

Spectral properties of the Laplacian on bond-percolation graphs

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Abstract Bond-percolation graphs are random subgraphs of the d -dimensional integer lattice generated by a standard bond-percolation process. The associated graph Laplacians, subject to Dirichlet or Neumann conditions at cluster boundaries, represent bounded, self-adjoint, ergodic random operators with off-diagonal disorder. They possess almost surely the non-random spectrum $[0, 4d]$ and a self-averaging integrated density of states. The integrated density of states is shown to exhibit Lifshits tails at both spectral edges in the non-percolating phase. While the characteristic exponent of the Lifshits tail for the Dirichlet (Neumann) Laplacian at the lower (upper) spectral edge equals $d/2$, and thus depends on the spatial dimension, this is not the case at the upper (lower) spectral edge, where the exponent equals $1/2$.

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Introduction

Spectral graph theory studies linear operators which are associated with graphs. The goal is to see how properties of the graph are reflected in properties of the operators and vice versa. This has attracted vivid interest in the last two decades [19, 13, 10, 12]. The kind of graphs we shall be concerned with in this paper are bond-percolation graphs [14], a special type of random subgraphs of the d -dimensional integer lattice. They are of popular use in Physics for modelling vari-

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ous types of random environments [26, 7]. On these graphs we consider Laplacians with different kinds of boundary conditions at cluster borders and study some of their spectral properties. Apart from the non-randomness of the spectrum, and of the spectral components in the Lebesgue decomposition, we show the existence and self-averaging of the integrated density of states. The main result establishes Lifshits tails of the integrated density of states at the lower and upper spectral edge in the non-percolating phase. Depending on the boundary condition and on the spectral edge, the Lifshits tail discriminates between stretched (i.e. linear) and condensed (i.e. cube- or ball-like) clusters which contribute the dominating eigenvalues. The crucial technical estimates in our proof are Cheeger [12] and Faber-Krahn [11] isoperimetric inequalities on graphs. Our analysis here is facilitated by the fact that in the non-percolating phase almost all graphs consist of infinitely many finite clusters. Yet, the non-percolating phase gives rise to interesting phenomena, because it supplies clusters of arbitrarily large size. In the percolating phase one has to cope with the infinite cluster, too, which requires a more intricate understanding. This case will be studied in [21].

Spectral properties of Laplacians on bond-percolation (or related) graphs have been studied in the Physics literature, see the general accounts [26, 7] or the recent examples [3, 6, 20] for applications to soft matter. The Lifshits tails, whose existence we prove here, were sought after in the numerical simulations [5] for the Neumann Laplacian on two- and three-dimensional bond-percolation graphs. The tails could not be observed there due to finite-size corrections and the considerable numerical effort needed to access such rare events. For the different case of Erdős–Rényi random graphs, however, the existence of Lifshits tails for the Neumann Laplacian was known on the basis of analytical, non-rigorous arguments [4, 5], which inspired our proof here. Moments of the eigenvalue density for this model were rigorously analysed in [15]. Other models in the physics literature deal with the adjacency operator on bond-percolation graphs [16, 24]. Quite often, this goes under the name quantum percolation. Yet, from a rigorous mathematical point of view, all of the above models with off-diagonal disorder have remained widely unexplored.

In contrast, Laplacians on *site*-percolation graphs of the d -dimensional integer lattice belong to the class of models with diagonal disorder. Therefore they are closer to the range of applicability of the highly developed theory of random Schrödinger operators [17, 8, 22, 27, 18]. This is partly of help for analysing their spectral properties with mathematical rigour. For finite-range hopping operators on site-percolation graphs, the non-randomness of the spectrum and existence of the integrated density of states was shown in [28, 29]. Particular emphasis is laid on the behaviour of the spectrum related to finitely supported eigenfunctions, see also [9], where the issue was first taken up from a mathematical point of view. Furthermore, a Wegner estimate is established in [29] for an Anderson model on site-percolation graphs.

This paper is organised as follows. In Section 1 we give the precise definitions of the objects we are dealing with and state our results. Theorem 1.14 in Subsection 1.3 contains the central result on Lifshits tails of the integrated density of states. All proofs are deferred to Section 2.

1 Definitions and Results

1.1 Bond-percolation graphs

A thorough and comprehensive account of (bond) percolation can be found in Grimmett's textbook [14], which serves as a standard reference on the subject. For $d \in \mathbb{N}$, a natural number, we denote by \mathbb{L}^d the (simple hypercubic) lattice in d dimensions. Being a graph, the lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ has a *vertex set*, which consists of the d -dimensional integer numbers \mathbb{Z}^d , and an *edge set* \mathbb{E}^d given by all unordered pairs $[x, y]$ of nearest-neighbour vertices $x, y \in \mathbb{Z}^d$, that is, those vertices which have Euclidean distance $|x - y| := (\sum_{v=1}^d |x_v - y_v|^2)^{1/2} = 1$. Here, elements of \mathbb{Z}^d are canonically represented as d -tuples $x = (x_1, \dots, x_d)$ with entries from \mathbb{Z} .

Given any subset of vertices $\mathcal{V} \subseteq \mathbb{Z}^d$ and a subset of edges $\mathcal{E} \subseteq \{[x, y] \in \mathbb{E}^d : x, y \in \mathcal{V}\}$ between them, we call the graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ a *subgraph* of \mathbb{L}^d . The *vertex degree*

$$d_{\mathcal{G}}(x) := |\{y \in \mathbb{Z}^d : [x, y] \in \mathcal{E}\}| \quad (1.1)$$

of $x \in \mathbb{Z}^d$ counts the number of edges in \mathcal{G} that share the vertex x as an endpoint. Here, $|\Lambda|$ denotes the cardinality of a subset $\Lambda \subset \mathbb{Z}^d$, and we use the convention $|\emptyset| = 0$ for the empty set. A graph is called *finite*, if $|\mathcal{V}| < \infty$.

A given graph \mathcal{G} consists of finitely or infinitely many *clusters* \mathcal{C}_j , $j = 1, 2, \dots, J \leq \infty$, which are the *maximally connected subgraphs* of \mathcal{G} . More precisely, $\mathcal{C}_j := (\mathcal{V}_j, \mathcal{E}_j)$ is a connected subgraph of \mathcal{G} , if $\mathcal{V}_j \subseteq \mathcal{V}$ with $|\mathcal{V}_j| \geq 2$, $\mathcal{E}_j \subseteq \mathcal{E}$ and if for every $x, y \in \mathcal{V}_j$ with $x \neq y$ there exists $K \in \mathbb{N}$ and $x_k \in \mathcal{V}_j$, $k = 0, 1, 2, \dots, K$, such that $x_0 := x$, $x_K := y$ and $[x_{k-1}, x_k] \in \mathcal{E}_j$ for all $k = 1, 2, \dots, K$. A connected subgraph \mathcal{C}_j of \mathcal{G} is maximal, and hence a cluster, if for every connected subgraph $\mathcal{C}' := (\mathcal{V}', \mathcal{E}')$ of \mathcal{G} obeying $\mathcal{V}' \supseteq \mathcal{V}_j$ and $\mathcal{E}' \supseteq \mathcal{E}_j$ one has $\mathcal{C}' = \mathcal{C}_j$. Apparently, the decomposition of \mathcal{G} into its clusters is unique – apart from enumeration.

Next, we consider the probability space $\Omega = \{0, 1\}^{\mathbb{E}^d}$, which is endowed with the usual product sigma-algebra, generated by finite cylinder sets, and equipped with a product probability measure \mathbb{P} . Elementary events in Ω are sequences of the form $\omega \equiv (\omega_{[x, y]})_{[x, y] \in \mathbb{E}^d}$, and we assume their entries to be independently and identically distributed according to a Bernoulli law

$$\mathbb{P}(\omega_{[x, y]} = 1) = p \quad (1.2)$$

with parameter $p \in]0, 1[$, the *bond probability*. To a given $\omega \in \Omega$ we associate the edge set

$$\mathcal{E}^{(\omega)} := \{[x, y] \in \mathbb{E}^d : \omega_{[x, y]} = 1\}. \quad (1.3)$$

Definition 1.1. *The mapping $\Omega \ni \omega \mapsto \mathcal{G}^{(\omega)} := (\mathbb{Z}^d, \mathcal{E}^{(\omega)})$ with values in the set of subgraphs of \mathbb{L}^d is called bond-percolation graph in \mathbb{Z}^d .*

The most basic properties of bond-percolation graphs are recalled in

Proposition 1.2. *For $d \geq 2$ there exists $p_c \in]0, 1[$, depending on d , such that*

(i) for every $p \in]0, p_c[$, the non-percolating phase, one has

$$\mathbb{P} \left\{ \omega \in \Omega : \begin{array}{l} \mathcal{G}^{(\omega)} \text{ consists of } \infty\text{-many} \\ \text{clusters, which are all finite} \end{array} \right\} = 1, \quad (1.4)$$

(ii) for every $p \in]p_c, 1[$, the percolating phase, one has

$$\mathbb{P} \left\{ \omega \in \Omega : \begin{array}{l} \mathcal{G}^{(\omega)} \text{ consists of exactly one infinite} \\ \text{cluster and } \infty\text{-many finite clusters} \end{array} \right\} = 1. \quad (1.5)$$

Remarks 1.3. (i) The proposition collects results from Thms. 1.10, 1.11, 4.2 and 8.1 in [14], which were mainly obtained by Hammersley in the late fifties. The uniqueness of the infinite cluster, however, was only proven thirty years later by Aizenman, Kesten and Newman, see [14].

(ii) In the one-dimensional situation, $d = 1$, one has $p_c = 1$ and part (i) of the proposition remains true.

1.2 Graph Laplacians

The subsequent definition introduces Laplacian-type operators associated with an arbitrary subgraph of the integer lattice. The particularisation to operators on bond-percolation graphs follows at the end of this subsection.

For a given subset $\Lambda \subseteq \mathbb{Z}^d$ let $\ell^2(\Lambda)$ denote the Hilbert space of complex-valued, square-summable sequences that are indexed by Λ .

Definition 1.4. *Given any subgraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of \mathbb{L}^d , we introduce the following bounded and self-adjoint linear operators on $\ell^2(\mathcal{V})$.*

(i) *The degree operator $D(\mathcal{G})$ is defined as the multiplication operator with the vertex-degree function $d_{\mathcal{G}} : \mathcal{V} \rightarrow \mathbb{N} \cup \{0\}$, $x \mapsto d_{\mathcal{G}}(x)$, that is,*

$$[D(\mathcal{G})\varphi](x) := d_{\mathcal{G}}(x) \varphi(x) \quad (1.6)$$

for all $\varphi \in \ell^2(\mathcal{V})$ and all $x \in \mathcal{V}$.

(ii) *The adjacency operator $A(\mathcal{G})$ is defined through its action*

$$[A(\mathcal{G})\varphi](x) := \sum_{y \in \mathcal{V} : [x,y] \in \mathcal{E}} \varphi(y) \quad (1.7)$$

for all $\varphi \in \ell^2(\mathcal{V})$ and all $x \in \mathcal{V}$. Here, we use the convention $\sum_{y \in \emptyset} \varphi(y) = 0$ for the empty sum.

(iii) *The Neumann Laplacian is defined by*

$$\Delta_{\mathbb{N}}(\mathcal{G}) := D(\mathcal{G}) - A(\mathcal{G}). \quad (1.8)$$

(iv) *The Pseudo-Dirichlet Laplacian is defined by*

$$\Delta_{\mathbb{D}}(\mathcal{G}) := (2d\mathbb{1} - D(\mathcal{G})) + \Delta_{\mathbb{N}}(\mathcal{G}) = 2d\mathbb{1} - A(\mathcal{G}), \quad (1.9)$$

where $\mathbb{1} \equiv \mathbb{1}_{\mathcal{V}}$ stands for the identity operator on $\ell^2(\mathcal{V})$.

(v) *The Dirichlet Laplacian is defined by*

$$\Delta_D(\mathcal{G}) := 2(2d\mathbb{1} - D(\mathcal{G})) + \Delta_N(\mathcal{G}) = 2d\mathbb{1} + (2d\mathbb{1} - D(\mathcal{G})) - A(\mathcal{G}). \quad (1.10)$$

Remarks 1.5. (i) The asserted boundedness and self-adjointness of the Laplacians $\Delta_X(\mathcal{G})$, $X \in \{\mathbb{N}, \tilde{\mathbb{D}}, \mathbb{D}\}$, follow from the corresponding properties of $D(\mathcal{G})$ and $A(\mathcal{G})$. Indeed, since $0 \leq d_{\mathcal{G}}(x) \leq 2d$ for all $x \in \mathcal{V}$, it is clear that $D(\mathcal{G})$ is self-adjoint and obeys $0 \leq D(\mathcal{G}) \leq 2d\mathbb{1}$ in the sense of quadratic forms. The operator $A(\mathcal{G})$ is symmetric, because

$$\langle \psi, A(\mathcal{G})\varphi \rangle = 2 \sum_{[x,y] \in \mathcal{E}} \psi^*(x) \varphi(y) \quad (1.11)$$

for all $\psi, \varphi \in \ell^2(\mathcal{V})$, where $\langle \psi, \varphi \rangle := \sum_{x \in \mathcal{V}} \psi^*(x) \varphi(x)$ denotes the standard Hilbert-space scalar product on $\ell^2(\mathcal{V})$. The factor 2 in (1.11) reflects that the sum is over *unordered* pairs. Moreover, applying the Cauchy–Schwarz inequality to (1.11), yields the upper bound $2d$ for the usual operator norm of $A(\mathcal{G})$, and self-adjointness follows from symmetry and boundedness.

(ii) The Neumann Laplacian $\Delta_N(\mathcal{G})$ is called *graph Laplacian* or *combinatorial Laplacian* in spectral graph theory, where it is commonly studied in various forms [19, 13, 10, 12].

(iii) The quadratic form

$$\langle \varphi, \Delta_N(\mathcal{G})\varphi \rangle = \sum_{[x,y] \in \mathcal{E}} |\varphi(x) - \varphi(y)|^2, \quad \varphi \in \ell^2(\mathcal{V}), \quad (1.12)$$

for the Neumann Laplacian reveals that

$$\Delta_N(\mathcal{G}) \geq 0. \quad (1.13)$$

Thus, a necessary and sufficient condition for $\varphi \in \ell^2(\mathcal{V})$ to belong to the zero-eigenspace of $\Delta_N(\mathcal{G})$ is that φ stays constant within each of the finite clusters of \mathcal{G} (separately). Consequently, each finite cluster of \mathcal{G} and each isolated vertex, i.e. each point in $\mathcal{I} := \{x \in \mathcal{V} : d_{\mathcal{G}}(x) = 0\}$, contributes exactly one zero eigenvalue to $\Delta_N(\mathcal{G})$. In contrast, zero is not an eigenvalue of $\Delta_{\tilde{\mathbb{D}}}(\mathcal{G})$ and $\Delta_D(\mathcal{G})$.

(iv) Let $X \in \{\mathbb{N}, \tilde{\mathbb{D}}, \mathbb{D}\}$ and let $\mathcal{C}_j := (\mathcal{V}_j, \mathcal{E}_j)$, $j = 1, \dots, J \leq \infty$ denote the clusters a graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ is composed of. Then $\Delta_X(\mathcal{G})$ is block-diagonal with respect to the clusters,

$$\Delta_X(\mathcal{G}) = \bigoplus_{j=1}^J \Delta_X(\mathcal{C}_j) \oplus \gamma_X \mathbb{1}_{\mathcal{I}} \quad (1.14)$$

on $\ell^2(\mathcal{V})$, where $\gamma_{\mathbb{N}} := 0$, $\gamma_{\tilde{\mathbb{D}}} := 2d$, respectively $\gamma_{\mathbb{D}} := 4d$.

(v) The Neumann and the Dirichlet Laplacian are related to each other. To see this we define a unitary involution $U = U^{-1} = U^*$ on $\ell^2(\mathcal{V})$ by setting

$$(U\varphi)(x) := (-1)^{\sum_{v=1}^d |x_v|} \varphi(x) \quad (1.15)$$

for all $x \in \mathcal{V}$ and all $\varphi \in \ell^2(\mathcal{V})$. This involution commutes with $D(\mathcal{G})$ and anti-commutes with $A(\mathcal{G})$ so that $D(\mathcal{G}) = UD(\mathcal{G})U$ and

$$A(\mathcal{G}) = -UA(\mathcal{G})U. \quad (1.16)$$

Hence, we infer the relation

$$\Delta_D(\mathcal{G}) = 4d\mathbb{1} - U\Delta_N(\mathcal{G})U. \quad (1.17)$$

Combining (1.17) with (1.13), (1.9) and (1.10), we arrive at the chain of inequalities

$$0 \leq \Delta_N(\mathcal{G}) \leq \Delta_{\tilde{D}}(\mathcal{G}) \leq \Delta_D(\mathcal{G}) \leq 4d\mathbb{1}. \quad (1.18)$$

(vi) Our terminology of the Laplacians is motivated by Simon [25]. Divide a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ into two subgraphs $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k)$, $k = 1, 2$, such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$. Then one gets super-, respectively subadditive behaviour

$$\begin{aligned} \Delta_N(\mathcal{G}) &\geq \Delta_N(\mathcal{G}_1) \oplus \Delta_N(\mathcal{G}_2), \\ \Delta_D(\mathcal{G}) &\leq \Delta_D(\mathcal{G}_1) \oplus \Delta_D(\mathcal{G}_2) \end{aligned} \quad (1.19)$$

as a consequence of (1.12) and (1.17). Thus, introducing a separating boundary surface lowers Neumann eigenvalues and raises Dirichlet eigenvalues – in analogy to the well-known behaviour of Laplacian eigenvalues of regions in continuous space, see e.g. Prop. 4 in Chap. XIII.15 of [23]. In contrast, the eigenvalues of the Pseudo-Dirichlet Laplacian $\Delta_{\tilde{D}}(\mathcal{G})$ behave indifferently with respect to this procedure. Though, $\Delta_{\tilde{D}}(\mathcal{G})$ is commonly termed a Dirichlet Laplacian in the literature.

Next, we associate Laplacians to the bond-percolation graphs of Definition 1.1.

Definition 1.6. *The mapping $\Delta_X : \Omega \ni \omega \mapsto \Delta_X^{(\omega)} := \Delta_X(\mathcal{G}^{(\omega)})$ with values in the bounded, self-adjoint operators on $\ell^2(\mathbb{Z}^d)$ is called Neumann or Pseudo-Dirichlet or Dirichlet Laplacian on bond-percolation graphs in \mathbb{Z}^d , depending on whether X stands for N or \tilde{D} or D.*

1.3 Results

To begin with we summarise the most basic spectral properties of the Laplacian on bond-percolation graphs in

Lemma 1.7. *Fix $X \in \{N, \tilde{D}, D\}$ and $p \in]0, 1[$. Then*

- (i) *the random operator Δ_X is ergodic with respect to \mathbb{Z}^d -translations.*
- (ii) *its spectrum is \mathbb{P} -almost surely non random, more precisely, it is given by $\text{spec}(\Delta_X) = [0, 4d]$ \mathbb{P} -almost surely.*
- (iii) *the components in the Lebesgue decomposition of the spectrum are also \mathbb{P} -almost surely non random. For every $\varkappa \in \{\text{pp}, \text{sc}, \text{ac}\}$ there exists a closed subset $\Sigma_X^{(\varkappa)} \subset \mathbb{R}$ such that $\text{spec}_{\varkappa}(\Delta_X) = \Sigma_X^{(\varkappa)}$ \mathbb{P} -almost surely.*

(iv) *in the non-percolating phase, $p \in]0, p_c[$, the spectrum of Δ_X is \mathbb{P} -almost surely only a dense pure-point spectrum with infinitely degenerate eigenvalues. The dense set of eigenvalues is also non random \mathbb{P} -almost surely.*

Remarks 1.8. (i) The lemma is proven in Section 2.

(ii) Part (ii) implies that the discrete spectrum of Δ_X is \mathbb{P} -almost surely empty.

(iii) As compared to the non-percolating phase considered in part (iv), there are additional spectral contributions from the percolating cluster if $p \in]p_c, 1[$. Among others, the percolating cluster contributes also infinitely degenerate, \mathbb{P} -almost surely non-random eigenvalues corresponding to compactly supported eigenfunctions. This can be established with the same mirror techniques as it was done for related models on site-percolation graphs [9, 28, 29]. Non-rigorous arguments [16, 24, 3] suggest the existence of continuous spectrum if p lies above the “quantum-percolation threshold” $p_q > p_c$.

We proceed with the existence and self-averaging of the integrated density of states of Δ_X . To this end let $\delta_x \in \ell^2(\mathbb{Z}^d)$ be the sequence which is concentrated at the point $x \in \mathbb{Z}^d$, i.e. $\delta_x(x) := 1$ and $\delta_x(y) := 0$ for all $y \neq x \in \mathbb{Z}^d$. Moreover, Θ stands for the Heaviside unit-step function, which we choose to be right continuous, viz. $\Theta(E) := 0$ for all real $E < 0$ and $\Theta(E) := 1$ for all real $E \geq 0$.

Definition 1.9. *For every $p \in]0, 1[$ and every $X \in \{N, \tilde{D}, D\}$ we call the function*

$$N_X : \mathbb{R} \ni E \mapsto N_X(E) := \int_{\Omega} \mathbb{P}(d\omega) \langle \delta_0, \Theta(E - \Delta_X^{(\omega)}) \delta_0 \rangle \quad (1.20)$$

with values in the interval $[0, 1]$ the integrated density of states of Δ_X .

Remarks 1.10. (i) Thanks to the ergodicity of Δ_X with respect to \mathbb{Z}^d -translations, one can replace δ_0 by δ_x with some arbitrary $x \in \mathbb{Z}^d$ in Definition 1.9 without changing the result.

(ii) The integrated density of states N_X is the right-continuous distribution function of a probability measure on \mathbb{R} . The set of its growth points coincides with the \mathbb{P} -almost-sure spectrum $[0, 4d]$ of Δ_X .

(iii) For $p < p_c$ the growth points of N_X form a dense countable set, where N_X is discontinuous. These jumps in N_X are due to the infinitely degenerate eigenvalues of Δ_X , which arise solely from the finite clusters, cf. Lemma 1.7(iv). For $p > p_c$ there are also contributions to the jumps that arise from the percolating cluster. In addition, the set of growth points of N_X should not be restricted to discontinuities for $p > p_c$, cf. Remark 1.8(iii).

(iv) Eqs. (1.16) and (1.17) imply the symmetries

$$\begin{aligned} N_{\tilde{D}}(E) &= 1 - \lim_{\varepsilon \uparrow 4d-E} N_{\tilde{D}}(\varepsilon), \\ N_{D(N)}(E) &= 1 - \lim_{\varepsilon \uparrow 4d-E} N_{N(D)}(\varepsilon) \end{aligned} \quad (1.21)$$

of the integrated densities of states for all $E \in \mathbb{R}$.

Definition 1.9 of the integrated density of states coincides with the usual one in terms of a macroscopic limit. To make this statement precise, we have to introduce restrictions of Δ_X to finite volume.

Definition 1.11. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a subgraph of \mathbb{L}^d and consider a subset $\Lambda \subseteq \mathbb{Z}^d$.

(i) The graph $\mathcal{G}_\Lambda := (\mathcal{V}_\Lambda, \mathcal{E}_\Lambda)$ with $\mathcal{V}_\Lambda := \mathcal{V} \cap \Lambda$ and $\mathcal{E}_\Lambda := \{[x, y] \in \mathcal{E} : x, y \in \mathcal{V}_\Lambda\}$ is called the restriction of \mathcal{G} to Λ . In particular, $\mathcal{G}_\Lambda^{(\omega)} = (\Lambda, \mathcal{E}_\Lambda^{(\omega)})$ is the restriction to Λ of a realisation $\mathcal{G}^{(\omega)} = (\mathbb{Z}^d, \mathcal{E}^{(\omega)})$ of the bond-percolation graph.

(ii) For $X \in \{N, \tilde{D}, D\}$ we define the restriction $\Delta_{X,\Lambda}$ of the Laplacian Δ_X to $\ell^2(\Lambda)$ as the random operator with realisations $\Delta_{X,\Lambda}^{(\omega)} := \Delta_X(\mathcal{G}_\Lambda^{(\omega)})$ for all $\omega \in \Omega$.

Lemma 1.12. Given $p \in]0, 1[$ and $X \in \{N, \tilde{D}, D\}$, there exists a set $\Omega' \subset \Omega$ of full probability, $\mathbb{P}(\Omega') = 1$, such that

$$N_X(E) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \left[\frac{1}{|\Lambda|} \text{trace}_{\ell^2(\Lambda)} \Theta(E - \Delta_{X,\Lambda}^{(\omega)}) \right] \quad (1.22)$$

holds for all $\omega \in \Omega'$ and all $E \in \mathbb{R}$, except for the (at most countably many) discontinuity points of N_X .

Remarks 1.13. (i) As to the limit $\Lambda \uparrow \mathbb{Z}^d$, we think of a sequence of cubes centred at the origin whose edge lengths tend to infinity. But there exist more general sequences of expanding regions in \mathbb{Z}^d for which the lemma remains true.

(ii) The proof of Lemma 1.12 for $X \in \{N, D\}$ follows from the Ackoglu–Krengel superergodic theorem on account of (1.19), see Thm. VI.1.7 in [8] and the discussion after Eq. (VI.16) there. For $X = \tilde{D}$, the proof follows from Lemma 4.5 in [22], which establishes weak convergence of the associated density-of-states probability measures, and Thm. 30.13 in [1].

(iii) The arguments in Sec. 6 of [29] show that the convergence in (1.22) holds whenever E is an algebraic number, that is the root of a polynomial with integer coefficients. Hence, the convergence (1.22) may even hold at discontinuity points of N_X . In particular, for $p < p_c$ it holds for all $E \in \mathbb{R}$.

The central result of this paper is

Theorem 1.14. Let $d \in \mathbb{N}$ and assume $p \in]0, p_c[$. Then the integrated density of states N_X of the Laplacian Δ_X on bond-percolation graphs in \mathbb{Z}^d exhibits Lifshits tails at both the lower spectral edge

$$\lim_{E \downarrow 0} \frac{\ln |\ln [N_X(E) - N_X(0)]|}{\ln E} = \begin{cases} -1/2 & \text{for } X = N, \\ -d/2 & \text{for } X = \tilde{D}, D \end{cases} \quad (1.23)$$

and at the upper spectral edge

$$\lim_{E \uparrow 4d} \frac{\ln |\ln [N_X^-(4d) - N_X(E)]|}{\ln(4d - E)} = \begin{cases} -1/2 & \text{for } X = D, \\ -d/2 & \text{for } X = N, \tilde{D}, \end{cases} \quad (1.24)$$

where $N_X^-(4d) := \lim_{E \uparrow 4d} N_X(E)$.

Remarks 1.15. (i) The Lifshits tails at the upper spectral edge are related to the ones at the lower spectral edge due to the symmetries (1.21).

(ii) Remark 1.5(iii), the symmetries (1.21), Lemma 1.12 and Remark 1.13(iii) imply the values

$$\begin{aligned} N_{\mathbb{D}}^{\sim}(0) = N_{\mathbb{D}}(0) = 0, \quad N_{\mathbb{N}}^-(4d) = N_{\mathbb{D}}^-(4d) = 1, \\ 1 - N_{\mathbb{D}}^-(4d) = N_{\mathbb{N}}(0) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{\text{trace}_{\ell^2(\Lambda)} \Theta(-\Delta_{\mathbb{N},\Lambda}^{(\omega)})}{|\Lambda|} = \kappa(p) + (1-p)^{2d} \end{aligned} \quad (1.25)$$

for the constants in Theorem 1.14. Here, $\kappa(p)$ is the mean number density of clusters, see e.g. Chap. 4 in [14], and $(1-p)^{2d}$ the mean number density of isolated vertices. Thanks to the right-continuity of the Theta-function, the operator $\Theta(-\Delta_{\mathbb{N},\Lambda}^{(\omega)})$ is nothing but the projector onto the null space of $\Delta_{\mathbb{N},\Lambda}^{(\omega)}$.

(iii) The Lifshits tails for $N_{\mathbb{N}}$ at the lower spectral edge – and hence the one for $N_{\mathbb{D}}$ at the upper spectral edge – is determined by the linear clusters of bond-percolation graphs. This explains why the associated Lifshits exponent $-1/2$ is not affected by the spatial dimension d . Technically, this relies on a Cheeger inequality [12] for the second-lowest Neumann eigenvalue of a connected graph.

(iv) If $d \geq 2$, then all other Lifshits tails of the theorem are determined by the most condensed clusters of bond-percolation graphs, like cubic clusters (see Remark 2.5 below for their definition), as they maximise the mean vertex degree among all clusters with a given number of vertices. In the proof of the theorem this will follow from a Faber-Krahn inequality [11] for the lowest (Pseudo-) Dirichlet eigenvalue of a connected graph. In contrast, for $d = 1$ there are no other clusters than linear ones, and the Lifshits exponent cannot discriminate between different boundary conditions.

2 Proofs

In this section we shall prove Lemma 1.7 and Theorem 1.14.

Proof (of Lemma 1.7). We follow the standard arguments laid down in [17, 8, 22]. The function $\Omega \ni \omega \mapsto \Delta_{\mathbb{X}}^{(\omega)}$, which takes on values in the set of bounded self-adjoint operators on $\ell^2(\mathbb{Z}^d)$, is measurable, and the probability measure \mathbb{P} is ergodic with respect to the group of translations $(\tau_z)_{z \in \mathbb{Z}^d}$ on Ω , which act as $\tau_z \omega := (\omega_{[x+z, y+z]})_{[x, y] \in \mathbb{Z}^d}$. Moreover, for every $z \in \mathbb{Z}^d$ let T_z be the unitary translation operator on $\ell^2(\mathbb{Z}^d)$, that is, $T_z \varphi(x) := \varphi(x - z)$ for all $\varphi \in \ell^2(\mathbb{Z}^d)$ and all $x \in \mathbb{Z}^d$. The operator identity $\Delta_{\mathbb{X}}^{(\tau_z \omega)} = T_z^{-1} \Delta_{\mathbb{X}}^{(\omega)} T_z$ holds for all $z \in \mathbb{Z}^d$ and all $\omega \in \Omega$ and renders $\Delta_{\mathbb{X}}$ an ergodic random operator [17], as claimed in part (i) of the lemma. Part (iii) is now a consequence of the general theory of ergodic random operators [17, 8, 22].

As to part (ii) it suffices to show the inclusion

$$\text{spec}(\Delta_{\mathbb{X}}^{(\omega)}) \supseteq [0, 4d] \quad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega \quad (2.1)$$

for all $X \in \{\mathbb{N}, \tilde{\mathbb{D}}, \mathbb{D}\}$, because the opposite inclusion is already supplied by (1.18). To verify (2.1), we define the event

$$\tilde{\Omega} := \left\{ \omega \in \Omega : \text{for every } l \in \mathbb{N} \text{ there exists a cube } \Lambda_l^{(\omega)} \subset \mathbb{Z}^d \right. \\ \left. \text{with } l^d \text{ points such that } \mathcal{G}_{\Lambda_l^{(\omega)}}^{(\omega)} = \mathbb{L}_{\Lambda_l^{(\omega)}}^d \right\}. \quad (2.2)$$

Here, we say that a subset of \mathbb{Z}^d is a cube with l^d points (or, equivalently, with edges of length $l - 1 \in \mathbb{N}$), if this subset is some translate of the d -fold Cartesian product $\{1, \dots, l\}^d$. Colloquially speaking, the condition in (2.2) requires all bonds inside of $\Lambda_l^{(\omega)}$ to be present.

Now, fix an arbitrary $E \in [0, 4d] = \text{spec}(\Delta)$ in the spectrum of the ordinary lattice Laplacian $\Delta = \Delta_X(\mathbb{L}^d)$. Then, there exists a Weyl sequence $(\psi_{E,n})_{n \in \mathbb{N}} \subset \ell^2(\mathbb{Z}^d)$ for Δ , that is, $\|\psi_{E,n}\| := \langle \psi_{E,n}, \psi_{E,n} \rangle^{1/2} = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n \rightarrow \infty} \|(\Delta - E\mathbb{1})\psi_{E,n}\| = 0. \quad (2.3)$$

We may also assume without loss of generality that the support $\text{supp } \psi_{E,n}$ is compact for all $n \in \mathbb{N}$, since Δ is bounded. Furthermore, if $(\psi_{E,n})_{n \in \mathbb{N}}$ is such a Weyl sequence, then so is $(T_{z_n} \psi_{E,n})_{n \in \mathbb{N}}$ with arbitrary $z_n \in \mathbb{Z}^d$. Thus, given any $\omega \in \tilde{\Omega}$ there exists a Weyl sequence $(\psi_{E,n}^{(\omega)})_{n \in \mathbb{N}}$ for Δ with the property that, loosely speaking, all the supports are contained well inside the cubes of (2.2). More precisely, we mean that given every $\omega \in \tilde{\Omega}$ and every $n \in \mathbb{N}$ there must exist an integer $l_n^{(\omega)} > 3$ and a cube $\Lambda_{l_n^{(\omega)}}^{(\omega)}$ from (2.2) such that $\min\{|x - y| : x \in \text{supp } \psi_{E,n}^{(\omega)}, y \in \mathbb{Z}^d \setminus \Lambda_{l_n^{(\omega)}}^{(\omega)}\} > 1$. This yields

$$\|(\Delta_X^{(\omega)} - E\mathbb{1})\psi_{E,n}^{(\omega)}\| = \|(\Delta - E\mathbb{1})\psi_{E,n}^{(\omega)}\| \quad (2.4)$$

for all $n \in \mathbb{N}$ and all $\omega \in \tilde{\Omega}$. Hence, $(\psi_{E,n}^{(\omega)})_{n \in \mathbb{N}}$ is also a Weyl sequence for $\Delta_X^{(\omega)}$, and we have shown the inclusion in (2.1) for all $\omega \in \tilde{\Omega}$. But $\mathbb{P}(\tilde{\Omega}) = 1$, as we shall argue now.

For every given integer $l \geq 2$ let $(\Lambda_{l,\mu})_{\mu \in \mathbb{N}} \subset \mathbb{Z}^d$ be a sequence of cubes in \mathbb{Z}^d with l^d points such that $\Lambda_{l,\mu_1} \cap \Lambda_{l,\mu_2} = \emptyset$, whenever $\mu_1 \neq \mu_2$. Then, the events $\Omega_{l,\mu} := \{\omega \in \Omega : \mathcal{G}_{\Lambda_{l,\mu}}^{(\omega)} = \mathbb{L}_{\Lambda_{l,\mu}}^d\}$ are pairwise statistically independent, and $\mathbb{P}(\Omega_{l,\mu}) > 0$ does not depend on $\mu \in \mathbb{N}$. So the Borel–Cantelli lemma implies $\mathbb{P}(\Omega_l) = 1$ for all integers $l \geq 2$, where $\Omega_l := \limsup_{\mu \rightarrow \infty} \Omega_{l,\mu}$. The proof of part (ii) is completed by noting that $\tilde{\Omega} \supset \bigcap_{l=1 \in \mathbb{N}} \Omega_l$.

Finally, we turn to part (iv) and assume $p < p_c$. We observe that in \mathbb{P} -almost every realisation of a bond-percolation graph the translates of any given finite cluster occur infinitely often. This follows from a Borel–Cantelli argument like the one in the previous paragraph. Hence, the block-diagonal structure (1.14) of Δ_X implies that (apart from the infinitely degenerate eigenvalue γ_X) the set of eigenvalues of Δ_X is \mathbb{P} -almost surely given by the union of the spectra of $\Delta_X(\mathcal{C})$, where \mathcal{C} runs through all possible finite clusters in \mathbb{L}^d . In particular, the set of eigenvalues is a non-random dense set and all eigenvalues are infinitely degenerate. \square

The remaining part of this section concerns the proof of Theorem 1.14. It relies on deterministic upper and lower bounds for small eigenvalues of clusters. The lower bounds are discrete versions of well-known isoperimetric estimates for Laplacian eigenvalues on manifolds.

Definition 2.1. For $X \in \{\mathbb{N}, \tilde{\mathbb{D}}, \mathbb{D}\}$ and a connected subgraph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ of \mathbb{L}^d let $E_X^{(1)}(\mathcal{G})$ denote the lowest non-zero eigenvalue of $\Delta_X(\mathcal{G})$.

Proposition 2.2 (Cheeger inequality). Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be a connected finite subgraph of \mathbb{L}^d . Then its lowest non-zero Neumann eigenvalue obeys

$$E_{\mathbb{N}}^{(1)}(\mathcal{G}) \geq \frac{[h_{\text{Ch}}(\mathcal{G})]^2}{4d}. \quad (2.5)$$

The quantity $h_{\text{Ch}}(\mathcal{G}) := \min_{\mathcal{K}} |\partial \mathcal{K}| / |\mathcal{K}|$ is the Cheeger constant, where the minimum is taken over all subgraphs \mathcal{K} of \mathcal{G} whose vertex set \mathcal{W} obeys $|\mathcal{W}| \leq |\mathcal{V}|/2$. Here, $|\partial \mathcal{K}| := \{[x, y] \in \mathcal{E} : x \in \mathcal{W}, y \in \mathcal{V} \setminus \mathcal{W}\}$ denotes the edge boundary of \mathcal{K} in \mathcal{G} .

Remarks 2.3. (i) Proposition 2.2 just quotes a special case of a more general, well-known result in graph theory, see e.g. Thm. 3.1(2) in [12].

(ii) The simple lower bound $h_{\text{Ch}}(\mathcal{G}) \geq 1/(|\mathcal{V}|/2)$ on the Cheeger constant yields

$$E_{\mathbb{N}}^{(1)}(\mathcal{G}) \geq \frac{d^{-1}}{|\mathcal{V}|^2}. \quad (2.6)$$

This bound produces asymptotically the correct $|\mathcal{V}|$ -dependence as $|\mathcal{V}| \rightarrow \infty$, if \mathcal{G} is a linear cluster \mathcal{L}_n , i.e. a connected subgraph of \mathbb{L}^d having 2 vertices with degree 1 and $n - 2$ vertices with degree 2. For highly connected clusters, such as cubic clusters in $d > 1$ dimensions (see Remark 2.5 below for their definition), the bound (2.6) is very crude as compared to (2.5). Though, (2.6) will suffice for our purpose.

The next lemma provides a Faber–Krahn inequality on graphs. In contrast to Cheeger inequalities, such estimates for graphs have not been known for a long time, see [11] for a detailed exposition. Lemma 2.4 adapts a result from [11], which is proven there for more general graphs, to the type of graph Laplacians we use here.

Lemma 2.4 (Faber–Krahn inequality). Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be a connected finite subgraph of \mathbb{L}^d . Then its lowest Pseudo-Dirichlet eigenvalue obeys

$$E_{\mathbb{D}}^{(1)}(\mathcal{G}) \geq \frac{h_{\text{FK}}}{|\mathcal{V}|^{2/d}}, \quad (2.7)$$

where $h_{\text{FK}} \in]0, \infty[$ is a constant that depends only on the spatial dimension d .

Remark 2.5. The Faber–Krahn inequality produces asymptotically the correct $|\mathcal{V}|$ -dependence as $|\mathcal{V}| \rightarrow \infty$, if, for example, \mathcal{G} is a *cubic cluster* \mathcal{Q}_l , that is, $\mathcal{Q}_l = \mathbb{L}_{\Lambda_l}^d$ for some finite cube $\Lambda_l \subset \mathbb{Z}^d$ with $|\Lambda_l| = l^d$ points, i.e. edges of length $l - 1 \in \mathbb{N}$.

Proof (of Lemma 2.4). We reduce the assertion to a particular case of Prop. 7.1 and Cor. 6.4 in [11] by choosing the weighted graph in Prop. 7.1 as \mathbb{L}^d with unit weights on all bonds – note that these results in [11] extend to $d = 1$. Given any $\Lambda \subset \mathbb{Z}^d$, this yields the inequality

$$\lambda_1(\Lambda) := \inf_{0 \neq \varphi \in c_0(\Lambda)} \frac{\langle \varphi, \Delta_N(\mathbb{L}^d)\varphi \rangle}{2d \langle \varphi, \varphi \rangle} \geq \frac{\beta_d^2}{2|\Lambda|^{2/d}}. \quad (2.8)$$

The constant $\beta_d \in]0, \infty[$ is the isoperimetric constant of Cor. 6.4 in [11], which is independent of $\Lambda \subset \mathbb{Z}^d$. Moreover, $c_0(\Lambda)$ stands for the $\ell^2(\mathbb{Z}^d)$ -subspace of real-valued sequences with support in Λ . In order to check that the above definition of $\lambda_1(\Lambda)$ matches the one in [11], we refer to Sect. 5.5 of that paper. The claim of the lemma follows from the estimate

$$E_{\mathbb{D}}^{(1)}(\mathcal{G}) \geq 2d \lambda_1(\mathcal{V}), \quad (2.9)$$

which we prove now. To this end we observe that the adjacency operator has only non-negative matrix elements $\langle \delta_x, A(\mathcal{G})\delta_y \rangle$ for all $x, y \in \mathcal{V}$ and that $A(\mathcal{G})$ is irreducible on $\ell^2(\mathcal{V})$ due to the connectedness of \mathcal{G} . Hence, it follows from the Perron–Frobenius theorem, see e.g. Thm. 2.1.4 in [2], that the eigenvector of $\Delta_{\mathbb{D}}(\mathcal{G}) = 2d\mathbb{1} - A(\mathcal{G})$ corresponding to the non-degenerate smallest eigenvalue $E_{\mathbb{D}}^{(1)}(\mathcal{G})$ can be chosen to have non-negative entries. This implies

$$E_{\mathbb{D}}^{(1)}(\mathcal{G}) = \inf_{\substack{0 \neq \phi \in \ell^2(\mathcal{V}) \\ \phi(x) \geq 0 \text{ for all } x \in \mathcal{V}}} \frac{\langle \phi, \Delta_{\mathbb{D}}(\mathcal{G})\phi \rangle}{\langle \phi, \phi \rangle}. \quad (2.10)$$

The proof of (2.9) is completed by noting that

$$\begin{aligned} \langle \phi, \Delta_{\mathbb{D}}(\mathcal{G})\phi \rangle &= \sum_{[x,y] \in \mathcal{E}} [\phi(x) - \phi(y)]^2 + \sum_{x \in \mathcal{V}} [2d - d_{\mathcal{G}}(x)](\phi(x))^2 \\ &= \sum_{[x,y] \in \mathbb{E}^d} [\phi(x) - \phi(y)]^2 + 2 \sum_{[x,y] \in \mathbb{E}^d \setminus \mathcal{E}} \phi(x)\phi(y) \\ &\geq \langle \varphi, \Delta_N(\mathbb{L}^d)\varphi \rangle \end{aligned} \quad (2.11)$$

for all ϕ as in (2.10), where $\varphi(x) := \begin{cases} \phi(x), & x \in \mathcal{V}, \\ 0, & x \in \mathbb{Z}^d \setminus \mathcal{V}. \end{cases} \quad \square$

As a last ingredient for the proof of Theorem 1.14 we need some simple upper estimates on $E_X^{(1)}(\mathcal{G})$ for special types of clusters. These estimates are obtained from the minmax-principle. For our purpose it is only important that they reproduce the asymptotically correct functional dependence on the number of vertices for large clusters.

Lemma 2.6. (i) *The lowest non-zero Neumann eigenvalue for a linear cluster \mathcal{L}_n , which was defined in Remark 2.3(ii), obeys*

$$E_N^{(1)}(\mathcal{L}_n) \leq \frac{12}{n^2} \quad (2.12)$$

for all $n \in \mathbb{N}$, $n \geq 2$.

(ii) *The lowest Dirichlet eigenvalue for a cubic cluster \mathcal{Q}_l , which was defined in Remark 2.5, obeys*

$$E_{\text{D}}^{(1)}(\mathcal{Q}_l) \leq \frac{27d}{l^2} \quad (2.13)$$

for all $l \in \mathbb{N}$, $l \geq 2$.

Proof. (i) The minmax-principle yields the upper estimate

$$E_{\text{N}}^{(1)}(\mathcal{L}_n) \leq \frac{\sum_{j=1}^{n-1} (u_{j+1} - u_j)^2}{\sum_{j=1}^n u_j^2} \quad (2.14)$$

for every $(u_1, \dots, u_n) \in \mathbb{R}^n$ subject to the orthogonality constraint $\sum_{j=1}^n u_j = 0$. Choosing $u_j := -j + (n+1)/2$ for $j \in \{1, \dots, n\}$, which is in accordance with the orthogonality constraint, proves part (i).

(ii) Appealing to the minmax-principle with a trial “function” that factorises with respect to the d Cartesian directions, gives

$$E_{\text{D}}^{(1)}(\mathcal{Q}_l) \leq d \frac{2u_1^2 + 2u_l^2 + \sum_{j=1}^{l-1} (u_{j+1} - u_j)^2}{\sum_{j=1}^l u_j^2} \quad (2.15)$$

for all $(u_1, \dots, u_l) \in \mathbb{R}^l$. Now, we choose $u_j := -|j - (l+1)/2| + (l-1)/2$ for all $j \in \{1, \dots, l\}$ so that $u_1 = u_l = 0$. If $l \geq 3$ is odd, an explicit calculation shows

$$E_{\text{D}}^{(1)}(\mathcal{Q}_l) \leq \frac{12d}{l^2 - 2l + 3} \leq \frac{27d}{l^2}, \quad (2.16)$$

while for an even integer $l \geq 2$ it yields

$$E_{\text{D}}^{(1)}(\mathcal{Q}_l) \leq \frac{12d}{l(l-1)} \leq \frac{24d}{l^2}, \quad (2.17)$$

and the lemma is proven. \square

The next two lemmas provide the key estimates for Theorem 1.14. While the lower bounds in Lemma 2.9 hold for all $p \in]0, 1[$, the upper bounds in Lemma 2.7 are restricted to the non-percolating phase.

Lemma 2.7 (Upper bounds). *Let $d \in \mathbb{N}$ and consider $p \in]0, p_c[$. Then there exist constants $\alpha_{\text{N}}^+, \alpha_{\text{D}}^+ \in]0, \infty[$ such that*

$$N_{\text{N}}(E) - N_{\text{N}}(0) \leq \exp\{-\alpha_{\text{N}}^+ E^{-1/2}\} \quad \text{for all } E \in]0, 4d], \quad (2.18)$$

$$N_{\text{D}}(E) \leq N_{\text{D}}^-(E) \leq \exp\{-\alpha_{\text{D}}^+ E^{-d/2}\} \quad \text{for all } E \in]0, 2d[. \quad (2.19)$$

Remark 2.8. It is only the right inequality in (2.19) whose validity is restricted to $E \in]0, 2d[$. The proof below will show that $N_{\text{D}}(E) \leq \exp\{-\alpha_{\text{D}}^+ E^{-d/2}\}$ holds for all $E \in]0, 4d[$.

Lemma 2.9 (Lower bounds). *Let $d \in \mathbb{N}$ and consider $p \in]0, 1[$. Then there exist constants $\alpha_{\tilde{N}}, \alpha_{\tilde{D}} \in]0, \infty[$ such that for every $E \in]0, 4d]$ one has*

$$N_{\tilde{N}}(E) - N_{\tilde{N}}(0) \geq \exp\{-\alpha_{\tilde{N}} E^{-1/2}\}, \quad (2.20)$$

$$N_{\tilde{D}}(E) \geq N_{\tilde{D}}(E) \geq \exp\{-\alpha_{\tilde{D}} E^{-d/2}\}. \quad (2.21)$$

Proof (of Theorem 1.14). Due to the symmetries (1.21), it suffices to prove the asserted Lifshits tails at the lower spectral edge. These follow from the estimates in Lemma 2.7 and Lemma 2.9, because after taking appropriate logarithms, the respective bounds coincide in the limit $E \downarrow 0$. \square

Proof (of Lemma 2.7). Fix $E \in]0, 4d]$ subject to $E < \gamma_X$ if $X \in \{\tilde{D}, D\}$. The constants γ_X were defined in Remark 1.5(iv). Definition 1.9 of the integrated density of states implies

$$N_X(E) - N_X(0) = \int_{\Omega} \mathbb{P}(d\omega) \langle \delta_0, [\Theta(E - \Delta_X^{(\omega)}) - P_X^{(\omega)}] \delta_0 \rangle, \quad (2.22)$$

where $P_X := \Theta(-\Delta_X)$ denotes the (random) projector onto the null space of Δ_X . Due to our assumptions on E and the block-diagonal form (1.14) of Δ_X , the right-hand side of (2.22) is only different from zero if the origin is part of a cluster. Let us call this event Ω_0 and the corresponding cluster $\mathcal{C}_0^{(\omega)} := (\mathcal{V}_0^{(\omega)}, \mathcal{E}_0^{(\omega)})$ for all $\omega \in \Omega_0$. Hence, we obtain

$$\begin{aligned} N_X(E) - N_X(0) &= \int_{\Omega_0} \mathbb{P}(d\omega) \langle \delta_0, [\Theta(E - \Delta_X(\mathcal{C}_0^{(\omega)})) - P_X(\mathcal{C}_0^{(\omega)})] \delta_0 \rangle \\ &\leq \int_{\Omega_0} \mathbb{P}(d\omega) \Theta(E - E_X^{(1)}(\mathcal{C}_0^{(\omega)})) \langle \delta_0, [\mathbb{1} - P_X(\mathcal{C}_0^{(\omega)})] \delta_0 \rangle \\ &\leq \mathbb{P}\{\omega \in \Omega_0 : E \geq E_X^{(1)}(\mathcal{C}_0^{(\omega)})\}. \end{aligned} \quad (2.23)$$

Before we make a distinction of the two cases $X = N$ and $X \in \{\tilde{D}, D\}$ in order to apply the Cheeger, respectively the Faber–Krahn inequality, we recall from Proposition 1.2 (and Remark 1.3(ii) for the case $d = 1$) that the cluster $\mathcal{C}_0^{(\omega)}$ is finite for \mathbb{P} -almost all $\omega \in \Omega_0$, since we assume $p < p_c$.

The Neumann case. Applying the weakened version (2.6) of Cheeger's inequality to (2.23), yields the claim

$$N_{\tilde{N}}(E) - N_{\tilde{N}}(0) \leq \mathbb{P}\{\omega \in \Omega_0 : |\mathcal{V}_0^{(\omega)}| \geq 1/(dE)^{1/2}\} \leq \exp\{-d^{-1/2}\zeta(p)E^{-1/2}\}. \quad (2.24)$$

The second inequality in (2.24) reflects the exponential decay of the cluster-size distribution in the non-percolating phase, see Thm. 6.75 in [14]. Here, $\zeta(p) > 0$ is some finite constant for every $p \in]0, p_c[$, which depends only on d . Formally, Thm. 6.75 in [14] does not cover the one-dimensional situation $d = 1$. But for $d = 1$ the exponential decay of the cluster-size distribution follows from elementary combinatorics.

The (Pseudo-) Dirichlet case. The inequalities (1.18), (2.23), the Faber–Krahn inequality of Lemma 2.4 and the exponential decay of the cluster-size distribution yield

$$N_{\mathbb{D}}(E) \leq N_{\tilde{\mathbb{D}}}(E) \leq \mathbb{P}\{\omega \in \Omega_0 : |\mathcal{V}_0^{(\omega)}| \geq (h_{\text{FK}}/E)^{d/2}\} \leq \exp\{-h_{\text{FK}}^{d/2} \zeta(p) E^{-d/2}\}. \quad (2.25)$$

Here we have used that $N_{\mathbb{D}}(0) = N_{\tilde{\mathbb{D}}}(0) = 0$, see Remark 1.15(ii). \square

Proof (of Lemma 2.9). Lemma 1.12, the isotony of the right-hand side of (1.22) in E , the right-continuity of $N_{\mathbb{X}}$ and (1.25) imply that

$$N_{\mathbb{X}}(E) - N_{\mathbb{X}}(0) \geq \limsup_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \text{trace}_{\ell^2(\Lambda)} [\Theta(E - \Delta_{\mathbb{X}, \Lambda}^{(\omega)}) - P_{\mathbb{X}, \Lambda}^{(\omega)}] \quad (2.26)$$

for all $E > 0$ and all $\omega \in \Omega'$. Concerning the limit in (2.26), we think of a sequence of expanding cubes that are centred at the origin, cf. Remark 1.13(i). Due to the block-diagonal form (1.14) of $\Delta_{\mathbb{X}}$, which continues to hold for $\Delta_{\mathbb{X}, \Lambda}^{(\omega)}$ with respect to the decomposition of $\mathcal{G}_{\Lambda}^{(\omega)}$ into clusters $\mathcal{C}_{\Lambda, j}^{(\omega)} := (\mathcal{V}_{\Lambda, j}^{(\omega)}, \mathcal{E}_{\Lambda, j}^{(\omega)})$, $j \in \{1, \dots, J_{\Lambda}^{(\omega)}\}$, we get

$$N_{\mathbb{X}}(E) - N_{\mathbb{X}}(0) \geq \limsup_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{j=1}^{J_{\Lambda}^{(\omega)}} \text{trace}_{\ell^2(\mathcal{V}_{\Lambda, j}^{(\omega)})} [\Theta(E - \Delta_{\mathbb{X}}(\mathcal{C}_{\Lambda, j}^{(\omega)})) - P_{\mathbb{X}}(\mathcal{C}_{\Lambda, j}^{(\omega)})]. \quad (2.27)$$

At this point we make again a distinction of the cases $\mathbb{X} = \mathbb{N}$ and $\mathbb{X} \in \{\tilde{\mathbb{D}}, \mathbb{D}\}$.

The Neumann case. Let \mathfrak{L}_n denote the set of all linear clusters with n vertices in \mathbb{L}^d and let $\chi_{\mathfrak{L}_n}$ be the characteristic function of this set of graphs. A crude lower bound on the j -sum in (2.27) for $\mathbb{X} = \mathbb{N}$ results from discarding all branched clusters, i.e. those which are not linear,

$$\begin{aligned} & \sum_{j=1}^{J_{\Lambda}^{(\omega)}} \sum_{n=2}^{\infty} \chi_{\mathfrak{L}_n}(\mathcal{C}_{\Lambda, j}^{(\omega)}) \text{trace}_{\ell^2(\mathcal{V}_{\Lambda, j}^{(\omega)})} [\Theta(E - \Delta_{\mathbb{N}}(\mathcal{C}_{\Lambda, j}^{(\omega)})) - P_{\mathbb{N}}(\mathcal{C}_{\Lambda, j}^{(\omega)})] \\ & \geq \sum_{n=2}^{\infty} \sum_{j=1}^{J_{\Lambda}^{(\omega)}} \chi_{\mathfrak{L}_n}(\mathcal{C}_{\Lambda, j}^{(\omega)}) \Theta(E - E_{\mathbb{N}}^{(1)}(\mathcal{C}_{\Lambda, j}^{(\omega)})) \\ & \geq \sum_{n=2}^{\infty} \Theta(E - 12/n^2) \sum_{j=1}^{J_{\Lambda}^{(\omega)}} \chi_{\mathfrak{L}_n}(\mathcal{C}_{\Lambda, j}^{(\omega)}). \end{aligned} \quad (2.28)$$

The first inequality in (2.28) follows from restricting the trace to the spectral subspace corresponding to $E_{\mathbb{N}}^{(1)}(\mathcal{C}_{\Lambda, j}^{(\omega)})$, the second inequality follows from the variational upper bound in Lemma 2.6(i). Thus, (2.27) and (2.28) yield

$$N_{\mathbb{N}}(E) - N_{\mathbb{N}}(0) \geq \limsup_{\Lambda \uparrow \mathbb{Z}^d} \sum_{n=2}^{\infty} \Theta(E - 12/n^2) n^{-1} L_n^{(\omega)}(\Lambda) \quad (2.29)$$

with

$$L_n^{(\omega)}(\Lambda) := \frac{n}{|\Lambda|} \sum_{j=1}^{J_\Lambda^{(\omega)}} \chi_{\mathfrak{L}_n}(\mathcal{C}_{\Lambda,j}^{(\omega)}) = \frac{1}{|\Lambda|} \left| \{x \in \Lambda : \mathcal{C}_\Lambda^{(\omega)}(x) \in \mathfrak{L}_n\} \right| \quad (2.30)$$

being the number of points in Λ that are vertices of a cluster of type \mathfrak{L}_n . Here, $\mathcal{C}_\Lambda^{(\omega)}(x)$ denotes the cluster of $\mathcal{G}_\Lambda^{(\omega)}$ that contains $x \in \Lambda$; if $d_{\mathcal{G}_\Lambda^{(\omega)}}(x) = 0$, we set $\mathcal{C}_\Lambda^{(\omega)}(x) = (\{x\}, \emptyset)$. For a given $n \in \mathbb{N}$, $n \geq 2$, and a sufficiently large bounded cube $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda|^{1/d} \geq 2n + 1$, let us also define the number density

$$\tilde{L}_n^{(\omega)}(\Lambda) := \frac{1}{|\Lambda|} \left| \left\{ x \in \Lambda : \min_{1 \leq v \leq d} \min_{y \in \mathbb{Z}^d \setminus \Lambda} |x_v - y_v| \geq n + 1 \text{ and } \mathcal{C}_\Lambda^{(\omega)}(x) \in \mathfrak{L}_n \right\} \right| \quad (2.31)$$

of vertices which are, in addition, sufficiently far away from the boundary of Λ . Clearly, one has

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} [L_n^{(\omega)}(\Lambda) - \tilde{L}_n^{(\omega)}(\Lambda)] = 0 \quad (2.32)$$

for all $\omega \in \Omega$ and all $n \in \mathbb{N}$, $n \geq 2$, since the difference in the two quantities results from a surface effect. The vertices that count for $\tilde{L}_n^{(\omega)}(\Lambda)$ are so far away from the boundary of Λ that the clusters they belong to cannot grow when enlarging Λ . Hence,

$$|\Lambda_1 \cup \Lambda_2| \tilde{L}_n^{(\omega)}(\Lambda_1 \cup \Lambda_2) \geq |\Lambda_1| \tilde{L}_n^{(\omega)}(\Lambda_1) + |\Lambda_2| \tilde{L}_n^{(\omega)}(\Lambda_2) \quad (2.33)$$

holds for all $\Lambda_1, \Lambda_2 \subset \mathbb{Z}^d$ provided $\Lambda_1 \cap \Lambda_2 = \emptyset$. Thus, $L_n(\Lambda)$ defines a superergodic process and the Ackoglu–Krengel superergodic theorem, see e.g. Thm. VI.1.7 in [8], and (2.32) imply

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} L_n^{(\omega)}(\Lambda) = \sup_{\Lambda \subset \mathbb{Z}^d} \int_{\Omega} \mathbb{P}(d\omega') \tilde{L}_n^{(\omega')}(\Lambda) = \mathbb{P}\{\omega' \in \Omega_0 : \mathcal{C}_0^{(\omega')} \in \mathfrak{L}_n\} \quad (2.34)$$

for all $n \in \mathbb{N}$, $n \geq 2$, and \mathbb{P} -almost all $\omega \in \Omega$. The event Ω_0 and the random cluster \mathcal{C}_0 were defined above (2.23).

Now, we neglect all terms in the n -sum in (2.29) except for the one which corresponds to the biggest integer $n(E)$ obeying $n(E) < (12/E)^{1/2} + 1$. From this we conclude together with (2.34) that

$$N_N(E) - N_N(0) \geq [n(E)]^{-1} \mathbb{P}\{\omega \in \Omega_0 : \mathcal{C}_0^{(\omega)} \in \mathfrak{L}_{n(E)}\}. \quad (2.35)$$

Elementary combinatorics shows that the probability on the right-hand side of (2.35) is bounded below by $\exp\{-n(E) f(p)\}$, where $f(p) \in]0, \infty[$ is a constant that depends only on d for a given $p \in]0, 1[$. This leads to the estimate

$$N_N(E) - N_N(0) \geq \frac{e^{-f(p)}}{(12/E)^{1/2} + 1} \exp\{-12^{1/2} f(p) E^{-1/2}\}, \quad (2.36)$$

which can be cast into the form (2.20) for $E \in]0, 4d]$.

The (Pseudo-) Dirichlet case. This case parallels exactly the previous one, except that here we retain cubic clusters instead of linear clusters. Let \mathfrak{Q}_l denote

the set of all cubic clusters in \mathbb{L}^d with l^d vertices, i.e. edges of length $l - 1 \in \mathbb{N}$, let χ_{Ω_l} be the characteristic function of this set of graphs and define the number density $Q_l^{(\omega)}(\Lambda) := (l^d / |\Lambda|) \sum_{j=1}^{J_\Lambda^{(\omega)}} \chi_{\Omega_l}(\mathcal{C}_{\Lambda,j}^{(\omega)})$ of points in Λ that are vertices of such a cubic cluster (when restricted to Λ). Now, the rôle of \mathfrak{L}_n in the previous case will be played by \mathfrak{Q}_l . Hence, the analogue of (2.29) reads

$$N_{\widetilde{D}}(E) \geq N_D(E) \geq \limsup_{\Lambda \uparrow \mathbb{Z}^d} \sum_{l=2}^{\infty} \Theta(E - 27d/l^2) l^{-d} Q_l^{(\omega)}(\Lambda), \quad (2.37)$$

where we have used Lemma 2.6(ii) instead of Lemma 2.6(i). The very same arguments that led to (2.34) imply in the present context

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} Q_l^{(\omega)}(\Lambda) = \mathbb{P}\{\omega' \in \Omega_0 : \mathcal{C}_0^{(\omega')} \in \mathfrak{Q}_l\} \quad (2.38)$$

for all $l \in \mathbb{N}$, $l \geq 2$, and \mathbb{P} -almost all $\omega \in \Omega$. By neglecting all terms in the l -sum in (2.37) except for the one which corresponds to the biggest integer $l(E)$ obeying $l(E) < (27d/E)^{1/2} + 1$, we conclude with (2.38) that

$$N_{\widetilde{D}}(E) \geq N_D(E) \geq [l(E)]^{-d} \mathbb{P}\{\omega \in \Omega_0 : \mathcal{C}_0^{(\omega)} \in \mathfrak{Q}_{l(E)}\}. \quad (2.39)$$

Again, there is an elementary combinatorial lower bound $\exp\{-[l(E)]^d g(p)\}$ for the probability in (2.39), where $g(p) \in]0, \infty[$ is a constant that depends only on d for a given $p \in]0, 1[$. So we arrive at

$$N_{\widetilde{D}}(E) \geq N_D(E) \geq [(27d/E)^{1/2} + 1]^{-d} \exp\{-[(27d/E)^{1/2} + 1]^d g(p)\}, \quad (2.40)$$

which can be cast into the form (2.21) for $E \in]0, 4d]$. \square

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