Directed Polymers in Random Environment are Diffusive at Weak Disorder[1]

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Abstract

In this paper, we consider directed polymers in random environment with discrete space and time. For transverse dimension at least equal to 3, we prove that diffusivity holds for the path in the full weak disorder region, i.e., where the partition function differs from its annealed value only by a non-vanishing factor. Deep inside this region, we also show that the quenched averaged energy has fluctuations of order 1. In complete generality (arbitrary dimension and temperature), we prove monotonicity of the phase diagram in the temperature.

Les Polymères Dirigés En Milieu Aléatoire Sont Diffusifs À Faible Désordre

Résumé: Ce travail porte sur les polymères dirigés en milieu aléatoire en temps et espace discrets. Lorsque la dimension transverse est supérieure à trois, nous prouvons que le polymère est diffusif dans toute la région de faible désordre, c'est-à-dire où la fonction de partition diffère de son espérance par un facteur multiplicatif de limite non nulle. Bien à l'intérieur de cette région, nous établissons aussi que les fluctuations de l'énergie moyenne du polymère (à environnement fixé) sont d'ordre un. Dans le cas général d'une dimension et d'une température quelconques, nous montrons que le diagramme de phase est monotone dans la température.

Key words and phrases. Directed polymers, random environment, weak disorder, diffusive behavior, invariance principle, FKG inequality
AMS 1991 subject classifications. Primary 60K35; secondary 60G42, 82A51, 82D30

[1] Draft October 2004

[2] Corresponding author. Partially supported by CNRS, UMR 7599 "Probabilités et Modèles aléatoires"
http://www.proba.jussieu.fr/pageperso/comets/comets.html

[3] Partially supported by JSPS Grant-in-Aid for Scientific Research, Wakatekenkyuu (B) 14740071
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1 Introduction

In this classical model, the polymer is a long chain of size n in the $d+1$-dimensional space, which is directed: It stretches in the first direction of $\mathbb{Z}^{d+1}$, and therefore is modelled as a nearest neighbor path in $\mathbb{Z}^d$. On the other hand, the environment describes locations which can be favorable or hostile to the monomers: it is given by independent identically distributed random variables $\eta = \{\eta(n,x); n \in \mathbb{N}, x \in \mathbb{Z}^d\}$, with all finite exponential moments. The polymer is attracted by large positive values of the environment, and repelled by large negative ones. Further motivations for the model can be found in the physics litterature [14], [23], and a rigourous survey in [10]. All these ingredients are incorporated in the **polymer measure** with environment η:

$$\mu_n(d\omega) = Z_n^{-1} \exp\{\beta H_n(\omega)\}\ P(d\omega)\ , \tag{1.1}$$

with

$$H_n(\omega) = \sum_{t=1}^{n} \eta(t, \omega_t)\ , \quad Z_n = P[\exp\{\beta H_n(\omega)\}]\ .$$

Here, P denotes the distribution of the simple random walk on the integer lattice $\mathbb{Z}^d$, $\beta > 0$ denotes the "temperature inverse" and prescribes how strongly the polymer path ω interacts with the medium, and the "partition function" $Z_n = P[\exp\{\beta H_n(\omega)\}]$ is the normalizing constant making μ_n a probability measure on the path space. The reader will make the distinction between P and the law Q of the environment η. Note also that the measure μ_n depends on n, β and on the environment η. We denote by λ the function

$$\lambda(\beta) = \ln Q[\exp\{\beta\eta(t,x)\}] \in \mathbb{R}\ , \quad \beta \in \mathbb{R}, \tag{1.2}$$

which is assumed to be finite on the whole real line.

Remark 1.1 We can naturally define the polymer measure (1.1) with negative β, not only with positive ones. However, considering negative β merely amounts to considering $-\eta$ with $|\beta|$ as the inverse temperature. For this reason, we can restrict ourselves to positive β without loss, as far as real β is concerned. Moreover, this restriction helps us simplify the exposition of some results in this paper, e.g., Theorem 3.2 below.

The issue is to understand the asymptotics of the polymer ω as $n \to \infty$ under the measure μ_n, for typical realization of the environment. In particular, one would like to determine the exponent $\xi = \xi(d,\beta) \in [1/2, 1)$ such that

$$|\omega_n| \text{ is of order } n^{\xi}$$

as $n \to \infty$. Another –but related– quantity of interest is the exponent $\chi = \chi(d,\beta) \in [0, 1/2]$ for the fluctuations of the normalizing constant, i.e. such that

$$\ln Z_n - a_n \text{ is of order } n^{\chi} \text{ for some constant } a_n$$

as $n \to \infty$. These exponents ξ, χ depend also on the distribution of the environment η.

The ground state of the model, defined as the limit when $\beta \to \infty$, is the so-called oriented last passage percolation model. For the ground state it is believed that the exponents $\xi(d,\infty), \chi(d,\infty)$ are universal, more precisely that they have the same value for all distributions of η. Recently, Johansson, together with Baik and Deift, rigourously calculated the

values of these exponents in dimension $d = 1$ and for specific distributions for η. More precisely, in dimension $d = 1$ and for exponential and geometric distributions, it is proven in [19] that $\chi(1, \infty) = 1/3$, together with the Tracy-Widom law for limit fluctuations. Also, for a one-dimensional Poissonized model, $\chi(1, \infty) = 1/3$ is obtained in [3] together with the Tracy-Widom limit, though $\xi(1, \infty) = 2/3$ is proved in [20]: the path is superdiffusive, in contrast with the underlying simple random walk which is diffusive (corresponding to $\xi = 1/2$).

A number of predictions, conjectures and numerical estimates can be found in the physical literature [23], on the values on such exponents, and relations between them. In particular, for all $\beta \in (0, \infty]$, the scaling relation

$$\chi = 2\xi - 1$$

is believed to hold in complete generality. This relation can be derived at a heuristic level as a scaling in the Kardar-Parisi-Zhang equation [22], which status is, unfortunately, not clear at a mathematical level. Instead, partial results have been obtained rigourously in specific situations [24], [29], [26], [7]. In fact, much is still open, especially for $d \geq 2$.

Bolthausen [5] placed the polymer model in the framework of martingales, and noticed that the almost-sure limit of the rescaled partition function is subject to a dichotomy:

$$\lim_n \frac{Z_n}{QZ_n} \quad \begin{cases} > 0 & Q\text{-a.s.} \\ \text{or} & \\ = 0 & Q\text{-a.s.} \end{cases} \tag{1.3}$$

A natural manner for measuring the disorder due to the random environment, is to call **weak disorder** the first case, and **strong disorder** the second one. Note that weak disorder can be defined as the region where $\chi = 0$ and $a_n = n\lambda(\beta)$. The terminology is justified by observing that the former case happens in large enough dimension for small β (including $\beta = 0$) and the latter case for large β and general unbounded environment. More precisely, a series of papers [18], [5], [1], [31] lead to the following.

Theorem A *Assume $d \geq 3$ and β small enough so that*

$$P(\exists n > 0 : \omega_n = 0) < \exp\{-[\lambda(2\beta) - 2\lambda(\beta)]\} \ . \tag{1.4}$$

Then, weak disorder holds and, for almost every realization of the environment, the rescaled path:

$$\omega^{(n)} = \left(\omega_{nt}/\sqrt{n}\right)_{t \geq 0} , \tag{1.5}$$

converges in law to the Brownian motion with diffusion matrix $d^{-1} I_d$.

This result was much a surprise for both mathematics and physics communities who did not expect that diffusivity could take place!

The second moment method was used to derive the theorem. The assumption on β means that the martingale Z_n/QZ_n is bounded in L^2, and it is far from being necessary: A weaker quantitative condition for weak disorder is obtained in [4] using size-biasing. Fifteen years were necessary to improve on it: The next result is a criterium for weak disorder, where the critical quantity is

$$I_n = \mu_{n-1}^{\otimes 2}(\omega_n = \tilde{\omega}_n) \ ,$$

i.e., the probability for two polymers ω and $\tilde{\omega}$ independently sampled from the polymer measure in the same environment, to meet at time n.

Theorem B *([6] for the Gaussian case, [9] for the general case). For non-zero β it holds*

$$\Big\{ \lim_n (Z_n/QZ_n) = 0 \Big\} = \Big\{ \sum_n I_n = \infty \Big\} \quad Q\text{-a.s.}$$

The result is obtained by writing the semi-martingale decomposition of $\ln Z_n/QZ_n$, and studying separately the terms. The above criterium is a refined (conditional) second moment condition, and the criterion can also be used to obtain quantitative infomation on the polymer measure itself, on its concentration and localization [9] in the strong disorder regime.

In the present paper, we first establish the monotonicity in β concerning the dichotomy (1.3):

Theorem 1.1 *There exists a critical value $\beta_{\rm c} = \beta_{\rm c}(d) \in [0, \infty]$ with*

$$\begin{aligned} \beta_{\rm c} &= 0, \qquad \text{for } d = 1, 2, \\ 0 < \beta_{\rm c} &\leq \infty \qquad \text{for } d \geq 3 \end{aligned}$$

such that the weak disorder holds if $\beta \in \{0\} \cup (0, \beta_{\rm c})$ and the strong disorder holds if $\beta > \beta_{\rm c}$.

We also prove monotonicity for the Lyapunov exponent, see Theorem 3.2. This result implies the absence of reentrant phase transition in the phase diagram of the model. The theorem follows from a correlation inequality, a natural ingredient which, however, appears here for the first time (as far as we know) in the field of directed polymers.

Now, we will focus on the regime of weak disorder. There, it is natural to expect that diffusive behavior takes place in the whole weak disorder region, not only under the stronger assumption (1.4). Our main result is indeed:

Theorem 1.2 *Assume $d \geq 3$ and weak disorder. Then, for all bounded continuous function F on the path space,*

$$\lim_n \mu_n[F(\omega^{(n)})] = \mathbf{E}F(B)$$

in probability, where $\omega^{(n)}$ is the rescaled path defined by (1.5) and B is the Brownian motion with diffusion matrix $d^{-1}I_d$. In particular, this holds for all $\beta \in [0, \beta_{\rm c})$.

Incidently, the statement shows that the scaling relation between exponents does hold in the full weak disorder region, with $\xi = 1/2$ and $\chi = 0$.

In this paper, we also consider the fluctuations of extensive thermodynamic quantities other than the partition function: we show that these are typically of order 1 – like $\ln Z_n$ itself –, but we can prove this result only in part of the weak disorder region:

Theorem 1.3 *Assume $d \geq 3$ and (1.4). Then, the energy averaged over the path*

$$\mu_n[H_n] - n\lambda'(\beta) \;\; \text{converges } Q\text{-a.s. to a finite random variable.}$$

as $n \to \infty$. A similar result holds for the entropy of μ_n with respect to P, see (6.5).

In the proof of theorem 1.2 we introduce an infinite time horizon measure on the path space which is a natural limit of the sequence μ_n. This measure is a time inhomogeneous Markov chain which depends on the environment. We cannot prove the central limit theorem for this Markov chain directly, but we need to average over the environment. In order to prove convergence in probability with respect to the environment, we use again a second moment method by introducing a second independent copy of the polymer before performing this average. All through, we use the convergence of the series $\sum I_n$ as a main technical quantitative ingredient.

To prove Theorem 1.3 we use analytic functions arguments. The crucial estimate is a bound on the second moment of some complex random variable, this explains why we do assume (1.4). It is well known that analytic martingales are powerful tools to study disordered systems (e.g., section 5 of [8]) in the regime of bounded second moment.

Our paper is organized as follows. After recalling some notations and basic facts, we prove the existence of the critical temperature, together with characterization of the weak disorder phase that we will use further on (section 3). We then introduce the Markov chain depending on the environment in section 4. Section 5 deals with Gaussian behavior of the polymer, and section 6 with limits of energy and entropy. In the last section, we illustrate the results in the case of Bernoulli environment, emphasizing their relations with (last passage) oriented percolation.

2 Notations and known facts

Let

$$\bar{\zeta}_n(\omega,\beta) = \exp\{\beta H_n(\omega) - n\lambda(\beta)\}$$

Then, for all β,

$$W_n = Z_n \exp\{-n\lambda(\beta)\} = Z_n/Q[Z_n] = P[\bar{\zeta}_n(\omega,\beta)]$$

is a positive martingale with respect to the σ-fields $\mathcal{G}_n = \sigma\{\eta(s,x), s \le n, x \in \mathbb{Z}^d\}$. By the martingale convergence theorem, it follows that

$$\lim_{n\to\infty} W_n = W_\infty \quad Q\text{-a.s.}\,,$$

where W_∞ is a non-negative random variable. It is easy to see that the event $\{W_\infty > 0\}$ is in the tail σ-field of $\{\mathcal{G}_n, n \ge 0\}$, hence it is trivial by Kolmogorov 0-1 law. This shows the dichotomy weak disorder versus strong disorder in (1.3), which reads, in our new notation,

$$Q\{W_\infty > 0\} \;=\; \begin{cases} 1 & \Longleftrightarrow \text{ weak disorder,} \\ \text{or} & \\ 0 & \Longleftrightarrow \text{ strong disorder.} \end{cases} \tag{2.1}$$

It is well known [6, 9] that the weak disorder can happen if the tranverse dimension is large enough, i.e., $d \ge 3$. For $d \ge 3$,

$$\pi_d := P[\exists n > 0 : \omega_n = 0] \in (0,1)\,, \tag{2.2}$$

and (1.4) can be rephrased as

$$\lambda(2\beta) - 2\lambda(\beta) < -\ln \pi_d \quad \Longrightarrow \quad W_\infty > 0 \;\; Q\text{-a.s.}$$

For $x \in \mathbb{Z}^d$, let P^x be the law of the simple random walk in $\mathbb{Z}^d$ starting at x. If $\theta_{n,x}$ denotes the shift operator given by

$$\theta_{n,x}\eta : (t,y) \mapsto \eta(n+t, x+y) \ ,$$

then we have by definition of W_n

$$W_n \circ \theta_{0,x} = P^x \left[\bar{\zeta}_n\right] \ .$$

By definition of W_n again, and by the simple Markov property, we have also

$$W_n \circ \theta_{0,x} = P^x \left[\exp\{\beta\eta(1,\omega_1) - \lambda(\beta)\} W_{n-1} \circ \theta_{1,\omega_1}\right] \tag{2.3}$$

and hence

$$W_\infty \circ \theta_{0,x} = P^x \left[\exp\{\beta\eta(1,\omega_1) - \lambda(\beta)\} W_\infty \circ \theta_{1,\omega_1}\right] \tag{2.4}$$

by taking the limit as $n \to \infty$.

3 Characterizations of the weak disorder phase and monotonicity

We start by gathering some useful characterizations of weak disorder, which should be compared to those in the case where $\mathbb{Z}^d$ is replaced by a regular tree [21, p.134]. Before stating the next proposition, we make a remark. For $\delta \in (0,1)$, (W_n^δ) is a uniformly integrable random variable. Therefore,

$$\lim_{n\to\infty} Q[W_n^\delta] = Q[W_\infty^\delta]. \tag{3.1}$$

Proposition 3.1 *The following statements are equivalent for any $\delta \in (0,1)$.*

(a1) *The martingale W_n is uniformly integrable.*

(a2) *The martingale W_n is L^1-convergent.*

(b1) *Weak disorder holds, i.e., $W_\infty > 0$, Q-a.s.*

(b2) *The limit (3.1) is positive.*

(c1) *There exists a process*

$$(X_n, e_n) = ((X_{n,x})_{x\in\mathbb{Z}^d}, (e_{n,x})_{x\in\mathbb{Z}^d}), \quad n \in \mathbb{N}$$

with values in $(\mathbb{R}^{\mathbb{Z}^d})^2$ such that

$$(e_n)_{n\in\mathbb{N}} \overset{\text{law}}{=} (\exp\{\beta\eta(n,\cdot) - \lambda(\beta)\})_{n\in\mathbb{N}}\,, \tag{3.2}$$

$$\text{For all } (n,x) \in \mathbb{N}\times\mathbb{Z}^d,\ Q[X_{n,x}] = 1, \tag{3.3}$$

$$\text{For all } (n,x) \in \mathbb{N}\times\mathbb{Z}^d,\ X_{n,x} = P^x[e_{n+1,\omega_1} X_{n+1,\omega_1}], \tag{3.4}$$

$$\text{For all } n \in \mathbb{N},\ X_n \text{ is independent of } e_1, \ldots, e_n. \tag{3.5}$$

(c2) *There exists a non-negative random field $X = (X_x)_{x\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$ such that $Q[X_x] = 1$ for all $x \in \mathbb{Z}^d$ and such that*

$$X \overset{\text{law}}{=} (P^x[e_{\omega_1} X_{\omega_1}])_{x\in\mathbb{Z}^d}$$

holds for any $\mathbb{R}^{\mathbb{Z}^d}$-valued random variable $e = (e_x)_{x\in\mathbb{Z}^d}$, independent of X, and

$$e \overset{\text{law}}{=} \exp(\beta\eta(1,\cdot) - \lambda(\beta)).$$

Remark 3.1 Statements (a-1,2), (b-1,2) are natural. We will see in sections 4 and 5, that (c1) is actually an important feature of the weak disorder phase, allowing us to construct the Markov chain μ in (4.2). The somewhat similar condition (c2) is in the flavor of "condition (γ)" in [21, Théorème 1]. □

Proof of Proposition 3.1: (a1) $\Longleftrightarrow$ (a2): This follows from standard martingale convergence results [13].
(b1) $\Longleftrightarrow$ (b2): This is obvious from the dichotomy (either $W_\infty = 0$, Q-a.s., or $W_\infty > 0$, Q-a.s.).
(a2) $\Longrightarrow$ (b1): The L^1-convergence implies $Q[W_\infty] = 1$, and hence (b1) by the dichotomy.
(b1) $\Longrightarrow$ (c1): Set

$$X_{n,x} = W_\infty \circ \theta_{n,x}/Q[W_\infty]\ , \quad e_{n,x} = \exp\{\beta\eta(n,x) - \lambda(\beta)\}. \tag{3.6}$$

We then have (3.2), (3.3) and (3.5). Moreover, we obtain (3.4) by (2.4).
(c1)$\Longrightarrow$ (a1): We will prove the uniform integrability by showing that

$$(Q[X_{0,0}|\widetilde{\mathcal{G}}_n])_{n\geq 1} = (W_n)_{n\geq 1} \tag{3.7}$$

where $\widetilde{\mathcal{G}}_n = \sigma[e_1, .., e_n]$. Iterating (3.4), we see from Markov property that

$$X_{0,0} = P[e_{1,\omega_1} \dots e_{n,\omega_n} X_{n,\omega_n}].$$

Taking the Q-expectation conditionally on $\widetilde{\mathcal{G}}_n$, and observing (3.3) and (3.5), we arrive at

$$Q[X_{0,0}|\widetilde{\mathcal{G}}_n] = P^x[e_{1,\omega_1} \cdots e_{n,\omega_n} Q[X_{n,\omega_n}]] = P^x[e_{1,\omega_1} \cdots e_{n,\omega_n}],$$

which proves (3.7).

(b1)$\Longrightarrow$ (c2): Define $X_n = (X_{n,x})_{x\in\mathbb{Z}^d}$ and $e_n = (e_{n,x})_{x\in\mathbb{Z}^d}$ by (3.6). We prove that X_1 is what we look for. Since X_1 is independent of e_1, we have

$$\begin{aligned} (P^x[e_{\omega_1} X_{1,\omega_1}])_{x\in\mathbb{Z}^d} &\overset{\text{law}}{=} (P^x[e_{1,\omega_1} X_{1,\omega_1}])_{x\in\mathbb{Z}^d} \\ &= (W_\infty \circ \theta_{0,x}/Q[W_\infty])_{x\in\mathbb{Z}^d} \\ &\overset{\text{law}}{=} X_1, \end{aligned}$$

where we have used (2.4) on the second line.
(c2)$\Longrightarrow$ (c1): Suppose that $\overline{e}_n = (\overline{e}_{n,x})_{x\in\mathbb{Z}^d}$ $(n \in \mathbb{N})$ are independent of X and

$$(\overline{e}_n)_{n\in\mathbb{N}} \overset{\text{law}}{=} (\exp(\beta\eta(n,\cdot) - \lambda(\beta)))_{n\in\mathbb{N}}\,.$$

We define $\overline{X}_n = (\overline{X}_{n,x})_{x\in\mathbb{Z}^d}$ $(n \in \mathbb{N})$ recursively by

$$\overline{X}_0 = X, \ \ \overline{X}_{n+1,x} = P^x[\overline{e}_{n,\omega_1} \overline{X}_{n,\omega_1}].$$

By the construction, $(\overline{X}_n, \overline{e}_n)$, $n = 0, 1, 2, \dots$ is a stationary process. Hence, the sequence of laws

$$\rho_n(ds_0 \cdots ds_n) = Q((\overline{X}_{n-j}, \overline{e}_{n-j}) \in ds_j,\ j = 0, \dots, n), \quad n \in \mathbb{N}$$

are consistent. Therefore, by Kolmogorov's extension theorem, there is a process (X_n, e_n), $n = 0, 1, 2, \ldots$ such that

$$Q((X_j, e_j) \in ds_j,\ j = 0, \ldots, n) = Q((\overline{X}_{n-j}, \overline{e}_{n-j}) \in ds_j,\ j = 0, \ldots, n), \quad n \in \mathbb{N}$$

Then, (3.2) and (3.3) are obvious, while the recursion for $\overline{X}_n$ implies (3.4). Finally, we see (3.5) from the fact that $\overline{X}_0$ and $\overline{e}_0, \ldots, \overline{e}_{n-1}$ are independent. □

We now turn to the monotonicity of the phase transition. We define the Lyapunov exponent by

$$\psi(\beta) = -\lim_{n \nearrow \infty} \frac{1}{n} Q[\ln W_n] = \lambda(\beta) - \lim_{n \nearrow \infty} \frac{1}{n} Q[\ln Z_n]. \tag{3.8}$$

The limit exists by subadditivity [9, Proposition 1.5]. We see from Jensen's inequality that $\psi(\beta)$ is non-negative. Moreover, ψ is continuous in β, since $\lim_{n \nearrow \infty} \frac{1}{n} Q[\ln Z_n]$ is convex in β.

Theorem 3.2 (a) *There exists a critical value $\beta_c = \beta_c(d) \in [0, \infty]$ with*

$$\beta_c = 0, \qquad \text{for } d = 1, 2, \tag{3.9}$$

$$0 < \beta_c \le \infty \qquad \text{for } d \ge 3 \tag{3.10}$$

such that

$$Q\{W_\infty > 0\} = \begin{cases} 1 & \text{if } \beta \in \{0\} \cup (0, \beta_c), \\ 0 & \text{if } \beta > \beta_c. \end{cases} \tag{3.11}$$

(b) *The Lyapunov exponent $\psi(\beta)$ is non-decreasing in $\beta \in [0, \infty)$. In particular, there exists $\beta_c^\psi = \beta_c^\psi(d)$ with*

$$\beta_c \le \beta_c^\psi \le \infty, \tag{3.12}$$

such that

$$\psi(\beta) \begin{cases} = 0 & \text{if } \beta \in \mathbb{R} \cap [0, \beta_c^\psi], \\ > 0 & \text{if } \beta \in \mathbb{R} \backslash [0, \beta_c^\psi] \end{cases} \tag{3.13}$$

Remark 3.2 It is natural to expect that $\beta_c = \beta_c^\psi$, i.e., the absence of the intermediate phase. However, this is an open problem at the moment, as well as whether weak or strong disorder hold at the critical value β_c.

Theorem 3.2 is a consequence of the monotonicity described in part (b) of the following lemma.

Lemma 3.3 (a) *Assume that $\phi : (0, \infty) \longrightarrow \mathbb{R}$ is $\mathcal{C}^1$ and that there are constants $C, p \in [1, \infty)$ such that*

$$|\phi'(u)| \le Cu^p + Cu^{-p}, \quad \text{for all } u > 0.$$

Then, $\phi(W_n), \frac{\partial \phi(W_n)}{\partial \beta} \in L^1(Q)$, $Q\phi(W_n)$ is $\mathcal{C}^1$ in $\beta \in \mathbb{R}$, and

$$\frac{\partial}{\partial \beta} Q\phi(W_n) = Q \frac{\partial}{\partial \beta} \phi(W_n).$$

(b) *Suppose in addition that ϕ is concave on $(0, \infty)$. Then,*

$$Q \frac{\partial}{\partial \beta} \phi(W_n) \le 0 \quad \text{for } \beta \ge 0. \tag{3.14}$$

Proof: (a): Let $I = [0, \beta_1]$ $(0 < \beta_1 < \infty)$ and

$$X_n = \frac{\partial W_n}{\partial \beta} = P[(H_n - n\lambda')\bar{\zeta}_n].$$

We first check that, for all n,

$$(*1) \qquad \sup_{\beta \in I} W_n, \ \sup_{\beta \in I} W_n^{-1}, \ \sup_{\beta \in I} |X_n| \in L^p(Q) \text{ for all } p \in [1, \infty),$$

and thereby that

$$(*2) \qquad \sup_{\beta \in I} \left| \frac{\partial \phi(W_n)}{\partial \beta} \right| \in L^1(Q).$$

For $(*1)$, we have

$$\begin{aligned} W_n^{-p} &\leq P[\bar{\zeta}_n]^{-p} \\ &\leq P[\bar{\zeta}_n^{-p}] \\ &\leq e^{pn\lambda} P \exp\left(p\beta \sum_{1 \leq s \leq n} |\eta(s, \omega_s)| \right). \end{aligned}$$

The property $(*1)$ claimed for W_n^{-1} is obvious from the above expression. W_n^p and $|X_n|^p$ are bounded similarly.

The claim $(*2)$ follows from $(*1)$ and from

$$\left| \frac{\partial \phi(W_n)}{\partial \beta} \right| = |\phi'(W_n) X_n| \leq (C W_n^p + C W_n^{-p}) |X_n|.$$

It is now, easy to conclude part (a) of the lemma. Since $\phi(W_n)$ is $\mathcal{C}^1$ in $\beta \in \mathbb{R}$, we have

$$\phi(W_n(\beta_1)) = \phi(1) + \int_0^{\beta_1} \frac{\partial \phi(W_n)}{\partial \beta} d\beta \quad \text{for all } \beta_1 \in \mathbb{R}.$$

The properties claimed in part (a) of the lemma follow from this expression, $(*1)$ and Fubini's theorem.

(b): We have

$$Q \frac{\partial}{\partial \beta} \phi(W_n) = Q[\phi'(W_n) X_n] = P\left[Q[\phi'(W_n)(H_n - n\lambda')\bar{\zeta}_n] \right]$$

Now, for a fixed path ω, the probability measure $\bar{\zeta}_n dQ$ is product, and therefore satisfies the FKG inequality [25, p.78]. The function $H_n - n\lambda'$ is increasing in η, while $\phi'(W_n)$ is a decreasing, since ϕ is concave. These imply

$$Q[\phi'(W_n)(H_n - n\lambda')\bar{\zeta}_n] \leq Q[\phi'(W_n)\bar{\zeta}_n] Q[(H_n - n\lambda')\bar{\zeta}_n] = 0,$$

and hence (3.14). □

Proof of Theorem 3.2: (a): By applying Lemma 3.3 to $\phi(x) = x^\delta$ $(0 < \delta < 1)$, it follows that the limit (3.1) is non-increasing in $\beta \in [0, \infty)$. This, together with Proposition 3.1, implies the existence of the values β_c with the property (3.11). We then see (3.9) from [9, Theorem 1.3(b)], and (3.10) from Theorem A in section 1.

(b): By applying Lemma 3.3 to $\phi(x) = \log x$, it follows that the limit (3.8) is non-decreasing in $\beta \in [0, \infty)$. This, together with the continuity of ψ, implies the existence of the values β_c^ψ with the property (3.13). We then see (3.12) from the obvious fact that $\psi(\beta) > 0$ implies $W_\infty = 0$, Q-a.s. □

4 The weak disorder polymer measure and its long time behavior

As a general fact, the measure μ_n is a (time-inhomogeneous) Markov chain, with transition probabilities

$$\mu_n(\omega_{i+1} = y|\omega_i = x) = \frac{\exp\{\beta\eta(i+1,y) - \lambda\}W_{n-i-1} \circ \theta_{i+1,y}}{W_{n-i} \circ \theta_{i,x}} \, P(\omega_1 = y|\omega_0 = x)$$

for $0 \leq i < n$, and $\mu_n(\omega_{i+1} = y|\omega_i = x) = P(\omega_1 = y|\omega_0 = x)$ for $i \geq n$. Indeed, one can check that, for any path $x_{[0,m]} = (x_0, \dots x_m)$ of length $m \leq n$,

$$\mu_n(\omega_{[0,m]} = x_{[0,m]}) = \bar{\zeta}_m \frac{W_{n-m} \circ \theta_{m,x_m}}{W_n} P(\omega_{[0,m]} = x_{[0,m]}) \; . \tag{4.1}$$

In the weak disorder regime, we denote by μ the (random, time-inhomogeneous) Markov chain starting at 0 with transition probabilities

$$\mu(\omega_{i+1} = y|\omega_i = x) = \frac{\exp\{\beta\eta(i+1,y) - \lambda\}W_\infty \circ \theta_{i+1,y}}{W_\infty \circ \theta_{i,x}} \, P(\omega_1 = y|\omega_0 = x) \; . \tag{4.2}$$

In other respects, for $A \in \mathcal{F}_\infty$ the limit

$$\mu_\infty(A) := \lim_{n\to\infty} \mu_n(A) \; .$$

exists by martingale convergence theorem for both numerator and denominator of $\mu_n(A)$.

The problem is that, it is not clear if the previous limit defines, for a.e. η, a probability measure on $\mathcal{F}_\infty$. But the Markov chain μ does. In the next result we relate these two objects μ_∞, μ, and we show that the latter yields a nice description of the limit, in a precise sense.

Proposition 4.1 *Assume weak disorder. Then,*

$$\mu(A) = \mu_\infty(A) \; Q\text{-a.s. for } A \in \cup_{n\geq 1}\mathcal{F}_n. \tag{4.3}$$

As a result,

$$Q\left\{\lim_{n\nearrow\infty} \mu_n = \mu \text{ weakly}\right\} = 1. \tag{4.4}$$

Moreover,

$$Q\mu(A) \;=\; Q\mu_\infty(A) \; , \quad \forall A \in \mathcal{F}_\infty \; , \tag{4.5}$$

$$P \;\ll\; Q\mu \;\ll\; P \;\; \text{on } \mathcal{F}_\infty. \tag{4.6}$$

To prove Proposition 4.1, the following simple observation is useful.

Lemma 4.2 *Suppose* $\{A_{m,n}\}_{m,n\geq 1} \subset \mathcal{F}_\infty$ *are such that* $\lim_{m\nearrow\infty} \sup_n P(A_{m,n}) = 0$. *Then*

$$\lim_{m\nearrow\infty} \sup_n Q\mu_n(A_{m,n}) = \lim_{m\nearrow\infty} \sup_n Q\mu_\infty(A_{m,n}) = 0.$$

Proof: We prove that $\lim_{m\nearrow\infty}\sup_n Q\mu_n(A_{m,n})=0$, the proof of the other one being similar. For $\delta>0$,

$$Q\mu_n(A_{m,n}) \le Q[\mu_n(A_{m,n}) : W_n \ge \delta] + Q[W_n \le \delta]$$

We have

$$\begin{aligned}\sup_n Q[\mu_n(A_{m,n}) : W_n \ge \delta] &\le \delta^{-1}\sup_n Q[W_n\mu_n(A_{m,n})] \\ &= \delta^{-1}\sup_n P(A_{m,n}),\end{aligned}$$

which vanishes as $m\nearrow\infty$. On the other hand, since W_n^{-1} converges Q-a.s., their distributions are tight:

$$\lim_{\delta\searrow 0}\sup_n Q[W_n\le\delta]=0.$$

These prove the lemma. □

Proof of Proposition 4.1: The first statement (4.3) follows from (4.1). The second statement (4.4) follows from (4.3) by noting that the set of continuous functions on Ω contains a dense countable set of cylindrical functions.

To see (4.5), we note that the averaged limit $Q\mu_\infty(A)$ is a probability measure on $\mathcal{F}_\infty$. Indeed, it is clearly finitely additive by definition, and we have also by Lemma 4.2, $\lim_m Q\mu_\infty(A_m)=0$ for all sequence $(A_m)_m$ in $\mathcal{F}_\infty$ which decreases to $\emptyset$. Therefore, we have (4.5) since the two probability measures $Q\mu$ and $Q\mu_\infty$ coincide on any $\mathcal{F}_n$.

We see from Lemma 4.2 that $Q\mu \ll P$. To show the converse, assume that $Q\mu(A)\equiv Q\mu_\infty(A)=0$. Then, $\mu_\infty(A)=0$ a.s. and $\mu_n(A)\to 0$ a.s. This implies that $W_n\mu_n(A)$ tends a.s. to 0 and, combined with the uniform integrability of (W_n), it also implies that this sequence is itself uniformly integrable (recall $\mu_n(A)\le 1$). Therefore, $W_n\mu_n(A)$ tends to 0 in $L^1(Q)$, that is,

$$P(A)=Q[W_n\mu_n(A)]\to 0\,,$$

which is the desired result. □

As a direct consequence, the polymer path inherits under μ the a.s. behavior of the simple random walk:

Remark 4.1 Assume weak disorder. Then, for Q-a.e. environment and μ-a.e. path,

$$\limsup_{n\to\infty}\frac{\omega_n}{\sqrt{2n\ln\ln n}}=1 \quad \text{(iterated logarithm law)}$$

$$\lim_n \frac{1}{\ln n}\sum_{j\le n}\frac{1}{j}\delta_{\omega_j/\sqrt{j}}=\mathcal{N}(0,\frac{1}{d}\mathrm{Id}) \quad Q-a.s. \quad \text{(a.s. central limit theorem)}$$

Regarding (4.3), we have a more quantative statement concerning the variational norm $\|\nu-\nu'\|_{\mathcal{F}_m}=2\sup\{\nu(A)-\nu'(A); A\in\mathcal{F}_m\}$.

Proposition 4.3 *In the weak disorder case,*

$$\lim_{k\to\infty}\sup_m Q\left[\|\mu_{m+k}-\mu\|_{\mathcal{F}_m}\right]=0$$

Remark 4.2 *In particular, the central limit theorem for μ_n would follow from the one for μ, but we could not prove the latter directly.*

Proof of Proposition 4.3: We start to prove that

$$\sup_m Q\left[W_\infty \|\mu_{m+k} - \mu\|_{\mathcal{F}_m}\right] \to 0 \ , \quad k \to \infty \ . \tag{4.7}$$

From (4.1) and the similar relation for μ, for $m, k \geq 0$, it holds

$$\begin{aligned} W_\infty \|\mu_{m+k} - \mu\|_{\mathcal{F}_m} &= W_\infty P\left[\bar{\zeta}_m \left| \frac{W_k \circ \theta_{m,\omega_m}}{W_{m+k}} - \frac{W_\infty \circ \theta_{m,\omega_m}}{W_\infty} \right| \right] \\ &= \frac{1}{W_{m+k}} P\left[\bar{\zeta}_m \left| W_\infty W_k \circ \theta_{m,\omega_m} - W_{m+k} W_\infty \circ \theta_{m,\omega_m} \right| \right] \\ &\leq |W_\infty - W_{m+k}| + P\left[\bar{\zeta}_m \left| W_k \circ \theta_{m,\omega_m} - W_\infty \circ \theta_{m,\omega_m} \right| \right] \end{aligned}$$

The Q-expectation of the first term in the right-hand side vanishes as $k \to \infty$, though for the second one,

$$\begin{aligned} &Q\left(P\left[\bar{\zeta}_m \left| W_k \circ \theta_{m,\omega_m} - W_\infty \circ \theta_{m,\omega_m} \right| \right]\right) = \\ &\qquad = Q\left(P\left[\bar{\zeta}_m Q(\left| W_k \circ \theta_{m,\omega_m} - W_\infty \circ \theta_{m,\omega_m} \right| | \mathcal{G}_m)\right]\right) \\ &\qquad = Q\left(P\left[\bar{\zeta}_m \|W_k - W_\infty\|_{L^1(Q)}\right]\right) \\ &\qquad = \|W_k - W_\infty\|_{L^1(Q)} \longrightarrow_{k\to\infty} 0 \ . \end{aligned}$$

This proves (4.7). Now, it suffices to write

$$\begin{aligned} Q\left[\|\mu_{m+k} - \mu\|_{\mathcal{F}_m}\right] &= Q\left[\|\mu_{m+k} - \mu\|_{\mathcal{F}_m} \left(\mathbf{1}_{W_\infty > \delta} + \mathbf{1}_{W_\infty \leq \delta}\right)\right] \\ &\leq \delta^{-1} Q\left[W_\infty \|\mu_{m+k} - \mu\|_{\mathcal{F}_m}\right] + 2Q\left[W_\infty \leq \delta\right] \ , \end{aligned}$$

and to optimize over positive δ's. □

5 Central limit theorems

Let $(\mathbb{W}, \mathcal{F}^{\mathbb{W}}, P^{\mathbb{W}})$ be the d-dimensional Wiener space:

$$\mathbb{W} = \{ w \in C([0,1] \to \mathbb{R}^d) \ ; \ w(0) = 0 \ \}$$

with the topology induced by the uniform norm $\|w\| = \sup_{0 \leq t \leq 1} |w_t|$, let $\mathcal{F}^{\mathbb{W}}$ be the Borel σ-field and $P^{\mathbb{W}}$ the Wiener measure . For $n = 1, 2, \ldots$, we define the diffusive rescaling $\omega \mapsto \omega^{(n)}$ $(\Omega \to \mathbb{W})$ by

$$\omega_t^{(n)} = \omega_{nt}/\sqrt{n}, \quad 0 \leq t \leq 1, \tag{5.1}$$

where $(\omega_t)_{t \in \mathbb{R}_+} \in \mathbb{W}$ is the linear interpolation of $(\omega_n)_{n \in \mathbb{Z}_+} \in \Omega$. This section is devoted to the proof of

Theorem 5.1 *Assume $d \geq 3$ and weak disorder. Then, for all $F \in C_{\mathrm{b}}(\mathbb{W})$,*

$$\lim_{n \nearrow \infty} \mu_n\left[F(\omega^{(n)})\right] = P^W[F(w/\sqrt{d})], \tag{5.2}$$

$$\lim_{n \nearrow \infty} \mu\left[F(\omega^{(n)})\right] = P^W[F(w/\sqrt{d})], \tag{5.3}$$

in Q-probability. In particular, these hold for all $\beta \in [0, \beta_{\mathrm{c}})$.

Remark 5.1 Since F is bounded, the convergence in Q-probability claimed for (5.2) and (5.3) is equivalent to $L^p(Q)$-convergence for any finite p.

As a first step we start with the following weaker statement, which proof is also much simpler:

Proposition 5.2 *Assume that weak disorder holds. Then,*

$$\lim_{n\nearrow\infty} Q\mu_n(\omega^{(n)} \in \cdot) \;=\; P^{\mathbb{W}}(w/\sqrt{d} \in \cdot), \quad \textit{weakly.} \tag{5.4}$$

$$\lim_{n\nearrow\infty} Q\mu(\omega^{(n)} \in \cdot) \;=\; P^{\mathbb{W}}(w/\sqrt{d} \in \cdot), \quad \textit{weakly.} \tag{5.5}$$

Remark 5.2 (i) As can be seen from the proof below, (5.5) is true for any probability measure R with $R \ll P$ instead of $Q\mu$.

(ii) Of course, it is unnecessary to state and prove Proposition 5.2 separately. However, the role of Lemma 5.3 below is made clearer in this way.

Proof: We write $\overline{F}(w) = F(w) - P^{\mathbb{W}}[F(\cdot/\sqrt{d})]$ for $F \in C_{\rm b}(\mathbb{W})$. We introduce the set $BL(\mathbb{W})$ of bounded Lipschitz functional on $\mathbb{W}$ by

$$BL(\mathbb{W}) = \{F : \mathbb{W} \to \mathbb{R}\,;\; \|F\|_{BL} \equiv \|F\| + \|F\|_L < \infty\,\},$$

where $\|F\| = \sup_{w\in\mathbb{W}} |F(w)|$ and

$$\|F\|_L = \sup\left\{ \frac{F(w) - F(\widetilde{w})}{\|w - \widetilde{w}\|}\;;\; (w,\widetilde{w}) \in \mathbb{W}\times\mathbb{W},\; w \neq \widetilde{w} \right\}.$$

Step 1: proof of (5.5). As is well known, (5.5) is equivalent to that

$$\lim_{n\nearrow\infty} Q\mu[\overline{F}(\omega^{(n)})] = 0, \quad \text{for all } F \in BL(\mathbb{W}), \tag{5.6}$$

e.g., [11, page 310, Theorem 11.3.3]. To show (5.6), we make use of an almost sure central limit for the simple random walk in the following form. If $\{N_k\}_{k\geq 1} \subset \mathbb{Z}_+$ is an increasing sequence such that $\inf_{k\geq 1} N_{k+1}/N_k > 1$, then for any fixed $F \in BL(\mathbb{W})$,

$$\lim_{n\nearrow\infty} \frac{1}{n} \sum_{1\leq k\leq n} \overline{F}(\omega^{(N_k)}) = 0, \quad P\text{-a.s.} \tag{5.7}$$

This follows from the argument in [2, pages 98 –100]. Now, for any convergent subsequence of $a_n = Q\mu[\overline{F}(\omega^{(n)})]$, we can find a further subsequence a_{N_k} with $\inf_{k\geq 1} N_{k+1}/N_k > 1$. The point is that, by (4.6), (5.7) holds with "P-a.s." replaced by "$Q\mu$-a.s." Thus, by integrating, we obtain that

$$\lim_{n\nearrow\infty} \frac{1}{n} \sum_{1\leq k\leq n} a_{N_k} = 0.$$

Therefore, we necessarily have (5.6).

Step 2: Now, we want to move from μ to μ_n in order to get (5.4). As before, we need only to prove that

$$\lim_{n\nearrow\infty} Q\mu_n[\overline{F}(\omega^{(n)})] = 0, \quad \text{for all } F \in BL(\mathbb{W}), \tag{5.8}$$

For $0 \le k \le n$, we write

$$\begin{aligned} Q\mu_n \left| \left[\overline{F}(\omega^{(n)})\right]\right| &\le Q\mu_n \left|F(\omega^{(n)}) - F(\omega^{(n-k)})\right| \\ &+ \|F\| \sup_m Q\left[\|\mu_{m+k} - \mu\|_{\mathcal{F}_m}\right] \\ &+ Q\mu \left| \left[\overline{F}(\omega^{(n-k)})\right]\right| \ . \end{aligned}$$

As $n \to \infty$ and for fixed k, the first and the last bounds vanish. In fact, we apply (5.5) to see that the last bound vanishes. For the first one, we note that F is uniformly continuous and that

$$\sup_{\omega\in\Omega} \max_{0\le t\le 1} |\omega_t^{(n)} - \omega_t^{(n-k)}| = O(k/\sqrt{n}).$$

Finally, letting $k \to \infty$, the middle bound vanishes due to (4.3). This proves (5.4). □

The following lemma is a key to prove Theorem 5.1.

Lemma 5.3 *For all $B \in \mathcal{F}_\infty^{\otimes 2}$, the following limit exists a.s. in the weak disorder region:*

$$\mu_\infty^{(2)}(B) = \lim_{n\to\infty} \mu_n^{\otimes 2}(B) \ . \tag{5.9}$$

Moreover,

$$\begin{aligned} \mu_\infty^{(2)}(B) &= \mu^{\otimes 2}(B) \ , \quad \forall B \in \cup_{n\ge 1}\mathcal{F}_n^{\otimes 2}, & (5.10)\\ Q\mu_\infty^{(2)}(B) &= Q\left[\mu^{\otimes 2}(B)\right] \ , \quad \forall B \in \mathcal{F}_\infty^{\otimes 2} \ , & (5.11)\\ Q\mu^{\otimes 2} &\ll P^{\otimes 2} \quad \text{on } \mathcal{F}_\infty^{\otimes 2} \ . & (5.12) \end{aligned}$$

Remark 5.3 It is tempting to think of $\mu_\infty^{(2)}$ as "$\mu_\infty^{\otimes 2}$", but since we do not know if μ_∞ is a.s. σ-additive, the notation is not appropriate.

Proof: Recall from Theorem B in section 1 (Theorem 2.1 in [9]), that the random series $\sum_n I_n$ either converges almost surely or diverges almost surely, according to weak or strong disorder. We therefore have that

$$\sum_n I_n < \infty \quad Q\text{-a.s.} \tag{5.13}$$

which is a main technical ingredient in the proof.

We start by proving that the limit (5.9) exists. For a sequence $(a_n)_{n\ge 0}$ (random or non-random), we set $\Delta a_n = a_n - a_{n-1}$ for $n \ge 1$. For $B \in \mathcal{F}_\infty^{\otimes 2}$ fixed, $X_n \overset{\text{def.}}{=} P^{\otimes 2}[\bar\zeta_n(\omega)\bar\zeta_n(\tilde\omega)\mathbf{1}_B]$ is a submartingale. The proof is based on the Doob's decomposition of the process X_n. We start by writting

$$X_n = P^{\otimes 2}(B) + M_n + A_n \ , \tag{5.14}$$

with M_n a martingale, $M_0 = A_0 = 0$, and A_n the increasing process defined by its increments

$$\begin{aligned} \Delta A_n &= Q[\Delta X_n | \mathcal{G}_{n-1}] \\ &= Q\left[P^{\otimes 2}\left[\bar\zeta_{n-1}(\omega)\bar\zeta_{n-1}(\tilde\omega)\mathbf{1}_B\left\{e(n,\omega_n)e(n,\tilde\omega_n) - 1\right\}\right] \Big| \mathcal{G}_{n-1}\right] \\ &= cP^{\otimes 2}\left[\bar\zeta_{n-1}(\omega)\bar\zeta_{n-1}(\tilde\omega)\mathbf{1}_B\mathbf{1}_{\omega_n=\tilde\omega_n}\right] \\ &= cW_{n-1}^2\mu_{n-1}^{\otimes 2}(B\cap\{\omega_n = \tilde\omega_n\}) & (5.15)\\ &\le cW_{n-1}^2 I_n \ , & (5.16) \end{aligned}$$

where $e(n,x) = \exp\{\beta\eta(n,x) - \lambda(\beta)\}$ and the constant $c = \exp\{\lambda(2\beta) - 2\lambda(\beta)\} - 1$ is finite. Hence the increasing process converges,

$$A_n \nearrow A_\infty \leq c(\sup_k W_k)^2 \sum_k I_k < \infty \quad Q\text{-a.s.} \tag{5.17}$$

We prove that the martingale (M_n) converges Q-a.s. by showing that

$$\langle\, M \,\rangle_\infty \leq 4C(\sup_k W_k)^4 \sum_k I_k < \infty \quad Q\text{-a.s.} \tag{5.18}$$

with a constant $C = C(\beta)$. Introducing

$$\varphi_n(\omega,\widetilde{\omega}) = e(n,\omega_n)e(n,\tilde{\omega}_n) - 1 - c\mathbf{1}_{\omega_n=\tilde{\omega}_n},$$

we have

$$\begin{aligned} \Delta M_n &= P^{\otimes 2}\left[\bar{\zeta}_{n-1}(\omega)\bar{\zeta}_{n-1}(\tilde{\omega})\varphi_n(\omega,\widetilde{\omega}) : B\right] \\ &= W_{n-1}^2\mu_{n-1}^{\otimes 2}[\varphi_n(\omega,\widetilde{\omega}) : B], \end{aligned} \tag{5.19}$$

and hence

$$\begin{aligned} \Delta\langle\, M \,\rangle_n &= Q\left[(\Delta M_n)^2\middle|\mathcal{G}_{n-1}\right] \\ &= W_{n-1}^4\mu_{n-1}^{\otimes 4}[Q[\varphi_n(\omega^1,\omega^2)\varphi_n(\omega^3,\omega^4)] : B\times B], \end{aligned} \tag{5.20}$$

where $\omega^1,\ldots,\omega^4$ are independent copies of the path ω. Note that $Q[\varphi_n(\omega,\widetilde{\omega})] = 0$ and that $C \stackrel{\text{def.}}{=} \sup_n Q[\varphi_n(\omega,\widetilde{\omega})^2]$ is a constant depending only on β. We see from these and Schwarz inequality that

$$\Delta\langle\, M \,\rangle_n \leq C\sum_{\substack{i=1,2\\ j=3,4}} W_{n-1}^4\mu_{n-1}^{\otimes 4}\left[\{\omega_n^i = \omega_n^j\}\cap(B\times B)\right] \tag{5.21}$$

$$\leq 4CW_{n-1}^4 I_n, \tag{5.22}$$

leading to (5.18). This proves that X_n, as well as $\mu_n^{\otimes 2}(B) = W_n^{-2}X_n$, converges Q-a.s.

As for (5.10), it follows from (4.1) directly that $\mu_\infty^{(2)}$ and $\mu^{\otimes 2}$ coincide on cylindric events. As in (4.5), the claims (5.11) and (5.12) boil down to proving that

$$\lim_{m\to\infty} Q\mu_\infty^{(2)}(B_m) = 0\ ,$$

for any $\{B_m\}\subset\mathcal{F}^{\otimes 2}$ with $\lim_{m\to\infty}P^{\otimes 2}(B_m) = 0$. It is enough to prove that

$$\lim_{m\to\infty}\mu_\infty^{(2)}(B_m) \equiv \lim_{m\to\infty}\lim_{n\to\infty}\mu_n^{\otimes 2}(B_m) = 0 \quad \text{in } Q\text{-probability},$$

and hence that

$$\lim_{m\to\infty}\sup_n X_n^{(m)} = 0 \quad \text{in } Q\text{-probability}, \tag{5.23}$$

where $X_n^{(m)} = P^{\otimes 2}[\bar{\zeta}_n(\omega)\bar{\zeta}_n(\tilde{\omega}) : B_m]$. Let

$$X_n^{(m)} = P^{\otimes 2}(B_m) + M_n^{(m)} + A_n^{(m)}$$

be the submartingale decomposition as (5.14). Of course, $\lim_{m\nearrow\infty} P^{\otimes 2}[B_m] = 0$. Observe that, similar to (5.17), it follows from (5.15) that

$$A_n^{(m)} \le c(\sup_k W_k)^2 P^{\otimes 2}(S : B_m) \ ,$$

where

$$S = \sum_{n\ge 1} \bar{\zeta}(\omega, n-1)\bar{\zeta}(\tilde{\omega}, n-1)\mathbf{1}_{\omega_n = \tilde{\omega}_n} \ .$$

Now, the weak disorder assumption (5.13) states that this variable S is $P^{\otimes 2}$-integrable for Q-almost every environment. Therefore,

$$\lim_{m\nearrow\infty} A_\infty^{(m)} = 0, \quad Q\text{-a.s.} \tag{5.24}$$

For $M_n^{(m)}$, we see from (5.21),(5.22) and the weak disorder assumption (5.13) that

$$\lim_{m\nearrow\infty} \langle\, M^{(m)} \,\rangle_\infty = 0, \quad Q\text{-a.s.} \tag{5.25}$$

This implies that

$$\lim_{m\nearrow\infty} \sup_n |M_n^{(m)}| = 0 \text{ in } Q\text{-probability} \ . \tag{5.26}$$

In fact, let $\tau(\ell) = \inf\{n \ge 0 \ ; \ \langle\, M^{(m)} \,\rangle_{n+1} > \ell\}$. Then,

$$Q\{\sup_n |M_n^{(m)}| \ge \varepsilon\} \le Q\{\langle\, M^{(m)} \,\rangle_\infty > \ell\} + Q\{\sup_n |M_n^{(m)}| \ge \varepsilon, \ \tau(\ell) = \infty\}.$$

Clearly, the first term on the right-hand-side vanishes as $m \nearrow \infty$ and so does the second term as can be seen from the following application of Doob's inequality:

$$\begin{aligned} Q\{\sup_n |M_n^{(m)}| \ge \varepsilon, \ \tau(\ell) = \infty\} &\le Q\{\sup_n |M_{n\wedge\tau(\ell)}^{(m)}| \ge \varepsilon\} \\ &\le 4\varepsilon^{-2} Q[\langle\, M^{(m)} \,\rangle_{\tau(\ell)}] \\ &\le 4\varepsilon^{-2} Q[\langle\, M^{(m)} \,\rangle_\infty \wedge \ell] \end{aligned}$$

By, (5.24) and (5.26), we conclude (5.23). □

Proof of Theorem 5.1: We write $\overline{F}(w) = F(w) - P^{\mathbb{W}}[F(\cdot/\sqrt{d})]$ for $F \in C_{\rm b}(\mathbb{W})$. We begin by proving (5.3). Repeating the same argument as in the step 1 of the proof of Proposition 5.5, but using (5.12) instead of (4.6), we obtain

$$\lim_{n\nearrow\infty} Q\mu^{\otimes 2}[G(\omega^{(n)}, \widetilde{\omega}^{(n)})] = (P^{\mathbb{W}})^{\otimes 2}[G(w/\sqrt{d}, \widetilde{w}/\sqrt{d})] \tag{5.27}$$

for any $G \in C_{\rm b}(\mathbb{W}\times\mathbb{W})$. Now, if we take $G(w,\widetilde{w}) = \overline{F}(w)\overline{F}(\widetilde{w})$, then (5.27) reads

$$\lim_{n\nearrow\infty} Q\left[\left(\mu\left[\overline{F}(\omega^{(n)})\right]\right)^2\right] = 0,$$

which proves (5.3).

To obtain (5.2) from (5.3), we show that

$$\lim_{n\nearrow\infty} Q\left|\mu_n[\overline{F}(\omega^{(n)})]\right| = 0 \quad \text{for all } F \in C_{\rm b}(\mathbb{W}).$$

This can be done by exactly the same approximation procedure as we used to deduce (5.4) from (5.5), see step 2 in the proof of Proposition 5.5. □

6 An analytic family of martingales

For β complex, $Q[\exp \beta\eta(n,x)]$ is well defined, but we also want its logarithm to be holomorphic. Let U_0 be the open set in the complex plane given by

$$U_0 = \text{ connected component of } 0 \text{ in } \{\beta \in \mathbb{C};\ Q[\exp \beta\eta(n,x)] \notin \mathbb{R}_-\} \ .$$

Then, U_0 is a neighborhood of the real axis, and $\lambda(\beta) = \log Q[\exp \beta\eta(n,x)]$ is an analytic function on U_0. Define, for $n \geq 0$ and $\beta \in U_0$,

$$W_n(\beta) = P\left[\exp\left(\beta \sum_{t=1}^{n} \eta(t,\omega_t) - n\lambda(\beta)\right)\right] \ . \tag{6.1}$$

Then, for all $\beta \in U_0$, the sequence $(W_n(\beta), n \geq 0)$ is a $(\mathcal{G}_n)_n$-martingale with complex values, and for fixed n, $W_n(\beta)$ is an analytic function of $\beta \in U_0$.

In view of the implication below (2.2), we introduce for $d \geq 3$, the real subset

$$U_1 = \Big\{\beta \in \mathbb{R} \ :\ \lambda(2\beta) - 2\lambda(\beta) < -\ln \pi_d\Big\} \ , \tag{6.2}$$

which is the set of $\beta \in \mathbb{R}$ such that the martingale $(W_n)_n$ is L^2-bounded. It is an open interval containing 0, where $W_\infty(\beta) > 0$.

Proposition 6.1 *Assume $d \geq 3$. Define U_2 as the connected component of the set*

$$\Big\{\beta \in U_0 \ :\ \lambda(2\,\mathrm{Re}\,\beta) - 2\,\mathrm{Re}\,\lambda(\beta) < -\ln \pi_d\Big\}$$

which contains the origin. Then, U_2 is a complex neighborhood of U_1, such that, as $n \to \infty$,

$$W_n(\beta) \to W_\infty(\beta) \ , \quad Q - a.s.,$$

where the convergence holds in the sense of analytic function. In particular, the limit $W_\infty(\beta)$ is holomorphic in U_2, and Q-a.s.,

$$\frac{d^k}{d\beta^k} W_n(\beta) \to \frac{d^k}{d\beta^k} W_\infty(\beta) \ , \quad k \geq 0$$

uniformly on compacts of U_2 ($k \geq 0$).

Proof of Proposition 6.1: From $\overline{(e^z)} = e^{\overline{z}}$ and $\overline{Q[f]} = Q[\overline{f}]$, we see that $\overline{\lambda(\beta)} = \lambda(\overline{\beta})$, and that

$$\begin{aligned}
Q\Big[|W_n(\beta)|^2\Big] &= Q\Big[P[\exp\{\beta H_n(\omega) - n\lambda(\beta)\}]P[\exp\{\overline{\beta} H_n(\tilde{\omega}) - n\overline{\lambda(\beta)}\}]\Big] \\
&= P^{\otimes 2}\Big[Q\Big[\exp\{\beta H_n(\omega) + \overline{\beta} H_n(\tilde{\omega}) - 2n\,\mathrm{Re}\,\lambda(\beta)\}\Big]\Big] \\
&= P^{\otimes 2}\Big[\exp\{[\lambda(2\,\mathrm{Re}\,\beta) - 2\,\mathrm{Re}\,\lambda(\beta)] \sum_{t=1}^{n} \mathbf{1}_{\omega_t = \tilde{\omega}_t}\}\Big] \\
&\nearrow\ P^{\otimes 2}\Big[\exp\{[\lambda(2\,\mathrm{Re}\,\beta) - 2\,\mathrm{Re}\,\lambda(\beta)] \sum_{t=1}^{\infty} \mathbf{1}_{\omega_t = \tilde{\omega}_t}\}\Big] < \infty
\end{aligned} \tag{6.3}$$

if $\beta \in U_2$.

Now, let a point $\beta \in U_2$, a radius $r > 0$ such that the closed disk $D(\beta, r) \subset U_2$. Choosing $\rho > r$ such that $D(\beta, \rho) \subset U_2$, we obtain by Cauchy's integral formula for all $\beta' \in D(\beta, r)$,

$$W_n(\beta') = \frac{1}{2i\pi} \int_{\partial D(\beta,\rho)} \frac{W_n(z)}{z - \beta'} dz = \int_0^1 \frac{W_n(\beta + \rho e^{2i\pi u}) \rho e^{2i\pi u}}{(\beta + \rho e^{2i\pi u}) - \beta'} du \ ,$$

hence

$$X_n := \sup\{|W_n(\beta')|; \beta' \in D(\beta, r)\} \leq \rho \int_0^1 \frac{|W_n(\beta + \rho e^{2i\pi u})|}{\rho - r} du$$

Letting $C = (\rho/\rho - r)^2$, we obtain by Schwarz inequality

$$\begin{aligned} (Q[X_n])^2 &\leq CQ[\int_0^1 |W_n(\beta + \rho e^{2i\pi u})|^2 du] \\ &\leq C \sup\{Q[|W_n(\beta'')|^2]; n, \beta'' \in D(\beta, \rho)\} \\ &< \infty \end{aligned}$$

in view of (6.3). Notice now that X_n, a supremum of positive submartingales, is itself a positive submartingale. Since $\sup Q[X_n] < \infty$, X_n converges Q-a.s. to a finite limit X_∞. Finally, $\sup\{|W_n(\beta')|; \beta' \in D(\beta, r)\} < \infty$ a.s., and W_n is uniformly bounded on compact subsets of U_2 on a set of environments of full probability. On this set, $(W_n, n \geq 0)$ is a normal sequence [28] which has a unique limit on the real axis: Since U_2 is connected, the sequence converges to some limit W_∞, which is holomorphic on U_2, and the derivatives also converges to those of W_∞. □

Note that we do not know that $W_\infty(\beta) \neq 0$ for general $\beta \in U_2$, except for $\beta \in U_1$ – and of course for some complex neighborhood around U_1 –. We draw now some consequences for real β's. We write $\mu_n = \mu_n^\beta$ to recall the dependence on the temperature.

Theorem 6.2 *Assume $d \geq 3$. Then W_∞ and $\ln W_\infty$ are analytic (real) function of $\beta \in U_1$. Moreover, as $n \to \infty$,*

$$\mu_n^\beta[H_n] - n\lambda'(\beta) \to (\ln W_\infty)'(\beta) \ , \tag{6.4}$$

though for the entropy $h(\mu_n^\beta|P) = \mu_n^\beta[\ln(d\mu_n^\beta/dP)]$,

$$h(\mu_n^\beta|P) - n[\beta\lambda'(\beta) - \lambda(\beta)] \to \beta(\ln W_\infty)'(\beta) - \ln W_\infty(\beta) \ , \tag{6.5}$$

for all $\beta \in U_1$.

On the other hand, for Q-a.e. environment,

$$\textit{the law of } \frac{H_n - n\lambda'(\beta)}{\sqrt{n}} \textit{ under } \mu_n \textit{ converges to the Gaussian } \mathcal{N}(0, \lambda''(\beta))$$

where $\lambda''(\beta) > 0$.

Comment: The *average energy* for the polymer measure, $\mu_n^\beta[H_n]$, scales like the annealed one $n\lambda'(\beta)$, but it has fluctuations of order one in this part of the weak disorder region. The entropy also has $\mathcal{O}(1)$ fluctuations. On the other hand, the last result shows that, due to variations from a path to another, the fluctuations of the *energy under the polymer measure* is normal and of order of magnitude $\mathcal{O}(\sqrt{n})$. □

Proof of Theorem 6.2: We have the identities

$$(\ln W_n)'(\beta) = \mu_n^\beta[H_n] - n\lambda'(\beta),$$
$$h(\mu_n^\beta|P) = \beta\mu_n^\beta[H_n] - n\lambda'(\beta) - \ln W_n(\beta)\ .$$

In view of Proposition 6.1, $(\ln W_n)'(\beta) = (W_n)'(\beta)/W_n(\beta)$ converges a.s. to $(W_\infty)'(\beta)/W_\infty(\beta) = (\ln W_\infty)'(\beta)$ for $\beta \in U_1$, which is the first result (6.4). The second one (6.5) follows easily. In order to prove the last one, we show the stronger statement that, for Q-a.e. environment,

$$\mu_n\left[\exp\{\frac{u(H_n - n\lambda'(\beta))}{\sqrt{n}}\}\right] \to \exp\{\frac{\lambda''(\beta)u^2}{2}\}$$

as $n \to \infty$ for all $u \in \mathbb{R}$ and $\beta \in U_2$. Write the left-hand side as

$$\frac{W_n(\beta + un^{-1/2})}{W_n(\beta)} \times \exp\left\{n[\lambda(\beta + un^{-1/2}) - \lambda(\beta) - un^{-1/2}\lambda'(\beta)]\right\}$$

Since $W_n \to W_\infty$ locally-uniformly on U_1, and since λ is smooth, the right-hand side converges Q-a.s. to $[W_\infty(\beta)/W_\infty(\beta)] \times \exp\{\lambda''(\beta)u^2/2\}$ as $n \to \infty$. □

7 Bernoulli environment

Let $p \in (0,1)$. In this section, we focus on the Bernoulli case, where

$$\eta(t,x) = \left\{ \begin{array}{c} 0 \\ -1 \end{array} \right. \quad \text{with } Q - \text{probability } \left\{ \begin{array}{c} p \\ 1-p \end{array} \right. ,$$

In this case, $\lambda(\beta) = \ln[p + (1-p)e^{-\beta}]$.

Consider also the site, oriented Bernoulli percolation (see [12], [16]), as follows: Call a site $(t,x) \in \mathbb{N} \times \mathbb{Z}^d$ open if $\eta(t,x) = 0$, and closed if $\eta(t,x) = -1$. Write $(n,x) \to^\eta (k,z)$ if there exists an oriented open path $((t,\omega_t); n \le t \le k)$ from (n,x) to (k,z), i.e., some path $((t,\omega_t); n \le t \le k)$ with nearest neighbors vertices ω_t and ω_{t+1} and $\eta(t,\omega_t) = 0$ for all t, and $\omega_n = x, \omega_k = z$. Write $(n,x) \to^\eta \infty$ if there exists an infinite oriented open path starting at (n,x), and denote by $\mathcal{C}$ the set of sites (n,x) such that $(n,x) \to^\eta \infty$ and $\|x\|_1 \le n$, $\|x\|_1 = n$ modulo 2. The set $\mathcal{C}$ is called the infinite cluster. It is well known that there exists some percolation threshold $\vec{p}_c(d) \in (0,1)$ such that for $p > \vec{p}_c(d)$ and $d \ge 1$, $\mathcal{C}$ is Q-a.s. non empty, and $\mathcal{C}$ is Q-a.s. empty for $p < \vec{p}_c(d)$. It is known (Theorem 2 in [17]), that $\mathcal{C}$ is a.s. connected, in the sense that a.s. on the set $\{(n,x) \to \infty, (m,y) \to \infty\}$, there exists some $(k,z) \to \infty$ such that both $(n,x) \to (k,z)$ and $(m,y) \to (k,z)$. Let H_n^* be the maximum value of H_n over all paths ω starting form $(0,0)$. In the last passage percolation problem, one is interested in the almost-sure limit

$$\tau = \lim_{n \nearrow \infty} -H_n^*/n,$$

(called the time constant), which exists and is constant by subadditivity [12], [16], and is non-negative. For directed polymers on the other hand, the a.s.-limit $\psi(\beta) = \psi(\beta,p)$ of $-(1/n)\ln W_n(\beta)$ exists, is constant by subadditivity and concentration [9], and is non-negative.

We have a commutative diagram, with β, n tending to $+\infty$:

$$\begin{array}{ccc} -\frac{1}{n\beta}\ln W_n(\beta) & \longrightarrow_\beta & -\frac{H_n^*}{n} \\ \downarrow_n & & \downarrow_n \\ \frac{\psi(\beta)}{\beta} & \longrightarrow_\beta & \tau \end{array}$$

The proofs of the horizontal limits are easy, and left to the reader. We have $\tau = 0$ for $p > \vec{p}_c(d)$ by definition of the percolation threshold, and $\tau > 0$ for $p < \vec{p}_c(d)$ in view of the exponential tails of the cluster of the origin [27]. Let us introduce another critical value,

$$p_c^{\psi} = \inf\{p \in [0,1] : \psi(\beta; p) = 0 \ , \forall \beta > 0\} \tag{7.1}$$

and recall π_d from (2.2). We have

$$\pi_d \geq p_c^{\psi} \geq \vec{p}_c(d) \ . \tag{7.2}$$

Indeed, it holds

$$\frac{\psi(\beta)}{\beta} = \frac{\lambda(\beta)}{\beta} - \lim_n \frac{1}{n\beta} \ln P[\exp \beta H_n] \geq \frac{\lambda(\beta)}{\beta} + \tau \ ,$$

which becomes strictly positive in the limit $\beta \to \infty$ if $p < \vec{p}_c(d)$. This proves the second inequality in our claim (7.2). Now, the first one follows from the observation that $[0, \infty[\subset U_1$ holds if $p > \pi_d$, (see example 2.1.1 in [10] for instance).

From now on, we assume that $d \geq 3$ and

$$p > \pi_d \ ,$$

which implies that weak disorder holds for all $\beta \geq 0$ and also $\tau = 0$. There are a strong analogies between our limiting fluctuations from the previous section for the directed polymer model, and the first passage time in oriented percolation. We now elaborate on these relations.

We have the identities

$$\lim_{\beta \to +\infty} \mu_n^{\beta}[H_n] = \lim_{\beta \to +\infty} \frac{1}{\beta} \ln W_n = H_n^* \ , \tag{7.3}$$

$$\lim_{n \to +\infty} H_n^* = H_{\infty}^* := -\mathrm{dist}(0, \mathcal{C}) \in (-\infty, 0] \ . \tag{7.4}$$

Here, 0 is the origin in $\mathbb{Z}^+ \times \mathbb{Z}^d$, and dist is the chemical "distance" given, for $s \leq t$, by $\mathrm{dist}((s,x),(t,y)) = \inf\{\sum_{s<u\leq t} \eta(u, x_u)\}$ where the infimum is taken over oriented nearest neighbor paths $((u, x_u); s < u \leq t)$ with $x_s = x, x_t = y$. We note that the convergence

$$\mu_n^{\beta}[H_n - n\lambda'(\beta)] \longrightarrow_n (\ln W_{\infty})'(\beta)$$

in (6.4) parallels that of (7.4), in the sense that $\mu_n^{\beta}(H_n)$ and H_n^*, which relates via (7.3), both have order one fluctuations.

As a related remark, let us recall the local limit theorem of Sinai [30]. Deep inside the region U_1,

$$P\left[\exp\{\beta H_n(\omega)\} \middle| \omega_n = x\right] = W_{\infty} \times \ W_{\infty} \circ \theta_{n,x}^{\leftarrow} + R_{n,x}$$

where $\theta_{n,x}^{\leftarrow}$ is given by $\theta_{n,x}^{\leftarrow}(\eta(\cdot,\cdot)) : (u,y) \mapsto \eta(n-u, x+y)$, and the error term $R_{n,x} \to 0$ in L^1 uniformly in $x : |x| \leq An^{1/2}$. The local limit theorem parallels the following observation in the percolation model

$$H_n^{*,x} \stackrel{\mathrm{def}}{=} \inf\{H_n(\omega); \omega_0 = 0, \omega_n = x\} = -\mathrm{dist}(0, \mathcal{C}) - \mathrm{dist}((n,x), \mathcal{C}) + o_Q(1)$$

for x not too large.

Acknowledgements: We thank Geoffrey R. Grimmett and Herbert Spohn for stimulating conversations and for indicating the reference [17].

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