f''(0)g''(0) - g''(0)^2 < 0$ as $f''(0) > 0$ and $g''(0) < 0$. Hence $(0,0)$ is a non-degenerate critical point of F, and, by Morse Lemma [Hi], there is a C^1 change of coordinates $(\alpha, \beta) \mapsto (t, u)$ around $(0,0)$ such that F takes the form

$$F(\alpha, \beta) = \alpha^2 - \beta^2$$

in the new coordinates. Thus, in a small neighborhood U of $(0,0)$, the level set $F^{-1}(0)$ consists of four smooth curves intersecting at $(0,0)$, one of them being the continuation of $F^{-1}(0) \cap \operatorname{int} I$. This finishes the proof. □

Let $n > 0$ be the number of components of Γ, and $K > 0$ be the sum of the numbers of all corners of Γ and all dispersing components of Γ. Define $I_1 = \{1, \dots, K\}$, and $I_j = \{1, \dots, n\}^{j-1} \times \{1, \dots, K\}$ for any $j \geq 2$. The set $\{P_1, \dots, P_K\}$ denotes the collection of all corners of Γ and dispersing components of Γ. For two points $q_1, q_2 \in \mathbb{R}^2$, let $[q_1, q_2]$ denote the segment joining q_1, q_2. Given an integer $1 \leq k \leq K$, we say that the vector $z \in \operatorname{int} M$ has a *k-singular collision* if

$$\begin{cases} q_1(z) = P_k & \text{if } P_k \text{ is corner,} \\ [\pi(z), q_1(z)] \text{ is tangent to } P_k & \text{if } P_k \text{ is a dispersing component.} \end{cases}$$

Definition A.2. *For any $j > 0$ and $\boldsymbol{i} = (i_0, \dots, i_j) \in I_j$, define $C_{\boldsymbol{i}}^j$ to be the set of all $z \in T^{-j+1} S_1^+$ such that $T^k z \in \operatorname{int} M_{i_k}$ for $k = 0, \dots, j-1$ and $T^{j-1} z$ has a i_j-singular collision. For $j < 0$, the set $C_{\boldsymbol{i}}^j$ is defined similarly, but S_1^+, T^{-1} have to be replaced by S_1^-, T.*

Lemma A.3. *Assume that $j > 0$, and $\boldsymbol{i} \in I_j$. When $j > 1$, we further assume that $M_{i_k} \subset M_0$ for any $1 \leq k \leq j-1$. Then $\overline{C_{\boldsymbol{i}}^j}$ has finitely many connected components, each being a smooth curve. In particular, $C_{\boldsymbol{i}}^j$ is regular.*

Proof. By the hypotheses of the lemma, the set $\overline{C_{\boldsymbol{i}}^j}$ coincides with the set $\overline{C_{\boldsymbol{i}'}^1}$ with a proper $\boldsymbol{i}'$, associated to the billiard obtained by unfolding the original billiard table along the trajectories of $C_{\boldsymbol{i}}^j$. The set $C_{\boldsymbol{i}'}^1$ is not connected when there is an "obstacle" which splits the orbits of elements of $C_{\boldsymbol{i}'}^1$. Since there might be only finitely many such obstacles (Γ has finitely many components), $C_{\boldsymbol{i}'}^1$ must have finitely many connected components. By Theorem 6.1 of [KS, Part V], these components are smooth, and Theorem A.1 tells us that their closure is smooth as well. This finishes the proof, we observe, as it is easy to check, that the closures of any two connected components of $C_{\boldsymbol{i}'}^1$ do not intersect. □

Lemma A.4. *All the indexes j, j' and $\boldsymbol{i}, \boldsymbol{i}'$ involved in the following propositions are assumed to satisfy the hypothesis of Lemma A.3. We have*

1. *for any $j, j' > 0, \boldsymbol{i} \in I_j, \boldsymbol{i} \in I_{j'}$ such that $(j, \boldsymbol{i}) \neq (j', \boldsymbol{i}')$, we have $\overline{C_{\boldsymbol{i}}^j} \cap \overline{C_{\boldsymbol{i}'}^{j'}} \subset \partial C_{\boldsymbol{i}}^j \cup \partial C_{\boldsymbol{i}'}^{j'}$,*
2. *for any $j > 0, \boldsymbol{i} \in I_j$, and any connected component ξ of $C_{\boldsymbol{i}}^j$, there exist $0 < j' \leq j, \boldsymbol{i}' \in I_{j'}$ with $i_0' = i_0$ such that $\partial \xi \subset C_{\boldsymbol{i}'}^{j'} \cup \partial M$,*
3. *for any $j > 0$, the set $(\cup_{1 \leq k \leq j} \cup_{\boldsymbol{i} \in I_k} \overline{C_{\boldsymbol{i}}^k}) \cup \partial M$ is connected.*

Proof. The proof of the first statement is an easy consequence of the definition of $C_{\boldsymbol{i}}^j$. The proof of the second and third propositions are essentially the same: it is enough to note that if the closure of a connected component of $C_{\boldsymbol{i}}^j$ does not intersect any connected component (distinct than itself) of $C_{\boldsymbol{i}'}^{j'}$ with $0 < j' \leq j$, then it reaches ∂M. □

We recall the definitions of several sets given in Section 4. Let $V_1^+ = S_1^+ \cap M_0$, and define inductively $V_k^+ = (T^{-1} V_{k-1}^+ \cap M_0) \cup V_1^+$ for $k > 1$. We similarly define V_n^- by replacing S_1^+, T^{-1} by S_1^-, T. The sets $V_k^\pm$ consist of elements of M_0 having at most $k-1$ consecutive collisions with Γ_0 before hitting a corner of Γ or having a tangential collision at Γ_-. Let $V_k = V_k^- \cup V_k^+$, and $\partial M_0 = \partial M \cap M_0$.

Corollary A.5. *In the following propositions, the unions are restricted to $\boldsymbol{i}$'s satisfying the hypothesis of Lemma A.3. We have*

1. *$S_1^+ = \cup_{\boldsymbol{i} \in I_1} C_{\boldsymbol{i}}^1$, and S_1^+ is neat,*
2. *$V_k^+ = \cup_{1 \leq m \leq k} \cup_{\boldsymbol{i} \in I_m} C_{\boldsymbol{i}}^m$ (where the union has to be further restricted to $\boldsymbol{i}$'s satisfying $M_{i_0} \subset M_0$), and V_k^+ is neat,*
3. *$T^{-1} V_k^+ = \cup_{2 \leq m \leq k} \cup_{\boldsymbol{i} \in I_m} C_{\boldsymbol{i}}^m$, and $T^{-1} V_k^+$ is neat.*

Similar propositions are valid for the sets $S_1^-, V_k^-, T V_k^-$.

Proof. It follows immediately from the definition of the sets involved and Lemmas A.3 and A.4. □

B Singular sets: $\mathcal{S}_k^{\pm}$

This appendix contains a result (Theorem B.4) which is crucial to prove the regularity of $\overline{\mathcal{S}_k^{\pm}}$ for any $k > 0$. Surprisingly we do not know whether the same result is valid for $\overline{S_k^{\pm}}$ for any $k > 0$ (Theorem A.1 tells us that this is the case for $k = 1$). This is one of the reasons for introducing the induced system (Ω, ν, Φ).

Definition B.1. *Let $N > 0$. A* smooth component *of $\overline{\mathcal{S}_N^+}$ if γ is a smooth compact curve γ which is the closure of a connected component of $C_{\boldsymbol{i}}^j \cap \mathcal{S}_N^+ \neq \emptyset$ for some $1 \leq j \leq N(2\bar{n} - 1)$ and $\boldsymbol{i} \in I_j$.*

Lemma B.2. *Consider a billiard satisfying B1. Let N be a non-negative integers, and γ_1, γ_2 be smooth components of $\overline{\mathcal{S}_N^+}$. Then either $\gamma_1 = \gamma_2$ or $\gamma_1 \cap \gamma_2 \subset \partial\gamma_1 \cup \partial\gamma_2$. A similar result is valid for smooth components of $\overline{\mathcal{S}_N^-}$.*

Proof. By the time-reversing symmetry of billiard systems, it is enough to prove the lemma for smooth components of $\overline{\mathcal{S}_N^+}$. Let j_1, j_2 be the integers as in the Definition B.1 referring to γ_1, γ_2, respectively. Suppose that $\gamma_1 \cap \gamma_2 \neq \emptyset$. Let $z \in \gamma_1 \cap \gamma_2$, and assume without loss of generality that $z \in \operatorname{int} \gamma_1$. In this case, we have $j_1 \leq j_2$. It is easy to see that if $j_1 = j_2$ and γ_1, γ_2 are generated by the same corner or dispersing component of Γ, then $\gamma_1 = \gamma_2$, otherwise $z \in \partial\gamma_2$. □

Lemma B.3. *Consider a billiard satisfying B1 and B2. Let N_1, N_2 be two non-negative integers, and γ_1, γ_2 be smooth components of $\overline{\mathcal{S}_{N_1}^+}$ and $\overline{\mathcal{S}_{N_2}^-}$, respectively. Then $\gamma_1 \cap \gamma_2$ is finite.*

Proof. For $i = 1, 2$, let Γ_i be the component of Γ for which $\pi(\gamma_i) \subset \Gamma_i$. If $\Gamma_1 \neq \Gamma_2$, then we obtain trivially that $\gamma_1 \cap \gamma_2 \subset \partial\gamma_1 \cap \partial\gamma_2$, and so $\gamma_1 \cap \gamma_2$ is finite. In the rest of this proof, we therefore consider the case $\Gamma_1 = \Gamma_2$, and study separately the two possibilities: $\gamma_1, \gamma_2 \in M_+$ and $\gamma_1, \gamma_2 \in M_- \cup \Delta$.

So suppose now that Γ_1 is a focusing component of Γ. The corresponding phase space is M_1. We will show, in this case, that $\mathcal{T}_z\gamma_1 \subset \operatorname{int} C'(z)$ and $\mathcal{T}_z\gamma_2 \subset \operatorname{int} C(z)$ for any $z \in \gamma_1 \cap \gamma_2$. This fact will implies that γ_1, γ_2 are transversal, and therefore finite because smooth components are by definition compact. To carried out our plan, we further split this part of the proof in the study of several cases, i.e, when $\gamma_1 \subset \overline{\mathcal{R}_{N_1}^+}$ ($\gamma_2 \subset \overline{\mathcal{R}_{N_2}^-}$), and when $\gamma_1 \subset \overline{\mathcal{S}_{N_1}^+ \setminus \mathcal{R}_{N_1}^+}$ ($\gamma_2 \subset \overline{\mathcal{S}_{N_2}^- \setminus \mathcal{R}_{N_2}^-}$). Before starting analyzing these cases, we observe that if $z \in \gamma_1 \cap \gamma_2$, then $z \notin S_2$. So let $\gamma_1 \subset \overline{\mathcal{R}_{N_1}^+}$. By the definition of a smooth component, and that the restriction of T to $M_1 \cap T^{-1}M_1$ is a diffeomorphism (see Section 5.3), we can find an integer $0 < n_1 \leq N_1$ such that $T^{n_1}\gamma_1$ is a smooth compact curve of $S_1 \cap M_1$ (the "vertical" boundary of M_1). It follows that if $0 \neq u \in \mathcal{T}_z\gamma_1$, then $D_zT^{n_1}u = a\partial/\partial\theta(T^{n_1}z)$ for some $a \neq 0$. We recall that, in the construction of C in Section 5.3, the vector $\partial/\partial\theta(y)$ generates the edge of $C(y)$ which gets mapped strictly inside the next cone. Thus by the invariance of C, we obtain that $u \in \operatorname{int} C'(z)$, i.e, $\mathcal{T}_z\gamma_1 \subset \operatorname{int} C'(z)$. Similarly one can show that $\mathcal{T}_z\gamma_2 \subset \operatorname{int} C(z)$ when $\gamma_2 \subset \overline{\mathcal{R}_{N_2}^-}$. We consider now the case $\gamma_1 \subset \overline{\mathcal{S}_{N_1}^+ \setminus \mathcal{R}_{N_1}^+}$. Let $I_1 \ni t \mapsto \gamma_1(t)$ be a regular parameterization of γ_1 where I_1 is a closed interval. As before, there exists an integer $0 \leq n_1 < N_1$ such that $\xi_1 := T^{n_1} \operatorname{int} \gamma_1$ is a smooth open curve contained in S_1^+ with a regular parameterization given by $\operatorname{int} I_1 \ni t \mapsto \xi_1(t) := T^{n_1}\gamma_1(t)$. Note that this time, we can only claim that T^{n_1} is a diffeomorphism on $\operatorname{int}\gamma_1$. Hence $\tau^+(\xi_1, \xi_1'(t))$ is equal to the length of the segment connecting $\pi(\xi_1)$ and the corner or dispersing component of Γ generating γ_1. Let $\tilde{\gamma}_1$ be the curve on M_1 which is the last iterate of γ_1 before this leaves M_1. This curve is smooth and compact because T is a diffeomorphism on $M_1 \cap T^{-1}M_1$. Let $t \mapsto \tilde{\gamma}_1(t)$ denote the obvious regular parametrization of $\tilde{\gamma}_1$. It is not difficult to see that Condition B2 and the invariance of C imply

$$\tau^+(\tilde{\gamma}_1(t), \tilde{\gamma}_1'(t)) > \tau_i^+ + \bar{\tau}, \quad t \in \operatorname{int} I_1. \tag{19}$$

Suppose now that $z = \gamma_1(t_1)$ for some $t_1 \in I_1$. Since $z \notin S_2$, passing to the limit as $t \to t_1$ in (19) , we obtain

$$\tau^+(\tilde{\gamma}_1(t_1), \tilde{\gamma}_1'(t_1)) > \tau_i^+ + \bar{\tau}.$$

This implies that $\tilde{\gamma}_1'(t_1) \in \operatorname{int} C'(\tilde{\gamma}_1(t_1))$, and, using the invariance of C, we obtain $\gamma_1'(t_1) \in \operatorname{int} C'(z)$, i.e., $\mathcal{T}_z\gamma_1 \subset \operatorname{int} C'(z)$. Similarly one can prove that $\mathcal{T}_z\gamma_2 \subset \operatorname{int} C(z)$ when $\gamma_2 \subset \overline{\mathcal{S}_{N_2}^- \setminus \mathcal{R}_{N_2}^-}$.

To finish the proof, we have to analyze the case $\gamma_1, \gamma_2 \in M_- \cup \Delta$. For $i = 1, 2$, let $I_i \ni t \mapsto \gamma_i(t)$ be a parameterization of γ_i as described above. A similar argument to the one above shows that $\gamma_1'(t) \in \operatorname{int} C'(\gamma_1(t))$ for every $t \in \operatorname{int} I_1$, and $\gamma_2'(t) \in \operatorname{int} C(\gamma_2(t))$ for every $t \in \operatorname{int} I_2$. In coordinates (s, θ), we have $\operatorname{int} C(z) = \{(s', \theta') : s'\theta' < 0\}$ for $z \in M_- \cup \Delta$, and so γ_1, γ_2 are, respectively, strictly increasing and decreasing. It follows that $\gamma_1 \cap \gamma_2$ can contain at most one element. We note that, as an bonus, we obtain that $\gamma_1'(t) \in C'(\gamma_1(t))$ for $t \in \partial I_1$ and $\gamma_2'(t) \in C(\gamma_2(t))$ for $t \in \partial I_2$. □

Theorem B.4. *Consider a billiard satisfying B1 and B2. Let $j > 0$, and γ be a smooth component of $\overline{\mathcal{S}_j^-}$ such that $\operatorname{int}\gamma \cap \mathcal{S}_1^+ = \emptyset$. Then $\overline{\Phi \operatorname{int}\gamma}$ is smooth. A similar conclusion is valid if we replace $\mathcal{S}_j^-, \mathcal{S}_1^+, \Phi$ by $\mathcal{S}_j^+, \mathcal{S}_1^-, \Phi^{-1}$.*

Proof. It is enough to prove the theorem only for images of Φ; by the time-symmetry of the billiard dynamics, the theorem extends automatically to images of Φ^{-1}.

To prove the theorem, we argue as follows. First of all, note that there is a $0 < k \le 2\bar{n} - 1$ such that $\Phi \operatorname{int}\gamma = T^k\gamma$. Let Γ_i, Γ_j be the boundary components of Γ, not necessarily distinct, such that $\gamma \in M_i$ and $T^k \operatorname{int}\gamma \in M_j$. Our goal is to show that $\overline{T^k \operatorname{int}\gamma}$ is smooth.

If we consider an orthogonal system of coordinates (x, y), then the curve Γ_i is the graph of a C^3 function $f : [0, a] \to \mathbb{R}$ for some $a > 0$ such that $f(0) = 0$. Note that the conditions that we have imposed on the components of Γ (see Section 2) imply that if f'' is zero at some point, then f is linear, i.e., Γ_i is a straight line. For any $t \in [0, a]$, the value $\alpha(t)$ denotes the angle that the x-axis forms with the tangent of the graph of f at $(t, f(t))$, i.e., $f'(t) = \tan\alpha(t)$. We consider the following C^1 parameterization of γ

$$t \mapsto \gamma(t) := ((t, f(t)), \theta(t)) \in M_i, \quad t \in [0, a']$$

where $\theta(t)$ is the angle formed by the vector $\gamma(t)$ with the oriented tangent of Γ_i at $(t, f(t))$, and $0 < a' \le a$. See Fig. 8.

For any $t \in [0, a']$, let $L(t)$ be the ray which is the continuation of the vector $\gamma(t)$. Denote by $\beta(t)$ the angle that x-axis forms with $L(t)$, and let $b(t) = \tan\beta(t)$. The functions β and b are C^1 on $[0, a']$. We may assume without loss of generality that $b(0) = 0$.

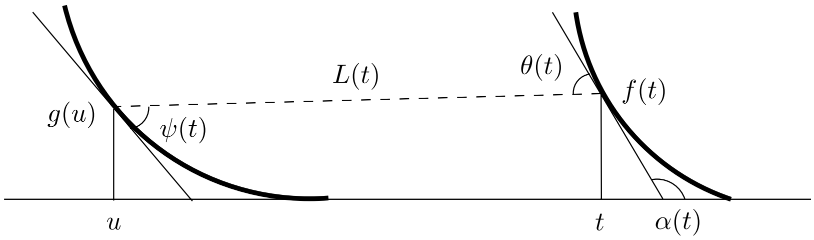

Figure 8: A billiard orbit between curves Γ_i and Γ_j

The curve Γ_j is the graph of a C^3 function $g : [-c, c] \to \mathbb{R}$ for some $c > 0$ and such that $g(u_0) = 0$ for some $-c < u_0 \leq 0$. When $u_0 < 0$, we rescale the billiard table Q so that $u_0 = -1$. For any $u \in [-c, c]$, denote by $\varphi(u)$ the angle that the x-axis forms with the tangent of the graph of g at $(u, g(u))$. There are two smooth functions $u, \psi : (0, a') \to \mathbb{R}$ such that $T^k \operatorname{int} \gamma$ has the following parameterization

$$t \mapsto \zeta(t) := ((u(t), g(u(t))), \psi(t)) \in M_j, \quad t \in (0, a').$$

The value $\psi(t)$ gives the angle formed by the oriented tangent of g at the point $(u(t), g(u(t)))$ with $L(t)$. See Fig. 8.

To prove the theorem, we need to show that either i) the parameterization $t \mapsto \zeta(t)$ is $C^1([0, a'])$ or ii) $t \mapsto u(t), t \in [0, a']$ is a homeomorphism and there exists an interval I given by $I = [u(a'), u(0)]$ or $I = [u(0), u(a')]$ such that the parameterization $u \mapsto \zeta(t(u))$ is $C^1(I)$. Note that in order to prove (i), it is enough to show that $u \in C^1([0, a'])$ since β is C^1 and ψ is an affine function of β and $\tan^{-1}(g' \circ u)$. Similarly to prove (ii), it is enough to show that $t \in C^1(I)$. In fact, we will only prove that either u or t have a C^1 extension up to one of the endpoints of their intervals of definition, since the extension up to the other endpoint can be proved in the same way. Accordingly, from now on the curve γ does not contain the endpoint $\gamma(a')$, i.e., γ is parameterized by $t \mapsto \gamma(t), t \in [0, a')$.

We have consider several cases depending on whether $\gamma(0) \in \overline{T^{-k+1}S_1^+}$ and whether Γ_i, Γ_j are dispersing, focusing or flat. If $\gamma(0) \notin \overline{T^{-k+1}S_1^+}$, then one can show that T^k has a smooth extension up to $\gamma(0)$ which, in this case, implies the theorem. Thus we assume now and for the rest of this proof that $\gamma(0) \in \overline{T^{-k+1}S_1^+}$. This means that the proper extension of the positive semi-trajectory of $\gamma(0)$ hits a corner of Γ or has a tangential reflection at a dispersing component of Γ at the kth collision, i.e., when $\gamma(0)$ lands on Ω. If, in the former case, the semi-trajectory does not hit Γ tangentially, then, as before, one can show that T^k has a smooth extension up to $\gamma(0)$ which proves the theorem. Therefore we further assume that the proper extension of the collision of the trajectory of $\gamma(0)$ always hits tangentially the component Γ_j. We will refer to this assumption by (*). This is the interesting case to study, because only in this case, the derivative of Φ blows up at $\gamma(0)$.

As we have already mentioned, there are several cases to consider depending on Γ_i, Γ_j are dispersing, focusing or flat. Furthermore these components may or not intersect. Note, however, that we do not have to consider all these configurations, because some of them are not compatible with Conditions B1 and B2, the definition Δ and Condition (*). Furthermore we will give a detailed proof of the theorem only for some of the allowed configurations; the proof for the remaining configurations follows the same lines.

Note that if $k > 0$, then $T^i \operatorname{int}\gamma \subset M_0$ for any $0 < i < k$ as a consequence of the definition of $\underline{(\Omega, \nu, \Phi)}$. These collisions with straight lines of Γ do not have any effect on the smoothness of $T^k \operatorname{int}\gamma$, and so we can neglect their presence in this proof. In fact, by properly unfolding the table Q, we obtain a new billiard table $\tilde{Q}$ with a billiard map $\tilde{T}$ such that $T^k \operatorname{int}\gamma = \tilde{T} \operatorname{int}\gamma$ and $\Gamma_i, \Gamma_j \subset \partial\tilde{Q}$. In the rest of this proof, we will have in mind the table $\tilde{Q}$ with the map $\tilde{T}$ rather than Q and T^k (and even in the Figures 9-12.

We will analyze the following configurations:

(a) Γ_i, Γ_j non-intersecting and both dispersing, i.e., $f''(0) > 0, g''(-1) < 0$. Condition (*) implies that $f'(0) = g'(-1) = 0$. See Fig 9,

(b) Γ_i, Γ_j intersecting and both dispersing [3], i.e., $f''(0) < 0, g''(0) > 0$. Condition (*) implies that $f'(0) = g'(0) = 0$. See Fig 10,

(c) Γ_i, Γ_j non-intersecting and both focusing, i.e., $f''(0) > 0, g''(-1) < 0$. Condition (*) implies that $g'(-1) = 0$, and $\operatorname{int}\gamma \subset \mathcal{S}_k^-$ implies that $f'(0) > 0$. See Fig 11,

(d) Γ_i is a straight line and Γ_j is focusing, i.e., $g''(-1) \neq 0$. Condition (*) implies that $g'(0) = 0$, and $\gamma \subset \Delta$ implies that $f' \equiv \mathit{const} > 0$, and that Γ_i, Γ_j are non-intersecting. Without loss of generality, we choose $g''(-1) > 0$. See Fig 12.

Configuration (a). We have $\beta(t) = 2\pi + \alpha(t) - \theta(t)$ for $t \in [0, a)$. By hypothesis, γ is a smooth curve such that its interior is contained in $\mathcal{S}_j^-$. It is not difficult to see that the tangent vector of $\operatorname{int}\gamma$ at any point $z \in \gamma$ is contained in $C(z)$ (see the proof of Lemma 8.2). A simple computation then shows that $\theta'(t) \leq 0$ for $[0, a)$. Since $\alpha'(t) = \cos^2\alpha(t) f''(t) > 0$ for $[0, a)$, we have $\beta'(t) > 0$ and $b'(t) = \beta'(t)/\cos^2\beta(t) > 0$ for $[0, a)$. For $(t, u) \in A := [0, a) \times (-1, \infty)$, let $F(t, u) = f(t) - g(u) - b(t)(t - u)$. We have $F(t, u(t)) = 0$ for every $t \in (0, a)$. Let

$$F_t(t, u) := \frac{\partial F}{\partial t} = f'(t) - b(t) - b'(t)(t - u),$$
$$F_u(t, u) := \frac{\partial F}{\partial u} = -g'(u) + b(t).$$

Since $g'(u) < 0$ for $u > -1$ and $b(t) > 0$ for $t \in [a, 0)$, $F_u(t, u) > 0$ for every $(t, u) \in A$. By the Implicit Function Theorem, $u \in C^1((0, a))$ and $u'(t) = -F_t(t, u(t))/F_u(t, u(t))$. From Fig. 9, we see that $\beta(t) > \alpha(t) \geq 0$ for $t \in [0, a)$. So $f'(t) - b(t) < 0$ for $t \in [0, a)$. From earlier observations, $b'(t)(t - u(t)) > 0$ for $t \in [0, a)$. Hence $F_t(t, u) < 0$ for $t \in (0, a)$, and so $u'(t) > 0$ for $t \in (0, a)$. This implies that u extends to a homeomorphism on $[0, a)$. Clearly $u(0) = -1$. If $t = t(u)$, then $t'(u) = -F_u(t(u), u)/F_t(t(u), u)$ and

$$\lim_{u \to -1^+} t'(u) = \frac{0}{b'(0) - f'(0)} = 0$$

because $b'(0) - f'(0) = b'(0) = \beta'(0) > 0$. We conclude that $t \in C^1(I)$ where $I = [-1, u(a))$.

[3] A similar configuration with Γ_j focusing, i.e., $g''(0) > 0$ and g defined for $-a' < u \leq 0$ is not allowed by Condition B2 (see Remark 6.3). Such a configuration is however studied in the proof of Theorem A.1.

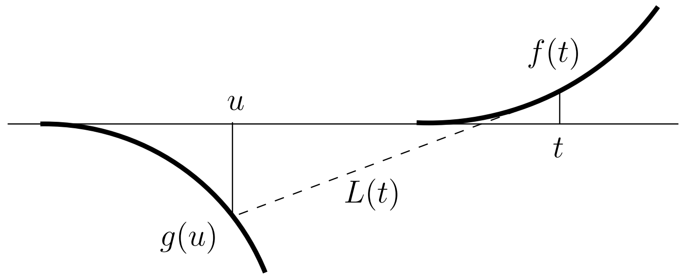

Figure 9: Configuration (a) (case $k = 1$)

Configuration (b). We have $\alpha(t) = 2\pi + f''(0)t + O(t^2)$ and $\theta(t) = \pi + \theta'(0)t + o(t)$. As for Configuration (a), one can show that $\theta'(t) \le 0$ for $t \in [0, a)$ so that $\theta'(0) \le 0$. The angle β is given by $\beta(t) = \alpha(t) + \theta(t) - 2\pi$ for $t \in [0, a)$, and therefore $\beta(t) = \pi + \eta t + o(t)$ and $b(t) = \eta t + o(t)$ where $\eta = f''(0) + \theta'(0) < 0$.

Let $g_2(u) = g''(0)u^2/2$ for $u \in \mathbb{R}$. Let $(t, u(t)) \in [0, a) \times [0, a)$ be the solution of $f(t) - g(u) = b(t)(t - u)$, and let $(t, u_*(t)) \in [0, a) \times [0, a)$ be the solution of $f(t) - g_2(u_*) = b(t)(t - u_*)$. By a straightforward computation, we obtain

$$\delta(t) := u(t) - u_*(t) = \left(1 - \frac{g'(\xi_1(t))}{b(t)}\right)^{-1} \frac{g'''(\xi_2(t))}{6b(t)} u_*^3(t) \tag{20}$$

where $\xi_1(t)$ is between $u(t)$ and $u_*(t)$, whereas $\xi_2(t)$ is between 0 and $u_*(t)$. Since $g'(\xi_1(t))/b(t) \le 0$ for $t \in [0, a)$, we have

$$|\delta(t)| \le M \frac{|u_*(t)|^3}{|b(t)|} \tag{21}$$

where $M = \max_{u \in [0,a)} |g'''(u)| > 0$. The solution $u_*(t)$ is given by (note that $tb(t) - f(t) < 0$)

$$\begin{aligned} u_*(t) &= \frac{b(t) + \sqrt{b^2(t) - 2g''(0)(tb(t) - f(t))}}{g''(0)} \\ &= \Lambda t + o(t), \end{aligned}$$

where $\Lambda = \left(\eta + \sqrt{\eta^2 - 2g''(0)\eta + g''(0)f''(0)}\right)/g''(0)$. Hence

$$\lim_{t \to 0^+} \frac{u_*(t)}{t} = \Lambda,$$

and therefore $u_* \in C^1([0, a))$. By using (21), we see that $|\delta(t)|/t = O(t^2)$. Hence $u \in C^1([0, a))$.

Configuration (c). We have $\beta(t) = \alpha(t) + \theta(t)$ for $t \in [0, a)$. From Fig. 11, it is clear that $g'(u(t)) > b(t)$ for $t \in (0, a)$ so that $F_u(t, u(t)) < 0$ for $t \in (0, a)$. A simple computation shows that the forward focusing time of $\gamma(t)$ (use Formula (3) with $q(t) = (t, f(t))$ and $v(t) = (\cos\beta(t), \sin\beta(t))$) is given by

$$\tau^+(\gamma'(t), \gamma(t)) = \frac{b(t) - f'(t)}{b'(t)\cos\beta(t)}, \quad t \in [0, a),$$

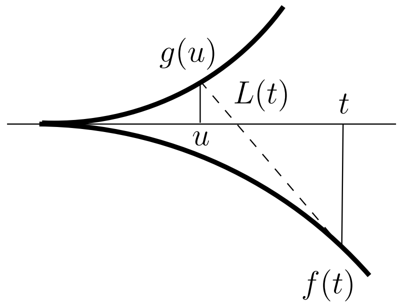

Figure 10: Configuration (b) (case $k = 1$)

and that Condition B2 implies

$$0 \leq \frac{b(t) - f'(t)}{b'(t)\cos\beta(t)} \leq t - u(t) - \frac{\bar{\tau}}{2}, \quad t \in (0, a), \tag{22}$$

for small $a > 0$. We recall that the number $\bar{\tau} > 0$ was introduced with Condition B2 (Section 6). From Fig. 11, we see that $(b(t) - f'(t))/\cos\beta(t) > 0$ for $t \in (0, a)$. Thus the first inequality of (22) implies that $b'(t) > 0$ for $t \in (0, a)$. Now the other inequality of (22) implies

$$\begin{aligned} f'(t) - b(t) - b'(t)(t - u(t)) &< f'(t) - b(t) + b'(t)\cos\beta(t)(t - u(t)) \\ &< \frac{\bar{\tau}}{2} b'(t)\cos\beta(t) < 0 \end{aligned}$$

for $t \in (0, a)$. Hence $F_t(t, u(t)) < 0$ for $t \in (0, a)$. By the Implicit Function Theorem, $u \in C^1((a, 0))$, and $u'(t) = -F_t(t, u(t))/F_u(t, u(t)) < 0$. Thus u extends to a homeomorphism on $[0, a)$ and $u(0) = -1$. By taking the limit of (22) as $t \to 0^+$, we obtain $0 \leq f'(0)/b'(0) < 1$. If $t = t(u)$, then $t'(u) = -F_u(t(u), u)/F_t(t(u), u)$ and

$$\lim_{u \to -1^-} t'(u) = \frac{0}{b'(0) - f'(0)} = 0.$$

We conclude that $t \in C^1(I)$ where $I = (u(a), -1]$.

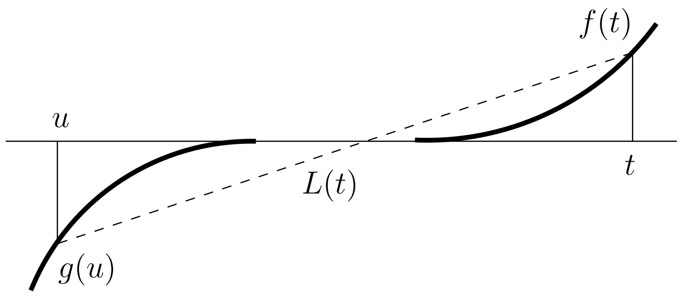

Figure 11: Configuration (c) (case $k = 1$)

Configuration (d). The analysis of this configuration is similar to the one of Configuration (a), and therefore we omit it. We observe that this reduction is possible only because $\gamma \subset \Delta$. This condition, in fact, implies that every vector tangent to γ is divergent, and therefore it tells us that γ essentially behaves as in Configuration (a).

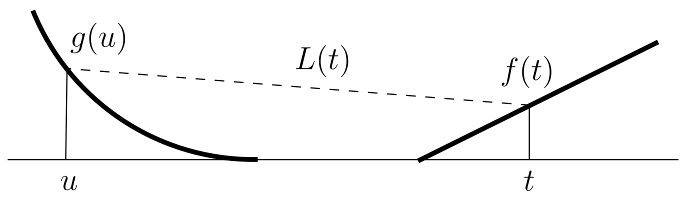

Figure 12: Configuration (d) (case $k = 1$)

The proof of the theorem is finished. □

Remark B.5. *One may wonder whether the previous theorem is valid for the singular sets of* T *as well. We do not know the answer: the proof of Theorem B.4 can be repeated word by word for the singular sets of* T *with the exception of the analysis of Configuration (d). The reason is that if* $\operatorname{int}\gamma \subset S_j^-$ *is contained in* $M_0 \setminus \Delta$*, then it is not true anymore that the tangent vectors of* γ *are divergent, and the reduction described in the analysis of Configuration (d) cannot be achieved.*

C Volume estimate

This section contains a proof of Proposition 7.4 of [LW].

Let M be a Riemannian (non degenerate metric) two-dimensional compact manifold M with boundary and corners. Let X be the union of finite number of smooth and compact curves in M. Denote by μ the Lebesgue measure on M, and denote by μ_X the measure induced on X by the Riemmanian metric.

Lemma C.1. *There exists a constant* C *such that for every closed* $Y \subset X$*, we have*

$$\limsup_{\delta \to 0^+} \frac{\mu(Y^\delta)}{\delta} \leq C\mu_X(Y).$$

Proof. It is enough to prove the statement when X is a smooth and compact curve. We also assume, without loss of generality, that M is endowed with the Euclidean metric. The general result then follows from the fact that any two Riemannian metrics on a compact manifold generate equivalent norms and volume forms.

For any $\delta > 0$ sufficiently small, we can always find $N = N(\delta)$ points $z_1, \dots, z_N \in Y$ ordered according to the orientation of X such

$$Y \subset \cup_{i=1}^N B(z_i, \delta).$$

Note that the finiteness of N is a consequence of the compactness of Y. It is easy to check that

$$Y^\delta \subset \cup_{i=1}^N B(z_i, 2\delta).$$

We claim that there exists an index set $J \subset \{1, \dots, N\}$ such if $l, m \in J$ and $l \neq m$, then

$$B(z_l, 2\delta) \cap B(z_m, 2\delta) = \emptyset,$$

and

$$\cup_{i=1}^N B(z_i, 2\delta) \subset \cup_{j \in J} B(z_j, 6\delta).$$

We give here a sketch of the proof of this claim, which, in fact, follows from a more general result (see for instance Lemma 6.8 of [F]). The construction of the subcover indexed by J is done inductively. Pick as the first element of the subcover any element of the original cover. Suppose that we have already chosen the first $k-1$ elements. The next element is chosen among the remaining balls of the original cover which do not intersect the $k-1$ already chosen elements. The process ends when no such ball remains. Now it is easy to check that the subcover just constructed has the required properties.

For sufficiently small $r > 0$, the measure of a ball of radius r is not larger than cr^2 for some constant $c > 0$ independent on the ball. Thus

$$\mu(Y^\delta) \leq \sum_{j \in J} \mu(B(z_j, 6\delta)) \leq 9c\delta \sum_{j \in J} \mathrm{diam}(B(z_i, 2\delta)).$$

Since

$$\sum_{j \in J} \mathrm{diam}(B(z_i, 2\delta)) \leq \mu_X(X \cap \bigcup_{j \in J} B(z_j, 2\delta)) \leq \mu_X(X \cap \bigcup_{i=1}^N B(z_i, 2\delta)),$$

and

$$\lim_{\delta \to 0^+} \mu_X(X \cap \bigcup_{i=1}^N B(z_i, 2\delta)) = \mu_X(Y),$$

we obtain

$$\limsup_{\delta \to 0^+} \sum_{j \in J} \mathrm{diam}(B(z_i, 2\delta)) \leq \mu_X(Y).$$

We conclude that

$$\limsup_{\delta \to 0^+} \frac{\mu(Y^\delta)}{\delta} \leq 9c\mu_X(Y).$$

□

Remark C.2. *The previous proof works also for degenerate metrics provided that the measure of a ball of radius r is not greater than cr^2 where c is a constant independent of the ball considered.*

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