The Fermi Golden Rule and its Form at Thresholds in Odd Dimensions

Arne Jensen*† Gheorghe Nenciu‡§

Abstract

Let H be a Schrödinger operator on a Hilbert space $\mathcal{H}$, such that zero is a non-degenerate threshold eigenvalue of H with eigenfunction Ψ_0. Let W be a bounded selfadjoint operator satisfying $\langle \Psi_0, W\Psi_0 \rangle > 0$. Assume that the resolvent $(H-z)^{-1}$ has an asymptotic expansion around $z = 0$ of the form typical for Schrödinger operators on odd-dimensional spaces. Let $H(\varepsilon) = H + \varepsilon W$ for $\varepsilon > 0$ and small. We show under some additional assumptions that the eigenvalue at zero becomes a resonance for $H(\varepsilon)$, in the time-dependent sense introduced by A. Orth. No analytic continuation is needed. We show that the imaginary part of the resonance has a dependence on ε of the form $\varepsilon^{2+(\nu/2)}$ with the integer $\nu \geq -1$ and odd. This shows how the Fermi Golden Rule has to be modified in the case of perturbation of a threshold eigenvalue. We give a number of explicit examples, where we compute the "location" of the resonance to leading order in ε. We also give results, in the case where the eigenvalue is embedded in the continuum, sharpening the existing ones.

1 Introduction

The main purpose of this paper is to study the following question. Consider a Schrödinger operator

$$H = -\Delta + V \quad \text{on } L^2(\mathbf{R}^3),$$

where for the moment we assume that $V \in C_0^\infty(\mathbf{R}^3)$. The essential spectrum of H is the half line $[0, \infty)$, and it is well-known that this spectrum is purely absolutely continuous. H may have a finite number of negative eigenvalues, and there may also be an eigenvalue

*Department of Mathematical Sciences, Aalborg University, Fredrik Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark. E-mail: matarne@math.aau.dk

†MaPhySto – A Network in Mathematical Physics and Stochastics, funded by the Danish National Research Foundation.

‡Department of Theoretical Physics, University of Bucharest, P. O. Box MG11, 76900 Bucharest, Romania. E-mail: nenciu@barutu.fizica.unibuc.ro

§Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O.Box 1-764, RO-014700 Bucharest, Romania. E-mail: Gheorghe.Nenciu@imar.ro

at the threshold zero. Suppose that zero is a nondegenerate eigenvalue with normalized eigenfunction Ψ_0. Let $W \in C_0^\infty(\mathbf{R}^3)$, and assume that it is nonnegative. Consider for small $\varepsilon > 0$ the family of Hamiltonians

$$H(\varepsilon) = H + \varepsilon W.$$

Since the perturbation is nonnegative, the zero eigenvalue cannot become an isolated negative eigenvalue, and since it is well-known that $H(\varepsilon)$ cannot have eigenvalues embedded in $(0, \infty)$, only two possibilities remain. Zero can remain an eigenvalue, or it can disappear. In the latter case one expects that it becomes a resonance. This is the kind of question that we will study.

The existence of the resonance can be verified in several ways, depending upon the "definition" of what a resonance is. One may look at this question in the spectral form. Thus one looks at a meromorphic continuation of the resolvent in some sense, and expects to find a pole close to zero in the complex energy plane. One can also study the question from the time-dependent point of view. Here one looks at the behavior of $\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0 \rangle$, which describes the survival probability for $e^{-itH(\varepsilon)}\Psi_0$, i.e. the probability to remain in the state Ψ_0 at time t. The resonance will then manifest itself in the form of a behavior of the type

$$\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0 \rangle = e^{-it\lambda(\varepsilon)} + \delta(\varepsilon, t), \quad t > 0, \tag{1.1}$$

i.e. corresponding to a metastable state. Here $\lambda(\varepsilon) = x_0(\varepsilon) - i\Gamma(\varepsilon)$ with $x_0(\varepsilon) > 0$ and $\Gamma(\varepsilon) > 0$ and, as far as a resonance defined in the spectral sense exists, should coincide with the resonance position. The error term in (1.1) should satisfy $\delta(\varepsilon, t) \to 0$ as $\varepsilon \to 0$.

Our main theorem gives conditions on H and W that lead to such results, with expressions for the leading terms in $x_0(\varepsilon)$ and $\Gamma(\varepsilon)$, as $\varepsilon \to 0$. In the case of an eigenvalue embedded in the interior of the absolutely continuous spectrum, formulae for computing the leading term in $\Gamma(\varepsilon)$ are often referred to as the Fermi Golden Rule. Thus we find versions of the Fermi Golden Rule in the case, where the eigenvalue is embedded at a threshold.

Let us give an outline of the main results, referring to the theorems for precise assumptions and conditions. The results are obtained in a semi-abstract framework. One of the main tools needed is the asymptotic expansion of the resolvent $R(z) = (H - z)^{-1}$ around zero. It is convenient to use the variable $\kappa = -i\sqrt{z}$, $z \in \mathbf{C} \setminus [0, \infty)$. We assume an expansion of the form

$$R(-\kappa^2) = \frac{1}{\kappa^2} P_0 + \sum_{j=-1}^{N} \kappa^j G_j + \mathcal{O}(\kappa^{N+1}) \tag{1.2}$$

as $\kappa \to 0$. This type of expansion is known to hold for Schrödinger operators in odd dimensions, with sufficiently rapidly decaying V. The expansion holds in the topology of bounded operators between weighted L^2-spaces. See [18, 17, 25, 19]. For the perturbation W we assume that it decays sufficiently rapidly, and as a crucial condition, we require

$$b = \langle \Psi_0, W\Psi_0 \rangle > 0. \tag{1.3}$$

We do not assume that W is nonnegative. We assume that there exists an odd integer ν, such that

$$g_\nu = \langle \Psi_0, W G_\nu W \Psi_0 \rangle \neq 0, \quad G_j = 0, \quad j = -1, 1, 3, \ldots, \nu - 2.$$

Our main abstract result then states that (1.1) holds. Furthermore, we have the estimate

$$|\delta(\varepsilon, t)| \leq C \varepsilon^{p(\nu)},$$

with $p(\nu) = \min\{2, (2+\nu)/2\}$. We have the expansions

$$\begin{aligned} \Gamma(\varepsilon) &= -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+(\nu/2)} (1 + \mathcal{O}(\varepsilon)), \\ x_0(\varepsilon) &= b\varepsilon(1 + \mathcal{O}(\varepsilon)), \end{aligned}$$

as $\varepsilon \to 0$. The proof of these results is based on the representation

$$\langle \Psi_0, e^{-itH(\varepsilon)} \Psi_0 \rangle = \lim_{\eta \searrow 0} \frac{1}{\pi} \int e^{-itx} \operatorname{Im} \langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1} \Psi_0 \rangle dx.$$

The idea is first to localize to an interval $I_\varepsilon \subseteq (\frac{1}{2} b\varepsilon, \frac{3}{2} b\varepsilon)$, depending on ε and ν, and then, by using the resolvent expansion, replace the term

$$\operatorname{Im} \langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1} \Psi_0 \rangle$$

in the integrand by a Lorentzian function

$$\frac{\Gamma(\varepsilon)}{(x - x_0(\varepsilon))^2 + \Gamma(\varepsilon)^2}.$$

To obtain this approximation we use the Schur-Livsic-Feshbach-Grushin (SLFG for short) formula to localize the essential terms. During all steps one has to control the errors. The detailed computations then lead to the result outlined above.

We apply these semi-abstract results to a number of cases involving Schrödinger operators in three and one dimensions, and on the half line. We consider both the one channel and the two channel cases. As an example of the type of results obtained, assume as above that $H = -\Delta + V$ on $L^2(\mathbf{R}^3)$ has zero as a nondegenerate eigenvalue (and no threshold resonance), with Ψ_0 a normalized real-valued eigenfunction. Let

$$X_j = \int_{\mathbf{R}^3} \Psi_0(\mathbf{x}) V(\mathbf{x}) x_j d\mathbf{x}, \quad j = 1, 2, 3.$$

Assume that at least one $X_j \neq 0$. Then $\nu = -1$, and we have

$$g_{-1} = \frac{b^2}{12\pi} (X_1^2 + X_2^2 + X_3^2).$$

There is a large literature concerned with establishing the spectral form of the Fermi Golden Rule in a rigorous framework. In particular, using dilation-analyticity, it was established in [30], in a large number of cases, including atoms and molecules. See also the discussion in [29]. The case of bound states embedded at a threshold is much less studied. The coupling constant case has been studied by several authors, see for example [28, 11, 12]. To compare with our results, one has to take $W = -V$, in order to satisfy (1.3), since $(-\Delta + V)\Psi_0 = 0$ implies $\langle \Psi_0, V\Psi_0 \rangle = \langle \Psi_0, \Delta\Psi_0 \rangle < 0$. Then the explicit results in [11] correspond to our case $\nu = -1$, and the results agree. The spectral form of the resonance problem has been studied near band edges for periodic Schrödinger operators in the semi-classical limit, in [23]. A general framework for a unified treatment of resonances and eigenvalues near thresholds has been given in [13], using meromorphic continuation of resolvents. The first paper to rigorously establish the time-dependent form above, with a remainder estimate, was the one by Skibsted [32].

The only work in the spectral form directly related to our study that we are aware of (even at a nonrigorous level), is that of Baumgartner [4], where some simplified two-channel models are considered. In these cases explicit computations can be performed, and one can explain how the usual Fermi Golden Rule has to be modified to be applied in the threshold case.

The time-dependent approach has been developed much later. Let us remark that here there is no need of an analytic continuation. The time-dependent approach, without analyticity, was initiated in [27] and continued in [21]. In [15] it was investigated how to get a better error term by using the perturbation theory in the spirit of Simon [30] in the dilation-analytic framework. In addition, again using the analytic continuation of the resolvent, the Fermi Golden Rule has been rigorously established in some models of quantum field theory and open systems (see [7, 16], and references therein). More recently, a number of authors have developed a general time-dependent approach, without analyticity, see for example [22, 34, 33, 24, 6, 35]. As far as we can determine, none of these approaches can be applied directly to the threshold eigenvalue case. (See however the remarks in [33, 6] concerning some examples of Schrödinger operators in high dimensions.)

It should be noted that all the time-dependent approaches (except [15]) use the SLFG formula in some form, or something equivalent to it. As we already said, we also rely heavily on the SLFG formula.

As in [4] one may describe the results stated above as explaining how the usual Fermi Golden Rule has to be modified to be applied in the threshold case. In the case of a resonance arising from the perturbation of an eigenvalue embedded in the continuum, one finds that the imaginary part of the resonance behaves generically like ε^2 as $\varepsilon \to 0$. We find the behavior $\varepsilon^{2+(\nu/2)}$, $\nu = -1, 1, \ldots$, which is quite different. In particular, for $\nu \geq 1$ the resonance arising from the threshold eigenvalue has a larger lifetime than one arising from an eigenvalue embedded in the continuum, i.e. one has an enhancement of the lifetime. On the contrary, for $\nu = -1$ the lifetime is smaller, i.e. one has an enhancement of the decay. This is clearly seen in the two channel case, when a threshold resonance is present in the open channel, and can be explained heuristically as the effect of the increase, due to the threshold resonance, of the density of states near the threshold.

Let us briefly outline the contents of the paper. In Section 2 we give the SLFG formula, and introduce the factorization method. In Section 3 we give our semi-abstract results, modelled on Schrödinger operators in odd dimensions. The main result is stated as Theorem 3.7. In Section 4 we give some results on the case where the eigenvalue is embedded in the continuum, sharpening (when no analytic continuation is available) the existing estimates on $\sup_{t>0} |\delta(\varepsilon, t)|$. In the differentiable case, as far as the dependence upon ε is concerned, our estimate on $\sup_{t>0} |\delta(\varepsilon, t)|$ is optimal. Let us note that these results are applicable to Schrödinger operators in both even and odd dimensions.

Then in Section 5 we verify the assumptions in Section 3 in a number of cases: (i) A Schrödinger operator on $L^2(\mathbf{R}^3)$, both in the one channel and the two channel cases. (ii) A Schrödinger operator on $L^2(\mathbf{R})$. Here it only makes sense to consider the two channel case, since for rapidly decaying potentials zero cannot be an eigenvalue. (iii) The operator $-d^2/dr^2 + \ell(\ell+1)r^{-2}$ on $L^2(\mathbf{R}_+)$. The results in this subsection relate directly to [4, Section V]. In all cases we determine the values of ν and compute g_ν explicitly. Finally, for the reader's convenience, in the Appendix we reobtain the results we need from [18], in the framework given in [19].

We conclude the introduction with the remark that in the case of a threshold eigenvalue embedded in the continuum, as can be seen in atoms and molecules, one can combine the approaches in Sections 3 and 4, to obtain results, which we refrain from stating in detail.

2 The Schur-Livsic-Feshbach-Grushin (SLFG) formula and the factorization method

Let H be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$ and E_0 a nondegenerate eigenvalue of H,

$$H\Psi_0 = E_0\Psi_0, \quad \|\Psi_0\| = 1. \tag{2.1}$$

We can without loss of generality in the sequel take $E_0 = 0$. Suppose now that a perturbation, described by the self-adjoint operator W, is added so the perturbed dynamics is generated by

$$H(\varepsilon) = H + \varepsilon W, \quad \varepsilon > 0. \tag{2.2}$$

Note that we only consider positive values of the parameter ε. For the sake of simplicity we shall assume that W is bounded, but all the results below extend to the case, when W is bounded with respect to H, with bound less than one.

At the heuristic level, it is argued that due to the perturbation, for ε small enough, 0 turns into a resonance having an (approximate) exponential decay law,

$$\langle \Psi_0, e^{-iH(\varepsilon)t}\Psi_0\rangle = e^{-i\lambda(\varepsilon)t} + \delta(\varepsilon, t), \tag{2.3}$$

with $\delta(\varepsilon, t) \to 0$, as $\varepsilon \to 0$, and

$$\lambda(\varepsilon) = E(\varepsilon) - i\Gamma(\varepsilon), \quad E(\varepsilon) = 0 + \varepsilon\langle \Psi_0, W\Psi_0\rangle + \mathcal{O}(\varepsilon^2). \tag{2.4}$$

The goal is to compute $\lambda(\varepsilon)$, and to obtain bounds on $|\delta(\varepsilon, t)|$. Again at the heuristic level, it is argued that the main contribution to the left-hand side of (2.3) is given by energies near to $E(\varepsilon)$, so one first considers (see [15])

$$A_{g_\varepsilon}(t) = \langle \Psi_0, e^{-iH(\varepsilon)t} g_\varepsilon(H(\varepsilon))\Psi_0\rangle, \tag{2.5}$$

where g_ε is the (possibly smoothed) characteristic function of a closed interval I_ε, containing the relevant energies. Let us remark that usually (see e.g. [15, 6, 24, 35]) I_ε is chosen to be a neighborhood of E_0, independent of ε. One of the key points of our approach is to make an appropriate ε-dependent choice of I_ε. Also, since we are interested in uniform (with respect to t) estimates on $\delta(\varepsilon, t)$, we shall take g_ε to be the characteristic function of I_ε.

From Stone's formula (suppose that the end points of I_ε are not eigenvalues of $H(\varepsilon)$) one gets

$$A_{g_\varepsilon}(t) = \lim_{\eta\searrow 0} \frac{1}{\pi}\int_{I_\varepsilon} dx\, e^{-ixt}\, \mathrm{Im}\langle \Psi_0, (H(\varepsilon) - x - i\eta)^{-1}\Psi_0\rangle. \tag{2.6}$$

As in [27], in order to compute the integrand in (2.6), we use the well-known Schur-Livsic-Feshbach-Grushin formula. More precisely, if P_0 is the orthogonal projection on Ψ_0, $Q_0 = 1 - P_0$, and $R_{0,\varepsilon}(z)$ is the resolvent of $Q_0H(\varepsilon)Q_0$ as an operator in $Q_0\mathcal{H}$, then in operator matrix form on $\mathcal{H} = P_0\mathcal{H} \oplus Q_0\mathcal{H}$, we have for $R_\varepsilon(z) = (H(\varepsilon) - z)^{-1}$ the representation

$$R_\varepsilon(z) = \begin{bmatrix} R_{\mathrm{eff}}(z) & -\varepsilon R_{\mathrm{eff}}(z)P_0WQ_0R_{0,\varepsilon}(z) \\ -\varepsilon R_{0,\varepsilon}(z)Q_0WP_0R_{\mathrm{eff}}(z) & R_{22} \end{bmatrix}, \tag{2.7}$$

with

$$R_{22} = R_{0,\varepsilon}(z) + \varepsilon^2 R_{0,\varepsilon}(z)Q_0WP_0R_{\mathrm{eff}}(z)P_0WQ_0R_{0,\varepsilon}(z),$$

and

$$R_{\mathrm{eff}}(z) = \big(P_0H(\varepsilon)P_0 - \varepsilon^2 P_0WQ_0R_{0,\varepsilon}(z)Q_0WP_0 - zP_0\big)^{-1},$$

where, with a slight abuse of notation, we write $R_{\mathrm{eff}}(z) = (H_{\mathrm{eff}}(z) - z)^{-1}$, and furthermore (remember that we assume $\mathrm{Rank}\, P_0 = 1$)

$$\begin{aligned} P_0(H_{\mathrm{eff}}(z) - z)P_0 &= F(z,\varepsilon)P_0 \\ &= \big(\varepsilon\langle\Psi_0, W\Psi_0\rangle - z - \varepsilon^2\langle\Psi_0, WR_{0,\varepsilon}(z)W\Psi_0\rangle\big)P_0. \end{aligned} \tag{2.8}$$

Using (2.8), (2.7), and (2.6), one obtains

$$A_{g_\varepsilon}(t) = \lim_{\eta\searrow 0} \frac{1}{2\pi i}\int_{I_\varepsilon} dx\, e^{-ixt}\Big(\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{F(x-i\eta,\varepsilon)}\Big). \tag{2.9}$$

The whole problem is to have a "nice" formula for $F(z,\varepsilon)$, so that the integral in (2.9) can be estimated. For that purpose we need some information on $R_{0,\varepsilon}(z)$. Let

$$W = A^*DA \tag{2.10}$$

be a factorization of W with D a self-adjoint involution. An example of such a factorization is the polar decomposition of W,

$$W = |W|^{1/2} D |W|^{1/2}, \tag{2.11}$$

where we take D to be unitary by defining it to be the identity on $\operatorname{Ker} W$.

Take $\operatorname{Im} z \to \infty$, and use regular perturbation theory to obtain

$$\begin{aligned} Q_0 R_{0,\varepsilon}(z) Q_0 &= Q_0 (H-z)^{-1} Q_0 \\ &\quad - \varepsilon Q_0 (H-z)^{-1} Q_0 W Q_0 (H-z)^{-1} Q_0 + \cdots \\ &= Q_0 (H-z)^{-1} Q_0 - \varepsilon Q_0 (H-z)^{-1} Q_0 \\ &\quad \times A^* [D + \varepsilon A Q_0 (H-z)^{-1} Q_0 A^*]^{-1} A Q_0 (H-z)^{-1} Q_0. \end{aligned} \tag{2.12}$$

With the notation

$$G(z) = A Q_0 (H-z)^{-1} Q_0 A^*, \tag{2.13}$$

one has for $\operatorname{Im} z \to \infty$

$$\begin{aligned} F(z,\varepsilon) &= \varepsilon \langle \Psi_0, W \Psi_0 \rangle - z \\ &\quad - \varepsilon^2 \langle \Psi_0, A^* D \{ G(z) - \varepsilon G(z) [D + \varepsilon G(z)]^{-1} G(z) \} D A \Psi_0 \rangle. \end{aligned} \tag{2.14}$$

Since $F(z,\varepsilon)$ is analytic in z, the equality (2.14) holds true for all z, for which either the right-hand side, or the left-hand side, exists. In particular, (2.14) holds true for $\operatorname{Im} z \neq 0$. The formulae (2.9) and (2.14) are the starting formulae of our approach. What is nedeed in each particular case is the behaviour of $G(z)$ in a neighbourhood of the energy of interest.

Remark 2.1. Let us comment briefly on our terminology. We call the formula (2.7) the Schur-Livsic-Feshbach-Grushin formula. For matrices, it goes back at least to Schur in a paper from 1917, and what we called the effective Hamiltonian, is known as the Schur complement. It is widely used in matrix theory and related areas, see [8] and also [9] for further references. In spectral theory related to quantum theories it is known as the Feshbach or Livsic formula, and references can be traced from [14, 7]. In a slightly different but equivalent form, the same formula appeared in the study of linear partial differential operators, see [31] and references therein. We named it the SLFG formula to emphasize the fact that all these scattered developments represent the same mathematical object: the Schur complement.

3 Threshold eigenvalues in the case of odd dimensions

In this case the ingredient is the expansion of $G(z)$ around $z = 0$. For the examples considered here, the corresponding expansions are provided by the results or methods in [18, 17, 25, 19]. In this section we shall use this expansion in a somewhat abstract setting, having in mind Schrödinger and Dirac operators in odd dimensions. More precisely, we assume H and W to satisfy the following conditions (A1)–(A5). Here $\rho(H)$ denotes the resolvent set, and $\sigma(H)$ the spectrum, with standard notation for the components of the spectrum.

Assumption 3.1. (A1) *There exists $a > 0$, such that $(-a, 0) \subset \rho(H)$ and $[0, a] \subset \sigma_{\mathrm{ess}}(H)$.*

(A2) *Assume that zero is a nondegenerate eigenvalue of H: $H\Psi_0 = 0$, with $\|\Psi_0\| = 1$, and there are no other eigenvalues in $[0, a]$. Let $P_0 = |\Psi_0\rangle\langle\Psi_0|$ be the orthogonal projection onto the one-dimensional eigenspace.*

(A3) *Assume*

$$\langle \Psi_0, W\Psi_0 \rangle = b > 0. \tag{3.1}$$

(A4) *For $\mathrm{Re}\,\kappa \geq 0$ and $z \in \mathbf{C} \setminus [0, \infty)$ we let*

$$\kappa = -i\sqrt{z}, \quad z = -\kappa^2. \tag{3.2}$$

There exist $N \in \mathbf{N}$ and $\delta_0 > 0$, such that for $\kappa \in \{\kappa \in \mathbf{C} \,|\, 0 < |\kappa| < \delta_0, \mathrm{Re}\,\kappa \geq 0\}$ we have

$$A(H + \kappa^2)^{-1}A^* = \frac{1}{\kappa^2}\widetilde{P}_0 + \sum_{j=-1}^{N} \widetilde{G}_j \kappa^j + \kappa^{N+1}\widetilde{G}_N(\kappa), \tag{3.3}$$

where

$$\widetilde{P}_0 = AP_0A^*, \tag{3.4}$$

$$\widetilde{G}_j \quad \text{are bounded and self-adjoint}, \tag{3.5}$$

$$\widetilde{G}_{-1} \quad \text{is of finite rank and self-adjoint}, \tag{3.6}$$

$$\widetilde{G}_N(\kappa) \quad \text{is uniformly bounded in } \kappa. \tag{3.7}$$

Taking into account that (remember that $Q_0 = 1 - P_0$)

$$(H + \kappa^2)^{-1} = \frac{1}{\kappa^2}P_0 + Q_0(H + \kappa^2)^{-1}Q_0, \tag{3.8}$$

one has from (3.3), (3.4), and (3.8) that

$$G(z) = \sum_{j=-1}^{N} \widetilde{G}_j \kappa^j + \kappa^{N+1}\widetilde{G}_N(\kappa). \tag{3.9}$$

From (3.9) we get

$$\langle \Psi_0, A^*DG(z)DA\Psi_0 \rangle = \sum_{j=-1}^{N} g_j \kappa^j + \kappa^{N+1} g_N(\kappa), \tag{3.10}$$

where

$$g_j = \langle \Psi_0, A^*D\widetilde{G}_jDA\Psi_0 \rangle, \tag{3.11}$$

$$g_N(\kappa) = \langle \Psi_0, A^*D\widetilde{G}_N(\kappa)DA\Psi_0 \rangle. \tag{3.12}$$

Notice that due to (3.5) we have

$$g_j = \overline{g}_j. \tag{3.13}$$

Finally, we need one further assumption.

Assumption 3.2. (A5) *There exists an odd integer, $-1 \le \nu \le N$, such that*

$$g_\nu \neq 0, \quad \widetilde{G}_j = 0 \quad \text{for } j = -1, 1, \ldots, \nu - 2. \tag{3.14}$$

A few remarks about the above assumptions: (A1) is nothing but the fact that we consider the perturbation of eigenvalues lying at a threshold, and that the threshold is not embedded in the essential spectrum. Assumptions (A2) is a simplifying "nondegeneracy" condition. There are many interesting cases from a physical point of view, where these two assumptions do not hold. The assumption (A3) is essential. It assures that the perturbation "pushes" the eigenvalue into the positive continuum at a rate of order ε, while (A5) implies (see below) that the "width" $\Gamma(\varepsilon)$ behaves as $\varepsilon^{2+(\nu/2)}$, as $\varepsilon \to 0$. If (A3) does not hold, and the perturbation pushes the eigenvalue into the continuum at a rate as say ε^2, then an exponential decay law may not exist. Let us note that for every odd $\nu \ge -1$ there are examples where (A5) holds with that choice of ν. See Section 5.3. We shall consider the problem of relaxing these assumptions in subsequent work. Assumption (A4) is our main tool. In particular, it implies that on $(0, \delta_0^2]$ the spectrum of H is absolutely continuous. We also notice that from (2.13), (3.10), and the first resolvent equation, it follows that

$$i^{\nu-1} g_\nu < 0. \tag{3.15}$$

By the heuristics of naive perturbation theory, one expects that the perturbation turns the zero eigenvalue into a "resonance", whose real part, up to errors of order ε^2, equals $b\varepsilon$. (Note that if (3.1) does not hold, then the eigenvalue may turn into an isolated eigenvalue of $H(\varepsilon)$.) This suggests to take the interval of "relevant energies" to be contained in $(\frac{1}{2}b\varepsilon, \frac{3}{2}b\varepsilon)$:

$$I_\varepsilon \subset (\tfrac{1}{2}b\varepsilon, \tfrac{3}{2}b\varepsilon). \tag{3.16}$$

Now the idea of the proof that $A_{g_\varepsilon}(t)$ has the form of the right-hand side of (2.3), is very simple: On the interval I_ε the function $\mathrm{Im}(F(x+i0,\varepsilon)^{-1})$ can be approximated by a Lorentzian function, whose parameters give $\lambda(\varepsilon)$. There are two error terms to be taken into account. The first one is coming from approximating $\mathrm{Im}(F(x+i0,\varepsilon)^{-1})$ on I_ε by a Lorentzian function, and the second one from the Lorentzian integral on $\mathbf{R} \setminus I_\varepsilon$. While the first one is increasing with the length of I_ε, the second one is decreasing. The length of I_ε will be chosen as to balance between the two.

In what follows *for ε sufficiently small* is a shorthand expression for *there exists ε_0 such that for $0 < \varepsilon \le \varepsilon_0$* the given statement holds. All the constants appearing below are finite and strictly positive. Consider

$$D_\varepsilon = \{z = x + i\eta \,|\, x \in (\tfrac{1}{2}b\varepsilon, \tfrac{3}{2}b\varepsilon),\ 0 < |\eta| < (\varepsilon b)^{2+\frac{\nu+1}{2}}\}. \tag{3.17}$$

Lemma 3.3. *Let*

$$p(\nu) = \min\{2,\ \frac{2+\nu}{2}\}. \tag{3.18}$$

Then for ε sufficiently small, and for $z \in D_\varepsilon$, we have

$$F(z,\varepsilon) = H(z,\varepsilon) + r(z,\varepsilon), \tag{3.19}$$

with

$$\sup_{\substack{0<\varepsilon<\varepsilon_0 \\ z\in D_\varepsilon}} \varepsilon^{-(2+p(\nu)+\frac{\nu}{2})}|r(z,\varepsilon)| < \infty, \tag{3.20}$$

and for $\nu = -1$

$$H(z,\varepsilon) = \varepsilon b - z - \varepsilon^2 g_{-1}\kappa^{-1}, \tag{3.21}$$

while for $\nu \geq 1$

$$H(z,\varepsilon) = \varepsilon b - z - \varepsilon^2\Big[a_\nu(\varepsilon)\kappa^\nu + g_{\nu+2}\kappa^{\nu+2} + \sum_{j=0}^{\frac{\nu+3}{2}} f_j(\varepsilon)\kappa^{2j}\Big]. \tag{3.22}$$

Here (*see* (3.11) *for* g_j)

$$a_\nu(\varepsilon) = g_\nu - \varepsilon\langle\Psi_0, A^* D(\widetilde{G}_\nu\widetilde{G}_0 + \widetilde{G}_0\widetilde{G}_\nu)DA\Psi_0\rangle, \tag{3.23}$$

the $f_j(\varepsilon)$ *are polynomials with real coefficients of degree at most* $2 + \frac{\nu-1}{2}$, *and*

$$f_0(\varepsilon) = g_0 + \mathcal{O}(\varepsilon). \tag{3.24}$$

Proof. The crucial point is that (and this is one of the reasons for our conditions on I_ε), since D_ε is "far" from the origin,

$$\sup_{z\in D_\varepsilon} \varepsilon\|G(z)\| \leq \begin{cases} C\varepsilon^{1/2} & \text{for } \nu = -1, \\ C\varepsilon & \text{for } \nu > -1. \end{cases}$$

Accordingly, for sufficiently small ε we have

$$\sup_{z\in D_\varepsilon} \|(D + \varepsilon G(z))^{-1}\| \leq 2, \tag{3.25}$$

and then from (2.14)

$$F(z,\varepsilon) = \varepsilon b - z - \varepsilon^2\Big\langle\Psi_0, A^*\Big(\sum_{k=0}^{m}(-\varepsilon)^k(DG(z)D)^{k+1}\Big)A\Psi_0\Big\rangle + q_m(z,\varepsilon). \tag{3.26}$$

By choosing $m = 0$ for $\nu = -1$, $m = 1$ for $\nu = 1$, and $m = 1 + \frac{\nu+3}{2}$ for $\nu > 1$, $q_m(z,\varepsilon)$ satisfies (3.20), i.e.

$$\sup_{\substack{0<\varepsilon<\varepsilon_0 \\ z\in D_\varepsilon}} \varepsilon^{-(2+p(\nu)+\frac{\nu}{2})}|q_m(z,\varepsilon)| < \infty. \tag{3.27}$$

Plug the expansion (3.10) into (3.26) with $N = -1$ for $\nu = -1$, $N = 3$ for $\nu = 1$, and $N = \nu+5$ for $\nu > 1$, and then keep in $H(z,\varepsilon)$ all the terms, which do not satisfy (3.27). □

Consider now the function $H(z,\varepsilon)$. From the definition of D_ε, it follows that for $z \in D_\varepsilon$ one has $|\operatorname{Im}\kappa^\nu| \geq C\varepsilon^{\nu/2}$ and $|\operatorname{Im}\kappa^2| \leq C\varepsilon^{2+\frac{\nu+1}{2}}$. Since all the coefficients appearing in the definition of $H(z,\varepsilon)$ are real, it follows that for sufficiently small ε we have

$$\inf_{z\in D_\varepsilon} |\operatorname{Im} H(z,\varepsilon)| \geq C\varepsilon^{2+\frac{\nu}{2}}. \tag{3.28}$$

Obviously, $H(z,\varepsilon)$ has limits as $\eta \to 0$,

$$H_\pm(x,\varepsilon) = \lim_{\eta\searrow 0} H(x \pm i\eta, \varepsilon). \tag{3.29}$$

Now (2.8) implies that $\eta \operatorname{Im} F(x+i\eta,\varepsilon) < 0$, and then from (3.28) and Lemma 3.3, it follows that

$$\eta \operatorname{Im} H(x+i\eta,\varepsilon) < 0. \tag{3.30}$$

Notice also that

$$\overline{H_+(x,\varepsilon)} = H_-(x,\varepsilon), \tag{3.31}$$

and on I_ε we have

$$|H_\pm(x,\varepsilon)| \geq |\operatorname{Im} H_\pm(x,\varepsilon)| \geq C\varepsilon^{2+\frac{\nu}{2}}. \tag{3.32}$$

Let $R(x,\varepsilon)$ and $I(x,\varepsilon)$ be the real and the imaginary parts of $H_+(x,\varepsilon)$, respectively, such that

$$H_\pm(x,\varepsilon) = R(x,\varepsilon) \pm iI(x,\varepsilon). \tag{3.33}$$

From (3.22) one has

$$R(x,\varepsilon) = \varepsilon b - x - \varepsilon^2 \sum_{j=0}^{\frac{\nu+3}{2}} (-x)^j f_j(\varepsilon) \tag{3.34}$$

(for $\nu = -1$ the sum in the right-hand side of (3.34) is zero, see (3.21)). For ε sufficiently small we have $R(\frac{\varepsilon b}{2},\varepsilon) > 0$ and $R(\frac{3\varepsilon b}{2},\varepsilon) < 0$, and for $x \in [\frac{\varepsilon b}{2}, \frac{3\varepsilon b}{2}]$,

$$-\frac{3}{2} < \frac{d}{dx} R(x,\varepsilon) < -\frac{1}{2}. \tag{3.35}$$

This implies that for sufficiently small ε the equation $R(x,\varepsilon) = 0$ has a unique solution $x_0(\varepsilon)$, i.e.

$$R(x_0(\varepsilon),\varepsilon) = 0. \tag{3.36}$$

In addition,

$$x_0(\varepsilon) = \varepsilon b + \mathcal{O}(\varepsilon^2). \tag{3.37}$$

Let now (see (3.33))

$$\Gamma(\varepsilon) = -I(x_0(\varepsilon),\varepsilon). \tag{3.38}$$

Notice that for sufficiently small ε (see (3.30) and (3.15)) we have $\Gamma(\varepsilon) > 0$. At this point we make the choice for I_ε as follows (notice that for all ν and sufficiently small ε, $I_\varepsilon \subset (\frac{\varepsilon b}{2}, \frac{3\varepsilon b}{2})$):

$$I_\varepsilon = \begin{cases} [x_0(\varepsilon) - \frac{1}{4}b\varepsilon,\ x_0(\varepsilon) + \frac{1}{4}b\varepsilon] & \text{for } \nu = -1, 1, \\ [x_0(\varepsilon) - \frac{\Gamma(\varepsilon)}{\varepsilon^2},\ x_0(\varepsilon) + \frac{\Gamma(\varepsilon)}{\varepsilon^2}] & \text{for } \nu > 1. \end{cases} \tag{3.39}$$

The next lemma estimates the error, when $F(z,\varepsilon)$ is replaced with $H(z,\varepsilon)$.

Lemma 3.4. *For sufficiently small* ε *we have*

$$\left| A_{g_\varepsilon}(t) - \frac{1}{2\pi i} \int_{I_\varepsilon} e^{-ixt} \left[\frac{1}{H_+(x,\varepsilon)} - \frac{1}{H_-(x,\varepsilon)} \right] dx \right| \leq C\varepsilon^{p(\nu)}. \tag{3.40}$$

Proof. For sufficiently small ε, and $z \in D_\varepsilon$, we have

$$|F(z,\varepsilon)| \geq \tfrac{1}{2}|H(z,\varepsilon)|. \tag{3.41}$$

Indeed, from Lemma 3.3 follows

$$|F(z,\varepsilon)| \geq |H(z,\varepsilon)| - C\varepsilon^{2+p(\nu)+\frac{\nu}{2}},$$

which together with (3.28), and the fact that $p(\nu) \geq \frac{1}{2}$, implies (3.41). Furthermore, from Lemma 3.3 and (3.41) follows

$$\left| \frac{1}{2\pi i} \int_{I_\varepsilon} e^{-ixt} \left[\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{H(x+i\eta,\varepsilon)} \right] dx \right| \leq C\varepsilon^{2+p(\nu)+\frac{\nu}{2}} \int_{I_\varepsilon} \frac{1}{|H(x+i\eta,\varepsilon)|^2} dx.$$

The estimate (3.28) implies that $\frac{1}{|H(x+i\eta,\varepsilon)|^2}$ is uniformly bounded for a fixed ε. Take the limit $\eta \searrow 0$ to get

$$\left| \lim_{\eta \searrow 0} \frac{1}{2\pi i} \int_{I_\varepsilon} e^{-ixt} \left[\frac{1}{F(x \pm i\eta,\varepsilon)} - \frac{1}{H(x \pm i\eta,\varepsilon)} \right] dx \right| \leq C\varepsilon^{2+p(\nu)+\frac{\nu}{2}} \int_{I_\varepsilon} \frac{1}{|H_\pm(x,\varepsilon)|^2} dx. \tag{3.42}$$

Now due to (3.32), (3.35), and (3.36) we have

$$|H_\pm(x,\varepsilon)| \geq C\sqrt{(x - x_0(\varepsilon))^2 + \varepsilon^{4+\nu}}, \tag{3.43}$$

and then

$$\varepsilon^{2+p(\nu)+\frac{\nu}{2}} \int_{I_\varepsilon} \frac{1}{|H_\pm(x,\varepsilon)|^2} dx \leq \varepsilon^{p(\nu)} \int_{-\infty}^{\infty} \frac{\varepsilon^{\frac{4+\nu}{2}}}{(x - x_0(\varepsilon))^2 + \varepsilon^{4+\nu}} dx = \varepsilon^{p(\nu)} \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx, \tag{3.44}$$

and the proof of the lemma is finished. □

We want to replace $H_\pm(x,\varepsilon)$ with the following function

$$L_\pm(x,\varepsilon) = -(x - x_0(\varepsilon)) \pm iI(x_0(\varepsilon),\varepsilon). \tag{3.45}$$

Inserted into (3.40), $L_\pm(x,\varepsilon)$ leads to a Lorentzian function. The next lemma estimates the error, when $H_\pm(x,\varepsilon)$ is replaced by $L_\pm(x,\varepsilon)$.

Lemma 3.5. *For sufficiently small* ε *we have*

$$\Big|\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{H_\pm(x,\varepsilon)}-\frac{1}{L_\pm(x,\varepsilon)}\right]dx\Big| \le C\varepsilon^{p(\nu)}. \tag{3.46}$$

Proof. As in the proof of Lemma 3.3 we have to estimate $H_\pm(x,\varepsilon)-L_\pm(x,\varepsilon)$. From (3.21), (3.22), and (3.29) we have

$$\sup_{x\in I_\varepsilon}\left\{\Big|\frac{d^2}{dx^2}R(x,\varepsilon)\Big|+\Big|\frac{d}{dx}R(x,\varepsilon)+1\Big|\right\}\le C\varepsilon^2 \tag{3.47}$$

and

$$\sup_{x\in I_\varepsilon}\Big|\frac{d}{dx}I(x,\varepsilon)\Big|\le C\varepsilon^{1+\frac{\nu}{2}};\;\; \sup_{x\in I_\varepsilon}\Big|\frac{d^2}{dx^2}I(x,\varepsilon)\Big|\le C\varepsilon^{\frac{\nu}{2}}. \tag{3.48}$$

Then from the Taylor expansion (with remainder) we get (see (3.47) and (3.48))

$$|H_\pm(x,\varepsilon)-L_\pm(x,\varepsilon)-(1+\tfrac{d}{dx}H_\pm(x_0(\varepsilon),\varepsilon))(x-x_0(\varepsilon))|\le C\varepsilon^{\min\{2,\frac{\nu}{2}\}}|x-x_0(\varepsilon)|^2, \tag{3.49}$$

$$|H_\pm(x,\varepsilon)-L_\pm(x,\varepsilon)|\le C\varepsilon^{p(\nu)}|x-x_0(\varepsilon)|. \tag{3.50}$$

Now

$$\frac{1}{H_\pm(x,\varepsilon)}-\frac{1}{L_\pm(x,\varepsilon)}=\frac{L_\pm(x,\varepsilon)-H_\pm(x,\varepsilon)}{L_\pm(x,\varepsilon)^2}+\frac{(L_\pm(x,\varepsilon)-H_\pm(x,\varepsilon))^2}{L_\pm(x,\varepsilon)^2H_\pm(x,\varepsilon)} \tag{3.51}$$

From (3.51), (3.49), (3.50), and the fact that (see (3.43) or (3.47))

$$|H_\pm(x,\varepsilon)|\ge|R(x,\varepsilon)|\ge C|x-x_0(\varepsilon)| \tag{3.52}$$

we get:

$$\Big|\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{H_\pm(x,\varepsilon)}-\frac{1}{L_\pm(x,\varepsilon)}\right]dx\Big|\le|1+\tfrac{d}{dx}H_\pm(x_0(\varepsilon),\varepsilon)|\Big|\int_{I_\varepsilon}e^{-ixt}\frac{(x_0(\varepsilon)-x)}{L_\pm(x,\varepsilon)^2}dx\Big|$$
$$+C\varepsilon^{\min\{2,\frac{\nu}{2}\}}\int_{I_\varepsilon}\frac{|x-x_0(\varepsilon)|^2}{|L_\pm(x,\varepsilon)|^2}dx+C\varepsilon^{2p(\nu)}\int_{I_\varepsilon}\frac{|x-x_0(\varepsilon)|}{|L_\pm(x,\varepsilon)|^2}dx. \tag{3.53}$$

With the notation $l(\varepsilon)=\frac{1}{2}$length I_ε we get (we always have $l(\varepsilon)\gg\Gamma(\varepsilon)$, see (3.39))

$$C\varepsilon^{\min\{2,\frac{\nu}{2}\}}\int_{I_\varepsilon}\frac{|x-x_0(\varepsilon)|^2}{|L_\pm(x,\varepsilon)|^2}dx+C\varepsilon^{2p(\nu)}\int_{I_\varepsilon}\frac{|x-x_0(\varepsilon)|^2}{|L_\pm(x,\varepsilon)|^3}dx$$
$$\le C\varepsilon^{\min\{2,\frac{\nu}{2}\}}l(\varepsilon)+C\varepsilon^{2p(\nu)}\ln\frac{l(\varepsilon)}{\Gamma(\varepsilon)}\le C\varepsilon^{p(\nu)}. \tag{3.54}$$

Furthermore, writing

$$\frac{(x_0(\varepsilon)-x)}{L_\pm(x,\varepsilon)^2}=\frac{1}{L_\pm(x,\varepsilon)}\pm\frac{i\Gamma(\varepsilon)}{L_\pm(x,\varepsilon)^2},$$

and using that $\frac{1}{\pi}\int_{\mathbf{R}} \frac{\Gamma(\varepsilon)}{x^2+\Gamma(\varepsilon)^2} dx = 1$, one has

$$\Big|\int_{I_\varepsilon} e^{-ixt} \frac{(x_0(\varepsilon)-x)}{L_\pm(x,\varepsilon)^2} dx\Big| \le \Big|\int_{I_\varepsilon} e^{-ixt} \frac{1}{L_\pm(x,\varepsilon)} dx\Big| + \pi. \tag{3.55}$$

Finally,

$$\Big|\int_{I_\varepsilon} e^{-ixt} \frac{1}{L_\pm(x,\varepsilon)} dx\Big| = \Big|\int_{-l(\varepsilon)/\Gamma(\varepsilon)}^{l(\varepsilon)/\Gamma(\varepsilon)} e^{-it\Gamma(\varepsilon)y} \frac{1}{y \mp i} dy\Big| \le C, \tag{3.56}$$

where the last inequality is obtained by estimating the last integral using the residue theorem. From (3.47) and (3.48) follows that

$$|1 + \tfrac{d}{dx} H_\pm(x_0(\varepsilon),\varepsilon)| \le C\varepsilon^{p(\nu)},$$

which together with (3.51), (3.54), (3.55), and (3.56) finishes the proof. □

We can now evaluate $A_{g_\varepsilon}(t)$.

Lemma 3.6. *For sufficiently small ε we have*

$$|A_{g_\varepsilon}(t) - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \le C\varepsilon^{p(\nu)}. \tag{3.57}$$

Proof. By direct computation

$$\frac{1}{2\pi i}\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{L_+(x,\varepsilon)} - \frac{1}{L_-(x,\varepsilon)}\right] dx = \frac{1}{\pi}\int_{I_\varepsilon} e^{-ixt} \frac{\Gamma(\varepsilon)}{(x-x_0(\varepsilon))^2+\Gamma(\varepsilon)^2} dx. \tag{3.58}$$

Due to (3.37), (3.23), and (2.2) we have

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\frac{\nu}{2}} \varepsilon^{2+\frac{\nu}{2}} + \mathcal{O}(\varepsilon^{3+\frac{\nu}{2}}), \tag{3.59}$$

which together with (3.39) implies

$$\Big|\Big(\int_{\mathbf{R}} - \int_{I_\varepsilon}\Big) e^{-ixt} \frac{\Gamma(\varepsilon)}{(x-x_0(\varepsilon))^2+\Gamma(\varepsilon)^2} dx\Big| \le C\int_{l(\varepsilon)}^\infty \frac{\Gamma(\varepsilon)}{x^2+\Gamma(\varepsilon)^2} dx \le C\varepsilon^{p(\nu)}. \tag{3.60}$$

Since by the residue theorem

$$\frac{1}{\pi}\int_{\mathbf{R}} e^{-ixt} \frac{\Gamma(\varepsilon)}{(x-x_0(\varepsilon))^2+\Gamma(\varepsilon)^2} dx = e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}, \tag{3.61}$$

(3.57) follows from Lemmas 3.4, 3.5, and the results (3.58) and (3.60). □

We are now in a position to formulate the main result.

Theorem 3.7. *Suppose* (A1)–(A5) *hold true. Then for sufficiently small ε we have*

$$|\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0\rangle - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \le C\varepsilon^{p(\nu)}. \tag{3.62}$$

Here $p(\nu) = \min\{2, (2+\nu)/2\}$, and

$$\Gamma(\varepsilon) = -i^{\nu-1} g_\nu b^{\nu/2} \varepsilon^{2+\nu/2}(1+\mathcal{O}(\varepsilon)), \tag{3.63}$$

$$x_0(\varepsilon) = b\varepsilon(1+\mathcal{O}(\varepsilon)). \tag{3.64}$$

Proof. The theorem follows from Lemma 3.6 by an argument due to Hunziker [15]. For completeness we reproduce it. Taking $t = 0$ in (3.57) one gets

$$|\langle \Psi_0, g_\varepsilon(H(\varepsilon))\Psi_0\rangle - 1| \leq C\varepsilon^{p(\nu)},$$

which gives (recall that $0 \leq g_\varepsilon(x) \leq 1$)

$$\|(1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}}\Psi_0\|^2 \leq C\varepsilon^{p(\nu)}. \tag{3.65}$$

Now

$$\begin{aligned} |\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0\rangle - A_{g_\varepsilon}(t)| &= |\langle (1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}}\Psi_0, e^{-itH(\varepsilon)}(1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}}\Psi_0\rangle| \\ &\leq \|(1 - g_\varepsilon(H(\varepsilon)))^{\frac{1}{2}}\Psi_0\|^2, \end{aligned}$$

which together with Lemma 3.6 and (3.65) finishes the proof. □

4 Continuum eigenvalues

While the starting formulae are again (2.9) and (2.14), here we assume that $G(z)$ is uniformly bounded and smooth (in the norm topology) in

$$D_a = \{z \,|\, |z| < a,\ \operatorname{Im} z \neq 0\} \tag{4.1}$$

for some $a > 0$. We use the standard notation, $C^{n,\theta},\ 0 \leq \theta \leq 1, n = 0, 1, \ldots$ for the class of functions whose n^{th} derivative is uniformly Hölder continuous of order θ in D_a. In particular, $C^{0,1}$ is the class of uniformly Lipschitz continuous functions in D_a. Suppose now that $G(z) \in C^{n,\theta}$ with $n + \theta > 0$. Then for $x \in (-a, a)$ and ε sufficiently small one can define (recall (2.14))

$$F_\pm(x, \varepsilon) = \lim_{\eta \searrow 0} F(x \pm i\eta, \varepsilon) = R(x, \varepsilon) \pm iI(x, \varepsilon). \tag{4.2}$$

Since for sufficiently small ε, $R(-a, \varepsilon) > 0$, $R(a, \varepsilon) < 0$, and $R(x, \varepsilon)$ is continuous, the equation

$$R(x, \varepsilon) = 0 \tag{4.3}$$

has at least one solution in $(-a, a)$. Moreover, from the fact that (see (2.14))

$$R(x, \varepsilon) = \varepsilon b - x - \varepsilon^2 S(x, \varepsilon), \tag{4.4}$$

where (uniformly for ε sufficiently small) $S(x, \varepsilon) \in C^{n,\theta}$, it is easy to see that any solution equals $\varepsilon b + \mathcal{O}(\varepsilon^2)$. In addition, for $n + \theta < 1$ (see also [6, Proposition 12]) any two solutions $x_0^1(\varepsilon), x_0^2(\varepsilon)$ satisfy $|x_0^1(\varepsilon) - x_0^2(\varepsilon)| \leq C\varepsilon^{\frac{2}{1-\theta}}$, while for $n + \theta \geq 1$ the solution is unique.

Let now $x_0(\varepsilon)$ be one of the solutions of (4.3) and, as before, define $\Gamma(\varepsilon)$ by

$$\Gamma(\varepsilon) = -I(x_0(\varepsilon), \varepsilon). \tag{4.5}$$

Notice that under our conditions

$$0 \leq \Gamma(\varepsilon) \leq C\varepsilon^2. \tag{4.6}$$

We are now in a position to formulate the analogue of the first part of Theorem 3.7 for embedded eigenvalues.

Theorem 4.1. *Assume that $G(z)$ is in $C^{n,\theta}$ on D_a. For sufficiently small ε we have the following two results:*

(i) *Assume $n = 0$, $0 < \theta < 1$, and*

$$\Gamma(\varepsilon) \geq C\varepsilon^\gamma \ \textit{ with } \ 2 \leq \gamma < \frac{2}{1-\theta}. \tag{4.7}$$

Then we have

$$|\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0\rangle - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \leq C\frac{1}{1-\theta}\varepsilon^\delta, \tag{4.8}$$

where

$$\delta = 2 - \gamma(1-\theta) > 0. \tag{4.9}$$

(ii) *For $n + \theta \geq 1$ we have*

$$|\langle \Psi_0, e^{-itH(\varepsilon)}\Psi_0\rangle - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \leq C \begin{cases} \varepsilon^2|\ln\varepsilon| & \textit{for } n = 0, \theta = 1, \\ \varepsilon^2 & \textit{for } n+\theta > 1. \end{cases} \tag{4.10}$$

Proof. The proof is similar to the proof of Theorem 3.7, but somewhat simpler, since now we assume smoothness in a whole neighborhood of zero, and not just an asymptotic expansion around zero.

Part (i): Choose in this case (notice that for sufficiently small ε we have $I_\varepsilon \subset (-a, a)$)

$$I_\varepsilon = [x_0(\varepsilon) - \frac{\Gamma(\varepsilon)}{\varepsilon^\delta}, x_0(\varepsilon) + \frac{\Gamma(\varepsilon)}{\varepsilon^\delta}], \tag{4.11}$$

and take, as before,

$$L_\pm(x, \varepsilon) = -(x - x_0(\varepsilon)) \pm iI(x_0(\varepsilon), \varepsilon). \tag{4.12}$$

Now we estimate directly $|F_\pm(x,\varepsilon) - L_\pm(x,\varepsilon)|$ on I_ε. Since we have $F_\pm(x_0(\varepsilon),\varepsilon) = L_\pm(x_0(\varepsilon),\varepsilon)$, from (4.4) and Hölder continuity one has on I_ε that

$$|F_\pm(x,\varepsilon) - L_\pm(x,\varepsilon)| \leq C\varepsilon^2|x - x_0(\varepsilon)|^\theta \tag{4.13}$$

In particular, from (4.7), (4.9), and (4.11), for all $x \in I_\varepsilon$ we have

$$|F_\pm(x,\varepsilon) - L_\pm(x,\varepsilon)| \leq C\varepsilon^2 \left(\frac{\Gamma(\varepsilon)}{\varepsilon^\delta}\right)^\theta$$

$$\le C\Gamma(\varepsilon)\frac{\varepsilon^2}{\Gamma(\varepsilon)^{1-\theta}\varepsilon^{\delta\theta}} \le C\Gamma(\varepsilon)\varepsilon^{2-(1-\theta)\gamma-\delta\theta} = C\Gamma(\varepsilon)\varepsilon^{\delta(1-\theta)}. \quad (4.14)$$

Since $0 < \delta < 2$, $0 < \theta < 1$, and $|L_\pm(x,\varepsilon)| \ge \Gamma(\varepsilon)$, from (4.14) one obtains that for all $x \in I_\varepsilon$ and ε sufficiently small

$$|F_\pm(x,\varepsilon)| \ge \tfrac{1}{2}|L_\pm(x,\varepsilon)|. \quad (4.15)$$

Again we estimate the error when replacing $F_\pm(x,\varepsilon)$ with $L_\pm(x,\varepsilon)$. From (4.11), (4.13) and (4.15):

$$\Big|\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{F_\pm(x,\varepsilon)} - \frac{1}{L_\pm(x,\varepsilon)}\right]dx\Big| \le C\varepsilon^2\int_{I_\varepsilon}\frac{|x-x_0(\varepsilon)|^\theta}{(x-x_0(\varepsilon))^2+\Gamma(\varepsilon)^2}dx$$
$$= C\varepsilon^2\left(\frac{1}{\Gamma(\varepsilon)}\right)^{1-\theta}\int_0^{\frac{1}{\varepsilon^\delta}}\frac{y^\theta}{y^2+1}dy \le C\frac{\varepsilon^2}{1-\theta}\left(\frac{1}{\Gamma(\varepsilon)}\right)^{1-\theta} \quad (4.16)$$

which together with (4.7) and (4.9) gives

$$\Big|\int_{I_\varepsilon} e^{-ixt}\left[\frac{1}{F_\pm(x,\varepsilon)} - \frac{1}{L_\pm(x,\varepsilon)}\right]dx\Big| \le C\frac{\varepsilon^\delta}{1-\theta}. \quad (4.17)$$

On the other hand, as in the proof of Lemma 3.6, we have

$$\Big|\Big(\int_{\mathbf{R}} - \int_{I_\varepsilon}\Big)e^{-ixt}\frac{\Gamma(\varepsilon)}{(x-x_0(\varepsilon))^2+\Gamma(\varepsilon)^2}dx\Big| \le C\varepsilon^\delta. \quad (4.18)$$

From the Lorentzian integral (3.61), (4.17), and (3.60) one obtains

$$|A_{g_\varepsilon}(t) - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \le C\frac{\varepsilon^\delta}{1-\theta}, \quad (4.19)$$

which together with Hunziker's argument (see the proof of Theorem 3.7) finishes the proof of the first part of the theorem.

Part (ii): At first sight the proof of the second part seems a bit more delicate, since we do not impose a lower bound as in (4.7) for $\Gamma(\varepsilon)$. Since for all $\eta > 0$, $\operatorname{Im} F(x \pm \eta, \varepsilon) \ne 0$ the idea is to make all the estimates in (2.9) before taking the limit $\eta \searrow 0$. So, consider $F_\pm(x,\eta,\varepsilon) = F(x \pm i\eta, \varepsilon)$ and subsequently (see (4.2), (4.3), (4.5)) $R(x,\eta,\varepsilon)$, $I(x,\eta,\varepsilon)$, $x_0(\eta,\varepsilon)$, $\Gamma(\eta,\varepsilon)$, $L_\pm(x,\eta,\varepsilon)$. Consider first the Lipschitz case, i.e. $n = 0$, $\theta = 1$. Fix ε sufficiently small. By choosing η as small as to assure that

$$I_{\eta,\varepsilon} = [x_0(\eta,\varepsilon) - \frac{\Gamma(\eta,\varepsilon)}{\varepsilon^2|\ln\varepsilon|}, x_0(\eta,\varepsilon) + \frac{\Gamma(\eta,\varepsilon)}{\varepsilon^2|\ln\varepsilon|}] \subset (-\frac{a}{2}, \frac{a}{2}), \quad (4.20)$$

one can mimic closely the estimates in the previous case. The only difference is that the integral $\int_0^{\frac{1}{\varepsilon^\delta}}\frac{y^\theta}{y^2+1}dy$ is replaced with $\int_0^{\frac{1}{\varepsilon^2|\ln\varepsilon|}}\frac{y}{y^2+1}dy \le C|\ln\varepsilon|$ and that in (3.60) ε^δ is replaced by $\varepsilon^2|\ln\varepsilon|$. One obtains that (uniformly as $\eta \searrow 0$)

$$\Big|\frac{1}{2\pi i}\int_{I_{\eta,\varepsilon}} dx\, e^{-ixt}\Big(\frac{1}{F(x+i\eta,\varepsilon)} - \frac{1}{F(x-i\eta,\varepsilon)}\Big) - e^{-it(x_0(\eta,\varepsilon)-i\Gamma(\eta,\varepsilon))}\Big| \le C\varepsilon^2|\ln\varepsilon|. \quad (4.21)$$

Taking $\eta \searrow 0$ in (4.21) and using $\lim_{\eta\searrow 0} x_0(\eta,\varepsilon) = x_0(\varepsilon)$ and $\lim_{\eta\searrow 0} \Gamma(\eta,\varepsilon) = \Gamma(\varepsilon)$, one obtains

$$|A_{g_\varepsilon}(t) - e^{-it(x_0(\varepsilon)-i\Gamma(\varepsilon))}| \leq C\varepsilon^2 |\ln \varepsilon|, \tag{4.22}$$

and again Hunziker's argument finishes the proof.

In the last case, namely $n+\theta > 1$, a more careful estimate is nedeed in order to kill the factor $|\ln \varepsilon|$, and we follow the proof of Lemma 3.5 in estimating the error due to the replace of $F_\pm(x,\eta,\varepsilon)$ with $L_\pm(x,\eta,\varepsilon)$. In this case we take (with a suitably small C)

$$I_{\eta,\varepsilon} = [x_0(\eta,\varepsilon) - C\frac{\Gamma(\eta,\varepsilon)}{\varepsilon^2}, x_0(\eta,\varepsilon) + C\frac{\Gamma(\eta,\varepsilon)}{\varepsilon^2}] \subset (-\frac{a}{2}, \frac{a}{2}). \tag{4.23}$$

In the proof of Lemma 3.5 one then makes the changes necessary to take into account that here

$$\sup_{x\in I_{\eta,\varepsilon}} \Big| \frac{d}{dx} I(x,\eta,\varepsilon) \Big| \leq C\varepsilon^2; \qquad \sup_{x\in I_{\eta,\varepsilon}} \Big| \frac{d^2}{dx^2} I(x,\eta,\varepsilon) \Big| \leq C\varepsilon^2,$$

$$|1 + \tfrac{d}{dx} F_\pm(x_0(\eta,\varepsilon),\eta,\varepsilon)| \leq C\varepsilon^2,$$

to obtain that

$$\Big| \int_{I_{\eta,\varepsilon}} e^{-ixt} \left[\frac{1}{F_\pm(x,\eta,\varepsilon)} - \frac{1}{L_\pm(x,\eta,\varepsilon)} \right] dx \Big| \leq C\varepsilon^2. \tag{4.24}$$

The rest of the argument remains unchanged. □

Remark 4.2. The regularity properties of $G(z)$ assumed above can be obtained using the Mourre method. See for example [27] and [2].

Remark 4.3. In the case $n+\theta \geq 1$, if actually $\Gamma(\varepsilon) = 0$, then $x_0(\varepsilon)$ will be an embedded eigenvalus, which can be seen as in [27] or [7].

Remark 4.4. Estimates uniform in time on the remainder term in the formula (1.1) for the survival probability in the non-analytic case can also be obtained from the general results in [6, 24, 33, 35]. Our estimates improve the previous ones; for example in the differentiable case (i.e. $n+\theta > 1$) we have an estimate of order ε^2, while all the previous results we are aware of have only an estimate of order ε. Actually in this case, as it can be seen from the discussion in [15], our result is optimal in the sense that a better estimate cannot hold, unless one replaces Ψ_0 with a better adapted wave function.

Remark 4.5. Aside from the condition that uniformly on D_a, $\varepsilon\|G(z)\| \leq \frac{1}{2}$ one can replace in all estimates leading to Theorem 4.1 ε by $\tilde{\varepsilon} = \varepsilon\|A\Psi_0\|$. This might be important in the cases when $\|A\Psi_0\|$ is very small, see [24].

5 Examples

As examples we consider one and two channel Schrödinger operators in odd dimensions. We shall restrict ourselves to the "physical" dimensions one and three. In the three dimensional case we consider various cases for both one and two channel Schrödinger operators. In the one dimensional case with local potentials we only consider the two channel case. We obtain explicit examples with $g_\nu \neq 0$ for ν arbitrarily large. In each case we find ν and g_ν, which gives the leading term in ε of $\Gamma(\varepsilon)$ (see (3.63)).

In the one channel case

$$H = -\Delta + V(\mathbf{x}), \tag{5.1}$$

$$(Wf)(\mathbf{x}) = W(\mathbf{x})f(\mathbf{x}), \tag{5.2}$$

in $L^2(\mathbf{R}^m)$, $m = 1, 3$, with V, W satisfying

$$\langle\cdot\rangle^\beta V \in L^\infty(\mathbf{R}^m), \tag{5.3}$$

$$\langle\cdot\rangle^\gamma W \in L^\infty(\mathbf{R}^m), \tag{5.4}$$

and β, γ are sufficiently large, in order to obtain the expansions below (see [19]), and we suppose that the singularity of $(H + \kappa^2)^{-1}$ at $\kappa = 0$ is coming from the existence of a nondegenerate eigenvalue at the threshold and/or a zero resonance. Note that we can allow singularities in V and W, but we have decided to omit the technicalities involved in dealing with such singularities.

In the two channel case we consider examples of a nondegenerate bound state of zero energy in the "closed" channel decaying due to the interaction with an odd dimensional Schrödinger operator in the open channel. Since only the bound state in the closed channel is relevant in the forthcoming discussion, we shall take $\mathbf{C}$ as the Hilbert space representing the closed channel, i.e. $\mathcal{H} = L^2(\mathbf{R}^m) \oplus \mathbf{C}$. As the unperturbed Hamiltonian we take

$$H = \begin{bmatrix} -\Delta + V & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.5}$$

where V satisfies (5.3), and as the perturbation we take

$$W = \begin{bmatrix} W_{11} & |W_{12}\rangle\langle 1| \\ |1\rangle\langle W_{12}| & b \end{bmatrix}, \tag{5.6}$$

which is a shorthand for

$$W \begin{bmatrix} f(\mathbf{x}) \\ \xi \end{bmatrix} = \begin{bmatrix} W_{11}(\mathbf{x})f(\mathbf{x}) + W_{12}(\mathbf{x})\xi \\ \int \overline{W_{12}(\mathbf{x})} f(\mathbf{x}) + b\xi \end{bmatrix}. \tag{5.7}$$

Here we assume

$$\langle\cdot\rangle^\gamma W_{11} \in L^\infty(\mathbf{R}^m), \quad \langle\cdot\rangle^{\gamma/2} W_{12} \in L^\infty(\mathbf{R}^m), \tag{5.8}$$

and furthermore that W_{11} is real-valued. In order to satisfy (3.1) we assume $b > 0$ in (5.6).

We use the following factorization of W. To simplify the notation below we introduce the weight function

$$\rho_\gamma = \langle\cdot\rangle^{-\gamma/2}. \tag{5.9}$$

In the one channel case we write

$$W = \rho_\gamma C \rho_\gamma, \tag{5.10}$$

i.e., C is the bounded operator of multiplication with $\langle \mathbf{x}\rangle^\gamma W(\mathbf{x})$. Writing the polar decomposition for C (with a self-adjoint D satisfying $D^2 = I$) as

$$C = |C|^{1/2} D |C|^{1/2}, \tag{5.11}$$

we have in this case

$$A = |C|^{1/2}\rho_\gamma. \tag{5.12}$$

In the two channel case let

$$B = \begin{bmatrix} \rho_{-\gamma} & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.13}$$

and

$$C = BWB = |C|^{1/2} D |C|^{1/2}, \tag{5.14}$$

where D is defined to be the identity on $\operatorname{Ker} C$, such that D is self-adjoint with $D^2 = I$. The operator C is bounded and self-adjoint, and we take

$$A = |C|^{1/2} B^{-1}, \tag{5.15}$$

i.e.

$$W = B^{-1}|C|^{1/2} D |C|^{1/2} B^{-1}. \tag{5.16}$$

Now, since $|C|^{1/2}$ is bounded, it is clear from (3.3) and (5.15) that we need the expansion of

$$B^{-1}(H+\kappa^2)^{-1}B^{-1} = \begin{bmatrix} \rho_\gamma(-\Delta + V + \kappa^2)^{-1}\rho_\gamma & 0 \\ 0 & \dfrac{1}{\kappa^2} \end{bmatrix}, \tag{5.17}$$

which, together with the fact that in our case

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.18}$$

reduces the problem of writing down (3.3) to the expansion of the resolvent in the scalar case. Summing up, in all cases the needed expansion of $(H+\kappa^2)^{-1}$ follows at once from the expansion of $\rho_\gamma(-\Delta+V+\kappa^2)^{-1}\rho_\gamma$. The expansions of $(-\Delta+V+\kappa^2)^{-1}$ near $\kappa = 0$ have been written down in [18, 17, 25, 19]. For the first example, although one can take the nedeed results from [18], for the reader's convenience we rederive them in the Appendix following the approach in [19] used in the rest of the paper . For the second example we need to carry the computations further than was done in [19], in order to get explicit expressions for the coefficients. The last example has not been treated previously, so we give some details of the computations.

5.1 Schrödinger operators in three dimensions

We first present the results in the case where the Schrödinger operator acts in three dimensions. The point zero is classified into four cases. It may be a regular point, in which case there is no singularity in the resolvent expansion. In the other three cases there exists at least one non-zero solution to $(-\Delta + V)\Psi = 0$, in the space $L^{2,-s}(\mathbf{R}^3)$, $1/2 < s \leq 3/2$. It turns out that $\Psi \in L^2(\mathbf{R}^3)$, if and only if $\langle V, \Psi \rangle = 0$, see Lemma A.2. In case there are solutions with $\langle V, \Psi \rangle \neq 0$, it is said that H has a resonance at the threshold. Among these solutions, one can choose a distinguished one, Ψ_c, called the *canonical zero resonance function* and all the others can be written as $\Psi = \alpha\Psi_c + \tilde{\Psi}$ with $\alpha \neq 0$ and $\tilde{\Psi} \in L^2(\mathbf{R}^3)$. See the Appendix for further details.

Our first result concerns the one channel case. Note that we take Ψ_0 to be real-valued.

Theorem 5.1 (One channel case). *Assume that V and W satisfy* (5.3) *and* (5.4) *with $\beta > 9$ and $\gamma > 5$, respectively. Assume that* (A1-3) *holds for $H = -\Delta + V$. Let*

$$X_j = \int_{\mathbf{R}^3} \Psi_0(\mathbf{x}) V(\mathbf{x}) x_j d\mathbf{x}, \quad j = 1, 2, 3. \tag{5.19}$$

Assume either that $X_j \neq 0$ for at least one j, or that $\langle \Psi_0, W\Psi_c \rangle \neq 0$. Then $\nu = -1$, and we have

$$g_{-1} = \frac{b^2}{12\pi}(X_1^2 + X_2^2 + X_3^2) + |\langle \Psi_0, W\Psi_c \rangle|^2. \tag{5.20}$$

If H does not have a resonance at the threshold, but still $X_j \neq 0$ for at least one j, then the second term in the right hand side of (5.20) *should be omitted, i.e.*

$$g_{-1} = \frac{b^2}{12\pi}(X_1^2 + X_2^2 + X_3^2). \tag{5.21}$$

Proof. Under our assumptions on V we have an asymptotic expansion (3.3) with $N = -1$, see Theorem A.4, and furthermore

$$G_{-1} = P_0 V G_3^0 V P_0 + |\Psi_c\rangle\langle\Psi_c|. \tag{5.22}$$

Insert this expression into (3.11) and use the explicit kernel (A.3) together with the result (see Lemma A.2)

$$\int_{\mathbf{R}^3} \Psi_0(\mathbf{x}) V(\mathbf{x}) d\mathbf{x} = 0 \tag{5.23}$$

to get (5.20).

If H does not have a resonance at the threshold, then (see Theorem A.4) the last term in the right hand side of (5.22) should be omitted. □

Remark 5.2. Let us explain through an example the significance of the conditions in the theorem. Take

$$V(\mathbf{x}) = \begin{cases} -V_0, & \text{if } |\mathbf{x}| \leq 1, \\ 0, & \text{if } |\mathbf{x}| > 1. \end{cases}$$

Here $V_0 > 0$ is a parameter. By adjusting this parameter, one can get a radial solution to $(-\Delta+V)\psi = 0$ for any angular momentum $\ell = 0, 1, \ldots$, which decays as $|\mathbf{x}|^{-\ell}$, as $|\mathbf{x}| \to \infty$. Thus for $\ell = 0$ we get a zero resonance. For $\ell = 1$ we get zero eigenvalues, such that at least one $X_j \neq 0$, see (5.19). For $\ell \geq 2$ all $X_j = 0$. For $\ell \geq 1$ the eigenvalue at zero is not simple. Examples with a simple zero eigenvalue can be obtained using only the radial part, see Remark 5.11. Note that in order to get $\langle \Psi_0, W\Psi_c\rangle \neq 0$ one will have to take a non-radial perturbation W.

Concerning the two channel case we have the following result.

Theorem 5.3 (Two channel case). *Assume that V and W satisfy* (5.3) *and* (5.4) *with $\beta > 9$ and $\gamma > 5$, respectively.*

(i) *Assume that $-\Delta + V$ has neither a threshold resonance nor a threshold eigenvalue. Then $\nu \geq 1$, and we have*

$$g_1 = \frac{-1}{4\pi} |\langle W_{12}, (I + G_0^0 V)^{-1} 1\rangle|^2. \tag{5.24}$$

(ii) *Assume that $-\Delta + V$ has a threshold resonance, and no threshold eigenvalue. Let Ψ_c denote the canonical zero resonance function. Assume that $\langle W_{12}, \Psi_c\rangle \neq 0$. Then $\nu = -1$, and*

$$g_{-1} = |\langle W_{12}, \Psi_c\rangle|^2. \tag{5.25}$$

Proof. In the two channel case we have $\Psi_0 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. We start with part (i). We get the required resolvent expansion (3.3) from Theorem A.3. Under the assumption on the potential V (3.3) holds for $N = 1$ and

$$G_1 = \begin{bmatrix} \frac{-1}{4\pi} |(I + G_0^0 V)^{-1} 1\rangle\langle (I + G_0^0 V)^{-1} 1| & 0 \\ 0 & 0 \end{bmatrix},$$

Now the proof consists in combining this expression with the definition (3.11) and the matrix W. This leads to the result stated in part (i). Concerning part (ii), then we use Theorem A.4 (see (A.19)) and perform the same computations as for part (i). $\square$

Remark 5.4. Note that the function $\psi = (I + G_0^0 V)^{-1} 1$ satisfies $(-\Delta + V)\psi = 0$ in the sense of distributions. Thus it is a generalized zero energy eigenfunction. Compare with the discussion in [4].

5.2 Schrödinger operators in one dimension

Since in the case dimension $m = 1$ and local short range potentials there is no bound state at the threshold, we can only consider the two channel case. For $m = 1$ the expansion of $\rho_\gamma(-\Delta + V + \kappa^2)^{-1}\rho_\gamma$ is much more complicated, due to the $1/\kappa$ singularity in the free resolvent. The result needed is obtained from [19]. Since it was not written down explicitly in [19], we reproduce some results needed to complete the computation.

The kernel of the free resolvent has the expansion

$$\begin{aligned}
(-\Delta+\kappa^2)^{-1}(\mathbf{x},\mathbf{y}) &= \frac{1}{2\kappa}e^{-\kappa|x-y|} \\
&= \frac{1}{2\kappa} - \frac{|x-y|}{2} + \kappa\frac{|x-y|^2}{4} + \mathcal{O}(\kappa^2) \\
&= \frac{1}{\kappa}G^0_{-1}(\mathbf{x},\mathbf{y}) + G^0_0(\mathbf{x},\mathbf{y}) + \kappa G^0_1(\mathbf{x},\mathbf{y}) + \mathcal{O}(\kappa),
\end{aligned} \tag{5.26}$$

where we also introduced the notation used here. Note that the G^0_j here are different from those defined in (A.3). We also use the notation $v(x) = |V(x)|^{1/2}$, $U(x) = 1$, if $V(x) \geq 0$, $U(x) = -1$, if $V(x) < 0$, such that the factorization used is $V = vUv$. We write $w = vU$.

The expansion results are obtained by studying the operator

$$M(\kappa) = U + v(-\Delta+\kappa^2)^{-1}v,$$

and its inverse, see [19, (4.3)]. We have

$$M(\kappa) = \tfrac{1}{2}\alpha P\kappa^{-1} + M_0 + M_1\kappa + \kappa^2 r(\kappa), \tag{5.27}$$

where

$$P = \alpha^{-1}|v\rangle\langle v|, \quad \alpha = \|v\|^2, \tag{5.28}$$

and $M_0 - U$ and M_1 are the integral operators given by the kernels

$$(M_0 - U)(x,y) = -\tfrac{1}{2}v(x)|x-y|v(y), \tag{5.29}$$

$$M_1(x,y) = \tfrac{1}{4}v(x)|x-y|^2 v(y), \tag{5.30}$$

and, for $\beta > 7$, the remainder $r(\kappa)$ is uniformly bounded in norm. Let $Q = 1 - P$, and let $S\colon QL^2(\mathbf{R}) \to QL^2(\mathbf{R})$ be the orthogonal projection onto $\operatorname{Ker} QM_0Q$. Then (see [19, Theorem 5.2 and (5.18)]) $\operatorname{Rank} S \leq 1$, and the formula for $M(\kappa)^{-1}$ is as follows.

$$\begin{aligned}
M(\kappa)^{-1} = {} & \frac{2\kappa}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1} \\
& + \frac{2}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1}Q(m_0+S+\kappa m_1(\kappa))^{-1}Q(1+\kappa\widetilde{M}(\kappa))^{-1} \\
& + \kappa^{-1}\frac{2}{\alpha}(1+\kappa\widetilde{M}(\kappa))^{-1}Q(m_0+S+\kappa m_1(\kappa))^{-1}Sq(\kappa)^{-1}S \\
& \times (m_0+S+\kappa m_1(\kappa))^{-1}Q(1+\kappa\widetilde{M}(\kappa))^{-1},
\end{aligned} \tag{5.31}$$

where we use the notation

$$\begin{aligned}
\widetilde{M}(\kappa) &= \frac{2}{\alpha}(M_0+\kappa M_1) + \mathcal{O}(\kappa^2), \\
m(\kappa) &= \frac{2}{\alpha}QM_0Q - \frac{2}{\alpha}\kappa Q(\frac{2}{\alpha}M_0^2 - M_1)Q + \mathcal{O}(\kappa^2)
\end{aligned}$$

$$\equiv m_0 + \kappa(m_1 + \kappa m_2(\kappa))$$
$$\equiv m_0 + \kappa m_1(\kappa), \tag{5.32}$$

and

$$q(\kappa) = q_0 + \mathcal{O}(\kappa) \tag{5.33}$$

as an operator in $SL^2(\mathbf{R})$, with

$$q(0) \equiv q_0 = Sm_1S. \tag{5.34}$$

In the formula (5.31), if QM_0Q is invertible as an operator in $QL^2(\mathbf{R})$, i.e. $S = 0$, the last term vanishes. If $S \neq 0$, we have the following result (see [19, Theorem 5.2]).

Proposition 5.5. *Assume $S \neq 0$. Let $\Phi \in SL^2(\mathbf{R})$, $\|\Phi\| = 1$. If Ψ is defined by*

$$\Psi(x) = \frac{1}{\alpha}\langle v, M_0\Phi\rangle + \frac{1}{2}\int_{\mathbf{R}} |x-y|v(y)\Phi(y)dy, \tag{5.35}$$

then

$$w\Psi = \Phi, \tag{5.36}$$

$\Psi \notin L^2(\mathbf{R})$, $\Psi \in L^\infty(\mathbf{R})$, and in the distribution sense

$$H\Psi = 0. \tag{5.37}$$

Conversely, if there exists $\Psi \in L^\infty(\mathbf{R})$ satisfying (5.37) *in the distribution sense, then*

$$\Phi = w\Psi \in SL^2(\mathbf{R}). \tag{5.38}$$

In addition,

$$q(0) = -\frac{2}{\alpha}\tilde{c}^2 S, \tag{5.39}$$

with

$$\tilde{c}^2 = \frac{2}{\alpha^2}|\langle v, M_0\Phi\rangle|^2 + \frac{1}{2}|\langle v, X\Phi\rangle|^2 > 0, \tag{5.40}$$

where X is the operator of multiplication with x. The function (unique up to a factor of modulus one)

$$\Psi_c = \frac{1}{\tilde{c}}\Psi \tag{5.41}$$

is called the canonical resonance function.

We are prepared to state the main result of this subsection.

Theorem 5.6 (Two channel case). *Assume V satisfies* (5.3) *with $\beta > 7$, and W satisfies* (5.8) *with $\gamma > 5$. Then we have the following results.*

(i) *If in the open channel there is no threshold resonance* (*i.e.* $S = 0$)*, then* $\nu \geq 1$*.*

(ii) *If there is a threshold resonance in the open channel* (*i.e.* $S \neq 0$)*, and* $\langle W_{12}, \Psi_c \rangle \neq 0$*, where* Ψ_c *is the canonical resonance function, then* $\nu = -1$*, and*

$$g_{-1} = |\langle W_{12}, \Psi_c \rangle|^2. \tag{5.42}$$

Proof. We have to insert the expansion (5.31) into

$$\rho_\gamma(-\Delta + V + \kappa^2)^{-1}\rho_\gamma = \rho_\gamma(-\Delta + \kappa^2)^{-1}\rho_\gamma \\ - \rho_\gamma(-\Delta + \kappa^2)^{-1} v M(\kappa)^{-1} v (-\Delta + \kappa^2)^{-1}\rho_\gamma, \tag{5.43}$$

and compute the $\frac{1}{\kappa}$ term. The main observation is that most of the singular terms vanish or cancel each other. Observe that

$$Q(1 + \kappa\widetilde{M}(\kappa))^{-1} v (-\Delta + \kappa^2)^{-1}\rho_\gamma \\ = \frac{1}{\kappa} Q|v\rangle\langle\rho_\gamma| - \frac{2}{\alpha} Q M_0 v G^0_{-1}\rho_\gamma - Q v G^0_0 \rho_\gamma + \mathcal{O}(\kappa) \\ = -\frac{2}{\alpha} Q M_0 v G^0_{-1}\rho_\gamma - Q v G^0_0 \rho_\gamma + \mathcal{O}(\kappa), \tag{5.44}$$

since by definition $Pv = v$ and $QP = 0$. Insertion of the expansion (5.31) into (5.43) gives four terms to be considered. From (5.44) follows that the third one is $\mathcal{O}(1)$. Computing the $\frac{1}{\kappa}$ contribution from the first two terms, one obtains (see (5.28))

$$\frac{1}{2}|\rho_\gamma\rangle\langle\rho_\gamma| - \frac{1}{2\alpha}\langle v^2, 1\rangle |\rho_\gamma\rangle\langle\rho_\gamma| = 0. \tag{5.45}$$

Since in the regular case (i.e. $S = 0$) the fourth term does not exist, the first part of the theorem follows from (5.44) and (5.45). Moreover, in the case $S \neq 0$, one has to consider only the fourth term. The computation of the $\frac{1}{\kappa}$ coefficient leads to (observe that $SQ = S$, and see also (5.26), (5.33), (5.39), (5.34), (5.35), and (5.40)),

$$\rho_\gamma(-\Delta + V + \kappa^2)^{-1}\rho_\gamma = \frac{1}{\tilde{c}^2\kappa}|\rho_\gamma\Psi\rangle\langle\rho_\gamma\Psi| + \mathcal{O}(1), \tag{5.46}$$

which gives (5.42), and the proof is finished. $\square$

5.3 Schrödinger operators on the half line with $\ell \geq 1$

In this subsection we consider the operator

$$H_{0,\ell} = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}, \quad \ell = 1, 2, \ldots, \tag{5.47}$$

on the space $\mathcal{H} = L^2(\mathbf{R}_+)$. It will provide us with examples of resolvent expansions, where we can verify Assumption (A5) with $\nu \geq 3$ odd and arbitrarily large. Note that the cases $\nu = -1$ and $\nu = 1$ were covered in the preceding sections.

It is well-known that the operator $H_{0,\ell}$ is essentially selfadjoint on $C_0^\infty((0,\infty))$. We now give the integral kernel of the resolvent $(H_{0,\ell}+\kappa^2)^{-1}$. To this end we need some results on special functions. We denote by $j_\ell(z)$ the spherical Bessel functions of the first kind, and by $h_\ell^{(1)}(z)$ the spherical Bessel functions of the third kind. We follow the notation and normalizations given in [1, Section 10.1]. We then define

$$u_\ell(z) = zj_\ell(z), \quad w_\ell(z) = izh_\ell^{(1)}(z).$$

We need the expansions of these two functions around zero. Using [1, (9.1.10),(10.1.1)], we get after some simplifications,

$$u_\ell(z) = z^{\ell+1}2^\ell \sum_{k=0}^{\infty} \frac{(-1)^k(k+\ell)!}{k!(2(k+\ell)+1)!} z^{2k} \tag{5.48}$$

For the function $w_\ell(z)$ we change the variable to get a simplified expression. Using [1, (10.1.16)], we get, computing as in [17],

$$w_\ell(i\zeta) = i^{-\ell}\zeta^{-\ell}\sum_{n=0}^{\infty} d_n\zeta^n, \tag{5.49}$$

$$d_n = (-1)^{n-1}\sum_{\substack{k=0\\k\geq \ell-n}}^{\ell} \frac{(\ell+k)!(-2)^{-k}}{k!(\ell-k)!}\frac{1}{(n-\ell+k)!}. \tag{5.50}$$

We recall from [17] the following result on the expansion coefficients of $h_1^{(1)}$. Note that we have not made the ℓ-dependence in d_n explicit, in order to avoid a complicated notation.

Lemma 5.7. *The coefficients* (5.50) *have the following property*

$$d_n = 0 \quad \textit{for } n = 1,3,\ldots,2\ell-1. \tag{5.51}$$

We now recall (see any standard text, for example [3, 26]) that the kernel of the resolvent is given as

$$(H_{0,\ell}+\kappa^2)^{-1}(r,r') = -\frac{i}{\kappa}u_\ell(i\kappa r_<)w_\ell(i\kappa r_>). \tag{5.52}$$

Here we have introduced the standard notation

$$r_> = \max\{r,r'\}, \quad r_< = \min\{r,r'\}. \tag{5.53}$$

The expansion results for u_ℓ and w_ℓ then lead to asymptotic expansions for the resolvents. We keep the same notation as in the previous subsection, so we introduce the weight function $\rho_\gamma(r) = \langle r\rangle^{-\gamma/2}$, now for $r \in \mathbf{R}_+$.

Proposition 5.8. *Assume that* $\gamma > 2p+3$. *We then have an expansion*

$$\rho_\gamma(H_{0,\ell}+\kappa^2)^{-1}\rho_\gamma = \sum_{j=0}^{p-1}\kappa^j\widetilde{G}_j + \kappa^p r_p(\kappa). \tag{5.54}$$

Here the expansion coefficients are bounded operators on $\mathcal{H}$, and the error term $r_p(\kappa)$ is uniformly bounded for κ small. We have

$$\widetilde{G}_j = 0, \quad j = 1, 3, \dots, 2\ell - 1. \tag{5.55}$$

We have the following integral kernel expressions (assuming $\gamma > 2\ell + 5$)

$$\widetilde{G}_0(r, r') = \rho_\gamma(r) \frac{(r_<)^{\ell+1}(r_>)^{-\ell}}{2\ell + 1} \rho_\gamma(r'), \tag{5.56}$$

$$\widetilde{G}_2(r, r') = \rho_\gamma(r) \frac{(r_<)^{\ell+1}(r_>)^{-\ell}}{2\ell + 1} \left[-\frac{1}{2}(r_<)^2 + \frac{\ell}{2\ell - 1}(r_>)^2 \right] \rho_\gamma(r'), \tag{5.57}$$

$$\widetilde{G}_{2\ell+1}(r, r') = 2^\ell \frac{\ell! d_{2\ell+1}}{(2\ell + 1)!} \rho_\gamma(r)(-r \cdot r')^{\ell+1} \rho_\gamma(r'). \tag{5.58}$$

Proof. The results (5.48), (5.49), and (5.52) yield, after some computations, the existence of an asymptotic expansion of the form given in (5.54). The result (5.55) is a consequence of Lemma 5.7, since the expansion of $z^{-\ell-1}u_\ell(z)$ only contains even powers of z. The kernel expressions follow after some tedious computations, which we omit. In the expression for (5.58) we used the relation $r_< \cdot r_> = r \cdot r'$. □

We can now describe our results. We consider the two channel set-up, where we now take the Hilbert space $\mathcal{H} = L^2(\mathbf{R}_+) \oplus \mathbf{C}$, and replace (5.5) by

$$H = \begin{bmatrix} H_{0,\ell} & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.59}$$

Theorem 5.9 (Two channel case). *Consider the two channel case with H given by* (5.59). *Assume that W given by* (5.6) *satisfies* (5.8) *with $\gamma > 2\ell + 5$. Assume that*

$$\langle W_{12}, r^{\ell+1} \rangle \neq 0.$$

Then we have $\nu = 2\ell + 1$ and

$$g_\nu = (-1)^{\ell+1} \left[\frac{\sqrt{\pi}}{2^{\ell+1}\Gamma(\ell + \frac{3}{2})} \right]^2 |\langle W_{12}, r^{\ell+1} \rangle|^2, \tag{5.60}$$

where Γ denotes the usual Gamma function.

Proof. We insert the expansion coefficients into (3.11), and after some simple computations, the result follows. A computer algebra computation using (5.50) yields the closed form of the coefficient, given in the theorem. □

Remark 5.10. The above result should be compared with the results in [4]. Here the same Hamiltonian is investigated using analytic continuation of the resolvent. The expression in (5.60) agrees with the one in [4].

Remark 5.11. One can also consider the operator

$$H = -\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + V(r), \tag{5.61}$$

where V decays sufficiently rapidly at infinity. The analysis of the threshold can be carried out along the same lines as above. In this case one can get a simple eigenvalue at the threshold for suitable V. The detailed analysis shows that also in the one channel case one can get examples, where (A5) is satisfied with ν arbitrarily large.

Remark 5.12. The results obtained here are similar to those in [17] for $-\Delta + V$ on $L^2(\mathbf{R}^m)$, $m \geq 5$ and odd. The free Schrödinger operator has an asymptotic expansion with coefficients having the same properties as above. The link between the two cases is given by $m = 2\ell + 3$. One could use the results in [17] to get one channel examples similar to those mentioned in the previous remark.

6 Further results

In this short section we list a few possible generalizations of the results obtained above.

(i) More examples, e.g., the one channel case in one dimension with nonlocal interactions, higher dimensions etc. The only problem is that the computations are more tedious.

(ii) Degenerate case, i.e. the case when 0 is a m-fold degenerate eigenvalue, $m < \infty$. If all the eigenvalues $b_1 < b_2 < \cdots < b_m$ of $P_0 W P_0$ on $P_0\mathcal{H}$ are strictly positive and nondegenerate, one can apply the method in Section 3 to each of them by replacing I_ε (see (3.16)) with $I_{j,\varepsilon} \subset (\varepsilon(b_j - a), \varepsilon(b_j + a))$, where

$$a = \frac{1}{2}\min_{j\neq k}\{b_1, |b_j - b_k|\}.$$

(iii) Even dimensions. In extending the theory developed in Section 3 one has to cope with the more complex asymptotic expansions for the resolvents [17, 19].

Acknowledgments G. Nenciu was partially supported by MaPhySto (The Danish National Research Foundation: Network in Mathematical Physics and Stochastics), and by Aalborg University. Both this support, and the hospitality of the Department of Mathematical Sciences, Aalborg University, are gratefully acknowledged. G. Nenciu was supported in part by CNCSIS under Grant 905-13A/2005. We thank the referee for suggesting that we include results on the continuum case.

A Resolvent Expansions in Three Dimensions

In this Appendix we show how to use the technique from [19] to derive the results on the asymptotic expansion of the resolvent of a Schrödinger operator in three dimensions and in particular to compute G_{-1} in (1.2).

This Appendix is included for the convenience of the reader. The results are also obtainable from the paper [18]. Moreover the expansion of the resolvent of a Schrödinger operator in three dimensions following the approach in [19] has been obtained in [10]; however the computation were not pushed far enough as to have an explicit formula for G_{-1}. We also refer to the survey paper [5] and the references therein. We start by introducing the setup. We refer to [18, 19] for several results, but provide an outline of the arguments needed to obtain the resolvent expansions. Let $H = -\Delta + V$ be a Schrödinger operator on $\mathcal{H} = L^2(\mathbf{R}^3)$. We assume that $V(\mathbf{x})$ is real-valued, and that for some suitably large $\beta > 0$ we have

$$\langle \cdot \rangle^{\beta} V \in L^{\infty}(\mathbf{R}^3). \tag{A.1}$$

Let $R_0^0(z) = (-\Delta - z)^{-1}$ be the resolvent of the free operator. Let $\mathcal{H}^s = L^{2,s}(\mathbf{R}^3)$, $s \in \mathbf{R}$, denote the weighted space. The expansion of $R_0^0(z)$ follows from the Taylor expansion of the well known kernel $\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}e^{-\kappa|\mathbf{x}-\mathbf{y}|}$ (see e.g. [18] for the mapping properties of the coefficients):

Proposition A.1. *Let $p \geq 1$ be an integer. Let $s > p + \frac{3}{2}$. Then we have the expansion*

$$R_0^0(-\kappa^2) = \sum_{j=0}^{p} \kappa^j G_j^0 + \mathcal{O}(\kappa^{p+1}) \tag{A.2}$$

as $\kappa \to 0$, $\operatorname{Re}\kappa \geq 0$, valid in the norm topology of $\mathcal{B}(\mathcal{H}^s, \mathcal{H}^{-s})$. The expansion coefficients are given by the integral kernels

$$G_j^0(\mathbf{x}, \mathbf{y}) = \frac{(-1)^j}{4\pi j!}|\mathbf{x} - \mathbf{y}|^{j-1}, \quad j = 0, 1, 2, \dots. \tag{A.3}$$

We have

$$G_0^0 \in \mathcal{B}(\mathcal{H}^{s_1}, \mathcal{H}^{-s_2}), \quad s_1, s_2 > \tfrac{1}{2}, \quad s_1 + s_2 \geq 2, \tag{A.4}$$
$$G_j^0 \in \mathcal{B}(\mathcal{H}^{s_1}, \mathcal{H}^{-s_2}), \quad s_1, s_2 > j + \tfrac{1}{2}, \quad j = 1, 2 \dots. \tag{A.5}$$

Following [19] we write the resolvent formula in the symmetric form

$$R(-\kappa^2) = (H + \kappa^2)^{-1} = R_0^0(-\kappa^2) - R_0^0(-\kappa^2) v M(\kappa)^{-1} v R_0^0(-\kappa^2), \tag{A.6}$$

where

$$M(\kappa) = U + vR_0^0(-\kappa^2)v = M_0 + \kappa M_1 + \kappa^2 M_2 + \kappa^3 M_3 + \mathcal{O}(\kappa^4), \tag{A.7}$$
$$M_0 = U + vG_0^0 v, \;\; M_j = vG_0^j v; \;\; j = 1, 2, ... \tag{A.8}$$

with $V = vUv$,

$$v(\mathbf{x}) = |V(\mathbf{x})|^{1/2};\ U(\mathbf{x}) = \begin{cases} +1 & \text{for } V(\mathbf{x}) \geq 0, \\ -1 & \text{for } V(\mathbf{x}) < 0, \end{cases} \tag{A.9}$$

and provided $\beta > 9$. Notice that $U^2 = I$. The analysis of the invertibility of $M(\kappa)$ starts with an analysis of $\operatorname{Ker} M_0$ (see also [18, 10]) .

Lemma A.2. (i) *Assume that* $\Phi \in \operatorname{Ker} M_0$. *Let* $\Psi = -G_0^0 v\Phi$. *Then* $\Psi \in \mathcal{H}^{-s}$ *for any* $s > \frac{1}{2}$, *and* $H\Psi = 0$ *in the sense of distributions. We have* $\Psi \in \mathcal{H}$, *if and only if* $\langle v1, \Phi\rangle = 0$ (*or equivalently* $\langle V1, \Psi\rangle = 0$).

(ii) *Assume* $\Psi \in \mathcal{H}^{-s}$ *for some* s, $\frac{1}{2} < s \leq \frac{3}{2}$, *and* $H\Psi = 0$ *in the sense of distributions. Let* $\Phi = Uv\Psi$. *Then* $\Phi \in \operatorname{Ker} M_0$.

Proof. Part (i): Let $\Phi \in \operatorname{Ker} M_0$ and $\Psi = -G_0^0 v\Phi$. Thus $(U + vG_0^0 v)\Phi = 0$, and we have $-\Delta\Psi = -v\Phi$ in the sense of distributions. Now $v\Psi = -vG_0^0 v\Phi = U\Phi$, such that $\Phi = Uv\Psi$. We conclude that $H\Psi = 0$ in the sense of distributions. The mapping properties of G_0^0 imply together with the decay of v that $\Psi \in \mathcal{H}^{-s}$ for any $s > \frac{1}{2}$.

Use the definitions to write

$$\begin{aligned}\Psi(\mathbf{x}) &= -\frac{1}{4\pi}\int \frac{1}{|\mathbf{x}-\mathbf{y}|}(v\Phi)(\mathbf{y})d\mathbf{y} \\ &= -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|}\int (v\Phi)(\mathbf{y})d\mathbf{y} - \frac{1}{4\pi}\int \Big(\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}|}\Big)(v\Phi)(\mathbf{y})d\mathbf{y}.\end{aligned}$$

Now the estimate

$$\Big|\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}|}\Big| \leq \frac{|\mathbf{y}|}{|\mathbf{x}-\mathbf{y}||\mathbf{x}|}$$

implies the last claim in Part (i).

Concerning Part (ii), then let Ψ satisfy the assumptions, and define $\Phi = Uv\Psi$. We have $-\Delta\Psi = -V\Psi = -v\Phi$. Define $\widetilde{\Psi} = -G_0^0 v\Phi$. Then $-\Delta\widetilde{\Psi} = -v\Phi$. We conclude that $-\Delta(\Psi - \widetilde{\Psi}) = 0$. Since $\Psi - \widetilde{\Psi} \in \mathcal{H}^{-s}$ for some s, $\frac{1}{2} < s \leq \frac{3}{2}$, it follows that $\Psi - \widetilde{\Psi} = 0$, since the only harmonic function in $\mathcal{H}^{-s}$ for these values of s is the zero function. We have shown that $\Psi = -G_0^0 v\Phi$. Thus $\Phi = Uv\Psi = -UvG_0^0 v\Phi$. Since $U^2 = I$, it follows that $\Phi \in \operatorname{Ker} M_0$. ☐

This lemma leads to the classification of the point zero in the spectrum of H. Notice first that since $M_0 = U + vG_0^0 v = U + \text{compact}$, $\operatorname{Ker} M_0$ is finite dimensional. Let S be the orthogonal projection on $\operatorname{Ker} M_0$ and S_1 be the orthogonal projection on $\operatorname{Ker} SM_1S$ (as an operator in $L^2(\mathbf{R}^3)$) . Now (see (A.8) and (A.3))

$$SM_1S = -\frac{1}{4\pi}|Sv1\rangle\langle Sv1| \tag{A.10}$$

which imply that $\dim S_1 \geq \dim S - 1$ and $\Phi \in \operatorname{Ker} SM_1S$, if and only if $\langle \Phi, v1\rangle = 0$. There are four cases to be considered.

(0) $\dim S = 0$. This is the regular case and $M(\kappa)^{-1}$ can be computed by Neumann expansion (see Theorem A.3).

(1) $\dim S = 1$ and $S_1 = 0$. In this case there are no bound states, but there is a threshold resonance described by the function $\Psi = -G_0^0 v\Phi$. We say that zero is an exceptional point of the first kind.

(2) $\dim S = \dim S_1 \geq 1$. Then zero is an eigenvalue of H of multiplicity $\dim S$. In this case we say that zero is an exceptional point of the second kind.

(3) $\dim S \geq 2$ and $\dim S_1 = \dim S - 1$. In this case zero is an eigenvalue of H of multiplicity $\dim S - 1$, and in addition there is a threshold resonance. In this case we say that zero is an exceptional point of the third kind. There is an ambiguity in writing the resonance function, since for all $\langle \Phi, v1\rangle \neq 0$, $\Psi = -G_0^0 v\Phi \notin L^2(\mathbf{R}^3)$ and is a candidate for a resonance function. However there is a distinguised one $\Psi_c = -G_0^0 v\Phi_c$ named the *canonical resonance function*. It is defined by the condition that the resonance contribution to the singular part of $R(-\kappa^2)$ takes the form $\frac{1}{\kappa}|\Psi_c\rangle\langle\Psi_c|$. The proof of Theorem A.4 gives the following formula for Ψ_c:

$$\Psi_c = \frac{\sqrt{4\pi}}{\|Sv1\|^2}(G_0^0 v|Sv1\rangle - P_0 V G_2^0 v|Sv1\rangle). \tag{A.11}$$

All the other resonance functions can be written as $\Psi = \alpha\Psi_c + \tilde{\Psi}$ with $\alpha \neq 0$ and $\tilde{\Psi} \in L^2(\mathbf{R}^3)$, satisfying $H\tilde{\Psi} = 0$ in the sense of distributions. In the rest of this appendix we write down the expansions in the cases of interest .

Theorem A.3. *Assume that zero is a regular point for H. Assume $\beta > 5$ in* (A.1). *Assume $s_1, s_2 > 5/2$. Then we have*

$$R(-\kappa^2) = G_0 + \kappa G_1 + \mathcal{O}(\kappa^2) \tag{A.12}$$

in the topology of $\mathcal{B}(\mathcal{H}^{s_1}, \mathcal{H}^{-s_2})$. We have

$$G_0 = G_0^0(I + VG_0^0)^{-1}, \tag{A.13}$$

$$G_1 = -\frac{1}{4\pi}|(I + G_0^0 V)^{-1}1\rangle\langle(I + G_0^0 V)^{-1}1|. \tag{A.14}$$

Proof. We only outline the main steps. Assume zero is a regular point for H. Thus we assume that $\operatorname{Ker} M_0 = \{0\}$. Since U has spectrum contained in $\{-1, +1\}$ and $vG_0^0 v$ is compact, it follows that M_0 is invertible. We want to compute explicitly the first two expansion coefficients in $R(-\kappa^2)$. If we assume $\beta > 5$, then we have, using the Neumann series,

$$M(\kappa) = M_0 + \kappa M_1 + \mathcal{O}(\kappa^2), \tag{A.15}$$

$$M(\kappa)^{-1} = M_0^{-1} - \kappa M_0^{-1} M_1 M_0^{-1} + \mathcal{O}(\kappa^2). \tag{A.16}$$

Now we insert this last expansion into (A.6) and compute to find that we have an expansion

$$R(-\kappa^2) = G_0 + \kappa G_1 + \mathcal{O}(\kappa^2),$$

with

$$G_0 = G_0^0 - G_0^0 v M_0^{-1} v G_0^0$$

and

$$G_1 = G_0^0 - G_0^0 v M_0^{-1} M_1 M_0^{-1} v G_0^0 - G_1^0 v M_0^{-1} v G_0^0 - G_0^0 v M_0^{-1} v G_1^0.$$

These coefficients can be rewritten in various ways. We show how to rewrite G_0. We have

$$\begin{aligned} G_0 &= G_0^0 - G_0^0 v (U + v G_0^0 v)^{-1} v G_0^0 \\ &= G_0^0 - G_0^0 v (I + U v G_0^0 v)^{-1} U v G_0^0 \\ &= G_0^0 - G_0^0 (I + v U v G_0^0)^{-1} v U v G_0^0 \\ &= G_0^0 - G_0^0 (I + V G_0^0)^{-1} V G_0^0 \\ &= G_0^0 (I + V G_0^0)^{-1}, \end{aligned}$$

by straightforward arguments. The coefficient is a bounded operator in $\mathcal{B}(\mathcal{H}^s, \mathcal{H}^{-s})$ for $s > 1$. An analogous, but somewhat longer computation, gives the expression (A.14) □

We also look at the case, when zero is an exceptional point of H. In this case we only carry the computations far enough to identify the coefficients of the singular terms in the expansion.

Theorem A.4. *Assume that zero is an exceptional point for H. Assume that $\beta > 9$ in* (A.1)*. Assume $s_1, s_2 > 9/2$. Then we have an asymptotic expansion*

$$R(-\kappa^2) = \frac{1}{\kappa^2} G_{-2} + \frac{1}{\kappa} G_{-1} + \mathcal{O}(1) \tag{A.17}$$

for κ small, in the topology of $\mathcal{B}(\mathcal{H}^{s_1}, \mathcal{H}^{-s_2})$, where

$$G_{-2} = 0, \tag{A.18}$$

$$G_{-1} = |\Psi_c\rangle\langle\Psi_c|, \tag{A.19}$$

for an exceptional point of the first kind,

$$G_{-2} = P_0, \tag{A.20}$$

$$G_{-1} = P_0 V G_3^0 V P_0, \tag{A.21}$$

for an exceptional point of the second kind, and

$$G_{-2} = P_0, \tag{A.22}$$

$$G_{-1} = P_0 V G_3^0 V P_0 + |\Psi_c\rangle\langle\Psi_c|, \tag{A.23}$$

for an exceptional point of the third kind. Here P_0 is the orthogonal projection onto the zero eigenspace, and Ψ_c is the canonical zero resonance function defined in (A.11)*. The operator G_3^0 is defined in* (A.3)*.*

Proof. We give the proof in the case of an exceptional point of third kind; the other two cases are particular cases when $S = S_1$ and $S_1 = 0$, respectively. The essential ingredient is [19, Corollary 2.2] (see also [20]). Rewrite (A.7) as

$$M(\kappa) = M_0 + \kappa M_1(\kappa). \tag{A.24}$$

Then by [19, Corollary 2.2] $M(\kappa)$ is invertible for small κ, if and only if the operator

$$m(\kappa) = \sum_{j=0}^{\infty} (-1)^j \kappa^j S\big[M_1(\kappa)(M_0 + S)^{-1}\big]^{j+1} S \tag{A.25}$$

is invertible in $S\mathcal{H}$. In the affirmative case we have

$$M(\kappa)^{-1} = (M(\kappa) + S)^{-1} + \frac{1}{\kappa}(M(\kappa) + S)^{-1} S m(\kappa)^{-1} S (M(\kappa) + S)^{-1}. \tag{A.26}$$

Now rewrite the expression for $m(\kappa)$ by inserting the expansion (A.7) and retaining terms up to order κ^2. The result is

$$m(\kappa) = m_0 + \kappa m_1 + \kappa^2 m_2 + \mathcal{O}(\kappa^3), \tag{A.27}$$

where

$$m_0 = S M_1 S, \tag{A.28}$$
$$m_1 = S M_2 S - S M_1 (M_0 + S)^{-1} M_1 S, \tag{A.29}$$
$$\begin{aligned} m_2 = {} & S M_3 S - S M_1 (M_0 + S)^{-1} M_2 S - S M_2 (M_0 + S)^{-1} M_1 S \\ & + S M_1 (M_0 + S)^{-1} M_1 (M_0 + S)^{-1} M_1 S. \end{aligned} \tag{A.30}$$

We iterate the procedure (the "telescoping principle") by applying [19, Corollary 2.2] to $m(\kappa)$. Notice that due to (A.28) and (A.10) the orthogonal projection on $\operatorname{Ker} m_0$ equals S_1 (see the discussion following Lemma A.2) . The result is

$$m(\kappa)^{-1} = (m(\kappa) + S_1)^{-1} + \frac{1}{\kappa}(m(\kappa) + S_1)^{-1} S_1 q(\kappa)^{-1} S_1 (m(\kappa) + S_1)^{-1}. \tag{A.31}$$

with

$$\begin{aligned} q(\kappa) &= q_0 + \kappa q_1 + \mathcal{O}(\kappa^2) \\ &= S_1 m_1 S_1 + \kappa[S_1 m_2 S_1 - S_1 m_1 (m_0 + S_1)^{-1} m_1 S_1] + \mathcal{O}(\kappa^2). \end{aligned} \tag{A.32}$$

Now q_0 is invertible, othervise one continues the procedure leading to stronger singularities than $\frac{1}{\kappa^2}$ for $M(\kappa)^{-1}$ and then for $R(-\kappa^2)$. Accordingly

$$q(\kappa)^{-1} = q_0^{-1} - \kappa q_0^{-1} q_1 q_0^{-1} + \mathcal{O}(\kappa^2), \tag{A.33}$$

It is a matter of computation to extract G_{-2} and G_{-1} from (A.6), (A.26), (A.31), and (A.33). Using

$$M_1S_1 = S_1M_1 = 0, \quad (M_0+S)^{-1}S = S(M_0+S)^{-1} = S,$$
$$(m_0+S_1)^{-1}S_1 = S_1(m_0+S_1)^{-1} = S_1, \quad \text{(A.34)}$$

and $SS_1 = S_1S = S_1$, a straightforward but somewhat tedious computation leads to:

$$G_{-2} = -G_0^0 vS_1q_0^{-1}S_1vG_0^0, \quad \text{(A.35)}$$
$$G_{-1} = G_0^0vS_1q_0^{-1}S_1m_2S_1q_0^{-1}S_1vG_0^0$$
$$- G_0^0v(S - S_1q_0^{-1}S_1m_1)(m_0+S_1)^{-1}(S - m_1S_1q_0^{-1}S_1)vG_0^0. \quad \text{(A.36)}$$

The last step is to check that (A.35) and (A.36) equal (A.22) and (A.23) respectively. As always in perturbation theory, there are a lot of identities originating in the resolvent equation which can be used to put the expansion coefficients in a simpler form. The one relevant here is given in Lemma A.5 below. We also include the proof from [18, Lemma 2.6].

Lemma A.5. *Assume that $f, g \in L^{2,s}(\mathbf{R}^3)$ for some $s > 5/3$, and that $\langle f, 1\rangle = 0$, $\langle g, 1\rangle = 0$. Then we have*

$$\langle f, G_2^0 g\rangle = -\langle G_0^0 f, G_0^0 g\rangle. \quad \text{(A.37)}$$

Proof. Since $f, g \in L^{2,s}(\mathbf{R}^3)$, it follows from (A.5) that $\langle f, G_2^0 g\rangle$ is well defined. Note also that the argument in the first part of the proof of Lemma A.2 shows that $G_0^0 f \in L^2(\mathbf{R}^3)$. Furthermore, $G_1^0 f = 0$. Thus it follows from the resolvent expansion (A.2) that

$$\langle f, G_2^0 g\rangle = \lim_{\kappa\to 0} \frac{1}{\kappa^2}\langle f, (R_0(-\kappa^2) - G_0^0)g\rangle \quad \text{(A.38)}$$
$$= \lim_{\kappa\to 0}\frac{1}{\kappa^2}\int \overline{\hat{f}}(\xi)[(\xi^2+\kappa^2)^{-1} - (\xi^2)^{-1}]\hat{g}(\xi)d\xi \quad \text{(A.39)}$$
$$= -\lim_{\kappa\to 0}\int (\xi^2+\kappa^2)^{-1}(\xi^2)^{-1}\overline{\hat{f}}(\xi)\hat{g}(\xi)d\xi \quad \text{(A.40)}$$
$$= -\int (\xi^2)^{-2}\overline{\hat{f}}(\xi)\hat{g}(\xi)d\xi = -\langle G_0^0 f, G_0^0 g\rangle. \quad \text{(A.41)}$$

Above we have assumed $\kappa > 0$. □

We start the simplification process by rephrasing the results in Lemma A.2. Let

$$T = -G_0^0 vS_1 \quad \text{and} \quad \widetilde{T} = UvP_0. \quad \text{(A.42)}$$

The operator T is a priori only bounded from $\mathcal{H}$ to $\mathcal{H}^{-s}$ for $s > 1/2$, but Lemma A.2 shows that it is actually bounded on $\mathcal{H}$, with $\operatorname{Ran} T = P_0\mathcal{H}$. We also have that $\widetilde{T}$ is bounded on $\mathcal{H}$ with $\operatorname{Ran}\widetilde{T} = S_1\mathcal{H}$. Now Lemma A.2 implies that

$$T\widetilde{T} = P_0 \quad \text{and} \quad \widetilde{T}T = S_1 \quad \text{(A.43)}$$

as operators on $\mathcal{H}$. The adjoint T^* is the closure of the operator $-S_1vG_0^0$. Combining these observations with Lemma A.5 one has the result that q_0 is invertible in $S_1\mathcal{H}$ with

$$S_1q_0^{-1}S_1 = -\widetilde{T}\widetilde{T}^*. \quad \text{(A.44)}$$

Now insert into (A.35) to get

$$G_{-2} = T\widetilde{T}\widetilde{T}^* T^* = P_0.$$

Similarly, using also (A.30), (A.34) and (A.8):

$$G_0^0 v S_1 q_0^{-1} S_1 m_2 S_1 q_0^{-1} S_1 v G_0^0 = \widetilde{T}^* S_1 m_2 S_1 \widetilde{T} = P_0 V G_3^0 V P_0. \tag{A.45}$$

As for the last term in (A.36), from (A.10) and (A.32) it follows that

$$(m_0 + S_1)^{-1} = S_1 - \frac{4\pi}{\|Sv1\|^4} |Sv1\rangle\langle Sv1|, \tag{A.46}$$

$$(S - S_1 q_0^{-1} S_1 m_1) S_1 = 0. \tag{A.47}$$

Using these results, (A.43), and (A.45), one obtains that (A.36) has the form (A.23) with

$$\Psi_c = \frac{\sqrt{4\pi}}{\|Sv1\|^2} (G_0^0 v |Sv1\rangle - P_0 V G_2^0 v |Sv1\rangle). \tag{A.48}$$

This concludes the proof of the theorm. □

Remark A.6. It is clear that if V decays sufficiently rapidly, then we can expand to any finite order, and in principle compute the coefficients explicitly in terms of the operators related to the zero resonance, zero eigenprojection, V, G_j^0, and operators constructed from these components. The computations rapidly get very complicated, and the many identities between the various expressions complicate matters further.

References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications Inc., New York, 1992, Reprint of the 1972 edition.

[2] S. Agmon, I. Herbst, and E. Skibsted, *Perturbation of embedded eigenvalues in the generalized N-body problem.* Comm. Math. Phys. **122** (1989), no. 3, 411–438.

[3] W. O. Amrein, J. M. Jauch, and K. B. Sinha, *Scattering theory in quantum mechanics*, W. A. Benjamin, Inc., Reading, Mass.-London-Amsterdam, 1977.

[4] B. Baumgartner, *Interchannel resonances at a threshold*, J. Math. Phys. **37** (1996), 5928–5938.

[5] D. Bollé, *Schrödinger operators at threshold* , In: *Ideas and methods in quantum and statistical physics* (Oslo, 1988), 173–196, Cambridge Univ. Press, Cambridge, 1992.

[6] O. Costin and A. Soffer, *Resonance theory for Schrödinger operators*, Comm. Math. Phys. **224** (2001), no. 1, 133–152.

[7] J. Dereziński and V. Jakšić, *Spectral theory of Pauli-Fierz operators*, J. Funct. Anal. **180** (2001), no. 2, 243–327.

[8] H. Dym and V. Katsnelson, *Contributions of Issai Schur to analysis*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), xci–clxxxviii, Progr. Math., 210, Birkhuser Boston, Boston, MA, 2003.

[9] D. Eschwé and M. Langer, *Variational principles for eigenvalues of self-adjoint operator functions*, Integral Equations Operator Theory **49** (2004), no. 3, 287–321.

[10] M. B. Ergodan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: I*, arXiv:math.AP/0410431.

[11] F. Gesztesy and H. Holden, *A unified approach to eigenvalues and resonances of Schrödinger operators using Fredholm determinants*, J. Math. Anal. Appl. **123** (1987), no. 1, 181–198.

[12] ______, *Addendum: "A unified approach to eigenvalues and resonances of Schrödinger operators using Fredholm determinants" [J. Math. Anal. Appl.* **123** *(1987), no. 1, 181–198]*, J. Math. Anal. Appl. **132** (1988), no. 1, 309.

[13] A. Grigis and F. Klopp, *Valeurs propres et résonances au voisinage d'un seuil*, Bull. Soc. Math. France **124** (1996), no. 3, 477–501.

[14] J. Howland, *The Livsic matrix in perturbation theory*, J. Math. Anal. Appl. **50** (1975), 415–437.

[15] W. Hunziker, *Resonances, metastable states and exponential decay laws in perturbation theory*, Comm. Math. Phys. **132** (1990), no. 1, 177–188.

[16] V. Jakšić, E. Kritchevski, and C.-A. Pillet, *Mathematical theory of the Wigner-Weisskopf atom*, mp-arc:05-83.

[17] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in $L^2(\mathbf{R}^m)$, $m \geq 5$*, Duke Math. J. **47** (1980), 57–80.

[18] A. Jensen and T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J. **46** (1979), 583–611.

[19] A. Jensen and G. Nenciu, *A unified approach to resolvent expansions at thresholds*, Rev. Math. Phys. **13** (2001), no. 6, 717–754.

[20] ______, *Erratum: "A unified approach to resolvent expansions at thresholds" [Rev. Math. Phys.* **13** *(2001), no. 6, 717–754]*, Rev. Math. Phys. **16** (2004), no. 5, 675–677.

[21] C. King, *Exponential decay near resonance, without analyticity*, Lett. Math. Phys. **23** (1991), 215–222.

[22] ———, *Resonant decay of a two state atom interacting with a massless non-relativistic quantised scalar field*, Comm. Math. Phys. **165** (1994), no. 3, 569–594.

[23] F. Klopp, *Resonances for perturbations of a semiclassical periodic Schrödinger operator*, Ark. Mat. **32** (1994), no. 2, 323–371.

[24] M. Merkli and I. M. Sigal, *A time-dependent theory of quantum resonances*, Comm. Math. Phys. **201** (1999), no. 3, 549–576.

[25] M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations*, J. Funct. Anal. **49** (1982), no. 1, 10–56.

[26] R. G. Newton, *Scattering theory of waves and particles*, second ed., Texts and Monographs in Physics, Springer-Verlag, New York, 1982.

[27] A. Orth, *Quantum mechanical resonance and limiting absorption: the many body problem*, Comm. Math. Phys. **126** (1990), no. 3, 559–573.

[28] J. Rauch, *Perturbation theory for eigenvalues and resonances of Schrödinger Hamiltonians*, J. Funct. Anal. **35** (1980), no. 3, 304–315.

[29] M. Reed and B. Simon, *Methods of modern mathematical physics. IV: Analysis of operators*, Academic Press, New York, 1978.

[30] B. Simon, *Resonances in N-body quantum systems with dilatation analytic potentials and the foundations of time-dependent perturbation theory*, Ann. of Math. (2) **97** (1973), 247–274.

[31] J. Sjöstrand and M. Zworski, *Elementary linear algebra for advanced spectral problems*, Preprint, December 2003.

[32] E. Skibsted, *Truncated Gamow functions, α-decay and the exponential law*, Comm. Math. Phys. **104** (1986), no. 4, 591–604.

[33] A. Soffer and M. Weinstein, *Time dependent resonance theory*, Geom. Funct. Anal. **8** (1998), 1086–1128.

[34] R. Waxler, *The time evolution of a class of meta-stable states*, Comm. Math. Phys. **172** (1995), no. 3, 535–549.

[35] K. Yajima, *Time-periodic Schrödinger equations*, In: *Topics in the theory of Schrödinger operators*, H. Araki and H. Ezawa (Eds.), World Scientific, 2004. Pages 9–69.