Ground State Energy of the Polaron in the Relativistic Quantum Electrodynamics

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Abstract

We consider the polaron model in the relativistic quantum electrodynamics(QED). We prove that the ground state energy of the model is finite for all values of the fine-structure constant and the ultraviolet cutoff Λ. Moreover we give an upper bound and a lower bound of the ground state energy.

Key words: relativistic QED; ground state energy, polaron model.

1 Introduction and Main Results

We consider the relativistic quantum electrodynamics(QED) for a fixed total momentum — the polaron model of the relativistic QED. The Hamiltonian, which describes a Dirac particle minimally coupled to the quantized radiation field, commutes with the total momentum operator and has a direct integral decomposition with respect to the total momentum operator([1], [2]). Each fibre in this direct integral decomposition is just the Hamiltonian of the polaron we consider. The Hilbert space of the polaron model is defined by

$$\mathcal{F} := \mathbb{C}^4 \otimes \mathcal{F}_{\mathrm{b}}(L^2(\mathbb{R}^3 \times \{1,2\})), \tag{1}$$

where

$$\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3{\times}\{1,2\})) := \bigoplus_{n=0}^{\infty}\left[\bigotimes_s^n (L^2(\mathbb{R}^3{\times}\{1,2\}))\right] \tag{2}$$

is the photon Fock space ($\otimes_s^n$ denotes n-fold symmetric tensor product). For a closable operator T on $L^2(\mathbb{R}^3{\times}\{1,2\})$ we denote by $\mathrm{d}\Gamma_{\rm b}(T)$, the second quantization operator of T (see [4]). Let $a(f)$, $f \in L^2(\mathbb{R}^3{\times}\{1,2\})$ be the annihilation operator on the photon Fock space. For a function $g_j \in L^2(\mathbb{R}^3{\times}\{1,2\})$, $j = 1,2,3$, we set

$$A_j := a(g_j) + a(g_j)^*, \quad j = 1,2,3. \tag{3}$$

Let $\{\alpha_1, \alpha_2, \alpha_3, \beta\}$ be the 4×4-Dirac matrices, i.e., $\{\alpha_i, \alpha_j\} = 2\delta_{i,j}$, $\{\alpha_i, \beta\} = 0, \beta^2 = 1$, $i,j = 1,2,3$. Here $\{A,B\} := AB + BA$. For three objects a_1, a_2, a_3 we set $\mathbf{a} = (a_1, a_2, a_3)$, and write $\mathbf{a}\cdot\mathbf{b} := \sum_{j=1}^3 a_j b_j$, provided that $a_j b_j$ and $\sum_{j=1}^3 a_j b_j$ are defined.

The Hamiltonian of the polaron model we consider is

$$H(\mathbf{p}) := \boldsymbol{\alpha}\cdot\mathbf{p} + M\beta + \mathrm{d}\Gamma_{\rm b}(\omega) - \boldsymbol{\alpha}\cdot\mathrm{d}\Gamma_{\rm b}(\mathbf{k}) - q\boldsymbol{\alpha}\cdot\mathbf{A}, \tag{4}$$

where $\mathbf{p} \in \mathbb{R}^3$ is the fixed total momentum, $M \geq 0$ is the mass of the Dirac particle, $q \in \mathbb{R}$ is a constant proportional to the fine-structure constant, and $\omega = |\mathbf{k}|$ is the 1-photon Hamiltonian ($\mathbf{k} \in \mathbb{R}^3$). Note that we omit the symbol $\otimes$ between the Hilbert space for the Dirac matrices $\mathbb{C}^4$ and the photon Fock space $\mathcal{F}_{\rm b}(L^2(\mathbb{R}^3{\times}\{1,2\}))$. The most important example of $\{g_j\}_{j=1}^3$ is of the form

$$f_j(\mathbf{k}, r) := \frac{\chi_\Lambda(\mathbf{k})}{|\mathbf{k}|^{1/2}} e_j^{(r)}(\mathbf{k}), \tag{5}$$

where the measurable functions $\mathbf{e}^{(1)}(\mathbf{k})$, $\mathbf{e}^{(2)}(\mathbf{k})$ are the polarization vectors:

$$\mathbf{k}\cdot\mathbf{e}^{(r)}(\mathbf{k}) = 0, \quad \mathbf{e}^{(r)}(\mathbf{k})\cdot\mathbf{e}^{(s)}(\mathbf{k}) = \delta_{r,s}, \quad \text{a.e.}\, \mathbf{k} \in \mathbb{R}^3, \quad r,s = 1,2, \tag{6}$$

and $\chi_\Lambda(\mathbf{k})$ is the characteristic function of the ball $\{\mathbf{k} \in \mathbb{R}^3 | |\mathbf{k}| < \Lambda\}$, $\Lambda > 0$.

We define

$$E_0(\mathbf{p}) := \inf_{\substack{\Psi \in \mathrm{Dom}(H(\mathbf{p})) \\ \|\Psi\|=1}} \langle \Psi, H(\mathbf{p})\Psi\rangle \tag{7}$$

the ground state energy of $H(\mathbf{p})$, where "Dom" means operator domain. We assume the following:

Hypothesis I. $g_j \in \mathrm{Dom}(\omega^{-1/2}) \cap \mathrm{Dom}(\omega)$**,** $\langle g_j, g_\ell\rangle \in \mathbb{R}$**,** $j, \ell = 1,2,3$**.**

It should be noted that it is highly non-trivial whether or not $E_0(\mathbf{p})$ is finite, because $H(\mathbf{p})$ contains the term $-\boldsymbol{\alpha}\cdot \mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k})$. This is the main problem discussed in the present paper.

We prove that the ground state energy $E_0(\mathbf{p})$ is finite under suitable conditions:

Theorem 1.1. *Assume Hypothesis I, and*

$$G(\mathbf{g}) := \inf_{\mathbf{k}\in\mathbb{R}^3\setminus\{0\}} \frac{1}{|\mathbf{k}|}\sum_{r=1,2}\int_{\mathbb{R}^3}\frac{|\mathbf{k}\cdot\mathbf{g}(\mathbf{k}',r)|^2}{|\mathbf{k}||\mathbf{k}'|-\mathbf{k}\cdot\mathbf{k}'}\mathrm{d}\mathbf{k}' < \infty. \tag{8}$$

Then, the ground state energy $E_0(\mathbf{p})$ is finite:

$$E_0(\mathbf{p}) > -\infty. \tag{9}$$

In particular, if $g_j = f_j$, $j = 1, 2, 3$, the ground state energy $E_0(\mathbf{p})$ is finite.

For a vector $u \in \mathbb{C}^4$, we set $a_j := \langle u, \alpha_j u\rangle_{\mathbb{C}^4}$, and

$$\mathcal{E}(\Lambda, u) := \mathbf{p}\cdot\mathbf{a} + M\langle u, \beta u\rangle + 4\pi\Lambda q^2\frac{1-|\mathbf{a}|^2}{|\mathbf{a}|}\log\left(\frac{1+|\mathbf{a}|}{1-|\mathbf{a}|}\right) - 4\pi\Lambda q^2.$$

In the physical case (i.e. the function g_j's are given by (5)), the lower bound of $E_0(\mathbf{p}) + \sqrt{|\mathbf{p}|^2 + M^2}$ are proportional to Λ:

Theorem 1.2. *Let $g_j = f_j$, $j = 1, 2, 3$. Then*

$$C_1\Lambda - \sqrt{|\mathbf{p}|^2 + M^2} \le E_0(\mathbf{p}), \tag{10}$$

$$E_0(\mathbf{p}) \le C_2(\Lambda) \tag{11}$$

where

$$C_1 := \inf_{\epsilon,\epsilon'>0}\left\{\epsilon|q| + 16\pi q^2 + \left(\epsilon' + \frac{1}{\epsilon'}\right)4\pi q^2, \sqrt{\frac{4\pi|q|}{3\epsilon} + \left(1+\frac{1}{\epsilon'}\right)4\pi q^2}\right\}, \tag{12}$$

$$C_2(\Lambda) := \inf_{\substack{u\in\mathbb{C}^4\\ \|u\|_{\mathbb{C}^4}=1}} \mathcal{E}(\Lambda, u). \tag{13}$$

2 Proof of Theorem 1.1 and 1.2

Lemma 2.1. *Let A be a positive self-adjoint operator on a Hilbert space $\mathcal{H}$. Let B be a symmetric operator with* $\mathrm{Dom}(A) \subset \mathrm{Dom}(B)$ *and*

$$\|B\Psi\| \le \|A\Psi\|, \quad \Psi \in \mathrm{Dom}(A). \tag{14}$$

Then, for all $\Psi \in D(A)$, $\langle \Psi, (A + B)\Psi\rangle \ge 0$.

Proof. By the Kato-Rellich theorem([4]), for all $\epsilon \in (-1,1)$, $A+\epsilon B$ is self-adjoint and $A+\epsilon B \geq 0$. Therefore $\langle \Psi, (A+B)\Psi \rangle \geq 0$ for all $\Psi \in \mathrm{Dom}(A)$. ∎

By this lemma, it suffices to show that there exists a constant $E \geq 0$ such that

$$\|(\mathrm{d}\Gamma_{\mathrm{b}}(\omega)+E)\Psi\|^2 \geq \|\boldsymbol{\alpha}\cdot(\mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k})+q\mathbf{A})\Psi\|^2, \quad \Psi \in \mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega)). \tag{15}$$

We use the following representation for $\boldsymbol{\alpha}$-matrices:

$$\alpha_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix}, \quad j=1,2,3,$$

with $(\sigma_1,\sigma_2,\sigma_3)$ being the Pauli matrices. Using anticommutation relation of α_1, α_2 and α_3, we have

$$\|\boldsymbol{\alpha}\cdot(\mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k})+q\mathbf{A})\Psi\|^2 = \sum_{j=1}^{3}\|(\mathrm{d}\Gamma_{\mathrm{b}}(k_j)+qA_j)\Psi\|^2 - q\big\langle \Psi, \mathbf{S}\cdot[a(i\mathbf{k}\times\mathbf{g})+a(i\mathbf{k}\times\mathbf{g})^*]\Psi\big\rangle, \tag{16}$$

where $S_j = \sigma_j \oplus \sigma_j$, $j=1,2,3$. The Hilbert space $\mathbb{C}^4 \otimes [\otimes_s^n L^2(\mathbb{R}^3\times\{1,2\})]$ is naturally embedded in $L^2(\mathbb{R}^3\times\{1,2\};\mathbb{C}^4\otimes[\otimes_s^{(n-1)}L^2(\mathbb{R}^3\times\{1,2\})])$. For a vector $\Psi \in \mathbb{C}^4 \otimes [\otimes_s^n L^2(\mathbb{R}^3\times\{1,2\})]$, we denote its value at point $(\mathbf{k},r)\in\mathbb{R}^3\times\{1,2\}$ by $\Psi(\mathbf{k},r,\cdot)$.

For $\Psi = (\Psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}$, we define

$$a^{(r)}(\mathbf{k})\Psi := \Big(\Psi^{(1)}(\mathbf{k},r), \sqrt{2}\Psi^{(2)}(\mathbf{k},r,\cdot), \cdots \sqrt{n}\Psi^{(n)}(\mathbf{k},r,\cdot),\cdots\Big) \in \mathcal{F}, \quad \mathbf{k}\in\mathbb{R}^3,\ r=1,2, \tag{17}$$

a Fock space valued function. This operator $a^{(r)}(\mathbf{k})$ is the distributional kernel of the annihilation operator.

Lemma 2.2. *For all $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$ and $\epsilon > 0$, the following inequality holds:*

$$\big|q\big\langle \Psi, \mathbf{S}\cdot[a(i\mathbf{k}\times\mathbf{g})+a(i\mathbf{k}\times\mathbf{g})^*]\Psi\big\rangle\big| \leq |q|\epsilon\langle\Psi,\mathrm{d}\Gamma_{\mathrm{b}}(\omega)\Psi\rangle + \frac{|q|}{\epsilon}\langle\mathbf{g},\omega\mathbf{g}\rangle\|\Psi\|^2, \tag{18}$$

where $\langle\mathbf{g},\omega\mathbf{g}\rangle := \sum_{j=1}^{3}\langle g_j,\omega g_j\rangle$.

Proof.

$$\begin{aligned}
\text{l.h.s of (18)} =& 2|q| \left| \operatorname{Re} \sum\!\!\!\!\!\!\int_{\mathbb{R}^3} \langle \Psi, -i\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}(\mathbf{k}, r)) a^{(r)}(\mathbf{k}) \Psi \rangle \mathrm{d}\mathbf{k} \right| \\
\leq & 2|q| \sum\!\!\!\!\!\!\int_{\mathbb{R}^3} \|\mathbf{S} \cdot (\mathbf{k} \times \mathbf{g}(\mathbf{k}, r)) \Psi\| \|a^{(r)}(\mathbf{k}) \Psi\| \mathrm{d}\mathbf{k} \\
=& 2|q| \sum\!\!\!\!\!\!\int_{\mathbb{R}^3} |\mathbf{k}|^{1/2} |\mathbf{g}(\mathbf{k}, r)| \cdot \| |\mathbf{k}|^{1/2} a^{(r)}(\mathbf{k}) \Psi\| \cdot \|\Psi\| \mathrm{d}\mathbf{k} \\
\leq & 2|q| \langle \mathbf{g}, \omega \mathbf{g} \rangle^{1/2} \left[\sum\!\!\!\!\!\!\int_{\mathbb{R}^3} \| |\mathbf{k}|^{1/2} a^{(r)}(\mathbf{k}) \Psi\|^2 \right]^{1/2} \|\Psi\| \\
\leq & |q| \epsilon \langle \Psi, \mathrm{d}\Gamma_{\mathrm{b}}(\omega) \Psi \rangle + \frac{|q|}{\epsilon} \langle \mathbf{g}, \omega \mathbf{g} \rangle \|\Psi\|^2,
\end{aligned}$$

where $\sum\!\!\!\!\!\!\int := \sum_{r=1,2} \int$. ∎

Lemma 2.3. *For all* $\Psi \in \operatorname{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$ *and* $\epsilon > 0$*, the following inequality holds:*

$$\left\langle \Psi, \mathbf{A}^2 \Psi \right\rangle \leq \left(2 + \epsilon + \frac{1}{\epsilon} \right) \langle \omega^{-1/2} \mathbf{g}, \omega^{-1/2} \mathbf{g} \rangle \langle \Psi, \mathrm{d}\Gamma_{\mathrm{b}}(\omega) \Psi \rangle + \left(1 + \frac{1}{\epsilon} \right) \langle \mathbf{g}, \mathbf{g} \rangle \|\Psi\|^2. \quad (19)$$

Proof.

$$\begin{aligned}
\langle \Psi, \mathbf{A}^2 \Psi \rangle \leq & \sum_{j=1}^{3} \left[(1+\epsilon) \|a(g_j) \Psi\|^2 + \left(1 + \frac{1}{\epsilon} \right) \|a(g_j)^* \Psi\|^2 \right] \\
\leq & \sum_{j=1}^{3} \Bigg[(1+\epsilon) \| |\mathbf{k}|^{-1/2} g_j \|^2 \cdot \| \mathrm{d}\Gamma_{\mathrm{b}}(\omega)^{1/2} \Psi \|^2 \\
& + \left(1 + \frac{1}{\epsilon} \right) \| |\mathbf{k}|^{-1/2} g_j \|^2 \cdot \| \mathrm{d}\Gamma_{\mathrm{b}}(\omega)^{1/2} \Psi \|^2 + \left(1 + \frac{1}{\epsilon} \right) \|g_j\|^2 \cdot \|\Psi\|^2 \Bigg].
\end{aligned}$$

∎

The following Lemma is the most important fact in the proof of Theorem 1.1:

Lemma 2.4. *For all* $\Psi \in \operatorname{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$*, the following inequality holds:*

$$\begin{aligned}
& \|\mathrm{d}\Gamma_{\mathrm{b}}(\omega) \Psi\|^2 - \sum_{j=1}^{3} \|\mathrm{d}\Gamma_{\mathrm{b}}(k_j) \Psi\|^2 - q \left\langle \mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k}) \Psi, \mathbf{A} \Psi \right\rangle - q \left\langle \mathbf{A} \Psi, \mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k}) \Psi \right\rangle \\
& \geq -4q^2 G(\mathbf{g}) \langle \Psi, \mathrm{d}\Gamma_{\mathrm{b}}(\omega) \Psi \rangle - q \langle \Psi, (a(\mathbf{k} \cdot \mathbf{g})^* + a(\mathbf{k} \cdot \mathbf{g})) \Psi \rangle. \quad (20)
\end{aligned}$$

Proof. We define

$$F := \frac{\mathbf{k} \cdot \mathbf{g}(\mathbf{k}', \mu)}{|\mathbf{k}| \cdot |\mathbf{k}| - \mathbf{k} \cdot \mathbf{k}'} \quad (21)$$

For all $\Psi \in \mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$, we have

$$\begin{aligned}
\text{l.h.s of (20)} = &\sum\!\!\!\!\!\!\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \sum\!\!\!\!\!\!\int_{\mathbb{R}^3}{}' \mathrm{d}\mathbf{k}' \left(|\mathbf{k}|\cdot|\mathbf{k}'| - \mathbf{k}\cdot\mathbf{k}'\right) \|(b - 2qF)a\Psi\|^2 \\
&- q\langle \Psi, [a(\mathbf{k}\cdot\mathbf{g}) + a(\mathbf{k}\cdot\mathbf{g})^*]\,\Psi\rangle \\
&- 4q^2 \sum\!\!\!\!\!\!\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \left[\sum\!\!\!\!\!\!\int_{\mathbb{R}^3}{}' \mathrm{d}\mathbf{k}' (|\mathbf{k}|\cdot|\mathbf{k}'| - \mathbf{k}\cdot\mathbf{k}')^{-1} |\mathbf{k}\cdot\mathbf{g}(\mathbf{k}',\mu)|^2\right] \|a\Psi\|^2,
\end{aligned}$$

where $a := a^{(r)}(\mathbf{k})$, $b := a^{(\mu)}(\mathbf{k}')$, and $\sum\!\!\!\!\!\!\int' := \sum_{\mu=1,2}\int$. Since $|\mathbf{k}|\cdot|\mathbf{k}'| - \mathbf{k}\cdot\mathbf{k}' \geq 0$, the inequality (20) holds. ∎

Proof of Theorem 1.1. Using Lemmas 2.2 - 2.4, we get

$$\begin{aligned}
&(\mathrm{d}\Gamma_{\mathrm{b}}(\omega) + E)^2 - \sum_{j=1}^{3} (\mathrm{d}\Gamma_{\mathrm{b}}(k_j) + qA_j)^2 - q\mathbf{S}\cdot[a(i\mathbf{k}\times\mathbf{g}) + a(i\mathbf{k}\times\mathbf{g})^*)] \\
&\geq 2E\,\mathrm{d}\Gamma_{\mathrm{b}}(\omega) + E^2 - 4q^2 G(\mathbf{g})\mathrm{d}\Gamma_{\mathrm{b}}(\omega) - q\,[a(\mathbf{k}\cdot\mathbf{g})^* + a(\mathbf{k}\cdot\mathbf{g})] - |q|\mathrm{d}\Gamma(\omega) \\
&\quad - |q|\langle \mathbf{g}, \omega\mathbf{g}\rangle - 4\langle \omega^{-1/2}\mathbf{g}, \omega^{-1/2}\mathbf{g}\rangle \mathrm{d}\Gamma_{\mathrm{b}}(\omega) - 2\langle \mathbf{g}, \mathbf{g}\rangle,
\end{aligned}$$

in the sense of quadratic form on $\mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$. Since $a(\mathbf{k}\cdot\mathbf{g}) + a(\mathbf{k}\cdot\mathbf{g})^*$ is $\mathrm{d}\Gamma_{\mathrm{b}}(\omega)^{1/2}$-bounded, for a large $E > 0$ we have

$$(\mathrm{d}\Gamma_{\mathrm{b}}(\omega) + E)^2 - \sum_{j=1}^{3} (\mathrm{d}\Gamma_{\mathrm{b}}(k_j) + qA_j)^2 - q\mathbf{S}\cdot[a(i\mathbf{k}\times\mathbf{g}) + a(i\mathbf{k}\times\mathbf{g}))] \geq 0.$$

By Lemma 2.1, for a large $E \geq 0$, we obtain

$$\mathrm{d}\Gamma_{\mathrm{b}}(\omega) - \boldsymbol{\alpha}\cdot\mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k}) - q\boldsymbol{\alpha}\cdot\mathbf{A} \geq -E, \tag{22}$$

in the sense of quadratic form on $\mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$. This inequality implies that $E_0(\mathbf{p})$ is finite.

Next we show that $G(\mathbf{f}) < \infty$ if $g_j = f_j (j = 1, 2, 3)$. By the definitions of $\mathbf{e}^{(r)}(\mathbf{k})$, the vectors $\mathbf{k}/|\mathbf{k}|, \mathbf{e}^{(1)}(\mathbf{k}), \mathbf{e}^{(2)}(\mathbf{k})$ are the orthonormal basis of $\mathbb{C}^3$. Therefore

$$G(\mathbf{f}) = \sup_{\mathbf{k}\in\mathbb{R}^3\setminus\{0\}} \frac{1}{|\mathbf{k}|} \int_{\mathbb{R}^3} \frac{\chi_\Lambda(\mathbf{k}')\mathrm{d}\mathbf{k}'}{|\mathbf{k}||\mathbf{k}'| - \mathbf{k}\cdot\mathbf{k}'} \cdot \frac{1}{|\mathbf{k}'|}\left[|\mathbf{k}|^2 - \frac{(\mathbf{k}\cdot\mathbf{k}')^2}{|\mathbf{k}'|^2}\right] = \int_{\mathbb{R}^3} \frac{\chi_\Lambda(\mathbf{k}')}{|\mathbf{k}'|^2}\mathrm{d}\mathbf{k}' = 4\pi\Lambda.$$

∎

Proof of Theorem 1.2. First we show (10). We set $g_j = f_j\ (j = 1, 2, 3)$. It is easy to see that

$$H(\mathbf{p}) \geq -\sqrt{|\mathbf{p}|^2 + M^2} + \mathrm{d}\Gamma_{\mathrm{b}}(\omega) - \boldsymbol{\alpha}\cdot\mathrm{d}\boldsymbol{\Gamma}_{\mathrm{b}}(\mathbf{k}) - q\boldsymbol{\alpha}\cdot\mathbf{A}. \tag{23}$$

By the definition of $\mathbf{e}^{(r)}(\mathbf{k})$, we have $\mathbf{k}\cdot\mathbf{f}(\mathbf{k},r)=0\,(\mathbf{k}\in\mathbb{R}^3, r=1,2)$. Therefore, using Lemmas 2.2 – 2.4, we have

$$\begin{aligned}
&(\mathrm{d}\Gamma_{\mathrm{b}}(\omega)+C_1\Lambda)^2-(\mathrm{d}\Gamma_{\mathrm{b}}(\mathbf{k})+q\mathbf{A})^2-q\mathbf{S}\cdot[a(i\mathbf{k}\times\mathbf{f})+a(i\mathbf{k}\times\mathbf{f})^*]\\
&\geq\left(2C_1\Lambda-2|q|\epsilon\Lambda-4q^2G(\mathbf{f})-\left(2+\epsilon'+\frac{1}{\epsilon'}\right)\langle\omega^{-1/2}\mathbf{f},\omega^{-1/2}\mathbf{f}\rangle\right)\mathrm{d}\Gamma_{\mathrm{b}}(\omega)\\
&\quad+C_1^2\Lambda^2-\frac{|q|}{2\epsilon\Lambda}\langle\mathbf{f},\omega\mathbf{f}\rangle-\left(1+\frac{1}{\epsilon'}\right)q^2\langle\mathbf{f},\mathbf{f}\rangle,\quad \epsilon,\epsilon'>0.
\end{aligned}\tag{24}$$

It is easy to see that $\langle\omega^{-1/2}\mathbf{f},\omega^{-1/2}\mathbf{f}\rangle=8\pi\Lambda$, $\langle\mathbf{f},\omega\mathbf{f}\rangle=8\pi\Lambda^3/3$, $\langle\mathbf{f},\mathbf{f}\rangle=4\pi\Lambda^2$. Hence, by the definition of C_1, the left hand side of (24) is positive for suitable $\epsilon,\epsilon'>0$. Thus, using Lemma 2.1 (and (16)), we have

$$H(\mathbf{p})\geq-\sqrt{|\mathbf{p}|^2+M^2}-C_1\Lambda.\tag{25}$$

For normalized vectors $u\in\mathbb{C}^4$, $\psi\in\mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))$ we define

$$\begin{aligned}
&a_j:=\langle u,\alpha_j u\rangle,\quad h(\mathbf{a}):=\mathrm{d}\Gamma_{\mathrm{b}}(\omega-\mathbf{a}\cdot\mathbf{k})-q\mathbf{a}\cdot\mathbf{A},\\
&\Psi:=u\otimes\psi\in\mathcal{F}.
\end{aligned}$$

Note that $\omega-\mathbf{a}\cdot\mathbf{k}\geq0$ and $\omega-\mathbf{a}\cdot\mathbf{k}$ is injective as a multiplication operator. We have

$$\langle\Psi,H(\mathbf{p})\Psi\rangle=\mathbf{a}\cdot\mathbf{p}+M\langle u,\beta u\rangle+\langle\psi,h(\mathbf{a})\psi\rangle\tag{26}$$

Since $h(\mathbf{a})$ is a van Hove type Hamiltonian, we have

$$\begin{aligned}
\inf\sigma(h(\mathbf{a}))&=-q^2\|(|\mathbf{k}|-\mathbf{a}\cdot\mathbf{k})^{-1/2}\mathbf{a}\cdot\mathbf{f}\|^2\\
&=-4\pi\Lambda q^2+q^2(1-|\mathbf{a}|^2)\int_{\mathbb{R}^3}\mathrm{d}\mathbf{k}\frac{\chi_\Lambda(\mathbf{k})}{|\mathbf{k}|^2-(\mathbf{a}\cdot\mathbf{k})^2}\\
&=-4\pi\Lambda q^2+2\pi\Lambda q^2(1-|\mathbf{a}|^2)\frac{1}{|\mathbf{a}|}\log\left(\frac{1-|\mathbf{a}|}{1+|\mathbf{a}|}\right),
\end{aligned}$$

where σ means the spectrum(e.g. [3]). Thus we have

$$E_0(\mathbf{p})\leq\inf_{u\in\mathbb{C}^4;\,\|u\|=1}\ \inf_{\psi\in\mathrm{Dom}(\mathrm{d}\Gamma_{\mathrm{b}}(\omega))}\langle\Psi,H(\mathbf{p})\Psi\rangle=\inf_{u\in\mathbb{C}^4;\,\|u\|=1}\mathcal{E}(u)$$

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