Ground State of a Model in the Relativistic Quantum Electrodynamics with a Fixed Total Momentum

Itaru Sasaki
Department of Mathematics,
Hokkaido University,
Sapporo 060-0810, Japan
e-mail: i-sasaki@math.sci.hokudai.ac.jp

December 21, 2005

Abstract

We consider the polaron model of the relativistic quantum electrodynamics with a fixed total momentum. We analyze some properties of the ground state energy of the model, and show that the polaron model has a ground state under a condition which includes conditions for an infrared and an ultraviolet cutoff for photon momenta. In particular, we show that the model with zero total momentum, a nonzero mass of the electron and an infrared cutoff has a ground state.

Key words: QED, relativistic QED, ground state energy, polaron, ground state.

1 INTRODUCTION AND MAIN RESULTS

We consider a system of an electron interacting with the quantized radiation field. In the full quantum electrodynamics the electron is described by the Dirac field. In this paper, however, we consider one electron case, and we suppose that the electron is described by the Dirac operator.

The total momentum of the system conserves if there is no external potential, i.e., the Hamiltonian of the system strongly commutes with the total momentum operator. Hence the Hamiltonian has a direct integral decomposition with respect to the total momentum [1, 2].

We study each fibre of the total Hamiltonian, which is a model in the relativistic quantum electrodynamics(QED) for a fixed total momentum — the polaron model of the relativistic QED. In the previous paper [7] we showed that the Hamiltonian of the polaron model of the relativistic QED is bounded from below under a natural condition. In this paper we analyze some properties of the ground state energy of the polaron Hamiltonian, and show that the polaron Hamiltonian has a ground state if the total momentum equals zero, the electron has a non-zero mass, and the radiation field satisfies an infrared regular condition.

1.1 Definition of the Model

In this paper we choose the Coulomb gauge for the electromagnetic field.

The Hilbert space for the photon field is the Boson Fock space over $L^2(\mathbb{R}^3 \times \{1,2\})$:

$$\mathcal{F}_{\mathrm{rad}} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_s^n L^2(\mathbb{R}^3 \times \{1,2\}) \right], \tag{1}$$

where $\otimes_s^n$ means the n-fold symmetric tensor product (see [9]). For a closable operator T on $L^2(\mathbb{R}^3 \times \{1,2\})$, we denote by $\mathrm{d}\Gamma(T)$, $\Gamma(T)$ the second quantization operators of T [9]. The Hilbert space for the total system is defined by

$$\mathcal{F} := \mathcal{H} \otimes \mathcal{F}_{\mathrm{rad}}, \tag{2}$$

where $\mathcal{H} := L^2(\mathbb{R}^3; \mathbb{C}^4)$ is the Hilbert space of the relativistic electron. For each vector $f \in L^2(\mathbb{R}^3 \times \{1,2\})$, we denote by $a(f)$, $a(f)^*$ the annihilation and the creation operator respectively(see [9]). Let $\mathbf{e}^{(\lambda)} : \mathbb{R}^3 \mapsto \mathbb{R}^3, \lambda = 1,2$, be polarization vectors of a photon:

$$\mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{e}^{(\mu)}(\mathbf{k}) = \delta_{\lambda,\mu}, \quad \mathbf{e}^{(\lambda)}(\mathbf{k}) \cdot \mathbf{k} = 0, \quad \mathbf{k} \in \mathbb{R}^3,\ \lambda, \mu \in \{1,2\}.$$

We write as $\mathbf{e}^{(\lambda)}(\mathbf{k}) = (e_1^{(\lambda)}(\mathbf{k}), e_2^{(\lambda)}(\mathbf{k}), e_3^{(\lambda)}(\mathbf{k}))$, and we suppose that each component $e_j^{(\lambda)}(\mathbf{k})$ is a Borel measurable function of $\mathbf{k}$. For three objects a_1, a_2, a_3, we set $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^3 a_j b_j$ if $\sum_{j=1}^3 a_j b_j$ is defined. For a linear object $F(\cdot)$ we set $F(\mathbf{a}) := (F(a_1), F(a_2), F(a_3))$. We choose a function $\rho \in L^2(\mathbb{R}^3) \cap \mathrm{Dom}(|\hat{\mathbf{k}}|^{-1/2})$, where "Dom" means the operator domain, and $|\hat{\mathbf{k}}|$ is a multiplication operator by the function $|\mathbf{k}|$. We set

$$g_j(\mathbf{k}, \lambda; \mathbf{x}) := |\mathbf{k}|^{-1/2} \rho(\mathbf{k}) e_j^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^3.$$

For each $\mathbf{x} \in \mathbb{R}^3$, $g_j(x) := g_j(\cdot\,; \mathbf{x}) \in L^2(\mathbb{R}^3 \times \{1,2\})$. We define

$$A_j(\mathbf{x}) := \frac{1}{\sqrt{2}} \overline{[a(g_j(\mathbf{x})) + a(g_j(\mathbf{x}))^*]}, \quad j = 1,2,3,$$

the quantized vector potential, where, for a closable operator T, $\bar{T}$ denotes its closure. $A_j(\mathbf{x})$ is a self-adjoint operator on $\mathcal{F}_{\mathrm{rad}}$(see [9]). The Hilbert space $\mathcal{F}$ can be identified as

$$\mathcal{F} = L^2(\mathbb{R}^3; \oplus^4 \mathcal{F}_{\mathrm{rad}}) = \int_{\mathbb{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\mathrm{rad}} \mathrm{d}\mathbf{x},$$

and we can define a self-adjoint decomposable operator on $\mathcal{F}$ by

$$\mathbf{A}(\hat{\mathbf{x}}) := \int_{\mathbb{R}^3}^{\oplus} \mathbf{A}(\mathbf{x}) \mathrm{d}\mathbf{x},$$

(see [1, 2]). The operator $\mathbf{A}(\hat{\mathbf{x}})$ is also called the quantized vector potential. The photon Hamiltonian is defined by

$$H_f := \mathrm{d}\Gamma(\omega),$$

where ω is a multiplication operator of the function $|\mathbf{k}|$ acting on $L^2(\mathbb{R}^3 \times \{1,2\})$. The Hamiltonian which describes one relativistic free electron interacting with the radiation field is given by

$$H := \boldsymbol{\alpha} \cdot (\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{x}})) + M\beta + H_f,$$

where $\hat{\mathbf{p}} = -i\nabla$ and ∇ is the gradient operator in $\mathcal{H}$, $\boldsymbol{\alpha}, \beta$ are 4×4 Dirac matrices, the constant $M \in \mathbb{R}$ is the mass of the electron, $q \in \mathbb{R}$ is a constant proportional to the fine-structure constant. For simplicity, we always omit the tensor product between $\mathcal{H}$ and $\mathcal{F}_{\mathrm{rad}}$. It is easy to see that H is symmetric. The essential self-adjointness of H was proven in the paper [2]:

Proposition 1.1. *Suppose that $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|^{-1})$. Then, $\bar{H}$ is a self-adjoint operator.*

Remark. The Hamiltonian $\bar{H}$ seems to depend on the choice of the polarization vectors $\mathbf{e}^{(1)}(\mathbf{k})$, $\mathbf{e}^{(2)}(\mathbf{k})$. However, we can show that $\bar{H}$ is unitarily equivalent to $\bar{H}'$ defined by using another polarization vectors $\mathbf{e}'^{(1)}(\mathbf{k})$, $\mathbf{e}'^{(2)}(\mathbf{k})$, see appendix. Therefore, all physical consequences of the model do not depend on the choice of polarization vectors.

★ Throughout the paper we assume that $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|^{-1})$.

We define the momentum operator $\mathbf{P}^{\mathrm{rad}}$ of the quantized radiation field by

$$\mathbf{P}^{\mathrm{rad}} = \mathrm{d}\Gamma(\hat{\mathbf{k}}).$$

Each component P_j^{rad} $(j = 1, 2, 3)$ is a self-adjoint operator on $\mathcal{F}_{\mathrm{rad}}$. The total momentum operator is defined by

$$\mathbf{P} := \overline{\hat{\mathbf{p}} + \mathbf{P}^{\mathrm{rad}}}.$$

The total momentum of the system governed by the Hamiltonian H is conserved, i.e., the Hamiltonian H strongly commutes with $\mathbf{P}$ (see [2]). To find the polaron Hamiltonian, we define a self-adjoint operator

$$Q := \overline{\hat{\mathbf{x}} \cdot \mathbf{P}^{\mathrm{rad}}},$$

where $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ are the multiplication operators of the functions x_1, x_2, x_3 acting on $\mathcal{H}$. Let U_F be the Fourier transform on $\mathcal{H}$, and we set $U := U_F \exp(iQ)$. By this unitary operator U, the Hamiltonian $\bar{H}$ and total momentum operator $\mathbf{P}$ are transformed as follows(see [1]):

$$\begin{aligned} U\bar{H}U^* &= \overline{\boldsymbol{\alpha} \cdot [\widetilde{\mathbf{p}} - \mathrm{d}\Gamma(\hat{\mathbf{k}}) - q\mathbf{A}] + M\beta + H_f}, \\ U\mathbf{P}U^* &= \widetilde{\mathbf{p}}, \end{aligned}$$

where $\mathbf{A} := \mathbf{A}(0)$ and $\widetilde{\mathbf{p}}$ is the multiplication operator by the (Fourier transformed) coordinates (p_1, p_2, p_3) on the Fourier transformed Hilbert space $U_F\mathcal{H}$. We can identify $U\mathcal{F}$ as

$$U\mathcal{F} := \int_{\mathbb{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\mathrm{rad}} \mathrm{d}\mathbf{p}. \tag{3}$$

The Hamiltonian of the polaron model we consider is of the form

$$H(\mathbf{p}) := \boldsymbol{\alpha} \cdot \mathbf{p} + M\beta + H_f - \boldsymbol{\alpha} \cdot \mathrm{d}\Gamma(\hat{\mathbf{k}}) - q\boldsymbol{\alpha} \cdot \mathbf{A}, \tag{4}$$

which acts on each fibre $\oplus^4 \mathcal{F}_{\mathrm{rad}}$ and $\mathbf{p} \in \mathbb{R}^3$ is a constant. The polaron Hamiltonian $H(\mathbf{p})$ is the fibre of $U\bar{H}U^*$ in the decomposition (3)(see [1, 2]):

Proposition 1.2. *Assume that $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|^{-1})$. Then $H(\mathbf{p})$ is essentially self-adjoint and*

$$U\bar{H}U^* = \int_{\mathbb{R}^3}^{\oplus} \overline{H(\mathbf{p})} \mathrm{d}\mathbf{p}.$$

Remark. Physically $\overline{H(\mathbf{p})}$ is the Hamiltonian of the fixed total momentum $\mathbf{p} \in \mathbb{R}^3$. We can show that all the spectral properties of $\overline{H(\mathbf{p})}$ do not depend on the choice of polarization vectors, because the Hamiltonians with different polarization vectors are unitarily equivalent each other. See Appendix.

The polaron Hamiltonian $\overline{H(\mathbf{p})}$ is bounded below [7], because we assume $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|^{-1})$. Therefore we can define the ground state energy of the polaron with total momentum $\mathbf{p}$:

$$E(\mathbf{p}) := \inf \sigma(\overline{H(\mathbf{p})}),$$

where $\sigma(\cdot)$ means the spectrum.

Let $N_{\mathrm{b}} := \mathrm{d}\Gamma(\mathbb{1})$ be the number operator on $\mathcal{F}_{\mathrm{rad}}$, where $\mathbb{1}$ is the identity on $L^2(\mathbb{R}^3 \times \{1, 2\})$. For a constant $m \geq 0$, we set

$$H_m(\mathbf{p}) := H(\mathbf{p}) + mN_{\mathrm{b}}, \quad E_m(\mathbf{p}) := \inf \sigma(\overline{H_m(\mathbf{p})}).$$

The case $m = 0$ is the original case. For $m > 0$, the massive Hamiltonian $H_m(\mathbf{p})$ had studied in [1, 2].

1.2 Some Properties of the Ground State Energy $E_m(\mathbf{p})$

In this subsection we assume that $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|^{-1})$. Therefore $\overline{H_m(\mathbf{p})}$ is self-adjoint and bounded from below (see [1, 2, 7]).

The ground state energy $E_m(\mathbf{p})$ depends on all the constants in $\overline{H_m(\mathbf{p})}$: the total momentum $\mathbf{p} \in \mathbb{R}^3$, the electron mass $M \in \mathbb{R}$, the virtual photon mass $m \geq 0$, the ultraviolet cutoff function ρ, and the fine-structure constant $q \in \mathbb{R}$. But $E_m(\mathbf{p})$ does not depend on the choice of polarization vectors. When the dependence of these variables is important, we write $E_m(\mathbf{p})$ as $E_m(\mathbf{p}, M, \cdots)$, making it explicitly.

Proposition 1.3 (Concavity of the ground state energy)**.** *$E_m(\mathbf{p})$ is concave function in the variables $(\mathbf{p}, M, m, q) \in \mathbb{R}^3 \times \mathbb{R} \times [0, \infty) \times \mathbb{R}$.*

Proposition 1.4 (Continuity of the ground state energy)**.** *$E_m(\mathbf{p}, M)$ is Lipschitz continuous function of $(\mathbf{p}, M)$, i.e.,*

$$|E_m(\mathbf{p}, M) - E_m(\mathbf{p}', M')| \leq \sqrt{|\mathbf{p} - \mathbf{p}'|^2 + |M - M'|^2}, \quad \mathbf{p}, \mathbf{p}' \in \mathbb{R}^3,\ M, M' \in \mathbb{R}.$$

Proposition 1.5. *The Hamiltonian $\overline{H_m(\mathbf{p}, M)}$ is unitarily equivalent to $\overline{H_m(\mathbf{p}, -M)}$. In particular $E_m(\mathbf{p}, M) = E_m(\mathbf{p}, -M)$, and $E_m(\mathbf{p}, M) \leq E_m(\mathbf{p}, 0)$ holds.*

A symmetry of a cutoff function ρ leads to a symmetry of the same kind with respect to the total momentum $\mathbf{p}$ of $\overline{H_m(\mathbf{p})}$:

Proposition 1.6 (Symmetry in the total momentum)**.** *Assume that $|\rho(\mathbf{k})| = |\rho(T\mathbf{k})|$ a.e. $\mathbf{k} \in \mathbb{R}^3$ for an orthogonal matrix $T \in O(3)$. Then $\overline{H_m(\mathbf{p})}$ is unitarily equivalent to $\overline{H_m(T\mathbf{p})}$, and $E_m(\mathbf{p}) = E_m(T\mathbf{p})$. In particular, if $|\rho(\mathbf{k})| = |\rho(-\mathbf{k})|$, a.e. $\mathbf{k} \in \mathbb{R}^3$, then $E_m(\mathbf{p}) = E_m(-\mathbf{p})$ and $E_m(\mathbf{p}) \leq E_m(0)$.*

A function $f(\mathbf{k})$ is called rotation invariant function if $f(\mathbf{k}) = f(T\mathbf{k})$ for all $T \in SO(3)$.

If the cutoff function ρ is rotation invariant, the spectral properties of $\overline{H_m(\mathbf{p})}$ does not depend on the direction of $\mathbf{p}$:

Proposition 1.7 (Spherical symmetry in the total momentum)**.** *Assume that $\rho(\mathbf{k})$ is a rotation invariant function. Then $\overline{H_m(\mathbf{p})}$ is unitarily equivalent to $\overline{H_m(\mathbf{p}')}$ for all $\mathbf{p}' \in \mathbb{R}^3$ with $|\mathbf{p}| = |\mathbf{p}'|$. In particular $E_m(\mathbf{p})$ is rotation invariant with respect to $\mathbf{p}$, and $E_m(\mathbf{p}) \geq E_m(\mathbf{p}')$ if $|\mathbf{p}| \leq |\mathbf{p}'|$.*

Proposition 1.8 (Massless limit of the ground state energy)**.** *$E_m(\mathbf{p})$ is monotone non-decreasing in $m \geq 0$ and*

$$\lim_{m \to +0} E_m(\mathbf{p}) = E_0(\mathbf{p}).$$

Generally, by Proposition 1.4, the following inequality holds:

$$0 \leq E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}|, \quad \mathbf{p}, \mathbf{k} \in \mathbb{R}^3.$$

If the electron mass M is not zero, we can get a stronger inequality:

Proposition 1.9. *In the case $m > 0$, the inequality $E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| > 0$ holds for all $M \neq 0$, $\mathbf{k} \in \mathbb{R}^3 \backslash \{0\}$ and $\mathbf{p} \in \mathbb{R}^3$. In the massless case $m = 0$, for all $\mathbf{p} \in \mathbb{R}^3$, there exists a constant $\mathsf{M} \geq 0$ such that the inequality $E(\mathbf{p} - \mathbf{k}) - E(\mathbf{p}) + |\mathbf{k}| > 0$ holds for all $|M| > \mathsf{M}$ and $\mathbf{k} \in \mathbb{R}^3 \backslash \{0\}$.*

When ρ is rotation invariant, by Proposition 1.7, the function $E_m(\mathbf{p}) - E_m(0) + |\mathbf{p}|$ is monotone non-decreasing, concave with respect to $|\mathbf{p}|$, and the following inequality holds

$$0 \leq E_m(\mathbf{p}) - E_m(0) + |\mathbf{p}| \leq |\mathbf{p}|.$$

For $C_1 > 0$, we set

$$a_1(C_1) := \inf_{0 \leq m \leq 1} (E_m(\mathbf{p}) - E_m(0) + |\mathbf{p}|)\Big|_{|\mathbf{p}|=C_1}$$

The following two propositions are important to derive the existence of a ground state of massless Hamiltonian $H(0)$ with zero total momentum.

Proposition 1.10. *Assume that $|M| > \mathsf{M}$(which is a constant in Proposition 1.9) and $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|)$ is a rotation invariant function. Then, for all $C_1 > 0$, the inequality $a_1(C_1) > 0$ holds, and*

$$\inf_{0 \leq m \leq 1} (E_m(\mathbf{p}) - E_m(0) + |\mathbf{p}|) \geq \frac{a_1(C_1)}{C_1} |\mathbf{p}|, \tag{5}$$

for all $|\mathbf{p}| \leq C_1$.

Proposition 1.11. *Assume that $M \neq 0$ and $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}})$ is a rotation invariant function. Suppose that $E(\mathbf{p}) - E(0) + |\mathbf{p}| > 0$ for all $\mathbf{p} \in \mathbb{R}^3 \backslash \{0\}$. Then for all $C_1 > 0$ the inequality $a_1(C_1) > 0$ holds, and*

$$\inf_{0 \leq m < 1} (E_m(\mathbf{p}) - E_m(0) + |\mathbf{p}|) \geq \frac{a_1(C_1)}{C_1} |\mathbf{p}|,$$

for all $|\mathbf{p}| \leq C_1$.

1.3 Existence of a Ground State

In this subsection we assume that $\rho \in \mathrm{Dom}(|\mathbf{k}|^{-1})$. For a bounded below self-adjoint operator T, we say that T has a ground state if $\inf \sigma(T)$ is an eigenvalue of T.

In paper [1] A. Arai studied a Hamiltonian in a class which includes $\overline{H_m(\mathbf{p})}$, $m > 0$ and prove the existence of a ground state of $\overline{H_m(\mathbf{p})}$, $m > 0$. — more precisely speaking, his criterion for the existence of a ground state does not include the case $\overline{H_m(\mathbf{p})}$, $m > 0$, but one can show that $\overline{H_m(\mathbf{p})}$, $m > 0$ has a ground state in the same manner as in [1].

In this subsection we give some criteria for the polaron Hamiltonian $\overline{H(\mathbf{p})}$ to have a ground state.

Theorem 1.12. *Suppose that*

$$\liminf_{m \to +0} \int_{\mathbb{R}^3} \frac{q^2}{(E_m(\mathbf{p}-\mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^2} \frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|} \mathrm{d}\mathbf{k} < 1. \tag{6}$$

Then the polaron Hamiltonian $\overline{H(\mathbf{p})}$ has a ground state.

The condition (6) has a restriction in q, and $E_m(\mathbf{p})$ depends on q. Therefore to check inequality (6) is difficult. In the case $\mathbf{p} = 0$, Theorem 1.12 is replaced by a somewhat simple one:

Theorem 1.13. *Assume that $M \neq 0$ and ρ is rotation invariant function. If the inequality*

$$\int_{\mathbb{R}^3} \frac{q^2}{(E(\mathbf{k}) - E(0) + |\mathbf{k}|)^2} \frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|} \mathrm{d}\mathbf{k} < 1 \tag{7}$$

holds, then $\overline{H(0)}$ has a ground state.

Proof of Theorems 1.12 and 1.13 are based on the photon number bound originated from [4, 5]. Therefore the inequalities (6) and (7) have restrictions on the coupling constant q. If one use the photon derivative bound, then one can remove the restriction on q.

Theorem 1.14. *Suppose that ρ is rotation invaliant and there is an open set $S \subset \mathbb{R}^3$ such that $\bar{S} := \operatorname{supp} \rho$ and ρ is continuously differentiable function in S. Assume that for all R, the set $S_R := \{\mathbf{k} \in S | |\mathbf{k}| < R\}$ has the cone property, and*

$$\limsup_{m \to +0} \int_S \frac{q^2}{(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^2} \frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|} \mathrm{d}\mathbf{k} < \infty, \tag{8}$$

and for all $p \in [1, 2)$ and $R > 0$, the following inequalities hold:

$$\sup_{0<m<1} \int_{S_R} \left[(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-2} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p \mathrm{d}\mathbf{k} < \infty,$$

$$\sup_{0<m<1} \int_{S_R} \left[(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \frac{|\nabla \rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p \mathrm{d}\mathbf{k} < \infty,$$

$$\sup_{0<m<1} \int_{S_R} \left[(E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \frac{1}{\sqrt{k_1^2 + k_2^2}} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p \mathrm{d}\mathbf{k} < \infty.$$

Then $\overline{H(\mathbf{p})}$ has a ground state.

Remember a property of the constant M(Propositions 1.9 and 1.10). In the case $\mathbf{p} = 0$ and $M \geq \mathsf{M}$, the conditions in Theorem 1.14 become simpler:

Theorem 1.15. *Let $|M| \geq \mathsf{M}$. Suppose that ρ is rotation invariant and there is an open set $S \subset \mathbb{R}^3$ such that $\bar{S} := \operatorname{supp} \rho$, and ρ is continuously differentiable in S. Assume that for all R, the set $S_R := \{\mathbf{k} \in S | |\mathbf{k}| < R\}$ has the cone property, and $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|^{-3/2})$, $|\mathbf{k}|^{-5/2}\rho(\mathbf{k}) \in L^p(S_R)$, and $|\mathbf{k}|^{-3/2}|\nabla \rho(\mathbf{k})| \in L^p(S_R)$, for all $p \in [1, 2)$ and $R > 0$. Then, the polaron Hamiltonian $\overline{H(0)}$ has a ground state.*

2 Proofs of Proposition 1.3-1.10

First we note that the operator $H_m(\mathbf{p})$ is essentially self-adjoint on any core for $H_f(m) := H_f + mN_{\mathrm{b}}$, because the term $-\boldsymbol{\alpha}\cdot \mathrm{d}\Gamma(\hat{\mathbf{k}}) - q\boldsymbol{\alpha}\cdot\mathbf{A}$ is $H_f(m)$-bounded with relative bound 1(see [7]) and one can apply Wüst's theorem (see [9]). Since $\mathcal{D} := \mathrm{Dom}(H_f)\cap\mathrm{Dom}(N_{\mathrm{b}})$ is a core for H_f, $\mathcal{D}$ is a common core for $H_m(\mathbf{p}), m\geq 0$. When we want to write the explicit dependence of variables $(\mathbf{p}, M, m, q)$, we write $H_m(\mathbf{p})$ as $H_m(\mathbf{p}, M, q)$.

Proof of Proposition 1.3. Let $(\mathbf{p}, M, m, q), (\mathbf{p}', M', m', q') \in \mathbb{R}^4\times[0,\infty)\times\mathbb{R}$, and $t\in[0,1]$. Then for all $\Psi\in\mathcal{D}$, we have

$$\begin{aligned}&\langle\Psi, H_{tm+(1-t)m'}(t\mathbf{p}+(1-t)\mathbf{p}', tM+(1-t)M', tq+(1-t)q')\Psi\rangle\\&= t\langle\Psi, H_m(\mathbf{p}, M, q)\Psi\rangle + (1-t)\langle\Psi, H_{m'}(\mathbf{p}', M', q')\Psi\rangle\\&\geq tE_m(\mathbf{p}, M, q) + (1-t)E_{m'}(\mathbf{p}', M', q').\end{aligned}$$

Thus we obtain

$$E_{tm+(1-t)m'}(t\mathbf{p}+(1-t)\mathbf{p}', tM+(1-t)M', tq+(1-t)q') \geq tE_m(\mathbf{p}, M, q)+(1-t)E_{m'}(\mathbf{p}', M', q'),$$

which implies that $E_m(\mathbf{p}, M, q)$ is a concave function. ∎

Proof of Proposition 1.4. By the equality $H_m(\mathbf{p}, M) = H_m(\mathbf{p}', M')+\boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{p}')+(M-M')\beta$, and variational principle, we have $E_m(\mathbf{p}, M)\leq H_m(\mathbf{p}', M')+\boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{p}')+(M-M')\beta$, in the sense of a quadratic form on $\mathcal{D}$. Hence we have

$$E_m(\mathbf{p}, M)\leq E_m(\mathbf{p}', M') + \sqrt{|\mathbf{p}-\mathbf{p}'|^2+(M-M')^2}.$$

Similarly, we have $E_m(\mathbf{p}', M')\leq E_m(\mathbf{p}, M)+\sqrt{|\mathbf{p}-\mathbf{p}'|^2+(M-M')^2}$. ∎

Proof of Proposition 1.5. We set $\gamma_5 := i\alpha_1\alpha_2\alpha_3$. It is easy to see that γ_5 is unitary operator and $\gamma_5\overline{H_m(\mathbf{p}, M)}\gamma_5^* = \overline{H_m(\mathbf{p}, -M)}$. Therefore $E_m(\mathbf{p}, M) = E_m(\mathbf{p}, -M)$. By Proposition 1.3, $M\mapsto E_m(\mathbf{p}, M)$ is concave. Hence $E_m(\mathbf{p}, 0) = E_m(\mathbf{p}, \frac{1}{2}M-\frac{1}{2}M)\geq E_m(\mathbf{p}, M)$. ∎

Proof of Proposition 1.6. For the matrix $T\in O(3)$, we define four 4×4 matrices:

$$\beta' := \beta,\quad \alpha_j' := \sum_{j=1}^{3} T_{j,l}\alpha_l,\quad j=1,2,3.$$

It is easy to see that $\{\alpha_j', \beta'\} = 0$, $\{\alpha_j', \alpha_l'\} = 2\delta_{j,l}$, $j,l=1,2,3$. Then there exists a 4×4 unitary matrix u_T such that (see [10, Lemma 2.25])

$$u_T\alpha_j u_T^{-1} = \sum_{k=1}^{3} T_{j,k}\alpha_k,\quad u_T\beta u_T^{-1} = \beta.$$

Therefore $u_T\boldsymbol{\alpha}\cdot\mathbf{p}u_T^{-1} = \sum_{k,l=1}^3 T_{l,k}\alpha_k p_l = \sum_{k,l=1}^3 \alpha_k (T^{-1})_{k,l}p_l = \boldsymbol{\alpha}\cdot(T^{-1}\mathbf{p})$. Similarly, we have

$$u_T(\boldsymbol{\alpha}\cdot d\Gamma(\hat{\mathbf{k}}))u_T^{-1} = \boldsymbol{\alpha}\cdot(T^{-1}\mathrm{d}\Gamma(\hat{\mathbf{k}})), \quad u_T\boldsymbol{\alpha}\cdot\mathbf{A}u_T^{-1} = \boldsymbol{\alpha}\cdot(T^{-1}\mathbf{A}) = (T\boldsymbol{\alpha})\cdot\mathbf{A}.$$

We define 1-photon rotation operator $\hat{T}$ by

$$(\hat{T}f)(\mathbf{k},\lambda) = f(T^{-1}\mathbf{k},\lambda), \quad (\mathbf{k},\lambda)\in\mathbb{R}^3\times\{1,2\}, \quad f\in L^2(\mathbb{R}^3\times\{1,2\}).$$

Then for all $f\in\mathrm{Dom}(\hat{k}_j\hat{T})$

$$\hat{T}^{-1}\hat{k}_j\hat{T}f(\mathbf{k},\lambda) = (k_j\hat{T}f)(T\mathbf{k},\lambda) = (T\mathbf{k})_j(\hat{T}f)(T\mathbf{k},\lambda) = (T\mathbf{k})_jf(\mathbf{k},\lambda).$$

Hence we obtain an operator equality $\hat{T}^{-1}\hat{k}_j\hat{T} = (T\hat{\mathbf{k}})_j$, $(j=1,2,3)$, and

$$\begin{aligned}
&\Gamma(\hat{T}^{-1})\mathrm{d}\Gamma(\hat{k}_j)\Gamma(\hat{T}) = \mathrm{d}\Gamma((T\hat{\mathbf{k}})_j) = (T\cdot\mathrm{d}\Gamma(\hat{\mathbf{k}}))_j,\\
&\Gamma(\hat{T}^{-1})H_f(m)\Gamma(\hat{T}) = H_f(m)\\
&\Gamma(\hat{T}^{-1})A_j\Gamma(\hat{T}), = \Phi_S(\hat{T}^{-1}g_j), \quad j=1,2,3,
\end{aligned}$$

where $\Phi_S(\cdot)$ is the Segal field operator(see [9]) and $g_j(\cdot) := g_j(\cdot,\mathbf{x}=0)\in L^2(\mathbb{R}^3\times\{1,2\})$. The operator $U := u_T\otimes\Gamma(\hat{T}^{-1})$ is a unitary operator on $\oplus^4\mathcal{F}_{\mathrm{rad}} = \mathbb{C}^4\otimes\mathcal{F}_{\mathrm{rad}}$ and

$$U\overline{H_m(\mathbf{p})}U^{-1} = \overline{(\boldsymbol{\alpha}\cdot(T^{-1}\mathbf{p}) + M\beta + H_f(m) - \boldsymbol{\alpha}\cdot\mathrm{d}\Gamma(\hat{\mathbf{k}}) - q(T\boldsymbol{\alpha})\cdot\Phi_S(\hat{T}^{-1}g))}. \tag{9}$$

Note that T is a 3×3-matrix and $\hat{T}$ is a unitary operator on $L^2(\mathbb{R}^3\times\{1,2\})$. Since $T\in O(3)$, we have $(T\boldsymbol{\alpha})\cdot\Phi_S(\hat{T}^{-1}\mathbf{g}) = \boldsymbol{\alpha}\cdot T^{-1}\Phi_S(\hat{T}^{-1}\mathbf{g})$, i.e.,

$$(T^{-1}\Phi_S(\hat{T}^{-1}\mathbf{g}))_j = \sum_{l=1}^3 (T^{-1})_{j,l}\Phi_S(\hat{T}^{-1}g_l), \quad j=1,2,3. \tag{10}$$

We define a functions

$$\mathbf{e}'^{(\lambda)}(\mathbf{k}) = T^{-1}\mathbf{e}^{(\lambda)}(T\mathbf{k}), \quad (\mathbf{k},r)\in\mathbb{R}^3\times\{1,2\}.$$

It is easy to see that $\mathbf{e}'^{(1)},\mathbf{e}'^{(2)}$ are a polarization vectors: $\mathbf{k}\cdot\mathbf{e}'^{(\lambda)}(\mathbf{k}) = 0$, $\mathbf{e}'^{(\lambda)}(\mathbf{k})\cdot\mathbf{e}'^{(\mu)}(\mathbf{k}) = \delta_{\lambda,\mu}$. Since $|\rho(\mathbf{k})| = |\rho(T\mathbf{k})|$, there exists a Borel measurable function $\mathbf{k}\mapsto\kappa(\mathbf{k})\in\mathbb{R}$ such that $\rho(T\mathbf{k}) = e^{i\kappa(\mathbf{k})}\rho(\mathbf{k})$, a.e. $\mathbf{k}\in\mathbb{R}^3$. Therefore, we have

$$\sum_{l=1}^3 (T^{-1})_{j,l}g_l(T\mathbf{k},\lambda) = \frac{e^{i\kappa(\mathbf{k})}\rho(\mathbf{k})}{|\mathbf{k}|^{1/2}}e_j'^{(\lambda)}(\mathbf{k}). \tag{11}$$

Let $H_m'(\mathbf{p})$ be $H_m(\mathbf{p})$ with replacing $\mathbf{e}^{(\lambda)}$ to $\mathbf{e}'^{(\lambda)}$. By (9),(10) and (11), we have

$$U\overline{H_m(\mathbf{p})}U^* = V\overline{H_m'(T^{-1}\mathbf{p})}V^*,$$

where $V := \Gamma(e^{i\kappa(\cdot)})$. By Theorem 4.2, $\overline{H'_m(T^{-1}\mathbf{p})}$ is unitarily equivalent to $\overline{H_m(T^{-1}\mathbf{p})}$. Therefore, $\overline{H(\mathbf{p})}$ is unitarily equivalent to $\overline{H_m(T^{-1}\mathbf{p})}$. Since $\mathbf{p} \in \mathbb{R}^3$ is arbitrary in the above argument, $\overline{H_m(\mathbf{p})}$ is unitarily equivalent to $\overline{H_m(T\mathbf{p})}$, and $E_m(\mathbf{p}) = E_m(T\mathbf{p})$. When $\rho(\mathbf{k}) = \rho(-\mathbf{k})$, a.e. $\mathbf{k} \in \mathbb{R}^3$, we have

$$E_m(0) = E_m(\tfrac{1}{2}\mathbf{p} - \tfrac{1}{2}\mathbf{p}) \geq \frac{1}{2}E_m(\mathbf{p}) + \frac{1}{2}E_m(-\mathbf{p}) = E_m(\mathbf{p}),$$

for all $\mathbf{p} \in \mathbb{R}^3$. ∎

Proof of Proposition 1.7. The first half of Proposition 1.7 is a direct consequence of Proposition 1.6. We show that $E_m(\mathbf{p})$ is non-increasing in $|\mathbf{p}|$. Temporally we assume that there exist vectors $\mathbf{p}, \mathbf{p}' \in \mathbb{R}^3$ such that $|\mathbf{p}| \leq |\mathbf{p}'|$ and $E_m(\mathbf{p}) < E_m(\mathbf{p}')$. Without loss of generality, we can assume that $\mathbf{p}' = t\mathbf{p}$ for a $t \geq 1$, because $E_m(\mathbf{p})$ does not depend on $\mathbf{p}/|\mathbf{p}|$. Since $E_m(s\mathbf{p}) \to E_m(0), (s \to 0)$, and the map $\mathbb{R} : s \mapsto E_m(s\mathbf{p})$ is continuous, there exists a constant $s' \in [0, 1)$ such that $E_m(s'\mathbf{p}) = E_m(\mathbf{p}')$. We set $r := (t-1)/(t-s') \in [0, 1)$. Then we get a contradiction

$$E_m(\mathbf{p}) = E_m(r(s'\mathbf{p}) + (1-r)\mathbf{p}') \geq rE_m(s'\mathbf{p}) + (1-r)E_m(\mathbf{p}') = E_m(\mathbf{p}').$$

Therefore $E_m(\mathbf{p})$ is a non-increasing function of $|\mathbf{p}|$. ∎

Proof of Proposition 1.8. Let $m \geq m' \geq 0$. Then we have $H_m(\mathbf{p}) \geq H_{m'}(\mathbf{p})$ in the sense of quadratic form on $\mathcal{D}$. Therefore $m \mapsto E_m(\mathbf{p})$ is monotone non-decreasing: $E_m(\mathbf{p}) \geq E_{m'}(\mathbf{p})$. It is easy to see that for all $\Psi \in \mathcal{D}$, $H_m(\mathbf{p})\Psi \to H(\mathbf{p})\Psi$ as $m \to 0$. Since $\mathcal{D}$ is a common core for all $H_m(\mathbf{p})$, $H_m(\mathbf{p}) \to H(\mathbf{p})$ in the strong resolvent sense (see [8, Theorem VIII. 25]). Using a fact about a strongly convergent operators [8, Theorem VIII. 24], we have $E_m(\mathbf{p}) \to E(\mathbf{p})$ as $m \to +0$. ∎

Proof of Proposition 1.9. We prove this proposition by absurd. First we prove it in the massive case $m > 0$. We fix a mass $M \neq 0$ and $\mathbf{p} \in \mathbb{R}^3$. Assume that

$$E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| = 0, \tag{12}$$

for a vector $\mathbf{k} \in \mathbb{R}^3 \setminus \{0\}$. Let $\Phi_m(\mathbf{p} - \mathbf{k})$ be a normalized ground state of $H_m(\mathbf{p} - \mathbf{k})$ — see the first six lines of Subsection 1.3. Then

$$\begin{aligned} E_m(\mathbf{p} - \mathbf{k}) &= \langle \Phi_m(\mathbf{p} - \mathbf{k}), \overline{H_m(\mathbf{p} - \mathbf{k})}\Phi_m(\mathbf{p} - \mathbf{k}) \rangle \\ &= \langle \Phi_m(\mathbf{p} - \mathbf{k}), \overline{H_m(\mathbf{p})}\Phi_m(\mathbf{p} - \mathbf{k}) \rangle + \langle \Phi_m(\mathbf{p} - \mathbf{k}), \boldsymbol{\alpha} \cdot \mathbf{k}\Phi_m(\mathbf{p} - \mathbf{k}) \rangle \\ &\geq E_m(\mathbf{p}) - |\mathbf{k}|. \end{aligned}$$

Hence, by assumption (12) we have $\langle \Phi_m(\mathbf{p} - \mathbf{k}), \overline{H_m(\mathbf{p})}\Phi_m(\mathbf{p} - \mathbf{k}) \rangle = E_m(\mathbf{p})$ and $\langle \Phi_m(\mathbf{p} - \mathbf{k}), \boldsymbol{\alpha} \cdot \mathbf{k}\Phi_m(\mathbf{p} - \mathbf{k}) \rangle = -|\mathbf{k}|$, which imply that $\Phi_m(\mathbf{p} - \mathbf{k})$ is a ground state of both $\overline{H_m(\mathbf{p})}$

and $\boldsymbol{\alpha}\cdot\mathbf{k}$. Since $\mathbf{k}$ is a non-zero vector, we have $\langle\Phi_m(\mathbf{p}-\mathbf{k}),\beta\Phi_m(\mathbf{p}-\mathbf{k})\rangle=0$, because $\boldsymbol{\alpha}\cdot\mathbf{k}\beta=-\beta\boldsymbol{\alpha}\cdot\mathbf{k}$. In what follows, to emphasise M-dependence, we write $H_m(\mathbf{p}-\mathbf{k})$ and $\Phi_m(\mathbf{p}-\mathbf{k})$ as $H_m(\mathbf{p}-\mathbf{k},M)$ and $\Phi_m(\mathbf{p}-\mathbf{k},M)$ respectively. Using the above facts, we have

$$E_m(\mathbf{p},M)=\langle\Phi_m(\mathbf{p}-\mathbf{k},M),\overline{H_m(\mathbf{p},0)}\Phi_m(\mathbf{p}-\mathbf{k},M)\rangle\geq E_m(\mathbf{p},0).$$

However, $E_m(\mathbf{p},M)\leq E_m(\mathbf{p},0)$ (Proposition 1.5). Hence we obtain $E_m(\mathbf{p},M)=E_m(\mathbf{p},0)$. Therefore $\Phi_m(\mathbf{p}-\mathbf{k},M)$ is a ground state of $\overline{H_m(\mathbf{p},0)}$, and

$$\begin{aligned}E_m(\mathbf{p},M)\Phi_m(\mathbf{p}-\mathbf{k},M)&=\overline{H_m(\mathbf{p},M)}\Phi_m(\mathbf{p}-\mathbf{k},M)\\&=M\beta\Phi_m(\mathbf{p}-\mathbf{k},M)+E_m(\mathbf{p},0)\Phi_m(\mathbf{p}-\mathbf{k},M).\end{aligned}$$

Since $E_m(\mathbf{p},M)=E_m(\mathbf{p},0)$, we have $M\beta\Phi_m(\mathbf{p},M)=0$. Therefore we get a contradiction: $\Phi_m(\mathbf{p},M)=\beta^2\Phi_m(\mathbf{p},M)=0$.

Next, we consider the case $m=0$. Suppose that there existed a vector $\mathbf{p}\in\mathbb{R}^3$ such that for all $\mathsf{M}\in\mathbb{R}_+$ the inequality $E(\mathbf{p}-\mathbf{k},M)-E(\mathbf{p},M)+|\mathbf{k}|=0$ holds for a constant $|M|\geq\mathsf{M}$ and a vector $\mathbf{k}\in\mathbb{R}^3\backslash\{0\}$. It is easy to see that $\lim_{m\to+0}\langle\Phi_m(\mathbf{p}-\mathbf{k},M),\overline{H(\mathbf{p}-\mathbf{k},M)}\Phi_m(\mathbf{p}-\mathbf{k},M)\rangle=E(\mathbf{p}-\mathbf{k},M)$. By the above assumption, we have

$$\begin{aligned}&\lim_{m\to+0}\langle\Phi_m(\mathbf{p}-\mathbf{k},M),\boldsymbol{\alpha}\cdot\mathbf{k}\Phi_m(\mathbf{p}-\mathbf{k},M)\rangle=-|\mathbf{k}|\\&\lim_{m\to+0}\langle\Phi_m(\mathbf{p}-\mathbf{k},M),\overline{H(\mathbf{p},M)}\Phi_m(\mathbf{p}-\mathbf{k},M)\rangle=E(\mathbf{p},M).\end{aligned}$$

Therefore $\lim_{m\to+0}\langle\Phi_m(\mathbf{p}-\mathbf{k},M),\beta\Phi_m(\mathbf{p}-\mathbf{k},M)\rangle=0$. This means that $E(\mathbf{p},M)=E(\mathbf{p},0)$. Since $\mathsf{M}\in\mathbb{R}_+$ is arbitrary and does not depend on $\mathbf{p}$, and $E(\mathbf{p},M)$ is concave, we have that $E(\mathbf{p},M)$ does not depend on M. On the other hand, one can easily show that $E(\mathbf{p},M)\to-\infty$ as $|M|\to\infty$. Therefore we get a contradiction. ∎

Proof of Proposition 1.10. Note that $[0,1]\ni m\mapsto E_m(\mathbf{p})$ is a continuous function. If $a_1(C_1)=0$ for a positive value $C_1>0$, then there exist a constant $m_1\in[0,1]$ such that $E_{m_1}(\mathbf{p})-E_{m_1}(0)+|\mathbf{p}|=0$ with $|\mathbf{p}|=C_1$. However, by Proposition 1.9, we have $E_m(\mathbf{p})-E_m(0)+|\mathbf{p}|>0$ for all $|\mathbf{p}|>0$. Hence we obtain $a_1(C_1)>0$ for all $C_1>0$. Note that $0=(E_m(\mathbf{p})-E_m(0)+|\mathbf{p}|)|_{\mathbf{p}=0}$ and $(E_m(\mathbf{p})-E_m(0)+|\mathbf{p}|)|_{|\mathbf{p}|=C_1}\geq a_1(C_1)$ for all $m\in[0,1]$, and $E_m(\mathbf{p})-E_m(0)+|\mathbf{p}|$ is a concave function of $|\mathbf{p}|$. Therefore inequality (5) holds. ∎

Proof of Proposition 1.11. Similar to the proof of Proposition 1.10 ∎

3 Proofs of Theorem 1.12-1.15

In the massive case $m>0$, $H_m(\mathbf{p})$ has a normalized ground state $\Phi_m(\mathbf{p})$. There exists a sequence $\{\Phi_{m_j}(\mathbf{p})\}_{j=1}^{\infty}$, $m_j\to0(j\to\infty)$ such that $\{\Phi_{m_j}\}$ has a weak limit $\Phi_0(\mathbf{p})$. If $\Phi_0(\mathbf{p})\neq0$, then the massless Hamiltonian $H(\mathbf{p})$ has a ground state(see [4, Lemma 4.9]).

In this section, we sometimes use the following identification

$$\oplus^4 \mathcal{F}_{\mathrm{rad}} = \bigoplus_{n=0}^{\infty} \mathbb{C}^4 \otimes \mathcal{F}^{(n)}, \quad \mathcal{F}^{(n)} := \otimes_s^n L^2(\mathbb{R}^3 \times \{1,2\}),$$

and

$$\mathbb{C}^4 \otimes \mathcal{F}^{(n)} \subset L^2(\mathbb{R}^3 \times \{1,2\}; \mathbb{C}^4 \otimes \mathcal{F}^{(n-1)}), \quad n = 1,2,3,\ldots.$$

where all vector $\Psi^{(n)} \in \mathbb{C}^4 \otimes \mathcal{F}^{(n)}$ is identified with a Hilbert space valued function $\Psi^{(n)}(\mathbf{k}, \lambda; \cdot) : \mathbb{R}^3 \times \{1,2\} \mapsto \mathbb{C}^4 \otimes \mathcal{F}^{(n-1)}$.

For a vector $\Psi \in \oplus^4 \mathcal{F}_{\mathrm{rad}}$, we define an object

$$a_\lambda(\mathbf{k})\Psi := (\Psi^{(1)}(\mathbf{k}, \lambda), \sqrt{2}\Psi^{(2)}(\mathbf{k}, \lambda; \cdot), \ldots, \sqrt{n}\Psi^{(n)}(\mathbf{k}, \lambda; \cdot), \ldots) \in \mathop{\times}_{n=0}^{\infty} \mathbb{C}^4 \otimes \mathcal{F}^{(n)},$$

where the symbol "$\times$" means the direct product. In general, $a_\lambda(\mathbf{k})\Psi \notin \oplus^4 \mathcal{F}_{\mathrm{rad}}$, but we can show that $a_\lambda(\mathbf{k})\Psi \in \oplus^4 \mathcal{F}_{\mathrm{rad}}$ for a class of vectors $\Psi \in \oplus^4 \mathcal{F}_{\mathrm{rad}}$. Let $\mathrm{w} : \mathbb{R}^3 \to [0, \infty)$ be an almost positive Borel measurable function, and we denote its multiplication operator on $L^2(\mathbb{R}^3 \times \{1,2\})$ by the same symbol. For $\Psi = (\Psi^{(n)})_{n=1}^{\infty} \in \mathrm{Dom}(\mathrm{d}\Gamma(\mathrm{w})^{1/2})$, the object $a_\lambda(\mathbf{k})\Psi$ is a $\oplus^4 \mathcal{F}_{\mathrm{rad}}$-valued function: $a_\lambda(\mathbf{k})\Psi \in \oplus^4 \mathcal{F}_{\mathrm{rad}}$, a.e. $\mathbf{k} \in \mathbb{R}^3, \lambda = 1,2$. Because, if $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma(\mathrm{w})^{1/2})$, then

$$\|\mathrm{d}\Gamma(\mathrm{w})^{1/2}\Psi\|^2 = \sum_{n=1}^{\infty} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} n\mathrm{w}(\mathbf{k}) \|\Psi^{(n)}(\mathbf{k}, \lambda; \cdot)\|^2_{\mathbb{C}^4 \otimes \mathcal{F}^{(n-1)}} \mathrm{d}\mathbf{k} < \infty,$$

which implies $a_\lambda(\mathbf{k})\Psi \in \oplus^4 \mathcal{F}_{\mathrm{rad}}$ for almost every $\mathbf{k} \in \mathbb{R}^3$ and $\lambda = 1,2$. Hence, for all $\Psi \in \mathrm{Dom}(N_{\mathrm{b}}^{1/2})$, $a_\lambda(\mathbf{k})\Psi$ is a $\oplus^4 \mathcal{F}_{\mathrm{rad}}$-valued function. For a self-adjoint operator T, we denote by $Q(T)$ the form domain of T. Note that $Q(\overline{H_m(\mathbf{p})}) \subset \mathrm{Dom}(N_{\mathrm{b}}^{1/2})$, $m > 0$, because $H_m(\mathbf{p}) - E(\mathbf{p}) \geq mN_{\mathrm{b}}$ in the sense of quadratic form on $\mathcal{D}$.

We set $\mathbf{g}(\mathbf{k}, \lambda) := \mathbf{g}(\mathbf{k}, \lambda; 0)$.

Proposition 3.1. *Let $m > 0$. Then $a_\lambda(\mathbf{k})\Phi_m(\mathbf{p}) \in \mathrm{Dom}(\overline{H_m(\mathbf{p})})$, a.e. $\mathbf{k} \in \mathbb{R}^3, \lambda = 1,2$ and*

$$a_\lambda(\mathbf{k})\Phi_m(\mathbf{p}) = \frac{q}{\sqrt{2}}(\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + \omega_m(\mathbf{k}))^{-1} \boldsymbol{\alpha} \cdot \mathbf{g}(\mathbf{k}, \lambda)\Phi_m(\mathbf{p}), \quad \text{a.e. } (\mathbf{k}, \lambda) \in \mathbb{R}^3 \times \{1,2\}. \tag{13}$$

Proof. Let $\omega_m(\mathbf{k}) := \omega(\mathbf{k}) + m = |\mathbf{k}| + m$. For all $f \in \mathrm{Dom}(\omega_m)$ and $\Psi \in \mathcal{D}$, we have

$$\langle (H_m(\mathbf{p}) - E_m(\mathbf{p}))\Psi, a(f)\Phi_m(\mathbf{p})\rangle = \langle \Psi, \big\{ -a(\omega_m f) + \boldsymbol{\alpha} \cdot a(\hat{\mathbf{k}} f) + \tfrac{q}{\sqrt{2}}\langle f, \mathbf{g}\rangle \big\}\Phi_m(\mathbf{p})\rangle.$$

Therefore,

$$\sum_{\lambda=1,2} \int_{\mathbb{R}^3} f(\mathbf{k}, \lambda)^* \langle (H_m(\mathbf{p}) - E_m(\mathbf{p}))\Psi, a_\lambda(\mathbf{k})\Phi_m(\mathbf{p})\rangle =$$

$$\sum_{\lambda=1,2} \int_{\mathbb{R}^3} f(\mathbf{k}, \lambda)^* \big\langle \Psi, -\omega_m(\mathbf{k}) a_\lambda(\mathbf{k})\Phi_m(\mathbf{p}) + \boldsymbol{\alpha} \cdot \mathbf{k} a_\lambda(\mathbf{k})\Phi_m(\mathbf{p}) + q\boldsymbol{\alpha} \cdot \mathbf{g}(\mathbf{k}, \lambda)\Phi_m(\mathbf{p}) \big\rangle.$$

Since the subspace $\mathrm{Dom}(\omega_m)$ is dense in $L^2(\mathbb{R}^3)\times\{1,2\}$, we obtain

$$\langle (H_m(\mathbf{p})-E_m(\mathbf{p}))\Psi, a_\lambda(\mathbf{k})\Phi_m(\mathbf{p})\rangle = \big\langle \Psi, (-\omega_m(\mathbf{k})a_\lambda(\mathbf{k})+\boldsymbol{\alpha}\cdot\mathbf{k}a_\lambda(\mathbf{k})+q\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda))\Phi_m(\mathbf{p})\big\rangle,$$

for almost every $\mathbf{k}\in\mathbb{R}^3, \lambda=1,2$, and all $\Psi\in\mathcal{D}$. This means that $a_\lambda(\mathbf{k})\Phi_m(\mathbf{p})\in D(\overline{H_m(\mathbf{p})})$ and

$$(\overline{H_m(\mathbf{p})}-E_m(\mathbf{p})+\omega_m(\mathbf{k})-\boldsymbol{\alpha}\cdot\mathbf{k})a_\lambda(\mathbf{k})\Phi_m(\mathbf{p}) = \frac{q}{\sqrt{2}}\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)\Phi_m(\mathbf{p}).$$

Hence (13) follows. ∎

Proof of Theorem 1.12. By Proposition 3.1 and the present assumption, we have

$$\liminf_{m\to+0}\|N_{\mathrm{b}}^{1/2}\Phi_m(\mathbf{p})\|^2 \le \liminf_{m\to+0}\sum_{\lambda=1}^{2}\int_{\mathbb{R}^3}\frac{q^2}{2}\frac{|\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)|^2}{(E_m(\mathbf{p}-\mathbf{k})-E_m(\mathbf{p})+|\mathbf{k}|+m)^2}\mathrm{d}\mathbf{k}<1.$$

Each component of the massive ground state $\Phi_{m_j}(\mathbf{p})^{(n)}$ converges to $\Phi_0(\mathbf{p})^{(n)}$ weakly as $j\to\infty$, and $\lim_{j\to\infty}\|N_{\mathrm{b}}^{1/2}\Phi_{m_j}\|<1$. Since $\oplus^4\mathbb{C}$ is a finite dimensional space, $\Phi_{m_j}(\mathbf{p})^{(0)}\to\Phi_0(\mathbf{p})^{(0)}$ strongly. It is easy to see that

$$\|\Phi_0(\mathbf{p})\|^2 \ge \|\Phi_0(\mathbf{p})^{(0)}\|^2 = \lim_{j\to\infty}\|\Phi_{m_j}(\mathbf{p})^{(0)}\|^2 = \lim_{j\to\infty}\langle\Phi_{m_j}(\mathbf{p}), P_\Omega\Phi_{m_j}(\mathbf{p})\rangle,$$

where P_Ω is the orthogonal projection in the Fock vacuum $(1,0,0,\ldots)\in\mathcal{F}_{\mathrm{rad}}$. Hence we have

$$\|\Phi_0(\mathbf{p})\|^2 \ge 1-\|N_{\mathrm{b}}^{1/2}\Phi_{m_j}(\mathbf{p})\|^2>0.$$

This means that $\Phi_0(\mathbf{p})(\neq 0)$ is a ground state of $\overline{H(\mathbf{p})}$. ∎

Proof of Theorem 1.13. By the assumption of Theorem 1.13, $q\rho=0$ or $E(\mathbf{k})-E(0)+|\mathbf{k}|>0$ for almost every $\mathbf{k}\in\mathbb{R}^3$. In the case $q\rho=0$, the Theorem 1.13 is trivial. Therefore we consider only the second case. Hence Proposition 1.11 holds. Note that $E(\mathbf{k})-E(0)+|\mathbf{k}|>0$ and (7) yield that $|\hat{\mathbf{k}}|^{-3/2}\rho\in L^2(\mathbb{R}^3)$. The right hand side of the inequality

$$(E_m(\mathbf{k})-E_m(0)+|\mathbf{k}|+m)^{-2}\frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|}\le\frac{C_1^2}{a_1(C_1)^2}\frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|^3}+\frac{1}{a(C_1)^2}\frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|}$$

is integrable and does not depend on $m>0$. By the Lebesgue dominated convergence theorem, we obtain

$$\lim_{m\to+0}\int_{\mathbb{R}^3}(E_m(\mathbf{k})-E_m(0)+|\mathbf{k}|+m)^{-2}\frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|}\mathrm{d}\mathbf{k}=\int_{\mathbb{R}^3}(E(\mathbf{k})-E(0)+|\mathbf{k}|)^{-2}\frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|}\mathrm{d}\mathbf{k}<1.$$

Hence the condition of Theorem 1.12 holds and $\overline{H(0)}$ has a ground state. ∎

For a Hilbert space $\mathcal{K}$, we denote by $\mathsf{B}(\mathcal{K})$ the set of bounded operators on $\mathcal{K}$.

One can easily prove the following proposition:

Lemma 3.2. *For each direction* $\mathbf{j} \in \mathbb{R}^3, |\mathbf{j}| = 1$, *the operator valued function* $\mathbb{R}^3\backslash\{0\} : \mathbf{k} \to (\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \in \mathsf{B}(\oplus^4\mathcal{F}_{\mathrm{rad}})$ *is differentiable in the norm resolvent sense, and*

$$\partial_{\mathbf{j}}(\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1}$$
$$= (\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1}\left(\boldsymbol{\alpha}\cdot\mathbf{j} + \frac{\mathbf{k}\cdot\mathbf{j}}{|\mathbf{k}|}\right)(\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1},$$

where $\partial_{\mathbf{j}}$ *means the differential for the* $\mathbf{j}$*-direction.*

We fix the following polarization vectors in the rest of this section.

$$\mathbf{e}^{(1)}(\mathbf{k}) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \mathbf{e}^{(2)}(\mathbf{k}) := \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}^{(1)}(\mathbf{k}). \tag{14}$$

For the set S in the Theorem 1.14, 1.15, we define $\Omega := S\backslash\{\mathbf{k} \in \mathbb{R}^3 | k_1 = k_2 = 0\}$, $\Omega_R := S_R \cap \Omega$. By Proposition 3.2, we get the following proposition.

Lemma 3.3. *Assume the assumptions in Theorem 1.14. Then* $a_\lambda(\mathbf{k})\Phi_m(\mathbf{p})$ *is strongly continuously differentiable in* Ω *and*

$$\partial_j a_\lambda(\mathbf{k})\Phi_m(\mathbf{p})$$
$$= \frac{q}{\sqrt{2}}(\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1}\left(\alpha_j + \frac{k_j}{|\mathbf{k}|}\right)$$
$$\times (\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1}\boldsymbol{\alpha}\cdot\mathbf{g}(\mathbf{k},\lambda)\Phi_m(\mathbf{p})$$
$$+ \frac{q}{\sqrt{2}}(\overline{H_m(\mathbf{p}-\mathbf{k})} - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1}\boldsymbol{\alpha}\cdot(\partial_j\mathbf{g}(\mathbf{k},\lambda))\Phi_m(\mathbf{p}),$$

where ∂_j *is a differential operator in* $k_j, (j = 1, 2, 3)$.

We set

$$\Psi_j(\mathbf{k},\lambda) = (\Psi_j^{(n)}(\mathbf{k},\lambda;\cdot))_{n=0}^\infty := \partial_j a_\lambda(\mathbf{k})\Phi_m(\mathbf{p}).$$

Lemma 3.4. *Assume the assumptions in Theorem 1.14. Then*

$$\partial_j\Phi_m^{(n)}(\mathbf{p})(\mathbf{k},\lambda;k_2,\ldots,k_n) = \frac{1}{\sqrt{n}}\Psi_j^{(n-1)}(\mathbf{k},\lambda;k_2,\ldots,k_n), \quad k_\ell = (\mathbf{k}_\ell,\lambda_\ell),$$

for all $\mathbf{k},\mathbf{k}_\ell \in \Omega$, $n \in \mathbb{N}$, $\lambda,\lambda_\ell = 1,2$, $j = 1,2,3$, *where* ∂_j *is the distributional derivative in* k_j.

Proof. In this proof, we omit the polarization coordinates λ, λ_ℓ and the total momentum $\mathbf{p}$. By the definition, for all $\psi(\mathbf{k},\mathbf{k}_2,\ldots,\mathbf{k}_n) \in C_0^\infty(\Omega^{n+1})$, we have

$$-\int_{\mathbb{R}^{3n}}(\partial_j\psi)(\mathbf{k},K)\Phi_m^{(n)}(\mathbf{k},K) = \lim_{h\to0}\int_{\mathbb{R}^{3n}}\psi(\mathbf{k},K)\frac{1}{|h|}[\Psi_m^{(n)}(\mathbf{k}+h\mathbf{j},K) - \Psi_m^{(n)}(\mathbf{k},K)]\mathrm{d}\mathbf{k}\mathrm{d}K,$$

where $K = (\mathbf{k}_2, \ldots, \mathbf{k}_n)$ and $\mathbf{j}$ is the unit vector of j-th axis. Hence we obtain

$$\left| \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \left[\int_{\mathbb{R}^{3(n-1)}} \mathrm{d}K \psi(\mathbf{k}, K) \left\{ \frac{1}{|h|} [\Phi_m^{(n)}(\mathbf{k} + h\mathbf{j}, K) - \Phi_m^{(n)}(\mathbf{k}, K)] - \frac{1}{\sqrt{n}} \Psi^{(n-1)}(\mathbf{k}, K) \right\} \right] \right|$$
$$\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \|\psi(\mathbf{k}, \cdot)\|_{L^2(\mathbb{R}^{3(n-1)})} \left\| \frac{\Delta_h}{|h|} \Phi_m^{(n)}(\mathbf{k}, \cdot) - \frac{1}{\sqrt{n}} \Psi^{(n-1)}(\mathbf{k}, \cdot) \right\|_{L^2(\mathbb{R}^{3(n-1)})}, \tag{15}$$

where the operator Δ_h is defined by $\Delta_h f(\mathbf{k}) := f(\mathbf{k} + h\mathbf{j}) - f(\mathbf{k})$ for all functions f. Note that $|h|^{-1} \Delta_h \Phi_m^{(n)}(\mathbf{k}, \cdot)$ converges to $\frac{1}{\sqrt{n}} \Psi^{(n-1)}(\mathbf{k}, \cdot)$ strongly in $L^2(\Omega^{3(n-1)})$ by Proposition 3.3. Since the function $\mathbf{k} \to \Psi^{(n-1)}(\mathbf{k}, \cdot)$ is strongly continuous in Ω, we have

$$\frac{\Delta_h}{|h|} \Phi_m^{(n)}(\mathbf{k}, \cdot) = \text{s-}\int_0^1 \frac{1}{\sqrt{n}} \Psi^{(n-1)}(\mathbf{k} + th\mathbf{j}, \cdot) \mathrm{d}t,$$

where s-$\int$ means the strong integral. By Proposition 3.3, $\|\Psi^{(n-1)}(\mathbf{k}, \cdot)\|_{L^2(\mathbb{R}^{3(n-1)})}$ is continuous in Ω, and therefore bounded in $\{\mathbf{k} \in \mathbb{R}^3 | \|\psi(\mathbf{k}, \cdot)\|_{L^2(\mathbb{R}^{3(n-1)})} \neq 0\}$. Hence

$$\begin{aligned}
&\left\| \frac{\Delta_h}{|h|} \Phi_m^{(n)}(\mathbf{k}, \cdot) - \frac{1}{\sqrt{n}} \Psi^{(n-1)}(\mathbf{k}, \cdot) \right\|_{L^2(\mathbb{R}^{3(n-1)})} \\
&\leq \sup_{|t| \leq 1} \frac{1}{\sqrt{n}} \|\Psi^{(n-1)}(\mathbf{k} + th\mathbf{j}, \cdot)\|_{L^2(\mathbb{R}^{3(n-1)})} + \frac{1}{\sqrt{n}} \|\Psi^{(n-1)}(\mathbf{k}, \cdot)\|_{L^2(\mathbb{R}^{3(n-1)})} \\
&\leq \text{const.}
\end{aligned}$$

Therefore, we can apply the Lebesgue dominated convergence theorem, and the right hand side of (15) converges to zero as $|h| \to 0$. ∎

By Proposition 3.2 and direct calculations, we obtain the following lemma:

Lemma 3.5. *Suppose that the assumption of Theorem 1.14 holds. Then*

$$\begin{aligned}
\|\partial_{(\mathbf{k})_j} \Phi_m^{(n)}(\mathbf{p}; \mathbf{k}, \lambda; \cdot)\| \leq& \|\partial_j a_\lambda(\mathbf{k}) \Phi_m(\mathbf{p})\| \\
\leq& \sqrt{2} |q| (E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-2} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \\
&+ \frac{|q|}{\sqrt{2}} (E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \frac{|\partial_j \rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \\
&+ \frac{|q|}{\sqrt{2}} (E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{3/2}} \\
&+ \frac{|q|}{\sqrt{2}} (E_m(\mathbf{p} - \mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}| + m)^{-1} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} |\partial_j \mathbf{e}^{(\lambda)}(\mathbf{k})|
\end{aligned}$$

for all $\mathbf{k} \in \Omega$, $\lambda = 1, 2$, $j = 1, 2, 3$.

Our polarization vectors (14) satisfy

$$|\partial_j \mathbf{e}^{(\lambda)}(\mathbf{k})| \leq \frac{2}{\sqrt{k_1^2 + k_2^2}}, \quad \mathbf{k} \in \mathbb{R}^3 \backslash \{\mathbf{k}' \in \mathbb{R}^3 | k_1' = k_2' = 0\}.$$

Lemma 3.6. *Suppose that the assumption of Theorem 1.14 holds. Then each component of the massive ground state is in a Sobolev space:* $\Phi_m^{(n)} \in \oplus^4 W^{1,p}((\Omega_R \times \{1,2\})^n)$ *for all* $p \in [1,2)$, $R > 0$ *and*

$$\sup_{0<m<1} \|\Phi_m^{(n)}(\mathbf{p})\|_{\oplus^4 W^{1,p}((\Omega_R\times\{1,2\})^n)} < \infty.$$

Proof. Similar to the proof of [5, page 557, **Step 2.**]. ∎

Proof of Theorem 1.14. By the assumption of Theorem 1.14 and Proposition 3.1 , there exists a sequence $\{m_j\}_{j=1}^\infty$ such that the subsequence $\{\Phi_{m_j}(\mathbf{p})\}_{j=1}^\infty$ and $\{N_{\mathrm{b}}^{1/2}\Phi_{m_j}(\mathbf{p})\}_{j=1}^\infty$ have weak limits as $m_j \to 0 (j \to \infty)$. We denote by $\Phi_0(\mathbf{p})$ the weak limit of $\{\Phi_{m_j}(\mathbf{p})\}_{j=1}^\infty$. If $\Phi_0(\mathbf{p}) \neq 0$, when $\Phi_0(\mathbf{p})$ is a ground state of $H(\mathbf{p})$ (see [4]).

Any vector $\Psi \in \oplus^4\mathcal{F}^n = \mathbb{C}^4 \otimes \mathcal{F}^n$ is a function of the particle spin coordinate $X \in \{1,2,3,4\}$, the n-photon wave number argument $(\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \mathbb{R}^{3n}$, and the photon helicity arguments $\lambda_1, \ldots, \lambda_n \in \{1,2\}$. For simplicity, we set

$$\Phi_j^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) := \Phi_{m_j}(\mathbf{p})^{(n)}(X; \mathbf{k}_1, \lambda_1; \cdots; \mathbf{k}_n, \lambda_n),$$
$$\Phi_0^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) := \Phi_0(\mathbf{p})^{(n)}(X; \mathbf{k}_1, \lambda_1; \ldots; \mathbf{k}_n, \lambda_n).$$

for $X \in \{1,2,3,4\}$ and $\lambda_1, \ldots, \lambda_n \in \{1,2\}$. Note that $\Phi_j^{(n)}, \Phi_0^{(n)} \in L^2(\mathbb{R}^{3n})$. We show that $\text{s-lim}_{j\to\infty} \Phi_j^{(n)} = \Phi_0^{(n)}$ for all $n \in \mathbb{N}$, $X \in \{1,2,3,4\}$ and $\lambda_1, \ldots, \lambda_n \in \{1,2\}$.

Since $|S_R| < \infty$ for all $R > 0$, $L^s(S_R) \subset L^2(S_R)$ for $s \geq 2$ for all $R > 0$. By this fact, for all $p \in [1,2)$, $\{\Phi_j^{(n)}\}_j$ weakly converges to $\Phi_0^{(n)}$ in the sense of $L^p(S_R^{3n})$. By Lemma 3.6, $\sup_j \|\Phi_j\|_{W^{1,p}(\Omega_R^n)} < \infty$ therefore, a subsequence of $\{\Phi_j^{(n)}\}_j$ converges to a vector $\widetilde{\Phi}_{0,R}^{(n)} \in W^{1,p}(\Omega_R^n)$ in the sense of the dual Sobolev space $W^{1,p}(\Omega_R^n)^*$, i.e., for all linear functionals $f \in W^{1,p}(\Omega_R^n)^*$, $f(\Phi_j^{(n)} - \widetilde{\Phi}_{0,R}^{(n)}) \to 0$ as $j \to \infty$. By a general fact of the dual of $W^{1,p}$(e.g. [6]), for all $f_0, f_1, \ldots, f_{3n} \in L^p(\Omega_R^n)^* = L^s(\Omega_R^n)$, $(p^{-1} + s^{-1} = 1)$, we have

$$\int_{\Omega_R^n} f_0(\Phi_j^{(n)} - \widetilde{\Phi}_{0,R}^{(n)}) + \sum_{i=1}^{3n} \int_{\Omega_R^n} f_i \partial_i (\Phi_j^{(n)} - \widetilde{\Phi}_{0,R}^{(n)}) \to 0, \quad (j \to \infty).$$

Therefore we have for all $R > 0$

$$\Phi_0^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n) = \widetilde{\Phi}_{0,R}^{(n)}(\mathbf{k}_1, \ldots, \mathbf{k}_n), \quad (\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \Omega_R^n.$$

Hence for all $p \in [1,2)$, we have $\Phi_j^{(n)} \to \Phi_0^{(n)}$, $(j \to \infty)$ in the weak sense of $W^{1,p}(\Omega_R^n)$. Now we assume that ρ is rotation invariant. Then S_R is a spherically symmetric region in $\mathbb{R}^3$, and Ω_R is a S_R without the k_3-axis. By the assumption of Theorem 1.14, Ω_R has the cone property. By using the Rellich-Kondrashov theorem([6, Theorem 8.9]), we have

$$\lim_{j\to\infty} \|\Phi_j^{(n)} - \Phi_0^{(n)}\|_{L^s(\Omega_R^n)} = 0,$$

for all $s < \frac{3np}{3n-p}$. Since $p \in [1,2)$ is arbitrary, we choose $p = 6n/(3n+2) \in [1,2)$, and we get $\lim_{j\to\infty} \|\Phi_j^{(n)} - \Phi_0^{(n)}\|_{L^2(\Omega_R^n)} = 0$, for all $R > 0$. We set $\Phi_j := (\Phi_j^{(n)})_{n=0}^{\infty}, \Phi_0 := (\Phi_0^{(n)})_{n=0}^{\infty} \in \oplus^4 \mathcal{F}_{\mathrm{rad}}$. Let χ_R be the characteristic function of the ball $\{\mathbf{k} \in \mathbb{R}^3 | |\mathbf{k}| < R\}$. We denote by P_n the orthogonal projection to the n-photon subspace $\mathbb{C}^4 \otimes \mathcal{F}^n$. Then we have

$$\begin{aligned}\|\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 &= \|P_n\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 + \|(1-P_n)\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 \\ &\leq \|P_n\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 + \frac{1}{n}\|N_{\mathrm{b}}^{1/2}\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2.\end{aligned}$$

Since each component $(\Gamma(\chi_R)\Phi_j)^{(n)}$ converges to $(\Gamma(\chi_R)\Phi_0)^{(n)}$ strongly as $j \to \infty$, we have

$$\lim_{j\to\infty} \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 \leq \frac{1}{n} \limsup_{j\to\infty} \|\Gamma(\chi_R)N_{\mathrm{b}}^{1/2}(\Phi_j - \Phi_0)\|^2,$$

for all $n \in \mathbb{N}$. Therefore we obtain

$$\operatorname*{s-lim}_{j\to\infty} \Gamma(\chi_R)\Phi_j = \Gamma(\chi_R)\Phi_0. \tag{16}$$

On the other hand, by Proposition 3.1 we have

$$\|H_f^{1/2}\Phi_j\|^2 = \sum_{\lambda=1,2}\int_{\mathbb{R}^3} |\mathbf{k}|\|a_\lambda(\mathbf{k})\Phi_j\|^2 \mathrm{d}\mathbf{k} \leq \int_{\mathbb{R}^3} \frac{q^2|\rho(\mathbf{k})|^2}{(E_{m_j}(\mathbf{p}-\mathbf{k}) - E_{m_j}(\mathbf{p}) + |\mathbf{k}| + m_j)^2} \mathrm{d}\mathbf{k} < \infty.$$

Note that a singular point of $\mathbf{k} \to (E_m(\mathbf{p}-\mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}|)^{-1}$ is only $\mathbf{k} = 0$(Proposition 1.9), $\lim_{|\mathbf{k}|\to\infty}(E_m(\mathbf{p}-\mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}|) = \infty$ and the map $\mathbf{k} \mapsto (E_m(\mathbf{p}-\mathbf{k}) - E_m(\mathbf{p}) + |\mathbf{k}|)$ is continuous. By the assumption of the theorem, we have

$$\lim_{j\to\infty} \|H_f^{1/2}\Phi_j\|^2 \leq \limsup_{j\to\infty} \int_{\mathbb{R}^3} \frac{q^2|\rho(\mathbf{k})|^2}{(E_{m_j}(\mathbf{p}-\mathbf{k}) - E_{m_j}(\mathbf{p}) + |\mathbf{k}| + m_j)^2} \mathrm{d}\mathbf{k} < \infty.$$

Therefore

$$\begin{aligned}\|\Phi_j - \Phi_0\|^2 &= \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 + \|(\Gamma(\chi_R) - 1)(\Phi_j - \Phi_0)\|^2 \\ &\leq \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 + \frac{1}{R}\|(1-P_0)\Gamma(\chi_R)H_f^{1/2}(\Phi_j - \Phi_0)\|^2 + \|P_0\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 \\ &\leq \|\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2 + \frac{\text{const.}}{R} + \|P_0\Gamma(\chi_R)(\Phi_j - \Phi_0)\|^2,\end{aligned}$$

where "const." means the constant independent of $R > 0$. By (16), we have

$$\operatorname*{s-lim}_{j\to\infty} \Phi_j = \Phi_0.$$

This means the Φ_0 is a normalized ground state of $\overline{H(\mathbf{p})}$. ∎

Proof of Theorem 1.15. We check the condition of Theorem 1.14. As in the proof of Theorem 1.13, the condition $\rho \in \mathrm{Dom}(|\hat{\mathbf{k}}|^{-3/2})$ and Proposition 1.10 imply that

$$\limsup_{m\to+0} \int_S \frac{q^2}{(E_m(\mathbf{k}) - E_m(0) + |\mathbf{k}| + m)^2} \frac{|\rho(\mathbf{k})|^2}{|\mathbf{k}|} \mathrm{d}\mathbf{k} < \infty.$$

By Proposition 1.10, for all $R > 0$ we have that $a_1(R) > 0$ and

$$\sup_{0 \le m \le 1} (E_m(\mathbf{k}) - E_m(0) + |\mathbf{k}|)^{-1} \le \frac{R}{a_1(R)|\mathbf{k}|}, \quad |\mathbf{k}| \le R.$$

Hence we have

$$\sup_{0<m<1} \int_{S_R} \left[(E_m(\mathbf{k}) - E_m(0) + |\mathbf{k}| + m)^{-2} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p \mathrm{d}\mathbf{k} \le \int_{S_R} \left[\frac{R^2}{a_1(R)^2} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{5/2}} \right]^p \mathrm{d}\mathbf{k} < \infty,$$

$$\sup_{0<m<1} \int_{S_R} \left[(E_m(\mathbf{k}) - E_m(0) + |\mathbf{k}| + m)^{-1} \frac{|\nabla\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p \mathrm{d}\mathbf{k} \le \int_{S_R} \left[\frac{R}{a_1(R)} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{3/2}} \right]^p \mathrm{d}\mathbf{k} < \infty,$$

$$\sup_{0<m<1} \int_{S_R} \left[(E_m(\mathbf{k}) - E_m(0) + |\mathbf{k}| + m)^{-1} \frac{1}{\sqrt{k_1^2 + k_2^2}} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{1/2}} \right]^p \mathrm{d}\mathbf{k}$$

$$\le \int_{S_R} \left[\frac{R}{a_1(R)} \frac{1}{\sqrt{k_1^2 + k_2^2}} \frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{3/2}} \right]^p \mathrm{d}\mathbf{k}$$

$$= 4\pi \left[\frac{R}{a_1(R)} \right]^p \int_{[0,\pi]} \sin\theta \mathrm{d}\theta \left[\frac{1}{\sin\theta} \right]^p \int_{[0,R)} |\mathbf{k}|^2 \mathrm{d}|\mathbf{k}| \left[\frac{|\rho(\mathbf{k})|}{|\mathbf{k}|^{5/2}} \right]^p < \infty.$$

Here we use the fact that ρ is rotation invariant. Therefore the condition of Theorem 1.14 holds. ∎

4 APPENDIX: A REMARK ON THE POLARIZATION VECTORS

In this appendix, we show that the quantum electrodynamics does not depend on the choice of polarization vectors, i.e., the Hamiltonians defined by different polarization vectors are unitarily equivalent each other. We show the equivalence only for the Hamiltonians H and $H(\mathbf{p})$, but one can apply our proof to the Pauli-Fierz model and various QED models. In the proof, we do not use the form of the cutoff function ρ and dispersion ω, and use only the facts that ρ and ω do not depend on the helicity argument λ.

We assume that the polarization vectors $\mathbf{e}^{(1)}(\mathbf{k})$, $\mathbf{e}^{(2)}(\mathbf{k})$ and $\mathbf{k}$ are a right-handed system;

$$\mathbf{k} \cdot \mathbf{e}^{(1)}(\mathbf{k}) = 0, \quad \|\mathbf{e}^{(1)}(\mathbf{k})\|_{\mathbb{R}^3} = 1, \quad \mathbf{e}^{(2)}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}^{(1)}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^3.$$

Next, we take any polarization vectors $\mathbf{e}'^{(1)}$, $\mathbf{e}'^{(2)}$:

$$\mathbf{k} \cdot \mathbf{e}'^{(\lambda)}(\mathbf{k}) = 0, \quad \mathbf{e}'^{(\lambda)}(\mathbf{k}) \cdot \mathbf{e}'^{(\mu)}(\mathbf{k}) = \delta_{\lambda,\mu}, \quad \mathbf{k} \in \mathbb{R}^3, \ \lambda, \mu \in \{1, 2\}.$$

Let H' and $H'(\mathbf{p})$ be the Hamiltonians H and $H(\mathbf{p})$ with $\mathbf{e}^{(\lambda)}$ replaced $\mathbf{e}'^{(\lambda)}$, $\lambda = 1, 2$, respectively.

The essential self-adjointness of Hamiltonians H and $H(\mathbf{p})$ does not depend on a choice of a polarization vectors, and $\bar{H}$ and $\overline{H(\mathbf{p})}$ are unitarily equivalent to $\bar{H}'$ and $\overline{H'(\mathbf{p})}$, respectively:

Theorem 4.1. *Assume that H is essentially self-adjoint. Then H' is essentially self-adjoint and $\bar{H}$ is unitarily equivalent to $\bar{H}'$.*

Theorem 4.2. *Assume that $H(\mathbf{p})$ is essentially self-adjoint. Then $H'(\mathbf{p})$ is essentially self-adjoint and $\overline{H(\mathbf{p})}$ is unitarily equivalent to $\overline{H'(\mathbf{p})}$.*

Proofs of Theorems 4.1 and 4.2. By the definition of polarization vectors, for each $\mathbf{k} \in \mathbb{R}^3$ it holds that $\mathbf{e}'^{(2)}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}'^{(1)}(\mathbf{k})$ or $\mathbf{e}'^{(2)}(\mathbf{k}) = -\frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}'^{(1)}(\mathbf{k})$. Let $O \subset \mathbb{R}^3$ be a set such that $\mathbf{e}'^{(2)}(\mathbf{k}) = -\frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}'^{(1)}(\mathbf{k})$, $\mathbf{k} \in O$ holds. We define a polarization vectors $\mathbf{e}''^{(\lambda)}$, $\lambda = 1, 2$,

$$\mathbf{e}''^{(1)}(\mathbf{k}) := \mathbf{e}'^{(1)}(\mathbf{k}), \quad \mathbf{e}''^{(2)}(\mathbf{k}) := \begin{cases} \mathbf{e}'^{(2)}(\mathbf{k}), & \mathbf{k} \in \mathbb{R}^3 \backslash O, \\ -\mathbf{e}'^{(2)}(\mathbf{k}), & \mathbf{k} \in O. \end{cases}$$

We define an operator H'' which is H with $\mathbf{e}^{(\lambda)}$ replacing $\mathbf{e}''^{(\lambda)}$, $\lambda = 1, 2$. Let

$$\mathbf{g}'(\mathbf{k}, \lambda; \mathbf{x}) := \frac{\rho(\mathbf{k})}{|\mathbf{k}|^{1/2}} \mathbf{e}'^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}, \quad \mathbf{g}''(\mathbf{k}, \lambda; \mathbf{x}) := \frac{\rho(\mathbf{k})}{|\mathbf{k}|^{1/2}} \mathbf{e}''^{(\lambda)}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}},$$

and we set

$$\mathbf{A}'(\hat{\mathbf{x}}) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^{\oplus} \overline{[a(\mathbf{g}'(\cdot, \mathbf{x})) + a(\mathbf{g}'(\cdot, \mathbf{x}))^*]} \mathrm{d}\mathbf{x}, \quad \mathbf{A}''(\hat{\mathbf{x}}) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3}^{\oplus} \overline{[a(\mathbf{g}''(\cdot, \mathbf{x})) + a(\mathbf{g}''(\cdot, \mathbf{x}))^*]} \mathrm{d}\mathbf{x},$$

self-adjoint operators on $\mathcal{F}$. Since $\mathbf{e}''^{(1)}(\mathbf{k})$, $\mathbf{e}''^{(2)}(\mathbf{k})$, $\mathbf{k}$ are a right-handed system: $\mathbf{k} \cdot \mathbf{e}''^{(1)}(\mathbf{k}) = 0$, $\mathbf{e}''^{(2)}(\mathbf{k}) = \frac{\mathbf{k}}{|\mathbf{k}|} \wedge \mathbf{e}''^{(1)}(\mathbf{k})$, for all $\mathbf{k} \in \mathbb{R}$ there exists $\theta(\mathbf{k}) \in [0, 2\pi)$ such that

$$\begin{bmatrix} \mathbf{e}^{(1)}(\mathbf{k}) \\ \mathbf{e}^{(2)}(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} \cos\theta(\mathbf{k}) & -\sin\theta(\mathbf{k}) \\ \sin\theta(\mathbf{k}) & \cos\theta(\mathbf{k}) \end{bmatrix} \begin{bmatrix} \mathbf{e}''^{(1)}(\mathbf{k}) \\ \mathbf{e}''^{(2)}(\mathbf{k}) \end{bmatrix}.$$

We define a unitary operator u_1 on $L^2(\mathbb{R}^3 \times \{1, 2\})$ by

$$\begin{bmatrix} (u_1 f)(\mathbf{k}, 1) \\ (u_1 f)(\mathbf{k}, 2) \end{bmatrix} := \begin{bmatrix} \cos\theta(\mathbf{k}) & -\sin\theta(\mathbf{k}) \\ \sin\theta(\mathbf{k}) & \cos\theta(\mathbf{k}) \end{bmatrix} \begin{bmatrix} f(\mathbf{k}, 1) \\ f(\mathbf{k}, 2) \end{bmatrix}, \quad \mathbf{k} \in \mathbb{R}^3.$$

The operator $U_1 := \Gamma(u_1)$ is a unitary operator on $\mathcal{F}_{\mathrm{rad}}$ and $U_1 \mathrm{d}\Gamma(\omega) U_1^* = \mathrm{d}\Gamma(\omega)$. By the equality $u_1 \mathbf{g}''(\cdot, \mathbf{x}) = \mathbf{g}(\cdot, \mathbf{x})$, we have $U_1 \mathbf{A}''(\hat{\mathbf{x}}) U_1^* = \mathbf{A}(\hat{\mathbf{x}})$. Therefore we get

$$U_1 \overline{H''} U_1^* = \overline{U_1 H'' U_1^*} = \overline{H}.$$

This means that the operator H'' is essentially self-adjoint and $\overline{H''}$ is unitarily equivalent to $\bar{H}$. Next we show that $\overline{H''}$ is unitarily equivalent to $\overline{H'}$. Let u_2 be a unitary operator on $L^2(\mathbb{R}^3 \times \{1, 2\})$ such that

$$(u_2 f)(\mathbf{k}, \lambda) := \begin{cases} -f(\mathbf{k}, 2), & \mathbf{k} \in S, \\ f(\mathbf{k}, \lambda), & \text{otherwise}. \end{cases}$$

It is easy to see that $u_1 g'_j(\cdot,\mathbf{x}) = g''_j(\cdot,\mathbf{x})$, $j=1,2,3$. Then $U_2 := \Gamma(u_2)$ is a unitary transformation on $\mathcal{F}_{\rm rad}$, and

$$U_2 {\rm d}\Gamma(\omega) U_2^* = {\rm d}\Gamma(\omega).$$

By the definition of u_2, the equality $U_2 \mathbf{A}'(\hat{\mathbf{x}}) U_2^* = \mathbf{A}''(\hat{\mathbf{x}})$ holds. Therefore we have

$$U_2 \overline{H'} U_2^* = \overline{U_2 H' U_2^*} = \overline{H''},$$

which implies that H' is essentially self-adjoint and $\overline{H'}$ is unitarily equivalent to $\overline{H''}$. Hence Theorem 4.1 is proved. The proof of Theorem 4.2 is similar to the proof of Theorem 4.1. ∎

Acknowledgements

We would like to thank A. Arai for useful discussions.

References

[1] A. Arai, Fundamental Properties of the Hamiltonian of a Dirac Particle Coupled to the Quantized Radiation Field, *Hokkaido Univ. Preprint Series in Math*, No. 447, 1999.

[2] A. Arai, A particle-field Hamiltonian in relativistic quantum electrodynamics, *J. Math. Phys.* **41** (2000), 4271-4283.

[3] A. Arai, Fock Spaces and Quantum Fields, Nippon-Hyouronsha, Tokyo, 2000(in Japanese).

[4] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* 151 (1997) 455-503.

[5] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. math.* **145** (2001), 557-595.

[6] E.H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics, American Mathematical Society, 1997

[7] I. Sasaki, Ground state energy of polaron in the relativistic quantum electrodynamics, *J. Math. Phys.* **46**, 1 (2005)

[8] M. Reed and B. Simon, Methods of Modern Mathematical Physics Vol.I, Academic Press, New York, 1972

[9] M. Reed and B. Simon, Methods of Modern Mathematical Physics Vol. II, Academic Press, New York, 1975.

[10] B. Thaller, The Dirac Equation, Springer-Verlag, Berlin, Heidelberg, 1992.