Hyperbolicity in multi-dimensional Hamiltonian systems with applications to soft billiards

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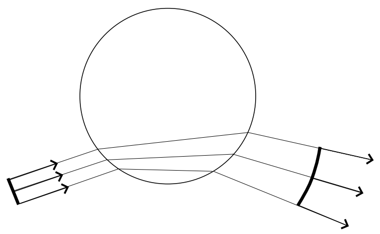
Abstract

When considering hyperbolicity in multi-dimensional Hamiltonian sytems, especially in higher dimensional billiards, the literature usually distinguishes between dispersing and defocusing mechanisms. In this paper we give a unified treatment of these two phenomena, which also covers the important case when the two mechanisms mix. Two theorems on the hyperbolicity (i.e. non-vanishing of the Lyapunov exponents) are proven that are hoped to be applicable to a variety of situations.

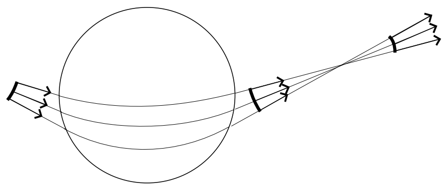
As an application we investigate soft billiards, that is, replace the hard core collision in dispersing billiards with disjoint spherical scatterers by motion in some spherically symmetric potential. Analogous systems in two dimensions have been widely investigated in the literature, however, we are not aware of any mathematical result in this multi-dimensional case. Hyperbolicity is proven under suitable conditions on the potential. This way we give a natural generalization of the hyperbolicity results obtained before in two dimensions for a large class of potentials.

Introduction

In this paper we give a method for proving hyperbolicity in multi-dimensional Hamiltonian systems. On the one hand, our motivation comes from multi-dimensional soft billiard systems, a situation to which our method is readily applied in Section 5. On the other hand, our work is in a great part inspired by the description of *many-dimensional* stadia given by Bunimovich and Reháček ([BR1] and [BR2]). As it has been observed even much earlier, a focusing phenomenon – which causes initially parallel trajectories being diverted towards each-other – can result in expansion in certain directions of the phase space, and thus ensure that the system is hyperbolic. The key to this effect is a suitably long free flight after such a focusing, which allows the trajctories to "meet at the focus point" and than diverge again. This phenomenon of "defocusing" has been well understood in 2 dimensions since the discovery of the Bunimovich stadium ([B]). Figure 1 shows (on a planar soft billiard example) how dispersing and focusing can cause trajectories to diverge.



(a) dispersing case



(b) defocusing case

Figure 1: mechanisms of hyperbolicity

However, in higher dimensions the picture is more complicated due to the phenomenon of "astigmatism", that is "different strength of focusing in different directions". Here a "direction" means that we perturb our reference trajectory in a well chosen – not arbitrary – way. Probably the most remarkable feature of Bunimovich-Reháček treatment of high-dimensional stadia is that it proves, despite of the above mentioned complications, the presence of hyperbolicity in many dimensions by understanding the phenomenon in the different directions separately.

In this paper we fight essentially the same difficulty – that is, different kinds of behaviour in different directions. In the context of dispersing vs. focusing, this can even mean dispersing in one direction and focusing in the other. We formulate the main results in a general context of Hamiltonian systems. We will require the first return map from some set to itself to have a decomposability property, that is, it should leave certain (lower dimensional) "directions" of perturbations invariant. Then we claim that understanding hyperbolicity in these directions separately gives hyperbolicity for the multi-dimensional system.

Our proof uses the elegant and transparent method of Wojtkowski using "Lagrangian sectors" in symplectic space. Throughout the paper, our main reference will be [LW], where this method is described excellently.

The paper is organized as follows. In the first section we repeat some notions and statements from [LW] concerning Lagrangian sectors in a symplectic space. Then in Section 2 we recall and compare two important ways of looking at hyperbolicity: the notions of cones and fronts. In Section 3 we give the definition of decomposability of the dymanics and state our main results in two theorems. Theorem 3.1 describes how Wojtkowski's approach to invariant cones can be translated to the language of fronts in a general multi-dimensional setting. Theorem 3.4 shows how it is possible to prove hyperbolicity for multi-dimensional systems by solving lower dimensional problems. We prove these two theorems in Section 4, where the technique of [LW], as recalled in Section 1, is applied.

As an application, in Section 5 we prove hyperbolicity of a large class of "soft" billiards in high dimensions. This means that we take a dispersing billiard with disjoint spherical scatterers, and than replace hard-core collisions by Hamiltonian motion in some spherically symmetric potential. In many of these examples it is indeed the case that the phenomena of dispersing and focusing/defocusing coexist.

1 Sectors in symplectic space

In this section we repeat notions and statements from [LW]. We will use these to prove Theorem 3.4 in Section 4.

Lagrangian subspaces in a symplectic linear space W are the maximal dimensional linear spaces on which the symplectic form identically vanishes. They always have half the dimension of the symplectic space. Given two transversal Lagrangian subspaces E_1 and E_2 there is a unique decomposition for any $w \in W$:

$$w = \pi_1 w + \pi_2 w = w_1 + w_2, \qquad w_i \in E_i,$$

where π_1 and π_2 denote the appropriate linear projections.

If E_1 and E_2 are two transversal Lagrangian subspaces, then the *cone above* E_1 *and below* E_2 is defined as

$$\mathcal{C}(E_1, E_2) = \{ \ w = w_1 + w_2 \mid w_i \in E_i, \ \omega(w_1, w_2) \geq 0 \ \}. \tag{1}$$

Equivalently, given the ordered pair of Lagrangian spaces (E_1, E_2) a quadratic form Q can be defined on W as

$$Q(w) = \omega(w_1, w_2) = \omega(\pi_1 w, \pi_2 w).$$

Then $\mathcal{C}(E_1, E_2) = \{w \mid Q(w) \geq 0\}$ and $\text{int}(\mathcal{C}(E_1, E_2)) = \{w \mid Q(w) > 0\}$.

The simplest example is the sector defined in $\mathbb{R}^d \times \mathbb{R}^d$ by

$$
\begin{aligned}
E_1 &= \mathbb{R}^d \times \{0\} \\
E_2 &= \{0\} \times \mathbb{R}^d \\
\mathcal{C} &= \mathcal{C}(E_1, E_2) = \{(dq, dv) \mid \langle dq, dv \rangle \geq 0\}.
\end{aligned}
\tag{2}
$$

This is called the standard sector. Since any two sectors are symplectically equivalent, one is always allowed to prove statements for the standard sector only.

A symplectic map $L : W \to W$ is called monotone (strictly monotone) with respect to the cone $\mathcal{C} = \mathcal{C}(E_1, E_2)$ if $L\mathcal{C} \subset \mathcal{C}$ ($L\mathcal{C} \subset \text{int}(\mathcal{C}) \cup \{0\}$).

The set of Lagrangian subspaces contained in the cone $\mathcal{C}$ is denoted by $\text{Lag}(\mathcal{C})$: for a Lagrangian subspace F we have $F \in \text{Lag}(\mathcal{C})$ iff $F \subset \text{int}(\mathcal{C}) \cup \{0\}$. There is a partial order on $\text{Lag}(\mathcal{C})$:

Definition 1.1. *For two Lagrangian subspaces $F_1, F_2 \in \text{Lag}(\mathcal{C})$ let*

$$F_1 < F_2 \text{ whenever } Q \circ (\pi_1|_{F_1})^{-1} < Q \circ (\pi_1|_{F_2})^{-1}.$$

The following theorem is Theorem 5.2 from [LW].

Theorem 1.2.

a.) For two transversal Lagrangian subspaces $E_1, E_2 \in \text{Lag}(\mathcal{C})$

$$E_1 < E_2 \text{ if and only if } \mathcal{C}(E_1, E_2) \subset \mathcal{C}(V_1, V_2)$$

b.) Furthermore if $E_1 < E_2$, then for a Lagrangian subspace $E \in \text{Lag}(\mathcal{C})$

$$E \subset \mathcal{C}(E_1, E_2) \text{ if and only if } E_1 \leq E \leq E_2.$$

This allows us to check invariance of a cone field (monotonicity of a symplectic map) by looking at the defining Lagrangian subspaces only: since a symplectic map $L : W \to W$ satisfies

$$L(\mathcal{C}(E_1, E_2)) = \mathcal{C}(L(E_1), L(E_2)),$$

we have that $L(\mathcal{C}(E_1, E_2))$ is strictly contained in $\mathcal{C}$ if and only if $L(E_1), L(E_2) \in \text{Lag}(\mathcal{C})$ and $L(E_1) < L(E_2)$.

2 Hyperbolicity, cones and fronts

In what follows we will consider a discrete time Hamiltonian dynamical system (M, T, μ). In the simplest case M is a smooth compact symplectic manifold, $T : M \to M$ is a symplectomorphism, and μ is the symplectic volume element. However, we may allow the presence of singularities, which will not affect our considerations in any way. For a precise definition of the dynamical system in this case, see [LW].

Our results can naturally be applied to Hamiltonian flows by taking some Poincaré section, or – as usual in billiard theory – orthogonal sections. An example will be shown in Section 5.

As the systems discussed througout the paper are Hamiltonian, the tangent planes are symplectic linear spaces, and the statements of Section 1 are relevant. In particular, when talking about a cone (in the tangent plane, or sometimes in a symplectic subspace of it) we think of a cone defined by an ordered pair of transversal Lagrangian subspaces, cf. formula (1).

One standard way of proving hyperbolicity is to show the existence of an invariant (and eventually strictly invariant) cone field $\mathcal{C}(x)$ on M. That is, we place a cone at μ-almost every point $x \in M$ into the tangent space of M at x: $\mathcal{C}(x) \subset \mathcal{T}_x(M)$. We call the cone field $\mathcal{C}(x)$ invariant if for μ-almost every point $x \in M$, $DT(\mathcal{C}(x)) \subset \mathcal{C}(Tx)$. (Here DT denotes the derivative of T.) A field $\mathcal{C}(x)$ of (closed) cones is said to be eventually strictly invariant if for μ-almost every point $x \in M$ there exists an $n \in \mathbb{N}$ such that $DT^n(\mathcal{C}(x)) \subset \text{int}(\mathcal{C}(T^n x)) \cup \{0\}$. It is standard (see e.g. [LW]) that the existence of an eventually strictly invariant cone field implies the non-vanishing of Lyapunov exponents (hyperbolicity). We may refer to these observations as the symplectic interpretation of hyperbolicity.

Another common way of understanding the phenomena that result in hyperbolicity, which we will refer to as the geometric interpretation, is by considering fronts. Fronts are sometimes called "local orthogonal manifolds", or simply "control surfaces" (e.g. in [BR2]). The name is related to a picture about billiard flows where a front is defined as a smooth 1-codim. submanifold E of the configuration space, every point of which is equipped (continuously) with a unit velocity orthogonal to that submanifold:

$$\mathcal{W} = \{(q, v(q)) \mid q \in E, v(q) \perp \mathcal{T}_q(E), |v| = 1\}.$$

Then the front is sometimes thought of as the submanifold E of the configuration space, and sometimes as the submanifold $\mathcal{W}$ of the phase space.

Although the concept of a front was born in the context of flows, we will see that in a construction of an invariant cone field, only the evolution of an infinitesimally small piece of a front is needed. More precisely, let us denote by B_q the derivative (matrix) of the function $v(q)$ at q. This is a linear map from the tangent space $\mathcal{T}_q(E)$ of E at q to the tangent space of the unit sphere at $v(q)$. However, these two can be identified, and thus B_q can be considered as a symmetric linear operator $B_q : \mathcal{T}_q(E) \to \mathcal{T}_q(E)$. B_q is exactly the curvature operator of E at q. We will see that the construction of invariant cones is related only to the evolution of the operator B under the dynamics.

In this infinitesimal sense we may very well speak of fronts in the context of a general discrete time Hamiltonian system (i.e. a symplectic map) $T : M \to M$ as well. At some point $x \in M$ we may view the tangent space $\mathcal{T}_x(M)$ as $\mathcal{T}_x(M) = \mathbb{R}^d \times \mathbb{R}^d$ equipped with the natural symplectic form $\omega((dq_1, dv_1), (dq_2, dv_2)) = \langle dq_1, dv_2 \rangle - \langle dq_2, dv_1 \rangle$. Corresponding to any given symmetric linear operator $B : \mathbb{R}^d \to \mathbb{R}^d$ there is a canonically defined d-dimensional subspace of $\mathcal{T}_x(M)$:

$$gB := \left\{(dq, Bdq) \mid dq \in \mathbb{R}^d\right\}. \tag{3}$$

We may view B as a map from the "configurational" part of $\mathcal{T}_x(M)$ to the "velocity" part. Then the graph of B, i.e. the subspace (3) is understood as the "local, infinitesimal part" or as the "tangent space of the front". As a consequence of the symmetricity of B, subspaces of the type (3) are the typical examples of Lagrangian subspaces of $\mathbb{R}^d \times \mathbb{R}^d$.

The above observations show that the notion of fronts, the basic tools in the geometric interpretation of hyperbolicity, can be directly translated into the symplectic language of [LW].

In view of these two parallel interpretations one may consider cones as collections of vectors tangent to fronts the curvatures of which satisfy some inequalities. In order to do so we introduce some notations: $B > 0$ ($B \geq 0$) means that the symmetric operator B is positive (semi-)definite while $B_1 > B_2$ ($B_1 \geq B_2$) means the same for the difference $B_1 - B_2$. By slightly abusing notation, given a real number c we simply refer to the operator $c\,Id$ as c.

The simplest and most common example of cones is the "cone of convex fronts":

$$\mathcal{C}_{B\geq 0} := \left\{(dq, Bdq) \mid dq \in \mathbb{R}^d, B \geq 0\right\} = \bigcup_{B\geq 0} gB.$$

However, in describing focusing-defocusing phenomena, one often has to use the more general "$c_* \leq B \leq c^*$" cone:

$$\mathcal{C}_{c_*\leq B\leq c^*} := \left\{(dq, Bdq) \mid dq \in \mathbb{R}^d, c_* \leq B \leq c^*\right\} = \bigcup_{c_*\leq B\leq c^*} gB. \tag{4}$$

Here c_* and c^* are typically finite real numbers, but one can allow symmetric matrices or $\pm\infty$ as well.

At first sight it may seem that we are abusing notation when we define cones both in the symplectic sense of Formula (1) and in the geometric sense of Formula (4). However, in Proposition 4.3 we will see that (4) is a special case of (1).

3 Statement of the theorems

In this section we state our main results about hyperbolicity of Hamiltonian systems. The first result can be stated without any further preparation. It states that for cones defined in terms of curvatures of fronts, invariance can be checked by looking only at the "boundaries" of the cones – that is, the fronts with extremal curvatures. This statement is a direct translation of the Theorem 5.2 from [LW] (repeated in this paper as Theorem 1.2) to the language of fronts. We use the notation introduced in (4).

Theorem 3.1. *Let the symmetric operators B_1 and B_2 describe fronts at the point $x \in M$ that evolve into fronts described by the operators B_1' and B_2' at the point $x' = Tx$. That is, $DT(gB_1) = gB_1'$ and $DT(gB_2) = gB_2'$. Let B_* and B^* also be operators of fronts at x'. Suppose that $B_1 < B_2$ and $B_* < B^*$. Use the notation $C = C_{B_1 \le B \le B_2} \subset \mathcal{T}_x(M)$ and $C^* = C_{B_* \le B' \le B^*} \subset \mathcal{T}_{x'}(M)$. Then,*

$$DT(C) \subset C^* \text{ if and only if } B_* \le B_1' < B_2' \le B^*$$

and

$$DT(C) \text{ is sctrictly contained in } C^* \text{ if and only if } B_* < B_1' < B_2' < B^*.$$

In both cases, $DT(C) = C_{B_1' \le B' \le B_2'}$. One can also replace B_1 and/or B_1' and/or B_* by $-\infty$ or B_2 and/or B_2' and/or B^* by $+\infty$.

This theorem will be proven in Section 4.

To state our second, more specific theorem, we first need a notion of decomposability.

Suppose that there is a subset U of the phase space M such that the trajectory of almost any point in M hits U in finite time. We plan to define our cones at points of U and than extend their definition to almost all of M via the dynamics. We wish to understand hyperbolicity of the dynamics through the first return map from U to U, thus U has to be chosen in such a way that this return map is simple enough to describe, but already shows hyperbolic features. With slight abuse of notation, we denote the first return map by T (just like the dynamics above). In billiard-like systems U typically consists of phase points that are just before or just after collision. We deal with the case when investigation of the first return map can be "decomposed" into several problems of lower dimension. We will prove that whenever such a decomposition is possible, it is sufficient to construct invariant cones for these lower dimensional problems in order to obtain the existence of invariant cones in the multi-dimensional system, and thus to prove hyperbolicity.

We will denote by J_x the "configurational part" of the tangent space of M at x, which we indentify with the "velocity part" as well. That is, $\mathcal{T}_x(M) = J_x \times J_x$.

Remark 3.2. *We note that in applications to flows, $J_x \times J_x$ will be only a subspace of the tangent space. That is, the flow direction is ignored, and J_x is tangent to the orthogonal section of the flow at x.*

In the definition to come, we denote the image (by the derivative of the dynamics) of a tangent vector (dq, dv) by (dq', dv'). That is,

$$(dq', dv') := DT((dq, dv)).$$

Definition 3.3. *Let $U \subset M$. We say that the first return map T from U to U can be decomposed into lower dimensional maps if for every $x \in U$ there exist subspaces $J_1, J_2, \ldots, J_m \subset J_x$ and $J_1', J_2', \ldots, J_m' \subset J_{Tx}$ so that $J_x = J_1 \oplus \cdots \oplus J_m$, $J_{Tx} = J_1' \oplus \cdots \oplus J_m'$ and for every $i \in \{1, \ldots, m\}$ if both $dq \in J_i$ and $dv \in J_i$ then $dq' \in J_i'$ and $dv' \in J_i'$. That is, $DT(J_i \times J_i) \subset J_i' \times J_i'$.*

Recall from Section 2 that as long as the differential aspects of the dynamics like hyperbolicity is concerned, when talking about a front, only its tangent space, i.e. the operator B is relevant. Thus, if such a decomposition exists, then we can talk about a "front" which is entirely in J_i and evolves into a "front" which is entirely in J_i'. This only means that a tangent vector $(dq, dv) \in J_i \times J_i$ evolves into a tangent vector $(dq', dv') \in J_i' \times J_i'$.

Now we are able to state our main theorem. The essence of the theorem roughly is that if the cone $c_* \leq B \leq c^*$ is invariant in all of the components, then it is also invariant for the multi-dimensional dynamics.

Theorem 3.4. *Suppose that $U \subset M$ and the first return map from U to U can be decomposed into lower dimensional maps as in Definiton 3.3. If there exist constants $-\infty < c_* < c^* < \infty$ ($c_* = -\infty$ and $c^* = \infty$ are also allowed, but not at the same time) such that the cone $\mathcal{C}_i = \{(dq, Bdq) \mid dq \in J_i, c_* \leq B \leq c^*\}$ is mapped into the cone $\mathcal{C}_i' = \{(dq', B'dq') \mid dq' \in J_i', c_* \leq B' \leq c^*\}$ for every $i \in \{1, \ldots, m\}$, then the cone $\mathcal{C} = \{(dq, Bdq) \mid dq \in J, c_* \leq B \leq c^*\}$ is mapped into the cone $\mathcal{C}' = \{(dq', B'dq') \mid dq' \in J', c_* \leq B' \leq c^*\}$. So the cone field defined by $c_* \leq B \leq c^*$ is invariant. The equivalent statement holds with strict inequalities and strict invariance.*

4 Proof of the theorems

In this section we prove Theorems 3.1 and 3.4. In order to do so we use the notions and facts recalled in Section 1. Note furthermore that the (strict) invariance of a cone field is exactly the (strict) monotonicity of the derivative of the dynamics (which is a symplectic linear map).

We start the proof with a small remark.

Remark 4.1. *For any sector $\mathcal{C}$,*

$$\mathcal{C} = \bigcup \{E \mid E \text{ is a Lagrangian subspace and } E \subset \mathcal{C}\}.$$

This is obvious for the standard sector (see (2)), and all sectors are symplectically equivalent.

The following lemma states that if a cone is part of another, then the partial order in the subcone coincides with the restriction (to the subcone) of the partial order in the containing cone.

Lemma 4.2. *Let $\mathcal{C}_1 = \mathcal{C}(E_1, E_2) \subset \mathcal{C}$. Denote the partial order on $\mathrm{Lag}(\mathcal{C})$ by $<_\mathcal{C}$ and the partial order on $\mathrm{Lag}(\mathcal{C}_1)$ by $<_{\mathcal{C}_1}$. Let $E_3, E_4 \in \mathrm{Lag}(\mathcal{C}_1)$ $[\subset \mathrm{Lag}(\mathcal{C})]$. Then*

$$E_3 <_\mathcal{C} E_4 \text{ if and only if } E_3 <_{\mathcal{C}_1} E_4.$$

Proof.

1.) First suppose that $\mathcal{C}_1 \subset \text{int}\mathcal{C} \cup \{0\}$. Then $E_1, E_2 \in \text{Lag}(\mathcal{C})$, and Theorem 1.2 a.) implies $E_1 <_{\mathcal{C}} E_2$. Furthermore, Theorem 1.2 b.) implies $E_1 \leq_{\mathcal{C}} E_3 \leq_{\mathcal{C}} E_2$ and $E_1 \leq_{\mathcal{C}} E_4 \leq_{\mathcal{C}} E_2$.

a.) If $E_1 \leq_{\mathcal{C}} E_3 <_{\mathcal{C}} E_4 \leq_{\mathcal{C}} E_2$, then Theorem 1.2 b.) and Remark 4.1 imply

$$\begin{aligned} \mathcal{C}(E_3, E_4) &= \bigcup \{E \mid E_3 \leq_{\mathcal{C}} E \leq_{\mathcal{C}} E_4\} \\ \mathcal{C}_1 &= \bigcup \{E \mid E_1 \leq_{\mathcal{C}} E \leq_{\mathcal{C}} E_2\}, \end{aligned}$$

so $\mathcal{C}(E_3, E_4) \subset \mathcal{C}_1$. Now Theorem 1.2 a.) implies $E_3 <_{\mathcal{C}_1} E_4$.

b.) If $E_3 <_{\mathcal{C}_1} E_4$ then Theorem 1.2 a.) implies $\mathcal{C}(E_3, E_4) \subset \mathcal{C}_1 \subset \mathcal{C}$. Applying Theorem 1.2 a.) again gives $E_3 <_{\mathcal{C}} E_4$.

So we see that the two partial orders coincide if $\mathcal{C}_1 \subset \text{int}\mathcal{C} \cup \{0\}$ (that is, $\mathcal{C}_1$ is strictly contained in $\mathcal{C}$).

2.) In the general case, let us choose a sector $\mathcal{C}_0$ which strictly contains $\mathcal{C}$. This is clearly possible if $\mathcal{C}$ is the standard sector: $\mathcal{C}(V_1, V_2)$ will do if $V_1 = \left\{(a, -a) \mid a \in \mathbb{R}^d\right\}$ and $V_2 = \left\{(a, -2a) \mid a \in \mathbb{R}^d\right\}$. For any other $\mathcal{C}$, any symplectic linear map which takes the standard sector into $\mathcal{C}$ will take the above $\mathcal{C}(V_1, V_2)$ into some $\mathcal{C}_0$ which strictly contains $\mathcal{C}$.

Having chosen such a $\mathcal{C}_0$, 1.) gives that on $\text{Lag}(\mathcal{C}_1)$, both $<_{\mathcal{C}}$ and $<_{\mathcal{C}_1}$ coincide with the partial order on $\text{Lag}(\mathcal{C}_0)$. This completes the proof.

□

The following proposition is an easy corollary of this lemma. It states that sectors defined in terms of curvatures of fronts are exactly the Lagrangian sectors defined by the tangents of the extremal fronts (graphs of the extremal curvature operators). Furthermore, the partial order on Lagrangian subspaces is exactly the usual partial order on symmetric matrices.

Proposition 4.3. *Let B_* and B^* be $d \times d$ symmetric (not necessarily positively defined) matrices, and let $B_* < B^*$. Also allow $B_* = -\infty$ or $B^* = \infty$ (but not both at the same time). Let $\mathcal{C} = \left\{(dq, Bdq) \mid dq \in \mathbb{R}^d, B_* \leq B \leq B^*\right\}$.*

Then $\mathcal{C} = \mathcal{C}(gB_*, gB^*)$, $\text{Lag}(\mathcal{C}) = \{gB \mid B_* < B < B^*\}$ and for $gB_1, gB_2 \in \text{Lag}(\mathcal{C})$ we have

$$gB_1 < gB_2 \text{ if and only if } B_1 < B_2.$$

When $B_* = -\infty$ or $B^* = \infty$, we mean $gB_* = \{0\} \times \mathbb{R}^d$ or $gB^* = \{0\} \times \mathbb{R}^d$, respectively.

Proof. First choose $B_* = c_* > -\infty$ (a scalar) and $B^* = \infty$. Then Lagrangian subspaces transversal to $gB^* = \{0\} \times \mathbb{R}^d$ are exactly the graphs gB of symmetric matrices B. An easy calculation shows that the quadratic form (used in the definition of the partial order on $\mathrm{Lag}(\mathcal{C})$, Definition 1.1) associated to a Lagrangian subspace gB has the simple form

$$(Q \circ (\pi_1|_{gB})^{-1})((dq, c_* dq)) = Q((dq, Bdq)) = \langle dq, (B - c_*)dq \rangle,$$

which immediately implies all the statements of the proposition. An analogous calculation works for $B_* = -\infty$ and $B^* = c^* < \infty$ (a scalar). In particular, on both of these sectors, the partial order coincides with the usual partial order on symmetric matrices.

So, by Theorem 1.2, for general $-\infty \leq B_* < B^* < \infty$ or $-\infty < B_* < B^* \leq \infty$, $\mathcal{C}(gB_*, gB^*)$ is contained in a sector of the above type, and by Lemma 4.2 it inherits the partial order. The rest of the statement follows from Theorem 1.2 and Remark 4.1. ☐

Remark 4.4. *Note that it would be possible to slightly extend the notion of such cones and define, for $B_1 < B_2$, beside $C_{B_1 \leq B \leq B_2}$, also*

$$C_{B_2 \leq B \leq B_1} := \left\{ (dq, Bdq) \mid dq \in \mathbb{R}^d, B_2 \leq B \textbf{ or } B \leq B_1 \right\}.$$

This would correspond to "closing" the space of matrices through infinity. With such a definition, $C_{B_2 \leq B \leq B_1} = C(gB_2, gB_1)$ would still hold, and the ordering of Lagrangian subspaces within the two "halves" of this cone would still coincide with the usual ordering of symmetric matrices.

Remark 4.5. *Note however, that (even with Remark 4.4 in mind) not all Lagrangian sectors can be defined in this way. For two symmetric matrices B_1 and B_2 it may very well happen that gB_1 and gB_2 are transversal, and so $C(gB_1, gB_2)$ is well defined, although neither $B_1 < B_2$ nor $B_2 < B_1$.*

Now we can turn to the proof of the main theorems.

Proof of Theorem 3.1. We use the notations of the theorem. Proposition 4.3 implies that $C = C(gB_1, gB_2)$ and $C^* = C(gB_*, gB^*)$. Suppose first that the operators are finite.

1. Suppose that $DT(C) \subset C^*$. Then obviously $gB_1', gB_2' \subset C^*$, so $B_* \leq B_1', B_2' \leq B^*$ holds. Choose a sector $\hat{C}$ which strictly contains C^*, and still has the form $\hat{C} = C_{\hat{B}_* \leq B' \leq \hat{B}^*}$. This ensures that $gB_1', gB_2', gB_*, gB^* \in \mathrm{Lag}(\hat{C})$ and the partial order on $\mathrm{Lag}(\hat{C})$ is still the partial order on matrices. Now by Theorem 1.2, $DT(C) \subset \hat{C}$ implies $B_1' < B_2'$. Now Proposition 4.3 implies that $DT(C) = C(B_1', B_2') = C_{B_1' \leq B' \leq B_2'}$.

2. Suppose that $B_* \leq B_1' < B_2' \leq B^*$. Then Proposition 4.3 immediately implies $DT(C) = C(B_1', B_2') = C_{B_1' \leq B' \leq B_2'}$, which implies $DT(C) \subset C^*$.

The statement with strict inequalities can be shown analogously, just there is no need to introduce $\hat{C}$. The cases when some of the operators are replaced by infinity can also be checked easily. ☐

The second main theorem is verified by checking the conditions of Theorem 3.1:

Proof of Theorem 3.4. We prove the statement about strict invariance – the non-strict version is completely analogous. We translate the theorem to the symplectic language. We use the notations

$$E_1^i = \{(dq, c_* dq) \mid dq \in J_i\} = gc_* \subset J_i \times J_i,$$

$$E_2^i = \{(dq, c^* dq) \mid dq \in J_i\} = gc^* \subset J_i \times J_i \ (i = 1, \dots, m),$$

$$\mathcal{C}_i = \mathcal{C}(E_1^i, E_2^i) \subset J_i \times J_i.$$

Here gB is the graph of the operator B as defined in (3). Subspaces and cones in $\mathcal{T}_{Tx}$, i.e. the objects with '-es (see the notation used in Definition 3.3 and in Theorem 3.4), and the higher dimensional cones and their boundary subspaces, i.e. the objects without i-s as indices are defined similarly. We also introduce B_1^i and B_2^i as the defining symmetric matrices for the image subspaces:

$$DTE_1^i = \{(dq, B_1^i dq) \mid dq \in J_i'\} = gB_1^i \subset J_i' \times J_i',$$

$$DTE_2^i = \{(dq, B_2^i dq) \mid dq \in J_i'\} = gB_2^i \subset J_i' \times J_i' \ (i = 1, \dots, m).$$

The reason why these operators B_1^i and B_2^i are well-defined is is exactly the decomposability required among the conditions of the theorem.

Furthermore, the conditions of the theorem say that $DT(\mathcal{C}_i)$ is strictly contained in $\mathcal{C}_i'$. By Theorem 3.1 this is equivalent to $c_* < B_1^i < B_2^i < c^*$.

The statement of the theorem is that $DT(\mathcal{C})$ is strictly contained in $\mathcal{C}'$. To prove this we only need to see – by Theorem 1.2 – that $DT(E_1), DT(E_2) \in \mathrm{Lag}(\mathcal{C}')$ and $DT(E_1) < DT(E_2)$.

Now the $DT(E_1^i)$ are orthogonal subspaces of $DT(E_1)$ which span $DT(E_1)$, and the $DT(E_2^i)$ are orthogonal subspaces of $DT(E_2)$ which span $DT(E_2)$. This means that $DT(E_1)$ and $DT(E_2)$ are also graphs of symmetric operators: $DT(E_1) = gB_1'$, $DT(E_2) = gB_2'$, where (in the appropriate base) B_1' and B_2' are block diagonal with blocks $B_1^1, \dots, B_1^m$ and $B_2^1, \dots, B_2^m$ respectively:

$$B_1' = \begin{pmatrix} B_1^1 & & & \\ & B_1^2 & & \\ & & \ddots & \\ & & & B_1^m \end{pmatrix}, \quad B_2' = \begin{pmatrix} B_2^1 & & & \\ & B_2^2 & & \\ & & \ddots & \\ & & & B_2^m \end{pmatrix}.$$

This obviously implies $c_* < B_1' < B_2' < c^*$. By Theorem 3.1 the proof of Theorem 3.4 is complete. □

5 Soft billiards in high dimensions

In this section we apply Theorem 3.4 to prove hyperbolicity for soft billiards in higher dimensions. We first give a definition of the dynamical system, than prove hyperbolicity under appropriate conditions. Finally, we discuss our conditions. We will see that for some classes of potentials (e.g. "high" repelling potentials) they are not much stronger than the "old" conditions necessary for the hyperbolicity of the $2D$ system. For other classes (e.g. continuous attracting potentials) they may be impossible to satisfy. We also show some specific examples of potentials that satisfy our conditions, and which demonstrate different combinations of dispersing and focusing/defocusing phenomena in the mechanism of hyperbolicity.

5.1 The soft billiard dynamical system

Consider finitely many disjoint spheres of radius R on the unit d-dimensional flat torus $\mathbb{T}^d$. The dynamical system of a point particle which processes uniform motion and interacts via elastic collisions with the scatterers, the hard dispersing billiard (with no corner points) is a paradigm of strongly chaotic motion, for all values of d (see [SCh], [Y], [BChSzT] and references therein). By a soft billiard we mean the following natural modification. The scatterers are no longer hard spheres, the point particle may enter them. The particle moves according to some spherically symmetric potential which vanishes identically outside the spheres.

In this paper we study the case $d \geq 3$. In contrast to this multi-dimensional situation, the planar system has been studied extensively in the literature. As to $d = 2$, results point into two different directions. On the one hand, for quite general softening of the potential, the chaotic behaviour is no longer present – stable orbits and islands appear in the phase space (see [RT], [D2], [D1] and references therein). However, when suitable conditions are satisfied (see Definition 5.3 below) the chaotic behaviour of the two dimensional hard billiard persists. The investigation of such *planar* soft billiards dates back to the pioneer works of Sinai ([S]) and Kubo ([K] and [KM]). A series of works (eg. [Kn1], [Kn2] and [M]) have resulted in [DL] which essentially identified property H1 (see Definition 5.3 below) as the one that ensures hyperbolicity in 2D. In [BT] even finer statistical properties have been proven for these planar soft billiards, however, in this paper we are only interested in the issue of hyperbolicity. In particular, we investigate what properties (in addition to property H1 from Definition 5.3) the potential should possess in order to extend the results of [DL] to the multi-dimensional case.

Let the Hamiltonian motion of the point particle be described by a potential which is identically zero outside and is some spherically symmetric function $V(r)$ inside the spherical scatterers (here r is the distance from the center of the scatterer). For simplicity we fix the mass and the full energy of our point particle as

$$m = 1, \qquad E = \frac{1}{2}.$$

This way the free flight velocity has unit length, $|v| = 1$ (in other words $v \in \mathbb{S}^d$ where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^d$).

To specify the state of our particle we should determine its location on $\mathbb{T}^d$ and its velocity (in $\mathbb{S}^d$ outside the scatterers). We denote the resulting flow phase space by M. The flow dynamics S^t on M is, on the one hand, free flight outside and, on the other hand, the Hamiltonian motion determined by the potential $V(r)$ inside the spherical scatterers. By the Hamiltonian nature of S^t the Liouville measure, to be denoted by μ, is a natural invariant measure for the flow. Outside the scatterers μ coincides with the product of the (normalized) Lebesgue measures on $\mathbb{T}^d$ and on $\mathbb{S}^d$.

Remark 5.1. *The Hamiltonian flow – even when restricted to the constant energy surface – naturally has one zero Lyapunov exponent: this corresponds to configurational perturbations in velocity direction. Hyperbolicity for the flow means that all other Lyapunov exponents are nonzero.*

We will prove hyperbolicity of the system by applying Theorem 3.4. In order to do so we need to specify the subset $U \subset M$ on which the dynamics is decomposable. Thus we will look at trajectory segments that start at a phase point just before collision, cross the potential, and than proceed by free flight until the next scatterer. This will be "one step" of the dynamics. We consider the orthogonal section to the flow at the pre-collision phase points both at the beginning and at the end of this step. This way the dynamics may be regarded as $T : U \to U$, a map from one such orthogonal section to the other.

By symmetry reasons a trajectory segment that crosses only one scatterer will be entirely contained in the plane defined by the initial trajectory line and the center of the scatterer. Within that plane, the collision angle φ (the angle of the initial velocity and the normal vector of the scatterer at the collision point) identifies the trajectory.

As long as $T : U \to U$ is concerned, the role played by the potential is completely described by the function $\Delta\Theta(\varphi)$ which gives the angle, within the plane of the trajectory, made by the two radii pointing to the points of entering and leaving the potential, as a function of the collision angle. Figure 2 shows the meaning of this function and the convention on signs.

Definition 5.2. *From here on we will refer to this function $\Delta\Theta(\varphi)$ as the* **rotation function**.

Being mainly interested in the differential aspects of T we introduce one more notation:

$$\kappa(\varphi) = \frac{d\Delta\Theta(\varphi)}{d\varphi}.$$

5.2 Hyperbolicity of multi-dimensional soft billiards

Below two properties are defined in terms of which our theorem on hyperbolicity is formulated.

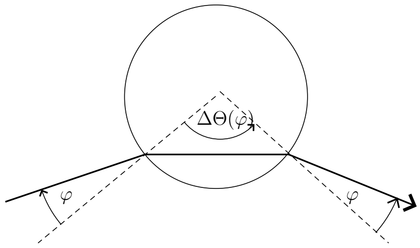


Figure 2: meaning of the rotation function

Definition 5.3. *The rotation function $\Delta\Theta(\varphi)$ satisfies property H1 if there exists $\delta > 0$ such that for all $\varphi \in [0, \frac{\pi}{2}]$, $|2 + \kappa(\varphi)| > \delta$.*

Definition 5.4. *The rotation function $\Delta\Theta(\varphi)$ satisfies property H2 if there exists $K < \infty$ such that $\alpha(\varphi) = \frac{\sin \Delta\Theta(\varphi)}{\sin(2\varphi+\Delta\Theta(\varphi))}$ and $\beta(\varphi) = \frac{\sin(\varphi+\Delta\Theta(\varphi))}{\sin(2\varphi+\Delta\Theta(\varphi))}$ satisfy $-K < \alpha(\varphi)$ and $-K < \beta(\varphi)$.*

Theorem 5.5. *Suppose that the rotation function satisfies conditions H1 and H2. Suppose also that the least distance between scatterers is at least τ_{min} where τ_{min} is large enough: $\tau_{min} > \tau_*(\delta, K, R)$. Then, the multi-dimensional soft billiard system is hyperbolic.*

Remark 5.6. *Property H1 (and suitably long free flight) is actually the condition used in [DL] and [BT] to prove (uniform) hyperbolicity in the 2-dimensional case. In higher dimensions, a new mechanism of scattering appears in the "new directions". Property H2 ensures exactly that we can handle this new phenomenon.*

Remark 5.7. *The minimum free flight we need depends on the parameters δ, K and R (the radius of the scatterer). We will not work out the explicit dependence, but it could be done by following the proof of the theorem.*

Remark 5.8. *The required minimum free flight $\tau_*(\delta, K, R)$ may be zero. This happens exactly when the potential causes immediate dispersing (no defocusing needed) in every direction. There are examples with this property (namely those for which the potential takes an arbitrary positive constant value inside the sphere), however, such a mechnanism is impossible for potentials that are continuous on $\mathbb{T}^d$. On details see Section 5.4.*

5.3 Proof of Theorem 5.5

We prove hyperbolicity by constructing an (eventually) strictly invariant cone field. As usual in billiard theory, we first define cones in the orthogonal section for a certain subset U of phase points – in this case, pre-collision points, – and then push these cones forward by the flow – either to the Poincaré section (discrete time case) or to every point of the trajectory until the next hitting of U (continuous time case). It is clear that since

μ-almost every phase point hits U in finite time, strict invariance of the cone (sub)field defined on U implies eventually strict invariance of the whole cone field (defined also μ-almost everywhere).

Consider a moment of free flight. Since the Lyapunov exponent (of the flow) corresponding to configurational perturbations in velocity direction is always zero, and velocity perturbations must be orthogonal to the velocity (in order to remain on the constant energy surface), the invariant cone field will be constructed in the orthogonal section.

$$J_x^q \times J_x^v = \{(dq, dv) \in \mathcal{T}_x(M) \mid dq \perp v, dv \perp v\}.$$

(Actually, $dv \perp v$ is automatic from the restriction on energy.) The configurational and velocity subspaces J_x^q and J_x^v can be naturally identified, and we denote both by J_x. When we apply Definition 3.3 and Theorem 3.4 to the flow, this is the J_x that appears there. Geometrically, $J_x \times J_x \subset \mathcal{T}_x(M)$ consists of tangent vectors of fronts.

It is important to note that time evolution of tangent vectors of fronts (vectors in $J_x \times J_x$) is described by not exactly the derivative of the dynamics. If both x and $S^t x$ are phase points of free flight, but the particle has crossed some scattering potential inbetween, then typically $D(S^t)_x(J_x \times J_x) \not\subset J_{S^t x} \times J_{S^t x}$, so fronts don't remain fronts under time evolution. To handle this situation, an extra projection of the configuration part in the flow direction may be necessary. This can also be thought of as a local reparametrization of trajectories in time - the perturbed orbit may need (positive or negative) extra time to "catch up". However, this kind of mapping from $J_x \times J_x$ to $J_{S^t x} \times J_{S^t x}$ is still symplectic. Keeping these observations in mind the proof of hyperbolicity goes along the following lines.

Proof of Theorem 5.5. In Subsection 5.3.1 we show that the natural mapping (induced by the dynamics) from one orthogonal section to the other is symplectic (Proposition 5.10). In Subsections 5.3.2 and 5.3.3 we show that this map can indeed be decomposed into lower dimensional problems in the sense of Definiton 3.3. (Actually, it can be fully decomposed – the subspaces J_i appearing are one-dimensional.) At the same time, the validity of the conditions of Theorem 3.4 are checked (Propositions 5.11 and 5.12). Then, Theorem 3.4 gives the existence of a strictly invariant cone field on the set U of pre-collision phase points. With the usual extension method (see above) this implies the existence of an eventually strictly invariant cone field μ-almost everywhere on the flow phase space (and almost everywhere on the usual Poincaré section, w.r.t the natural invariant measure). This proves hyperbolicity of both the soft billiard flow and the soft billiard map. □

5.3.1 Symplecticity of the orthogonal section mapping

Consider a phase point $x = (q, v)$ and its image along the flow, $S^t x = x' = (q', v')$ with both time moments 0 and t corresponding to free flight, i.e.

$$|v'| = |v| = 1. \tag{5}$$

The tangent planes of the phase space at the points x and x', $\mathcal{T}_x M$ and $\mathcal{T}_{x'} M$, can be both identified with $\mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d$. The $d-1$ dimensional planes in $\mathbb{R}^d$, perpendicular to v and v', will be denoted by J and J', respectively. Tangent planes to fronts at x are $d-1$ dimensional subspaces of the $2(d-1)$ dimensional linear space $J \times J$ given as

$$F = \{ \ (dq, Bdq) \mid dq \in J \ \}$$

where $B : J \to J$ is a symmetric operator. Our main concern is to investigate the evolution of fronts up to linear level which we describe by a linear map

$$D : J \times J \to J' \times J'; \qquad (dq', dv') = D(dq, dv).$$

To calculate D we need to consider another linear map,

$$DS^t : \mathcal{T}_x M \to \mathcal{T}_{x'} M; \qquad (\overline{dq}, \overline{dv}) = DS^t(dq, dv),$$

the tangent map of the flow. Now consider $(\overline{dq}, \overline{dv}) = DS^t(dq, dv)$ with $(dq, dv) \in J \times J$. On the one hand, $\overline{dv} \in J'$ by the conservation of energy (5). Thus we may put $dv' = \overline{dv}$. On the other hand, $\overline{dq}$ may have components pointing out of the plane J'. To get a suitable front at x', we need to cut the neighbouring trajectories orthogonally, that is, should modify $\overline{dq}$ along the flow direction. Thus we have

$$dq' = \overline{dq} - \langle \overline{dq}, v' \rangle v'$$

where we use the notation $\langle w, z \rangle$ for the Euclidean inner product of two vectors $w, z \in \mathbb{R}^d$. All in all we have

$$(dq', dv') = D(dq, dv) = (\overline{dq} \ - \ \langle \overline{dq}, v' \rangle v' \ , \ \overline{dv}) \in J' \times J'.$$

Remark 5.9. *We could have altered the configurational part of the perturbation along the flow direction of the perturbed trajectory and define dq' as $\overline{dq} - \langle \overline{dq}, v' + \overline{dv} \rangle \, (v' + \overline{dv})$. However, this would only differ from the previously defined dq' in a second order term, thus at the linear level the two definitions coincide.*

The even dimensional linear space $\mathcal{T}_x M$ has a natural symplectic structure:

$$\tilde{\omega}((dq_1, dv_1), (dq_2, dv_2)) = \langle dq_1, dv_2 \rangle - \langle dq_2, dv_1 \rangle \quad \text{ for } dq_i, dv_i \in \mathbb{R}^d.$$

Similarly, the natural symplectic structure for $J \times J$ is

$$\omega((dq_1, dv_1), (dq_2, dv_2)) = \langle dq_1, dv_2 \rangle - \langle dq_2, dv_1 \rangle \text{ for } dq_i, dv_i \in J.$$

With slight abuse of notation the natural symplectic structures of $\mathcal{T}_{x'} M$ and $J' \times J'$ will also be referred to as $\tilde{\omega}$ and ω, respectively.

Proposition 5.10. *The map $D : J \times J \to J' \times J'$ is symplectic (i.e. preserves the symplectic form ω).*

Proof. Consider $dx_i = (dq_i, dv_i)$, $i = 1, 2$ and their images:

$$\overline{dx}_i = (\overline{dq}_i, \overline{dv}_i) = DS^t(dq_i, dv_i) \quad \text{and} \quad dx'_i = (dq'_i, dv'_i) = D(dq_i, dv_i).$$

By the Hamiltonian nature of the flow

$$\tilde{\omega}(\overline{dx}_1, \overline{dx}_2) = \tilde{\omega}(dx_1, dx_2). \tag{6}$$

Furthermore

$$\omega(dx_1, dx_2) = \langle dq_1, dv_2 \rangle - \langle dq_2, dv_1 \rangle = \tilde{\omega}(dx_1, dx_2), \tag{7}$$

while

$$\begin{aligned} \omega(dx'_1, dx'_2) &= \langle dq'_1, dv'_2 \rangle - \langle dq'_2, dv'_1 \rangle = \langle \overline{dq}_1 - \langle \overline{dq}_1, v' \rangle v', \overline{dv}_2 \rangle - \langle \overline{dq}_2 - \langle \overline{dq}_2, v' \rangle v', \overline{dv}_1 \rangle \\ &= \langle \overline{dq}_1, \overline{dv}_2 \rangle - \langle \overline{dq}_2, \overline{dv}_1 \rangle = \tilde{\omega}(\overline{dx}_1, \overline{dx}_2) \end{aligned} \tag{8}$$

as $\overline{dv}_i = dv'_i \in J'$ for $i = 1, 2$. Putting Formulas (6), (7) and (8) together gives the Proposition. ☐

Now we can turn to the decomposability of the map D in the sense of Definition 3.3 and to the construction of invariant cones in the resulting components.

Recall that we would like to prove hyperbolicity by applying Theorem 3.4 to the (orthogonal section of) the first return map to U, where U is the set of precollision phase points. In order to do so throughout the next two subsections we will consider the situation sketched on Figure 3 and use the notations indicated on this figure.

5.3.2 in-plane scattering

Let s be the plane parallel to n (the normal vector of the scatterer at the considered point of collision) and v (the velocity), which contains O (the center of the scatterer). s is actually the plane of the (reference) trajectory.

Suppose first that a tangent vector of a front just before collision has the form (dr, dv) where both dr and dv are parallel to s. This means that $dr \parallel dv$, and that both the reference trajectory and the perturbed trajectory are in s. That is, the scattering is reduced to the 2-dimensional case, which has been discussed already in [DL], [BT], and references therein.

In this case, in the relation $dv = Bdr$, B is a *scalar* multiplier.

Let v', dr', dv', B' be the corresponding quantities just before the next collision. Note that $dr' \parallel dv'$, even though they are not (necessarily) "in-plane" for the next collision – the trajectory segments for two consecutive collisions typically don't lie in the same plane.

It is known (e.g. from [BT]) that if condition H1 holds and the free flight is long enough, then there exists a $C_1 > 0$ such that the "$0 \le B \le C_1$" cone is strictly invariant – the precise formulation of this statement will be given in Proposition 5.11 below. The construction of the invariant cone field in the $2D$ case is based on this fact.

Furthermore, if the free flight is long enough, then C can be chosen to be small – actually, even the same B gives a smaller B' if the free flight is longer.

We have just proven the following

Proposition 5.11. *Let $J_1 = \{dq \mid dq \in s, dq \perp v\}$, $J_1' = \{dq' \mid dq' \in s, dq' \perp v'\}$. If $(dq, dv) \in J_1 \times J_1$, then $(dq', dv') \in J_1' \times J_1'$. If the free flight is long enough, then there exists a $C_1 > 0$ such that*

$$\text{if } 0 \leq B \leq C_1 \text{ then } 0 < B' < C_1.$$

Also, if the free flight is long enough, then C_1 can be chosen to be small.

5.3.3 orthogonal scattering

Suppose now that a tangent vector of the front (just before collision) has the form (dr, dv) where $dr \parallel dv \perp s$. Also suppose that n and v are not parallel, that is, the collision is not central (in other words, the particle is not heading towards the center of the scatterer, or the collision angle φ is nonzero) [1] .

Let e denote the line parallel to v which crosses the center of the scatterer (see Figure 3). We may think of e as the 'optical axis' since the dynamics is invariant under rotations around this line.

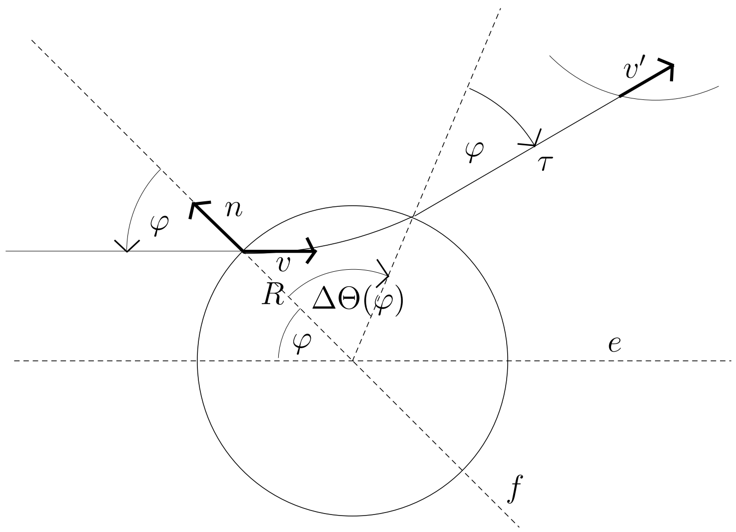


Figure 3: notation for the scattering

Notice that the perturbation (dr, dv) described above leaves the shape of the trajectory invariant, because the collision angle (and the distance of the incoming path-line

[1]If the collision is central, the discussion of the previous subsection applies. Actually, in this case there is no difference between in-plane and orthogonal scattering. The reader may also check that the two discussions give the same conditions in the limit as $\varphi \to 0$.

from O) are (up to linear terms) unchanged: $d\varphi = 0$. Moreover, every configuration point of the trajectory is displaced orthogonally to the plane s. We will see that actually $dr' \parallel dv' \parallel dr \parallel dv \perp s$.

To simplify considerations, we first discuss two special cases.

(a) If $dv = 0$ (the perturbation has the form $dx_a = (dr, 0)$ where $dr \perp s$), then the perturbation is equivalent (up to linear terms) to a small rotation "around" e by the angle $\frac{dr}{R \sin\varphi}$. [2] Consequently, the whole trajectory is rotated around e by the same angle. The resulting perturbations of r' and v' can then be read from Figure 3. Basic geometry gives

$$\begin{aligned} dr'_a &= (R\sin(\varphi+\Delta\Theta) + \tau\sin(2\varphi+\Delta\Theta))\frac{dr}{R\sin\varphi} \\ dv'_a &= \sin(2\varphi+\Delta\Theta)\frac{dr}{R\sin\varphi}. \end{aligned}$$

(b) If $dr = 0$ (the perturbation has the form $dx_b = (0, dv)$ where $dv \perp s$), then the perturbation is equivalent (up to linear terms) to a small rotation around f by the angle $\frac{dv}{|v|\sin\varphi} = \frac{dv}{\sin\varphi}$. Consequently, the whole trajectory is rotated around f by the same angle. The resulting perturbations of r' and v' can, again, be read from Figure 3. Basic geometry gives

$$\begin{aligned} dr'_b &= (R\sin\Delta\Theta + \tau\sin(\varphi+\Delta\Theta))\frac{dv}{\sin\varphi} \\ dv'_b &= \sin(\varphi+\Delta\Theta)\frac{dv}{\sin\varphi}. \end{aligned}$$

Now we consider a general perturbation with $dr \parallel dv \perp s$ and to investigate the evolution of fronts we put $B = \frac{dv}{dr}$. By $(dr, dv) = (dr, 0) + (0, dv) = dx_a + dx_b$, we have $dx' = (dr', dv') = (dr'_a, dv'_a) + (dr'_b, dv'_b) = dx'_a + dx'_b$ where dx'_a and dx'_b have been calculated above. We see that $dr' \parallel dv' \parallel dr \parallel dv \perp s$. We may obtain the "curvature" of the perturbation at the next point of income as $B' = \frac{dv'}{dr'}$. To decrease the number of symbols used, it is convenient to "scale out" the radius of the scatterer. That is, we introduce $A := RB$ and $l := \frac{\tau}{R}$. All in all, we get

$$A' = \frac{\sin(2\varphi+\Delta\Theta) + \sin(\varphi+\Delta\Theta)\,A}{\sin(\varphi+\Delta\Theta) + \sin(2\varphi+\Delta\Theta)\,l + \sin\Delta\Theta\,A + \sin(\varphi+\Delta\Theta)\,lA}. \tag{9}$$

We introduce the notation

$$\begin{aligned} \alpha &= \frac{\sin\Delta\Theta}{\sin(2\varphi+\Delta\Theta)} \\ \beta &= \frac{\sin(\varphi+\Delta\Theta)}{\sin(2\varphi+\Delta\Theta)}. \end{aligned}$$

[2] By rotation around e we mean a rotation leaving e invariant. If $d > 3$, then there are many such rotations – just like there are many directions for dr such that $dr \perp s, v$. Given dr there is a unique way to choose the appropriate rotation.

With these, (9) can be rewritten as

$$A' = \frac{1 + \beta A}{\alpha A + \beta(1 + lA) + l}. \tag{10}$$

Looking at this formula, it is clear that if both α and β are bounded from below by $-K$ (which is exactly Condition H2) then $0 \le A \le \frac{1}{2K}$ implies that $1 + \beta A \ge \frac{1}{2}$, so the numerator is positive and the denominator is at least

$$-K\frac{1}{2K} - K + \frac{1}{2}l = \frac{l}{2} - K - \frac{1}{2},$$

which is positive if l is large enough. Furthermore, $A' < \frac{1}{2K}$ can be guaranteed by an even larger l. (Notice that the numerator can be arbitrarily large if $A > 0$ and β is large, but in this case, the coefficient of l in the denominator is large as well.)

Putting this together, we get that for a suitable $C > 0$ and $l^* < \infty$, $l \ge l^*$ implies

$$\text{if } 0 \le A \le C \text{ then } 0 < A' < C.$$

It is not hard to see that (even if different scatterers have different radii), the same implication hold with B instead of A.

Again, as in the case of "in plane" scattering, if the free flight is long enough, then C can be chosen to be small.

We have just proven the following

Proposition 5.12. *Let $J_2, J_3, \dots, J_{d-1}$ be pairwise orthogonal 1-dimensional subspaces such that $J_i \perp s, v$ $(i = 2, \dots, d-1)$. Let $J_i' = J_i$ $(i = 2, \dots, d-1)$. For every $i \in \{2, \dots, d-1\}$ if $(dr, dv) \in J_i \times J_i$ then $(dr', dv') \in J_i' \times J_i'$. If the free flight is long enough, then there exists a $C_2 > 0$ such that with $dr \in J_i$, $dv = Bdr$, $dv' = B'dr'$*

$$\text{if } 0 \le B \le C_2 \text{ then } 0 < B' < C_2.$$

Also, if the free flight is long enough, then C_2 can be chosen to be small.

Looking at Propositions 5.11 and 5.12, we see that if the free flight is long enough, we can choose $C_1 = C_2$. So, the conditions of Theorem 3.4 are satisfied with $c_* = 0$ and $c^* = C_1 = C_2$.

This completes the proof of Theorem 5.5.

5.4 Discussion, specific potentials

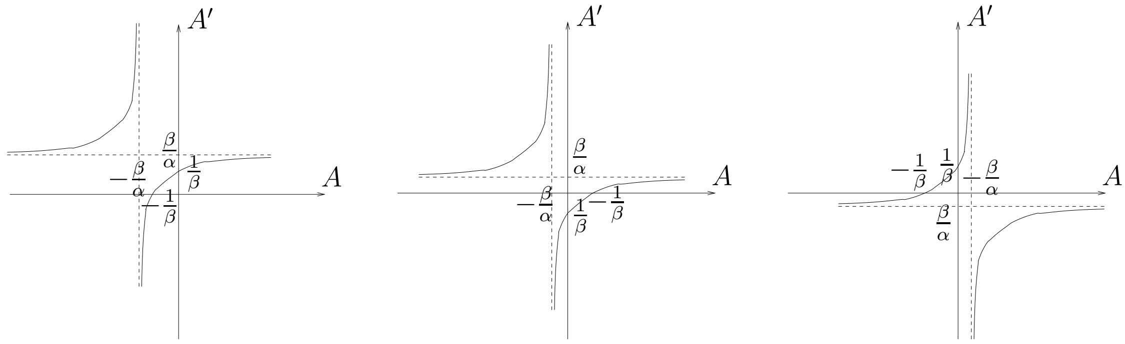
In this section we give some examples to which Theorem 5.5 applies. Note that, as the trajectory stays within a plane inside the potential, the calculation of $\Delta\Theta(\varphi)$ from $V(r)$ is a two dimensional problem. For the examples we consider below this has been carried out in [DL] and in [BT]. The validity of property H1 will also follow from the results of these works. Thus we only need to check Property H2 (see Definition 5.4). In all our examples, $\Delta\Theta$ is a continuous function of φ. In this case, Property H2 can be guaranteed if for any φ at least one of the three conditions below holds:

1. $\sin(\Delta\Theta + 2\varphi) \neq 0$;
2. $\sin(\Delta\Theta + 2\varphi)$ becomes zero at some φ_0, however, $\sin\Delta\Theta$ and/or $\sin(\Delta\Theta + \varphi)$ have the same sign as $\sin(\Delta\Theta + 2\varphi)$ in a neighbourhood of φ_0, so α and/or β can be seen to be positive;
3. $\sin(\Delta\Theta + 2\varphi)$ becomes zero at some φ_0, however, $\sin\Delta\Theta$ and/or $\sin(\Delta\Theta + \varphi_0)$ are/is zero simultaneously: in this case property H2 can be checked by L'Hospital's rule as a condition on κ: $\alpha \to \frac{\kappa}{2+\kappa}$ and/or $\beta \to \frac{1+\kappa}{2+\kappa}$, both of which are bounded if only Property H1 holds. [3]

Before turning to the specific examples we make several more remarks. Let us consider the formula (10) in case $\tau = 0$ (and thus $l = 0$), i.e. the relation of the curvatures just before (A) and just after (A') crossing the potential in the orthogonal scattering:

$$A' = \frac{1 + \beta A}{\beta + \alpha A}.$$

Then A' is a (locally) strictly increasing function of A. This can be seen by calculating the derivative and using that $\beta^2 > \alpha$, which comes from the definitions of α and β, and the concavity of the function $f(x) = \log(\sin x))$. See Figure 4 for the sketch of this function for the different values that α and β can take. See also Figure 5 to see where these possible values occur.



(a) $\alpha > 0$, $\beta > 0$ (b) $\alpha < 0$, $\beta < 0$ (c) $\alpha < 0$, $\beta > 0$

Figure 4: The graph of $A' = \frac{1+\beta A}{\beta+\alpha A}$ for the possible values of α and β

In case $\beta > 0$ and $\alpha \geq 0$, we have $A' > 0$ whenever $A \geq 0$. In other words, incoming convex perturbations turn into outgoing convex perturbations and there is no need on

[3]Note in these cases the two required conditions, H1 and H2, are equivalent. This happens, for example, when $\varphi = 0$ – in this particular case the reference plane is not well defined, any incoming front gives rise to the same type of perturbation and it makes no sense to distinguish between in-plane and orthogonal scattering.

a lower bound on τ to obtain the invariance of the cone field. We will refer to this as the dispersing mechanism in the othogonal scattering. On the other hand, if $\beta \leq 0$ or even if $\beta > 0$ but $\alpha < 0$ it may happen that $A' < 0$ for $A > 0$, thus we need a positive lower bound on τ. We will refer to this as the defocusing mechanism in the orthogonal scattering.

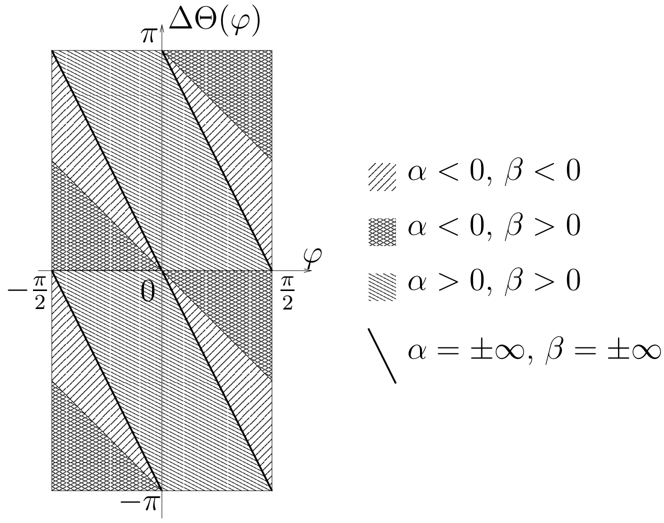


Figure 5: values of α and β in the different regions

On the other hand, a similar analysis for the in-plane scattering can be given based on κ (see e.g. [DL]): in case $\kappa \geq 0$ or $\kappa < -2$, we have dispresing mechanism and thus no lower bound on τ is needed, while $-2 < \kappa < 0$ results in defocusing mechanism and there is a need for a lower bound on τ to obtain the invariance of cones.

5.4.1 Repelling versus attracting potentials

In this subsection we consider potentials for which $V(R) = 0$, which implies $\Delta\Theta(\frac{\pi}{2}) = 0$ as the Hamiltonian flow is continuous in this case.

First we give a negative result. Assume $\kappa(\varphi) < -2$ for all φ. Note that this happens for attracting potentials, e.g. for potentials with Coulomb type singularities (cf. [DL] and references therein). Our assumptions imply $\pi - 2\varphi < \Delta\Theta < \pi - \varphi$ for φ close to $\frac{\pi}{2}$ and thus $\beta \to -\infty$ as $\varphi \to \frac{\pi}{2}$. This means Property H2 can not be satisfied. We have infinitely weak focusing in the orthogonal scattering, even though the dispersing mechanism works for the in-plane scattering. It is hard to tell what the dynamical behaviour of such multi-dimensional soft billiards can be: the different directions may mix up and result in an asymptotic expansion on a long term. Nonetheless, it is possible one can construct examples with arbitrary long free flights for which periodic orbits with zero Lyapunov exponents arise. This is an interesting area of further research.

Assume now, on the contrary, that the potential is repelling: $V'(r) < 0$ for all r. Assume also that $\Delta\Theta(\varphi)$ is continuously differentiable, and that Property H1 is satisfied.

Then, by repulsion, the potential naturally satisfies $0 < \Delta\Theta(\varphi) < \pi$ and, by property H1, $\Delta\Theta(\varphi) \neq \pi - 2\varphi$ for $0 < \varphi < \frac{\pi}{2}$ (see Remark 5.15). This would contradict an assumption that $\kappa(\varphi) < -2$ for all φ, so we must have $\kappa(\varphi) > -2$ for all φ. ($2 + \kappa(\varphi)$ cannot change signs, because it is continuous and never zero.)

Our last assumption is $V(0) \geq E(= \frac{1}{2})$ – this ensures $\Delta\Theta(0) = 0$. This means that the potential is "high": impossible for the particle to climb to the top (and reach the center).

Remark 5.13. *Actually, $V(0) < E = \frac{1}{2}$ would imply $\Delta\Theta(0) = \pi$ which would also contradict Property H1 for a repelling continuous potential with continuously differentiable $\Delta\Theta(\varphi)$. So, $V(0) \geq E$ is needed for nice repelling potentials to be hyperbolic, already in 2D. Note however, that the literature discusses several hyperbolic potentials for which $\Delta\Theta(0) = \pi$, and, simultaneously, $\Delta\Theta(\varphi)$ is not continuously differentiable.*

So, Property H2 holds (see Figure 6 (a))

- for $0 < \varphi < \frac{\pi}{2}$ because $0 < \Delta\Theta < \pi - 2\varphi$, condition 1. above is satisfied;
- in the limit as $\varphi \to 0$, by either condition 2 or 3 above;
- in the limit as $\varphi \to \frac{\pi}{2}$, α is bounded by condition 3 and $\beta > 0$ (actually $\beta \to +\infty$) by condition 2.

That is, in this case, the high dimensional study brings no new condition. We have just proven the following corollary of Theorem 5.5:

Corollary 5.14. *Assume that the scatterers of a multi-dimensional soft billiard system are described by a high repelling potential: $V'(r) < 0$ for all r and $V(R) = 0$, $V(0) \geq \frac{1}{2}$. Assume also that the rotation function $\Delta\Theta(\varphi)$ is continuously differentiable and it satisfies Property H1. Then, if the free flight is long enough, the system is hyperbolic.*

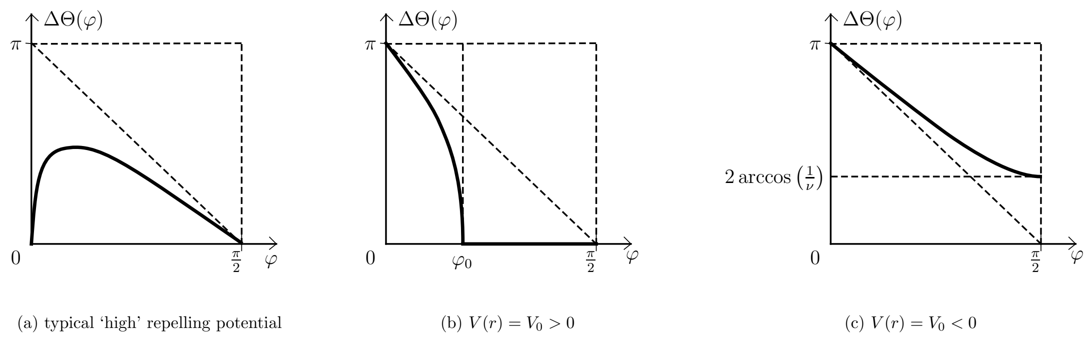


Figure 6: rotation function for three examples

Note furthermore that we have dispersing mechanism for the orthogonal scattering (since $\alpha > 0$, $\beta > 0$) and defocusing mechanism for the in-plane scattering, at least for large collision angles (since $-2 < \kappa < 0$).

By [DL], an important family for which the rotation function has the above properties is Kubo's class of repelling potentials, i.e. potentials with

$$\frac{d}{dr}(rV'(r)) \leq 0, \qquad V'(r) < 0$$

for all $0 \leq r \leq R$.

Remark 5.15. *The range of the function $\Delta\Theta(\varphi)$ is (topologically) a circle, so it is in principle possible to have $\Delta\Theta$ continuously differentiable with $\Delta\Theta(\varphi_1) = a$, $\Delta\Theta(\varphi_2) = b$ but $\kappa(\varphi) \neq \frac{b-a}{\varphi_2-\varphi_1}$ for any $\varphi \in [\varphi_1, \varphi_2]$ (the graph of $\Delta\Theta(\varphi)$ can "go around" the circle). In particular, it is possible that $\Delta\Theta(\frac{\pi}{2}) = 0$ and $\Delta\Theta(\varphi_0) = \pi - 2\varphi_0$ (so $\sin(2\varphi_0 + \Delta\Theta(\varphi_0)) = 0$) for some $\varphi_0 \in (0, \frac{\pi}{2})$ although $\kappa(\varphi) \neq -2$ for any φ.*

However, that never happens for a repelling potential: It's easy to see that if $V'(r) < 0$ for all r, then $0 \leq \Delta\Theta(\varphi) \leq \pi$ in the sense that the graph of $\Delta\Theta(\varphi)$ is contained in the semicircle $\{\gamma \mid \gamma \in [0, \pi] (\mathrm{mod}\ 1)\}$. So, if $\Delta\Theta(\varphi)$ is continuously differentiable and $V(R) = 0$ (so $\Delta\Theta(\frac{\pi}{2}) = 0$), then Property H1 automatically implies Property H2 for $0 < \varphi < \frac{\pi}{2}$ via condition 1. See Figure 7.

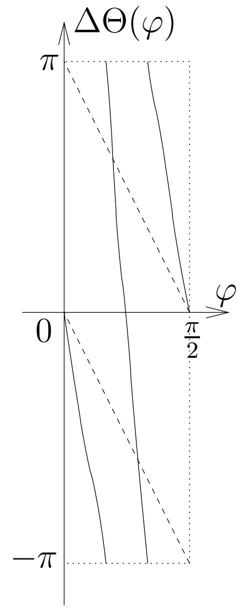


Figure 7: what never happens for a repelling potential

5.4.2 Constant potentials

Now let us consider constant potentials, i.e. the case when $V(r) = V_0$ for some constant value $V_0 < E$ where E is the full energy of the particle ($V_0 \geq E$ corresponds to hard billiards). The rotation function can be explicitely calculated (see [BT]). It is worth introducing $\nu = \sqrt{2E - 2V_0}$. We distinguish between the cases $V_0 > 0$ ($\nu < 1$) and $V_0 < 0$ ($\nu > 1$).

Let us consider the case of positive V_0 first and introduce furthermore the angle φ_0 for which:

$$\nu = \sin \varphi_0.$$

We have

$$\Delta\Theta(\varphi) = \begin{cases} 2 \arccos\left(\frac{\sin\varphi}{\nu}\right) & \text{if } \varphi \leq \varphi_0 \\ 0 & \text{if } \varphi > \varphi_0 \end{cases}.$$

From this Formula (see also Figure 6 (b))

- if $0 < \varphi < \frac{\pi}{2}$, then $0 \leq \Delta\Theta < \pi - 2\varphi$, condition 1. above is satisfied;
- in the limit as $\varphi \to \frac{\pi}{2}$, $\alpha = 0$ and β becomes $+\infty$ as condition 2. above applies (actually, for $\varphi > \varphi_0$ the scattering is an elastic reflection like in hard billiards),
- in the limit as $\varphi \to 0$, both α and β are bounded (and positive) by condition 3 above.

Note furthermore that we have dispersing mechanism in both scattering problems: by $\alpha > 0$ and $\beta > 0$ on the one hand (orthogonal scattering) and by $\kappa = 0$ or $\kappa < -2$ (in-plane scattering). In particular, the lower bound on τ can be arbitrarily small (independently of the potential) to obtain hyperbolicity.

Now let us turn to the case of $V_0 < 0$ (i.e. $\nu > 1$). We have (see also Figure 6 (c))

$$\Delta\Theta(\varphi) = 2 \arccos\left(\frac{\sin\varphi}{\nu}\right)$$

for all φ. Property H1 is satisfied by $0 > \kappa \geq -\frac{2}{\nu}$ where the minimum is obtained at $\varphi = 0$.

- Note that the flow is not continuous at $\varphi = \frac{\pi}{2}$ as $\Delta\Theta(\frac{\pi}{2}) \neq 0$. This makes it possible that condition 1. applies for all $\varphi \neq 0$.
- When $\varphi \to 0$, conidition 3. applies as $\alpha = \frac{-1}{\nu-1}$ and $\beta = \frac{\nu-2}{2\nu-2}$.

Thus Property H2 is satisfied. It is worth noting $-K < \alpha < 0$ for all φ while β may change its sign depending on the value of ν.

Nonetheless, in both one-dimensional scattering problems we have defocusing mechanism.

We have just proven the following corollary of Theorem 5.5:

Corollary 5.16. *Assume that the scatterers of a multi-dimensional soft billiard system are described by a (nonzero) constant potential:* $V(r) = V_0$ *for all* r*. Then, if the free flight is long enough, the system is hyperbolic. If* $V_0 > 0$*, any positive lower bound on the free flight is enough.* [4]

6 Concluding remarks

In this paper we have presented a method for proving hyperbolicity in multi-dimensional Hamiltonian systems (Theorem 3.4) and have applied it to the case of soft billiards in high dimensions (Theorem 5.5). Two directions of further research closely related to our results are as follows.

On the one hand we hope that our method could be applied to discuss the issue of hyperbolicity in other multi-dimensional Hamiltonian systems. Possible candidates are, for example, multi-dimensional (hard) billiards with focusing and possibly also dispersing boundary components.

On the other hand, having proven the non-vanishing of Lyapunov exponents for certain classes of multi-dimensional soft billiards, questions that naturally arise are on the ergodic and statistical properties of these systems. Properties to be proven are, in increasing order of difficulty, uniform hyperbolicity, ergodicity (that, via hyperbolicity, would automatically imply K-mixing and Bernoulli property) and finally exponential decay of correlations. However, in order to prove ergodicity, or even more, exponential mixing, one needs to have fine estimates on the singularities of the system, objects highly non-trivial in multi-dimensional billiards – both hard and soft (see [BChSzT]). Nonetheless, as already mentioned in [BT], it seems possible that property H1 can be slightly relaxed and thus allow for a broader class of potentials in which the singularities are easier to handle. We will turn back to this question in a separate paper.

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[4] When both mechanisms are dispersing (as in the $V_0 > 0$ case) usually no lower bound on the free flight is needed at all – scatterers can be allowed to touch. However, we don't discuss such systems now.

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