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Necessary and sufficient condition of the completeness and minimality for one system of exponents with degeneration.

The following system of exponents with "degenerated" coefficients $\omega^{\pm}$ is considered:

$$\left\{A^{+}(t)\cdot\omega^{+}(t)e^{int};A^{-}(t)\cdot\omega^{-}(t)e^{-i(n+1)t}\right\}_{n\geq 0}, \quad (1)$$

where $A^{\pm}(t)\equiv\left|A^{\pm}(t)\right|e^{i\alpha^{\pm}(t)}$-are complex-valued functions on the segment $[-\pi,\pi]$; $\omega^{\pm}(t)$ have the presentations

$$\omega^{\pm}(t)\equiv\prod_{i=1}^{l^{\pm}}\left\{\sin\left|\frac{t-\tau_i^{\pm}}{2}\right|\right\}^{\beta_i^{\pm}}, \quad (2)$$

$\left\{\tau_i^{\pm}\right\}\subset(-\pi,\pi);\left\{\beta_i^{\pm}\right\}\subset R$ are the sets of real numbers. Earlier we obtained the completeness and minimality of the system (1) in the space $L_p\equiv L_p(-\pi,\pi),1<p<+\infty$ for definite conditions on the functions $A^{\pm}(t)$ and the coefficients $\omega^{\pm}(t)$. In offered paper we obtain the necessary and sufficient condition of the completeness and minimality for this system in L_p for concrete conditions on the functions $A^{\pm};\omega^{\pm}$. We require the fulfilling of the following conditions:

1) $\alpha^{\pm}(t)$ are piecewise-Helder functions on the segment $[-\pi,\pi]$, $\{s_i\}_1^r$ is the set of discontinuity points of the function $\theta(t)\equiv\alpha^{-}(t)-\alpha^{+}(t)$ on $[-\pi,\pi]$, and moreover

$$\left\{\tau_i^{+},\tau_i^{-}\right\}\cap\{s_i\}_1^r=\{\varnothing\};$$

2) $\left|A^{\pm}(t)\right|$ are measurable functions on $[-\pi,\pi]$, and satisfy the condition

$$\sup_{(-\pi,\pi)} vrai\left\{\left|A^{+}(t)\right|^{\pm 1};\left|A^{-}(t)\right|^{\pm 1}\right\}\leq M<+\infty$$

Denote by $\{h_i\}_1^r$ the jumps of the function $\theta(t)$ at the points s_i, i.e. $h_i=\theta(s_i+0)-\theta(s_i-0), i=\overline{1,r}$.

Integer numbers n_i, $i=\overline{1,r}$ we define from the following correlations:

$$
\begin{gathered}
-\frac{1}{q} < \frac{h_i}{2\pi} + n_{i-1} - n_i \le \frac{1}{p}, \; i = \overline{1,r};\; n_0 = 0; \\
\omega = \theta(-\pi) - \theta(\pi) + 2\pi \cdot n_r
\end{gathered} \tag{3}
$$

Let $\frac{1}{p} + \frac{1}{q} = 1$. The following theorem takes place.

Theorem. Let the functions $A^{\pm}(t)$ satisfy the conditions 1), 2); the coefficients $\omega^{\pm}(t)$ have the presentations (2), moreover, the inequalities

$$
-\frac{1}{p} < \beta_i^{\pm} < \frac{1}{q}, \; i = \overline{1, l^{\pm}}
$$

are fulfiled. Then the system (1) is complete in L_p if and only if $\omega \le \frac{2\pi}{p}$; is minimal in L_p if and only if $\omega > -\frac{2\pi}{q}$; where the value ω is defined from (3).

Before proving this theorem we give some earlier known facts, which will be used further.

Statement 1 [2]. Let the system $\{x_i\}_0^{\infty} \subset B_1$ is minimal in B_1 and system $\{x_i\}_{-n}^{\infty} \subset B_2 \subset B_1$ is complete and minimal in B_2 for some $n \in N$, where B_i, $i = 1,2$ are Banach spaces, moreover, from the convergence in B_2 it follows the convergence in B_1. Then if $L\left[\{y_i\}_{i=-n}^{-1}\right] \cap B_1^* = \{0\}$ then $\{x_i\}_0^{\infty}$ is complete in B_1 where $\{y_i\}_{-n}^{\infty} \subset B_2^*$ is biorthoqonal to $\{x_i\}_{-n}^{\infty}$ system in B_2, $B_i^*, i = \overline{1,2}$ are conjugate spaces.

Statement 2. Let all conditions of theorem are fulfiled. If the inequalities

$$
-\frac{2\pi}{q} < h_k < \frac{2\pi}{p}, \; k = \overline{1,r};
$$

take place, then the system (1) forms the basis in L_p, $1 < p < +\infty$.

Proof of theorem. Not restricting generality, we can consider that the jumps h_i, $i = \overline{1,r}$ satisfy the conditions

$$
-\frac{2\pi}{q} < h_i \le \frac{2\pi}{p}, \; i = \overline{1,r}
$$

Really, otherwise we introduce the following function:

$$g(t) \equiv \begin{cases} 1, \ -\pi \le t < s_1, \\ e^{in_1\pi}, \ s_1 \le t < s_2, \\ e^{in_r\pi}, s_r \le t \le \pi \end{cases}$$

For simplicity we consider that $0 < s_1 < s_2 < \ldots < s_r < \pi$. We multiply each member of system (1) on this function and consider the new system:

$$\left\{ \widetilde{A}^{+}(t) \cdot \omega^{+}(t) \cdot e^{int}; \widetilde{A}^{-}(t) \cdot \omega^{-}(t) \cdot e^{-ikt} \right\}_{n \ge 0, k \ge 1},$$

where $\widetilde{A}^{\pm}(t) \equiv g(t) \cdot A^{\pm}(t)$. It is not difficult to verify that for this system all conditions of theorem are fulfiled, and all corresponding values n_i, $i = \overline{1, r}$; are equal to zero.

We follow the scheme of the work [2].

So, first of all we suppose that $-\frac{2\pi}{q} < \omega \le \frac{2\pi}{p}$. Denote by $\{s_{ik}\}$, $k = \overline{1, m}$ the points from the set $\{s_i\}_1^r$, at which for the corresponding jumps $\{h_i\}_1^r$, in the conditions (3) the sign of equality is reached; i.e. $h_{ik} = \frac{2\pi}{p}$. Then it is not difficult to note that for sufficiently small $\varepsilon > 0$ the inequalities

$$-\frac{2\pi}{q_\varepsilon} < h_i < \frac{2\pi}{p - \varepsilon}, \quad -\frac{1}{p - \varepsilon} < \beta_k^{\pm} < \frac{1}{q_\varepsilon},$$
$$-\frac{2\pi}{q_\varepsilon} < \omega < \frac{2\pi}{p - \varepsilon}, \quad i = \overline{1, r}; \quad k = \overline{1, l^{\pm}};$$

where $\frac{1}{q_\varepsilon} + \frac{1}{p - \varepsilon} = 1$ and $p - \varepsilon > 1$, take place. In this case according to statement 2 system (1) forms basis in $L_{p-\varepsilon}$ and, consequently, it is minimal in L_p.

Further we introduce the following functions:

$$\alpha(t) \equiv \begin{cases} -2\pi k, \ t \in [s_{ik}, s_{ik+1}), \ k = \overline{1, m}, \\ 0, \ t \notin [s_i, \pi], \ s_{im+1} = \pi; \end{cases}$$

$$A_0^{-}(t) \equiv e^{i\alpha(t)} \cdot A^{-}(t), \ \alpha_0^{-}(t) \equiv \arg A_0^{-}(t) = \alpha^{-}(t) + \alpha(t),$$

$$A_0^+(t) \equiv A^+(t) \cdot \begin{cases} e^{-imt}, if\ \omega \neq \dfrac{2\pi}{p}, \\ e^{-i(m+1)t}, if\ \omega = \dfrac{2\pi}{p}; \end{cases}$$

where

$$\alpha_0^+(t) \equiv \arg A_0^+(t) \equiv \begin{cases} \alpha^+(t) - m \cdot t,\ if\ \omega \neq \dfrac{2\pi}{p}, \\ \alpha^+(t) - (m+1)t,\ if\ \omega = \dfrac{2\pi}{p}; \end{cases}$$

Obviously, the jumps of the functions $\theta_0(t) \equiv \alpha_0^-(t) - \alpha_0^+(t)$ and $\theta(t)$ at the points $\{s_i\}_1^r$ are connected by correlations: $h_{i_k}^0 = h_{i_k} - 2\pi$, $k = \overline{1,m}$; and $h_i^0 = h_i$ for $i \notin \{i_k\}_1^m$, where $\{h_i^0\}$ are the jumps of $\theta_0(t)$ at the points s_i, $i = \overline{1,r}$.

We introduce into consideration the new system:

$$\left\{A_0^+(t)\omega^+(t)e^{int}; A_0^-(t)\omega^-(t)e^{-ikt}\right\}_{n\geq 0, k\geq 1}, \tag{4}$$

If we denote by ω_0 the value, corresponding to this system, defined from (3), then it will be equal to: $\omega_0 = \theta_0(-\pi+0) - \theta_0(\pi-0)$ for $\omega \neq \dfrac{2\pi}{p}$ and $\omega_0 = \omega - 2\pi$ for $\omega = \dfrac{2\pi}{p}$. Consequently, $\dfrac{h_{ik}^0}{2\pi} = -\dfrac{1}{q}$, $k = \overline{1,m}$ and

$$\frac{\omega_0}{2\pi} = \begin{cases} \dfrac{1}{2\pi}[\theta_0(-\pi+0) - \theta_0(\pi-0)],\ if\ \omega \neq \dfrac{2\pi}{p}, \\ -\dfrac{1}{q}, \qquad if\ \omega = \dfrac{2\pi}{p}. \end{cases}$$

Then for sufficiently small $\varepsilon > 0$ we have:

$$-\frac{2\pi}{q_\varepsilon} < h_i^0 < \frac{2\pi}{p+\varepsilon},\ -\frac{2\pi}{q_\varepsilon} < \omega_0 < \frac{2\pi}{p+\varepsilon},$$

where $\dfrac{1}{q_\varepsilon} + \dfrac{1}{p+\varepsilon} = 1$.

Further, consider the weight Hardi class $H^{\pm}_{p,\nu}$, introduced in $[\cdot]$. Following the work $[\cdot]$, we consider conjugation problem in classes $H^{\pm}_{p,\nu^{\pm}}$:

$$\begin{cases} F^{+}(\tau) + G(\tau)\cdot F^{-}(\tau) = q(\arg\tau),\ |\tau| = 1, \\ F^{-}(\infty) = 0, \end{cases}$$

where $\nu^{\pm} \equiv [\omega^{\pm}]^{p}$, $G(e^{it}) \equiv \dfrac{\omega^{-}(t)\cdot A_0^{-}(t)}{\omega^{+}(t)\cdot A_0^{+}(t)}, g \in L_{p,\nu^{+}}$ is arbitrary function, $L_{p,\mu}$ is usual Lebesque class with the weight μ. Denote by:

$$X_1^{\pm}(z) = \exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\frac{\omega^{-}(t)}{\omega^{+}(t)}\cdot\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2^{\pm}(z) = \exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|\frac{A_0^{-}(t)}{A_0^{+}(t)}\right|\cdot\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_3^{\pm}(z) = \exp\left\{\pm\frac{i}{4\pi}\int_{-\pi}^{\pi}\theta_0(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}.$$

Let

$$Z_i(z) \equiv \begin{cases} X_i^{+}(z), & |z| < 1, \\ [X_i^{-}(z)]^{-1}, & |z| > 1,\ i = \overline{1,3} \end{cases}$$

and

$$Z(z) \equiv \prod_i Z_i(z)$$

We present the function $\theta_0(t)$ in the form: $\theta_0(t) = \theta_0^0(t) + \theta_1(t)$, where $\theta_0^0(t)$ is continuous part, $\theta_1(t)$ is the function of jumps, which is defined by the formula:

$$\theta_1(-\pi) = 0,\ \theta_1(t) = \sum_{-\pi < s_k < t} h_k,\ -\pi < t \le \pi;$$

(not restricting generality, we consider that the function $\theta_0(t)$ is continuous from the left side).

Let $h_0 = h_0^{(1)} - h_0^{(0)}$, where $h_0^{(1)} = \theta_1(-\pi) - \theta_1(\pi)$, $h_0^{(0)} = \theta_0^0(\pi) - \theta_0^0(-\pi)$ Denote by

$$U(t) \equiv \prod_k \left\{ \sin\left|\frac{t-s_k}{2}\right| \right\}^{-\frac{h_k}{2\pi}},$$

$$U_0(t) \equiv \left\{ \sin\left|\frac{t-\pi}{2}\right| \right\}^{-\frac{h_0}{2\pi}} \cdot \exp\left\{ -\frac{1}{4\pi} \int_{-\pi}^{\pi} \theta_0^0(s) ctg \frac{t-s}{2} ds \right\}$$

Then the boundary values of the function $Z_i(z), i=\overline{1,3}$ have the following presentations:

$$\left|Z_1^-(e^{it})\right| = \left[\frac{\omega^+(t)}{\omega^-(t)}\right]^{\frac{1}{2}}, \left\|Z_2^-(e^{it})\right\|_\infty^{\pm 1} < +\infty,$$

$$\left|Z_3(e^{it})\right| = U_0(t) \cdot U(t) \cdot \left\{ \sin\left|\frac{t-\pi}{2}\right| \right\}^{-\frac{h_0}{2\pi}}$$

Applying these presentations, taking into account the inequality $\left(\frac{1}{p+\varepsilon} + \frac{1}{q_\varepsilon} = 1\right)$:

$$-\frac{1}{p+\varepsilon} < 1 - \frac{1}{p} = -\frac{h(\tau_{ik})}{2\pi} + \beta_{ik}^{\pm} = \frac{1}{q} < \frac{1}{q_\varepsilon},$$

for sufficiently small $\varepsilon > 0$, and doing analogously the work [1] we obtain, that the system (4) forms the basis in $L_{p+\varepsilon}$, and in this case biorthogonal system has the form:

$$\overline{h}_n^+(t) = \frac{\varphi_n^+(t)}{Z^+(e^{it})}, n \geq 0;\ \overline{h}_n^-(t) = \frac{\varphi_n^-(t)}{Z^+(e^{it})}, n \geq 1;$$

where

$$\varphi_n^{\pm}(t) = \frac{\sum_{k=1}^{n} b_n^{\pm} \cdot e^{\mp it}}{2\pi\ A_0^+(t)}, \left\{b_n^{\pm}\right\}$$

are definite coefficients. Applying the boundary value $Z^{\pm}(e^{it}), \left|h_n^+(t)\right|$ can be presented in the form:

$$\left|h_n^+(t)\right| = \frac{h(t)\cdot\left|\sum_{k=0}^{n} b_n^+ \cdot e^{-ikt}\right|}{\prod_{k=1}^{m}\left|e^{it} - e^{is_{ik}}\right|^{\frac{1}{q}}},$$

where the function $h(t) \geq \delta > 0$ in sufficiently small neighbourhoods of the points $\{s_{ik}\}_1^m$. From here it follows that the linear cover $L\left[\{h_n^+\}_{n=0}^m\right]$ doesn't belong to the space L_q. Then according to the statement 1 the system (1) is complete and minimal in L_p. And now, let $\omega \leq -\frac{2\pi}{q}$, for example, $-\frac{2\pi}{q} - 2\pi < \omega \leq -\frac{2\pi}{q}$. Then from the previous arguments it follows that in this case the system

$$\left\{A^+(t)\omega^+(t)e^{int}; A^-(t)\omega^-(t)e^{-\mathrm{int}}\right\}_{n\geq 1},$$

is complete and minimal, and, as a result, the system (1) is complete, but is not minimal in L_p. The other cases are proved analogously.

Theorem is proved.

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