LOWER TRANSPORT BOUNDS FOR ONE-DIMENSIONAL CONTINUUM SCHRÖDINGER OPERATORS

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Dedicated to Joachim Weidmann on the occasion of his 65th birthday.

Abstract. We prove quantum dynamical lower bounds for one-dimensional continuum Schrödinger operators that possess critical energies for which there is slow growth of transfer matrix norms and a large class of compactly supported initial states. This general result is applied to a number of models, including the Bernoulli-Anderson model with a constant single-site potential.

1. Introduction

We study one-dimensional Schrödinger operators associated to

$$\varphi \mapsto -\varphi'' + V\varphi, \tag{1}$$

where $V : \mathbb{R} \to \mathbb{R}$ satisfies

$$\|V\|_{1,\mathrm{unif}} := \sup_{x\in\mathbb{R}} \int_x^{x+1} |V(t)|\, dt < \infty. \tag{2}$$

We will have results for the associated whole-line operator, denoted by H, as well as for the associated selfadjoint operator H_D on $[0,\infty)$ with Dirichlet boundary condition at 0 (which could be easily adjusted to other boundary conditions).

We are interested in situations where non-trivial quantum transport for systems governed by the above Hamiltonians can be established. To this end we will consider the time averaged p-th moments of the position operator $(X\varphi)(x) = x\varphi(x)$ with given initial state f:

$$M_f(T,p) := \frac{2}{T}\int_0^\infty \exp\left(-\frac{2t}{T}\right) \left\| |X|^{p/2} \exp(-itH) f \right\|^2 dt. \tag{3}$$

In the same way we define $M_{f,D}(T,p)$ if H is replaced by H_D. The presence of transport will be proven through lower bounds for the lower growth exponents

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(lower *diffusion exponents*)

$$\beta_f^-(p) := \liminf_{T\to\infty} \frac{\log M_f(T,p)}{\log T}, \tag{4}$$

and similarly $\beta_{f,D}^-(p)$.

Using Abelian means in (3) is convenient for our proofs. This is done by most authors and may in most applications be replaced by Cesaro means

$$\frac{1}{T}\int_0^T \left\| |X|^{p/2} \exp(-itH) f \right\|^2 \, dt, \tag{5}$$

without changing the value of the diffusion exponents. This is easy to see if an a-priori upper bound $\||X|^{p/2}\exp(-itH)f\|^2 \le C|t|^N$ is available. The latter arises for example in the form of ballistic upper bounds on quantum transport (i.e. $N = p$), which hold in great generality, see e.g. [21] for $p = 2$ and $p = 4$.

For discrete one-dimensional Schrödinger operators on $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$, respectively, Damanik and Tcheremchantsev [7] have developed a general method which allows one to derive lower bounds on diffusion exponents from upper bounds on the growth of norms of transfer matrices. In Section 2 we will present an extension of their method to continuum operators. Due to our intended applications we will focus on results which arise from transfer matrix bounds in the vicinity of a single "critical" energy.

The most interesting issue which arises in this extension is the question for the proper choice of the initial state f. The paper [7] only considers the case $f = \delta_1$, a discrete unit mass. While this is somewhat natural in discrete space, there is no corresponding choice in the continuum (at least if one wants to stay in Hilbert space). Our results will allow for any compactly supported initial state as long as it "feels" the critical energy in the sense that it is not orthogonal to the eigensolutions of the Schrödinger equation at this energy.

Our main motivation for extending the methods of [7] to the continuum comes from applications to random Schrödinger operators, specifically continuum Bernoulli-Anderson models where the random coupling constants take only two possible values. These operators are known to almost surely exhibit *spectral localization*, that is they have pure point spectrum with exponentially decaying eigenfunctions [4]. On the other hand, it was also observed in [4] that these operators may have a discrete set of critical energies at which the Lyapunov exponent vanishes. Dynamical localization (in the sense of time-boundedness of all moments of the position operator) was obtained in [4] only after projecting onto energy intervals which have positive distance from the critical energies.

In Section 4 we will show that the existence of critical energies in continuum Bernoulli-Anderson models indeed gives rise to quantum transport in the sense that almost surely

$$\beta_f^-(p) \ge p - \frac{1}{2} \tag{6}$$

for all $p > 0$ and suitable f.

As a prototype of a continuum Bernoulli-Anderson model consider

$$-\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} \omega_n \chi_{[n,n+1]}, \tag{7}$$

where the i.i.d. random variables ω_n only take the values 0 or 1. This operator has not just one but infinitely many critical energies at which the Lyapunov exponent vanishes. We find that (6) holds almost surely for all square-integrable f with support in $[0, 1]$.

Thus (7) provides a model for the co-existence of spectral localization and dynamical delocalization in the form of super-diffusive transport. The latter is best characterized by the mean-square deviation, i.e. $p = 2$ in (3): $\beta(2) = 2$ would be ballistic transport, $\beta(2) = 1$ diffusive, while we get $\beta(2) \geq 3/2$. Our work provides a continuum analogue of results established previously for the discrete *dimer model* (see [1] for spectral localization and [18] for the lower transport bound (6)).

We point out that the co-existence phenomenon arises for continuum Anderson models already in the prototypical case (7), while in the discrete case the standard Anderson model (with independent sites) has no critical energies and is spectrally and dynamically localized for any non-trivial distribution of the single site couplings [2, 11].

Obtaining the probabilistic transfer matrix bounds which are necessary to deduce (6) for Bernoulli-Anderson models is quite subtle (while $\beta_f^-(p) \geq p-1$ follows from a much simpler deterministic bound). To get (6), which is physically expected to be the exact diffusion exponent (at least for the dimer and $p = 2$, see [9]), we employ a law of large numbers type result from [18].

In Section 5 we add another application of our general results in Section 2. Here we consider self-similar potentials which are generated by means of a substitution rule. We discuss potentials generated by the Thue-Morse substitution or the period doubling substitution in detail and prove the existence of critical energies at which the norms of transfer matrices remain uniformly (resp., linearly) bounded. This then yields the lower bound $p - 1$ (resp., $(p - 5)/2$) for diffusion exponents.

We conclude this introduction by comparing our approach to dynamical lower bounds for continuum Schrödinger operators with previous ones.

The first general method to prove dynamical lower bounds for Schrödinger operators with singular spectral measures goes back to Guarneri [14] and was further developed by Combes [3] and Last [20]. Dynamical lower bounds are found in terms of continuity properties of spectral measures with respect to Hausdorff measures. While this correspondence holds in arbitrary dimension, this approach is particularly useful in one dimension since the required input can be established using the Jitomirskaya-Last extension [16, 17] of Gilbert-Pearson theory [13]. Nevertheless, proofs of Hausdorff-absolute continuity of spectral measures are often quite involved or even impossible. For example, within the class of self-similar potentials, only (discrete) potentials of Fibonacci type could be handled, models associated with Thue-Morse or period doubling symmetry are as yet outside the scope of this approach. Moreover, the required spectral continuity may not hold at all for interesting models. For example, the Bernoulli-Anderson model has pure point spectrum and hence no useful spectral continuity properties.

Another method was recently developed by Germinet, Kiselev, and Tcheremchantsev [12]. While their approach is similar in spirit to ours, namely that upper bounds on transfer matrix norms imply lower bounds for diffusion exponents, our results give better bounds for the applications we have in mind. Their method is particularly suitable for models that admit power-law upper bounds on transfer matrix norms for large sets of energies, and hence their applications establish good dynamical bounds for models with this feature, such as random decaying potentials and sparse potentials. For models with small (e.g., finite) sets of such energies, our method gives better dynamical bounds. For example, their method combined with our Theorem 4 below gives the bound $\beta_f^-(p) \geq \frac{1}{2}(p-1)$ in the case of the Bernoulli-Anderson model with a critical energy, whereas we obtain the stronger bound (6), which is conjectured to be optimal. Moreover, their proof of the general dynamical bound is more involved than ours. Thus, while the results of [12] and this paper are somewhat related, the scopes in terms of applications are almost disjoint.

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2. The main result

In this section, we develop the continuum analog of the approach to quantum dynamical lower bounds from [7]. As discussed in the introduction, working in the continuum requires to come up with a better understanding of which initial states will generate transport. The key technical input which settles this issue is contained in Lemma 2.6. As a bonus, this observation also suggests how the dynamical lower bound in the discrete case can be extended to arbitrary (finitely supported) initial states. We will discuss this extension in Section 7.

A central role will be played by solutions of

$$-u'' + (V-z)u = 0 \tag{8}$$

for $z \in \mathbb{C}$. To be more precise, define for differentiable v on a subinterval $I \subset \mathbb{R}$, and $x \in I$, the vector $\underline{v}(x)$ by $\underline{v}(x) = (v(x), v'(x))^t$, where t denotes the transpose. Then, for arbitrary $x, y \in \mathbb{R}$ and $z \in \mathbb{C}$, the transfer matrix is the unique 2×2-matrix $M(x,y,z)$ with $M(x,y,z)\underline{u}(y) = \underline{u}(x)$ for every solution u of (8). $M(x,y,z)$ has columns $(u_N(x), u_N'(x))^t$ and $(u_D(x), u_D'(x))^t$, where u_N and u_D are the solutions of (8) which satisfy initial conditions $(1,0)^t$ and $(0,1)^t$ at x, respectively.

As usual, the Wronski determinant $W(u_1,u_2)$ of solutions u_1, u_2 of (8) is defined by $W(u_1,u_2) = u_1(x)u_2'(x) - u_1'(x)u_2(x)$ and this expression does not depend on x. This implies that $\det M(x,y,z) = 1$.

Let $\mathrm{u}_{1,z}$ be the solution of (8) with $\underline{\mathrm{u}}_{1,z}(0) = (1,0)^t$, $\mathrm{u}_{0,z}$ the solution of (8) with $\underline{\mathrm{u}}_{0,z}(0) = (0,1)^t$. For $z \in \mathbb{C}$ with positive imaginary part, let $\mathrm{u}_{\infty,z}$ be the solution of (8), which is square integrable (at ∞) and satisfies $\mathrm{u}_{\infty,z}(0) = 1$.

Finally, for a measurable locally bounded g and $f \in L^2(\mathbb{R})$ with compact support we write

$$\langle g, f\rangle := \int_{\mathbb{R}} g(t)\overline{f(t)}dt.$$

In the situation we have in mind, g will be a solution of (8).

We will first state our result for the half line. For $\alpha > 0$, $C > 0$, and $N > 1$, we define

$$P(\alpha, C, N) := \{E \in \mathbb{R} : \|M(x,y,E)\| \leq CN^\alpha \text{ for all } 0 \leq x, y \leq N\}.$$

We can now give a precise version of our main theorem.

Theorem 1. *Suppose $E_0 \in \mathbb{R}$ is such that there exist $C > 0$ and $\alpha > 0$ with $E_0 \in P(\alpha, C, N)$ for all sufficiently large N. Let $A(N)$ be a subset of $P(\alpha, C, N)$ containing E_0 such that* $\operatorname{diam}(A(N)) \longrightarrow 0$ *as $N \to \infty$. Then, for every compactly* *supported $f \in L^2(0,\infty)$ with $\langle u_{0,E_0}, f\rangle \neq 0$, there exists $\widetilde{C} > 0$ such that for T large enough,*

$$M_{f,D}(T,p) \geq \widetilde{C}|B(T)|T^{\frac{p-3\alpha}{1+\alpha}},$$

where $B(T)$ is the $1/T$ neighborhood of $A(T^{\frac{1}{1+\alpha}})$.

We will now state our result for the whole line. In this case, for $\alpha > 0$, $C > 0$, and $N > 1$, we define

$$P(\alpha, C, N) := \{E \in \mathbb{R} : \|M(x,y,E)\| \leq CN^\alpha \text{ for all } -N \leq x, y \leq N\}.$$

Theorem 2. *Suppose $E_0 \in \mathbb{R}$ is such that there exist $C > 0$ and $\alpha > 0$ with $E_0 \in P(\alpha, C, N)$ for sufficiently large N. Let $A(N)$ be a subset of $P(\alpha, C, N)$ containing E_0 such that $\mathrm{diam}(A(N)) \longrightarrow 0$ as $N \to \infty$. Let $f \in L^2(\mathbb{R})$ be compactly supported and satisfy $\langle u, f\rangle \neq 0$ for at least one solution u of (8) with $z = E_0$. Then, there exists $\widetilde{C} > 0$ such that for T large enough,*

$$M_f(T,p) \geq \widetilde{C}|B(T)|T^{\frac{p-3\alpha}{1+\alpha}},$$

where $B(T)$ is the $1/T$ neighborhood of $A(T^{\frac{1}{1+\alpha}})$.

As in [7], the previous theorems have the following immediate consequences. We only state them for the half-line case. They hold with obvious modifications for the whole line.

Corollary 2.1. *Suppose there is an energy $E_0 \in \mathbb{R}$ such that $\|M(x,y,E_0)\| \leq CN^\alpha$ for all N large enough and $0 \leq x, y \leq N$. Then, for every compactly supported $f \in L^2(0,\infty)$ with $\langle u_{0,E_0}, f\rangle \neq 0$, we have*

$$\beta^-_{f,D}(p) \geq \frac{p-1-4\alpha}{1+\alpha},$$

Proof. We only need to take $A(N) = \{E_0\}$ for every N. Then $|B(T)| = 2T^{-1}$ and the assertion follows. □

Corollary 2.2. *Suppose that there exist $C > 0$, $E_0 \in \mathbb{R}$, $0 < \theta \leq 1$ such that $\|M(x,y,E)\| \leq C$ for all N, $0 \leq x,y \leq N$ and $E \in [E_0 - N^{-\theta}, E_0 + N^{-\theta}]$. Then, for every compactly supported $f \in L^2(0,\infty)$ with $\langle u_{0,E_0}, f\rangle \neq 0$, we have*

$$\beta^-_{f,D}(p) \geq p - \theta.$$

Proof. Let $A(N) = [E_0 - N^{-\theta}, E_0 + N^{-\theta}]$. Then $|B(T)| \geq |A(T)| = 2T^{-\theta}$ and the assertion follows. □

Since compactly supported perturbations of V (and even perturbations with a suitable power-decay; compare [6]) leave the power-law bounds of the form above unchanged, these results immediately extend to all these perturbed models, whenever they apply. We refer the reader to [6, 7] for a more detailed discussion of stability issues.

The proofs of Theorem 1 and Theorem 2 will be given at the end of this section. We first gather a series of preliminary results that we will need in the proof.

Lemma 2.3. *Let $\mathcal{H}$ be a separable Hilbert space, S a selfadjoint operator on $\mathcal{H}$ and A a closed operator on $\mathcal{H}$. Then,*

$$2\pi \int_0^\infty \exp\left(-\tfrac{2t}{T}\right) \| A\exp(-itS)f\|^2\, dt = \int_{\mathbb{R}} \left\| A(S - E - \tfrac{i}{T})^{-1} f\right\|^2 dE$$

for every $f \in \mathcal{H}$ and $T > 0$.

Proof. This identity is well known. For example, it falls well within the discussion in [22, pp. 142–144]. For the convenience of the reader we give a sketch of the proof:

Let $T > 0$ be given. Define $\varphi : \mathbb{R} \longrightarrow \mathcal{H}$ by

$$\varphi(t) := \begin{cases} \exp(-\frac{t}{T})\exp(-itS)f & : \ t > 0, \\ 0 \in \mathcal{H} & : \ t \leq 0, \end{cases}$$

and $\widehat{\varphi} : \mathbb{R} \longrightarrow \mathcal{H}$ by $\widehat{\varphi}(E) := \int_{\mathbb{R}} \exp(-itE)\varphi(t)dt$. Then,

$$\exp\left(-\tfrac{2t}{T}\right) \| A\exp(-itS)f\|^2 = \|A\varphi(t)\|^2$$

and

$$\widehat{\varphi}(E) = \int_0^\infty \exp(-itE)\exp(-\tfrac{t}{T})\exp(-itS)f dt = i(S + E - \tfrac{i}{T})^{-1}f.$$

Now, (32) on [22, p. 143] says $2\pi \int_{\mathbb{R}} \|A\varphi(t)\|^2 dt = \int_{\mathbb{R}} \|A\widehat{\varphi}(E)\|^2 dE$ and the desired equality follows. □

Lemma 2.4. *For each $M \in (0, \infty)$, there is $C = C(M) < \infty$ such that for every $q : \mathbb{R} \to \mathbb{C}$ with $\|q\|_{1,\mathrm{unif}} \leq M$ and each solution u of $-u'' + qu = 0$,*

$$|u'(x)|^2 \leq C \int_{x-1}^{x+1} |u(s)|^2 ds \quad \textit{for every } x \in \mathbb{R}, \tag{9}$$

and

$$\int_{a-1}^{a+1} \left(|u(x)|^2 + |u'(x)|^2\right) dx \leq (1 + 2C) \int_{a-2}^{a+2} |u(x)|^2 \, dx \quad \textit{for every } a \in \mathbb{R}. \tag{10}$$

Proof. The first statement is well known, see for example Lemma A.3 in [4]. It implies that

$$\begin{aligned} \int_{a-1}^{a+1} |u'(x)|^2 dx &\leq C \int_{a-1}^{a+1} \int_{x-1}^{x+1} |u(t)|^2 dt\, dx \\ &\leq C \int_{a-1}^{a+1} \int_{a-2}^{a+2} |u(t)|^2 dt\, dx \\ &= 2C \int_{a-2}^{a+2} |u(t)|^2 dt, \end{aligned}$$

which gives (10). □

Lemma 2.5. *Let $E \in \mathbb{R}$ and $N \geq 0$ be given. Define*

$$L(N) := \sup_{0 \leq x,y \leq N} \|M(x, y, E)\|.$$

Then, for every $\delta \in \mathbb{C}$ and $0 \leq x, y \leq N$, we have

$$\|M(x, y, E + \delta)\| \leq L(N)\exp\left(L(N)|x - y||\delta|\right). \tag{11}$$

In particular, if $\|M(x, y, E)\| \leq CN^\alpha$ for all $0 \leq x, y \leq N$, then

$$\|M(x, y, E + \delta)\| \leq C\exp(C)N^\alpha \tag{12}$$

whenever $0 \leq x, y \leq N$ and $0 \leq |\delta| \leq N^{-1-\alpha}$.

Proof. Essentially, (11) is [23, Eq. (3.2)]. The estimate (12) is an immediate consequence of (11). □

The following lemma is the crucial new ingredient in our treatment of the half-line operator.

Lemma 2.6. *For $z \in \mathbb{C} \setminus \mathbb{R}$, define $u_{f,z} = (H_D - z)^{-1} f$. Suppose $E \in \mathbb{R}$ and $f \in L^2(0,\infty)$ with* $\operatorname{supp} f \subset [0,s]$ *are such that*

$$0 = \lim_{\delta \to 0+} \inf\{\|\underline{u}_{f,z}(s)\| : z \in \mathbb{C}_+,\ |z - E| \le \delta\}. \tag{13}$$

Then, $0 = \langle u_{0,E}, f \rangle$.

Proof. By (13), there exists a sequence (z_n) in $\mathbb{C}_+$ with $z_n \to E$ and $\underline{u}_{f,z_n}(s) \to (0,0)^t$ for $n \to \infty$. By $u_{f,z_n}(0) = 0$ for all n and continuity, the inhomogeneous equation

$$-u'' + (V - E)u = f$$

has a solution v with $v(0) = v(s) = v'(s) = 0$. Let $Y(t)$ be the fundamental matrix of the homogeneous equation at $x = s$ (i.e., the columns of Y, $(v_1, v_1')^t$ and $(v_2, v_2)^t$ are solutions v_1, v_2 of the homogeneous equation which satisfy $\underline{v}_1(s) = (1,0)^t$ and $\underline{v}_2(s) = (0,1)^t$). Then,

$$\underline{v}(x) = Y(x) \int_s^x Y(t)^{-1} (0, f(t))^t dt.$$

Restricting our attention to the first component, we obtain

$$0 = v(0) = \int_s^0 [-v_1(0)v_2(t) + v_2(0)v_1(t)] f(t) dt. \tag{14}$$

Now, obviously,

$$u(t) := [-v_1(0)v_2(t) + v_2(0)v_1(t)]$$

is a solution of the homogenous equation with $u(0) = 0$. As v_1 and v_2 are linearly independent, u does not vanish identically. Thus, u agrees up to a non-vanishing factor with $u_{0,E}$. The assertion of the lemma therefore follows from (14). □

To treat the whole line operator we will use a variant of the lemma. It is given as follows.

Lemma 2.7. *For $z \in \mathbb{C} \setminus \mathbb{R}$, define $\underline{u}_{f,z} = (H - z)^{-1} f$. Let $E \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ with* $\operatorname{supp} f \subset [-s,s]$ *be such that*

$$0 = \lim_{\delta \to 0+} \inf\{\|\underline{u}_{f,z}(s)\| + \|\underline{u}_{f,z}(-s)\| : z \in \mathbb{C}_+,\ |z - E| \le \delta\}. \tag{15}$$

Then, $\langle u, f \rangle = 0$ for every solution u of $-u'' + Vu = Eu$.

Proof. By (15), there exists a sequence (z_n) in $\mathbb{C}_+$ with $z_n \to E$ and $\underline{u}_{f,z_n}(s) \to (0,0)^t$ and $\underline{u}_{f,z_n}(-s) \to (0,0)^t$ for $n \to \infty$. Let v be the solution of $-v'' + (V - E) = f$ with $\underline{v}(s) = (0,0)^t$. Then, by continuous dependence of solutions on initial conditions, $\underline{u}_{f,z_n}(x) \longrightarrow \underline{v}(x)$ for every $x \in \mathbb{R}$. In particular, $\underline{v}(-s) = (0,0)^t$. Let $Y(t)$ be as in the proof of the previous lemma. Then,

$$\underline{v}(x) = Y(x) \int_s^x Y(t)^{-1} (0, f(t))^t dt = \int_s^x \begin{pmatrix} -v_1(x)v_2(t) + v_2(x)v_1(t) \\ -v_1'(x)v_2(t) + v_2'(x)v_1(t) \end{pmatrix} f(t) dt.$$

Thus,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underline{v}(-s) = \int_{-s}^{s} \begin{pmatrix} v_1(-s)v_2(t) - v_2(-s)v_1(t) \\ v_1'(-s)v_2(t) - v_2'(-s)v_1(t) \end{pmatrix} f(t) dt.$$

Now, $u_1 := v_1(-s)v_2 - v_2(-s)v_1$ and $u_2 := v_1'(-s)v_2 - v_2'(-s)v_1$ are solutions of $-u'' + Vu = Eu$. They satisfy $\underline{u}_1(-s) = (0, W(v_1, v_2))^t = (0, 1)$ and $\underline{u}_2(-s) = (-W(v_1, v_2), 0)^t = (-1, 0)^t$. Thus, u_1, u_2, are a fundamental system for $-u''+Vu = Eu$ and we have shown that $\langle u_i, f\rangle = 0$, $i = 1, 2$. This completes the proof of the lemma. □

Proof of Theorem 1. As before we set $u_{f,z} := (H_D-z)^{-1}f$. Let $s > 0$ with $\operatorname{supp} f \subset [0, s]$ and define $N(T) := T^{\frac{1}{1+\alpha}}$.

Now, we can apply Lemma 2.3 with $S = H_D$ and $A = |\cdot|^{\frac{p}{2}}$ and Lemma 2.4 with $u = u_{f,z}$.

For T large enough, this gives

$$\begin{aligned}
M_{f,D}(T, p) &= \tfrac{2}{T}\int_0^\infty \exp\left(-\tfrac{2t}{T}\right)\Big\| |\cdot|^{p/2}\exp(-itH_D)f\Big\|^2 dt \\
\text{(Lemma 2.3)} \quad &= \int_{\mathbb{R}} |x|^p \tfrac{1}{\pi T}\int_{\mathbb{R}} |u_{f,E+\frac{i}{T}}(x)|^2 dE\, dx \\
&\geq \frac{1}{4}\sum_{n=s+2}^{\infty}(n-2)^p \int_{n-2}^{n+2}\tfrac{1}{\pi T}\int_{\mathbb{R}}|u_{f,E+\frac{i}{T}}(x)|^2 dE\, dx \\
&\geq \frac{1}{4}\sum_{n=s+2}^{\infty}(n-2)^p \int_{B(T)}\int_{n-2}^{n+2}\tfrac{1}{\pi T}|u_{f,E+\frac{i}{T}}(x)|^2 dx\, dE \\
&\geq \frac{c}{T}\sum_{n=s+2}^{\infty}(n-2)^p \int_{B(T)}\int_{n-1}^{n+1}\|\underline{u}_{f,E+\frac{i}{T}}(x)\|^2 dx\, dE. \qquad (16)
\end{aligned}$$

In the last step, Lemma 2.4 was used, based on the fact that $u_{f,E+i/T}$ is a solution of $-u'' + Vu = (E + i/T)u$ on $[n-2, n+2]$. Observe that the constant $c > 0$ can be chosen uniformly for all sufficiently large T. Using that the transfer matrices satisfy $\|M^{-1}\| = \|M\|$, we can further bound (16) from below by

$$\begin{aligned}
&\geq \tfrac{c}{T}\sum_{n=s+2}^{\infty}(n-2)^p\int_{B(T)}\int_{n-1}^{n+1}\|M(x, s, E+\tfrac{i}{T})\|^{-2}\|\underline{u}_{f,E+\frac{i}{T}}(s)\|^2 dx\, dE \\
&\geq \tfrac{c}{T}\sum_{n=\frac{N(T)}{2}+2}^{N(T)-1}(n-2)^p\int_{B(T)}\int_{n-1}^{n+1}\|M(x, s, E+\tfrac{i}{T})\|^{-2}\|\underline{u}_{f,E+\frac{i}{T}}(s)\|^2 dx\, dE \\
&\geq \tfrac{2c}{T}\sum_{n=\frac{N(T)}{2}+2}^{N(T)-1}\left(\tfrac{N(T)}{2}\right)^p\int_{B(T)}(C\exp(C)N(T)^\alpha)^{-2}\,\|\underline{u}_{f,E+\frac{i}{T}}(s)\|^2 dE \\
&\geq \tfrac{2^{1-p}c}{T}\tfrac{N(T)}{3}N(T)^p|B(T)|(C\exp(C)N(T)^\alpha)^{-2}\inf_{\operatorname{dist}(z,B(T))\leq\frac{1}{T}}\|\underline{u}_{f,z}(s)\|^2.
\end{aligned}$$

Here, we used Lemma 2.5 in the second to the last step.

By Lemma 2.6 and $\langle u_{0,E}, f\rangle \neq 0$, there exists $\kappa > 0$ and $\delta > 0$ with $\inf\{\|\underline{u}_{f,z}(s)\|^2 : |z-E_0| \leq \delta\} \geq \kappa$. By $\operatorname{diam}(A(N)) \to 0$ as $N \to \infty$ and $E_0 \in A(N)$ for all N, we obtain

$$\inf\{\|\underline{u}_{f,z}(s)\|^2 : \operatorname{dist}(z, B(T)) \leq \tfrac{1}{T}\} \geq \kappa > 0$$

for T sufficiently large.

Thus, we can summarize the above estimates as

$$M_{f,D}(T,p) \geq \widetilde{C}|B(T)|T^{\frac{p-3\alpha}{1+\alpha}}.$$

This proves the theorem. □

Proof of Theorem 2. We set $u_{f,z} := (H-z)^{-1}f$. By Lemma 2.7, there exists $\epsilon \in \{-1,1\}$, $\kappa > 0$ and $\delta > 0$ such that

$$\inf\{\|\underline{u}_{f,z}(\epsilon s)\|^2 : |z-E_0| \leq \delta\} \geq \kappa.$$

Now, we consider the positive half-line if $\epsilon = 1$ and the negative half-line if $\epsilon = -1$. The proof is then a simple modification of the argument used in the proof of Theorem 1. □

3. The Bernoulli-Anderson model: Basic setting and deterministic results

In the next three sections we consider the following situation: Let g_0 and g_1 be two real-valued, locally integrable potentials with support in $[0,1]$. Also, let $\omega = (\omega_n)_{n\in\mathbb{Z}}$ be a two sided sequence with $\omega_n \in \{0,1\}$ for all n. Define the Schrödinger operator

$$H_\omega = -\frac{d^2}{dx^2} + V_\omega, \tag{17}$$

where

$$V_\omega(x) = \sum_{n\in\mathbb{Z}} g_{\omega_n}(x-n) \tag{18}$$

in $L^2(\mathbb{R})$. We may equivalently write

$$H_\omega = -\frac{d^2}{dx^2} + V_{per}^{(0)}(x) + \sum_{n\in\mathbb{Z}} \omega_n g(x-n), \tag{19}$$

with deterministic periodic background potential $V_{per}^{(0)}(x) = \sum_n g_0(x-n)$ and single-site potential $g = g_1 - g_0$. In the case where the ω_n are independent, identically distributed random variables, the family H_ω then represents a continuum Bernoulli-Anderson-type model. This case will be considered in Section 4, while we state a simple deterministic bound in this section.

Define also $V_{per}^{(1)}(x) = \sum_n g_1(x-n)$ and consider the two periodic Schrödinger operators $H^{(j)} = -d^2/dx^2 + V_{per}^{(j)}$. Let $T_j(E)$ be the transfer matrix for $H^{(j)}$ at energy E from 0 to 1.

We say that $E_0 \in \mathbb{R}$ is a critical energy for H_ω if

(20)
(i) $T_0(E_0)$ and $T_1(E_0)$ commute,
(ii) E_0 is contained in the interior of the spectra of $H^{(0)}$ and of $H^{(1)}$.

The same definition was used for discrete polymer models in [18]. With the help of Theorem 2 we can now extend the results obtained in [18] to continuum operators, starting with a continuum analog of Theorem 1 in [18].

Lemma 3.1. *If E_0 is a critical energy for H_ω, then the transfer matrix of H_ω at E_0 is globally bounded: There exists $C < \infty$ such that*

$$\|M(x,y,E_0)\| \leq C \tag{21}$$

for all $x, y \in \mathbb{R}$.

By the whole-line version of Corollary 2.1 (with $\alpha = 0$), this has the following immediate consequence.

Theorem 3. *If $f \in L^2(-s,s)$ is not orthogonal in $L^2(-s,s)$ to the space of solutions of $-u'' + V_\omega u = E_0 u$, then*

$$\beta_f^-(p) \geq p - 1. \tag{22}$$

Proof of Lemma 3.1. By (20)(ii), the two transfer matrices $T_j(E_0)$ each have two different complex-conjugate eigenvalues $e^{\pm i\eta_j} \notin \{\pm 1\}$ or are $\pm I$ (in which case we set $\eta_j = 0$ or $\eta_j = \pi$). Due to commutation, there exists a real invertible matrix M such that

$$MT_j(E_0)M^{-1} = \begin{pmatrix} \cos\eta_j & -\sin\eta_j \\ \sin\eta_j & \cos\eta_j \end{pmatrix} \tag{23}$$

simultaneously for $j = 0$ and $j = 1$. This shows that the transfer matrix of H_ω at E_0 between two given integers x and y is similar (via M) to a product of rotations, and thus has norm bounded by $C = \|M\|\|M^{-1}\|$. Standard arguments (e.g., [4, Appendix A]) imply that (21) holds for arbitrary $x, y \in \mathbb{R}$ and suitably enlarged C. □

Note that Lemma 3.1 is an entirely deterministic result: If a critical energy E_0 exists (which only depends on g_0 and g_1), then (21) holds for *every* choice of the sequence ω. Similarly, the dependence on ω enters Theorem 3 only through the non-orthogonality condition on f, and thus involves only finitely many ω_n, ($n = -s, \ldots, s-1$ if s is an integer). The bound (22) then holds uniformly in the values of all other ω_n.

4. The Bernoulli-Anderson model: Almost sure results

We now consider the model (17), (18) for independent, identically distributed Bernoulli random variables ω_n, i.e. we equip $\Omega := \{0,1\}^{\mathbb{Z}}$ with the measure $P = \prod_{j\in\mathbb{Z}} \mu$, where μ is a Bernoulli probability measure on $\{0,1\}$, $\mu(\{0\}) = p$, $\mu(\{1\}) = 1 - p$ for some $0 < p < 1$.

For this case, under a slight restriction on the phases η_0 and η_1 from the proof of Lemma 3.1, we will improve the result from the previous section and show that the lower diffusion exponents almost surely satisfy the lower bound $p - 1/2$. This is a continuum analog of Theorem 4 in [18], which establishes the same almost sure lower bound for discrete random polymer models.

This will be achieved by combining Corollary 2.2 with a Borel-Cantelli argument and using a large deviations analysis of the growth of transfer matrices for energies near a critical energy. Here we follow the ideas developed for discrete models in [18].

Our aim is to analyze the growth behavior of the transfer matrices

$$T_\omega(k,m,E) := T_{\omega_{k-1}}(E) \ldots T_{\omega_m}(E)$$

for $\omega \in \{0,1\}^{\mathbb{Z}}$ and E close to E_c. This can very conveniently be done by a Prüfer type decomposition, i.e. by decomposing the action of the transfer matrices into a rotation and a scaling.

To do this simultaneously for all E close to E_c, we need the following lemma.

Lemma 4.1. *Let E_c be a critical energy. Then, there exists an interval I around E_c, $\eta_j \in \mathbb{R}$, and analytic functions $a_j, b_j : I \longrightarrow \mathbb{C}$, $j = 0, 1$, $F : I \longrightarrow \mathrm{GL}(2, \mathbb{C})$ such that*

$$\widetilde{T}_j(E) := F(E)^{-1} T_j(E) F(E) = \begin{pmatrix} a_j(E) & \overline{b_j(E)} \\ b_j(E) & \overline{a_j(E)} \end{pmatrix} \quad \text{for } E \in I \tag{24}$$

with

$$b_j(E_c) = 0, \ \ \text{and} \ a_j(E_c) = e^{i\eta_j} \ \text{for} \ \eta_j \in [0, 2\pi), \ \ j = 0, 1.$$

In fact, we may choose $b_j(E) = 0$ and $|a_j(E)| = 1$ for all $E \in I$ and either $j = 0$ or $j = 1$. Moreover, $1 = \det \widetilde{T}_j(E) = |a_j(E)|^2 - |b_j(E)|^2$.

Proof. As E_c is in the interior of both periodic spectra, we have

$$D_j(E_c) := \operatorname{tr} T_j(E_c) \in [-2, 2]$$

for $j = 1, 2$. W.l.o.g. we may assume that either $D_0(E_c) \in (-2, 2)$ or that $|D_j(E_c)| = 2$ for both values of j. In the latter case E_c is a degenerate gap for $H^{(0)}$ and $H^{(1)}$ and thus $T_0(E_c)$ and $T_1(E_c)$ are both either I or $-I$.

By Lemma A.1, there is an open neighborhood I of E_c and complex conjugate analytic $v_{\pm}(E)$ which for each $E \in I$ are linearly independent eigenvectors of $T_0(E)$ to complex conjugate analytic eigenvalues $\rho_{\pm}(E)$.

The matrix $F(E) := (v_+(E), v_-(E))$ is invertible and

$$F(E)^{-1} T_0(E) F(E) = \begin{pmatrix} \rho_+(E) & 0 \\ 0 & \overline{\rho_+(E)} \end{pmatrix}.$$

Thus (24) holds for $j = 0$ with $b_0(E) = 0$ and $a_0(E) = \rho_+(E)$ for all $E \in I$. Moreover, as $T_1(E)$ is real and the columns of $F(E)$ are complex conjugates of each other, there exists a_1 and b_1 with

$$F(E)^{-1} T_1(E) F(E) = \begin{pmatrix} a_1(E) & \overline{b_1(E)} \\ b_1(E) & \overline{a_1(E)} \end{pmatrix}.$$

As F and T_j are analytic, so are a_j and b_j. As T_j has determinant equal to one by constancy of the Wronskian, we have

$$1 = \det \widetilde{T}_j(E) = |a_j(E)|^2 - |b_j(E)|^2.$$

Finally, the linearly independent eigenvectors $v_+(E_c)$ and $v_-(E_c)$ of $T_0(E_c)$ are eigenvectors of $T_1(E_c)$ as well (this is trivial if $|D_0(E_c)| = |D_1(E_c)| = 2$ and in the other case follows from the fact that $T_0(E_c)$ and $T_1(E_c)$ commute and that $T_0(E_c)$ has one-dimensional eigenspaces). We infer $b_1(E_c) = 0$ and $|a_1(E_c)| = 1$. □

Given this lemma, we can give the Prüfer type analysis of the action of the transfer matrices mentioned above. This will be carried out on the level of the $\widetilde{T}_j$. We identify $\mathbb{R}/2\pi\mathbb{Z}$ with $[0, 2\pi)$. For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we define the unit vector $\widetilde{e}_\theta$ by

$$\widetilde{e}_\theta := \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(i\theta) \\ \exp(-i\theta) \end{pmatrix}.$$

Then,

$$\widetilde{T}_j(E) \widetilde{e}_\theta = \frac{1}{\sqrt{2}} \begin{pmatrix} a_j(E) \exp(i\theta) + \overline{b_j(E)} \exp(-i\theta) \\ b_j(E) \exp(i\theta) + \overline{a_j(E)} \exp(-i\theta) \end{pmatrix}.$$

Obviously, the first and the second component of $\widetilde{T}_j(E)\widetilde{e}_\theta$ are complex conjugates of each other. Thus, for each $\delta := E - E_c$, there exists a unique map

$$\mathcal{S}_{\delta,j} : \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$$

with

$$\widetilde{T}_j(E)\widetilde{e}_\theta = \|\widetilde{T}_j(E)\widetilde{e}_\theta\|\widetilde{e}_{\mathcal{S}_{\delta,j}(\theta)}$$

for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Moreover, by $|a_j|^2 - |b_j|^2 = 1$, we find

$$\|\widetilde{T}_j(E)\widetilde{e}_\theta\|^2 = 1 + 2\mathrm{Re}(a_j(E)b_j(E)e^{2i\theta}) + 2|b_j(E)|^2. \tag{25}$$

In order to study the transfer matrices it will be convenient to define iterates of the $\mathcal{S}_{\delta,j}$. More precisely, for $l, m \in \mathbb{Z}$ with $l \geq m$ we define inductively

$$\mathcal{S}_{\delta,\omega}^{m,m}(\theta) = \theta, \;\; \mathcal{S}_{\delta,\omega}^{l+1,m}(\theta) = \mathcal{S}_{\delta,\omega_l}(\mathcal{S}_{\delta,\omega}^{l,m}(\theta)).$$

Proposition 4.2. *Let M be a matrix of the form $\begin{pmatrix} a & \overline{b} \\ b & \overline{a} \end{pmatrix}$. Then,*

$$\|M\| = \sup_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \|M\widetilde{e}_\theta\|.$$

Proof. Let $Q := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. As Q is unitary, we have $\|M\| = \|Q^{-1}MQ\|$. By assumption on M, the matrix $Q^{-1}MQ$ is real. Thus, setting $e_\theta := (\cos(\theta), \sin(\theta))^t$ and using $\widetilde{e}_\theta = Qe_\theta$, we obtain

$$\|Q^{-1}MQ\| = \sup_\theta \|Q^{-1}MQe_\theta\| = \sup_\theta \|Q^{-1}M\widetilde{e}_\theta\| = \sup_\theta \|M\widetilde{e}_\theta\|.$$

This finishes the proof. □

We are now in a position to provide the key expression for the norm of the transfer matrix.

Proposition 4.3. *Let $\omega \in \{0,1\}^{\mathbb{Z}}$, $\delta \in \mathbb{R}$, $k, m \in \mathbb{Z}$ with $k > m$ be given. Then,*

$$\log \|T_\omega(k,m,E_c+\delta)\|^2 = 2\delta \sup_\theta \{\mathrm{Re} \sum_{l=m}^{k-1} c_{\omega_l} e^{2i\mathcal{S}_{\delta,\omega}^{l,m}(\theta)}\} + O(\delta^2(k-m), 1),$$

where $c_{\omega_l} := e^{i\eta_{\omega_\ell}} \frac{db_{\omega_l}}{dE}(E_c)$.

Proof. Let $E = E_c + \delta$. We begin by estimating $\|F(E)^{-1}T_\omega(k,m,E)F(E)\|^2$:

$$\begin{aligned}
\|F(E)^{-1}T_\omega(k,m,E)F(E)\|^2 &= \|\widetilde{T}_{\omega_{k-1}}(E)\dots\widetilde{T}_{\omega_m}(E)\|^2 \\
\text{(Prop 4.2)} \quad &= \sup_\theta \|\widetilde{T}_{\omega_{k-1}}(E)\dots\widetilde{T}_{\omega_m}(E)\widetilde{e}_\theta\|^2 \\
&= \sup_\theta \prod_{l=m}^{k-1} \|\widetilde{T}_{\omega_l}(E)\widetilde{e}_{\mathcal{S}_{\delta,\omega}^{l,m}(\theta)}\|^2 \\
(25) \quad &= \sup_\theta \prod_{l=m}^{k-1} (1 + 2\mathrm{Re}(a_{\omega_l}(E)b_{\omega_l}(E)e^{2i\mathcal{S}_{\delta,\omega}^{l,m}(\theta)}) + 2|b_{\omega_l}(E)|^2).
\end{aligned}$$

As b is analytic around E_c with $b_j(E_c) = 0$ for $j = 0, 1$, by Lemma 4.1, we have $b(E_c + \delta) = O(\delta)$. Thus, taking logarithms and invoking $\log(1 + x) = x + O(x^2)$, we obtain from the previous formula

$$\log \|F(E)^{-1}T_\omega(k, m, E)F(E)\|^2 = 2 \sup_\theta \mathrm{Re} \sum_{l=m}^{k-1} a_{\omega_l}(E)b_{\omega_l}(E)e^{2i\mathcal{S}^{l,m}_{\delta,\omega}(\theta)} + O(\delta^2(k-m)).$$

By analyticity of b_j and a_j around E_c we further have $b_j(E) = \delta \frac{db_j}{dE}(E_c) + O(\delta^2)$ and $a_j(E) = a_j(E_0) + O(\delta) = e^{i\eta_j} + O(\delta)$. Thus, we end up with

$$\log \|F(E)^{-1}T_\omega(k, m, E)F(E)\|^2 = 2\delta \sup_\theta \mathrm{Re} \sum_{l=m}^{k-1} c_{\omega_l} e^{2i\mathcal{S}^{l,m}_{\delta,\omega}(\theta)} + O(\delta^2(k - m)).$$

The statement of the proposition follows as $F(E)$ and its inverse $F(E)^{-1}$ are uniformly bounded in a neighborhood of E_c. □

The proposition gives a closed expression for the norm of the transfer matrices in terms of sums of the form

$$\sum_{l=m}^{k-1} c_{\omega_l} e^{2i\mathcal{S}^{k,m}_{\delta,\omega}(\theta)}.$$

For sums of this form, a large deviation estimate has been proven in [18] in the case where $\{0, 1\}^{\mathbb{Z}}$ is equipped with a Bernoulli measure. This estimate carries over to our situation almost immediately. Here are the details:

Definition 4.4. *Set* $\mathcal{S}^k_{\delta,\omega} := S^{k,0}_{\delta,\omega}$. *Let* $\alpha > 0$, $N \in \mathbb{N}, \omega \in \{0, 1\}^{\mathbb{Z}}$, $\delta \in \mathbb{R}$ *and* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ *be given. Define* $I_{\omega,N}(\theta, \delta) := \sum_{k=0}^{N-1} c_{\omega_k} e^{2i\mathcal{S}^k_{\delta,\omega}(\theta)}$ *and*

$$\Omega_N(\alpha, \delta, \theta) := \{\omega \in \Omega : \exists k \le N \ \ s.t.\ \ |I_{\omega,k}(\theta, \delta)| \ge N^{\alpha+\frac{1}{2}}\}.$$

We consider Ω as a probability space with the Bernoulli measure P defined at the beginning of this section. As above η_0 and η_1 are the phases of $a_0(E_c)$ and $a_1(E_c)$.

Lemma 4.5. *Assume that* $\eta_0 - \eta_1$ *is not an integer multiple of* π. *Then, for every* $\alpha > 0$, *there exist* $C_1 < \infty$ *and* $C_2 > 0$ *such that for all* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ *and all* $\delta \in \mathbb{R}$ *and* $N \in \mathbb{N}$ *with* $\delta^2 N \le 1$, *the estimate*

$$P(\Omega_N(\alpha, \delta, \theta)) \le C_1 e^{-C_2 N^\alpha}$$

holds.

Proof. Let $E = E_c + \delta$. By definition of the action $\mathcal{S}$ we have $\widetilde{T}_j(E)\widetilde{e}_\theta = \|\widetilde{T}_j(E)\widetilde{e}_\theta\|\widetilde{e}_{\mathcal{S}_{\delta,j}(\theta)}$. Combining this with (25), we find

$$\widetilde{T}_j(E)\widetilde{e}_\theta = \widetilde{e}_{\mathcal{S}_{\delta,j}(\theta)} + O(|b_j(E)|).$$

On the other hand, the analyticity shown in Lemma 4.1 gives

$$\widetilde{T}_j(E)\widetilde{e}_\theta = \widetilde{e}_{\theta+\eta_j} + O(\delta).$$

Combining the last two equalities and using $b_j(E_c + \delta) = O(\delta)$, we obtain

$$\widetilde{e}_{\mathcal{S}_{\delta,j}(\theta)} = \widetilde{e}_{\theta+\eta_j} + O(\delta),$$

from which we conclude

$$e^{2i\mathcal{S}_{\delta,j}(\theta)} = e^{2i(\theta+\eta_j)} + O(\delta).$$

This formula is the crucial input in the proof of Theorem 6 of [18]. Thus, we can now follow this proof line by line to obtain the desired statement. We only note that the condition $|pe^{2i\eta_0}+(1-p)e^{2i\eta_1}|<1$ used in this context in [18] is equivalent to our condition $\eta_0 \neq \eta_1 \mod \pi$ (as $pe^{2i\eta_0}+(1-p)e^{2i\eta_1}$ is a convex-combination of two numbers on the unit circle). □

We can now state our main result on Bernoulli-type models.

Theorem 4. *Assume that $\eta_0 - \eta_1$ is not an integer multiple of π. Let $\alpha > 0$ be arbitrary. Then, there are $c > 0$ and $C < \infty$ such that for every $N \in \mathbb{N}$, there is a set $\Omega_N(\alpha) \subset \{0,1\}^{\mathbb{Z}}$ with $P(\Omega_N(\alpha)) \le Ce^{-cN^\alpha}$ and*

$$\|T_\omega(x,y,E)\| \le C$$

for all $\omega \in \Omega \setminus \Omega_N(\alpha)$, $-N \le x,y \le N$ and $|E-E_c| \le N^{-\alpha-1/2}$.

In particular, $\beta_f^-(p) \ge p - 1/2$ holds for almost every ω and every compactly supported f that is not orthogonal to all solutions of $-u'' + V_\omega u = E_0 u$.

Proof. The first claim is established by following the proof of Theorem 6 in [18]: By translation invariance it suffices to consider $0 \le x,y \le 2N$. Let $\Omega_N(\alpha,\delta) := \Omega_{2N}(\alpha,\delta,0) \cap \Omega_{2N}(\alpha,\delta,\pi/2)$. Then, by Lemma 4.5, $P(\Omega_N(\alpha,\delta)) \le C_1' e^{-C_2' N^\alpha}$. From Proposition 4.3 it follows that there is a constant $C' < \infty$ such that for all $N \in \mathbb{N}$, integers $0 \le k,m \le 2N$, $|\delta| \le N^{-\alpha-1/2}$ and $\omega \in \Omega_N(\alpha,\delta)$ it holds that

$$\|T_\omega(k,m,E_c+\delta)\| \le C'. \tag{26}$$

Here we have also used that $T(k,m) = T(k,0)T(m,0)^{-1}$ and that $\|A\| = \sup_\theta \|Ae_\theta\| \le \sqrt{2}\{\|Ae_0\|, \|Ae_{\pi/2}\|\}$ for every 2×2-matrix A. As remarked at the end of the proof of Theorem 3, the bound (26) extends to transfer matrices between arbitrary real $x,y \in [0,2\pi]$.

Note that so far δ is fixed in $\Omega_N(\alpha,\delta)$. To find a set $\Omega_N(\alpha)$ such that transfer matrices for $\omega \in \Omega_N(\alpha)^c$ are bounded uniformly for all $|\delta| \le N^{-\alpha-1/2}$ we use Lemma 2.5. Set $\varepsilon = N^{-\alpha-1/2}$ and

$$\Omega_N(\alpha) = \bigcup_{\ell=-N}^{N} \Omega_N(\alpha, \ell\epsilon/N).$$

For fixed ℓ, Lemma 2.5 shows that $\|T_\omega(x,y,E_c+\delta)\|$ is uniformly bounded for $\omega \in \Omega_N(\alpha,\ell\varepsilon/N)$, $\delta \in [\ell\varepsilon/N - \frac{1}{N}, \ell\varepsilon/N + \frac{1}{N}]$ and $0 \le x,y \le 2N$. This establishes the first part of the theorem as $P(\Omega_N(\alpha)) \le 2NC_1' e^{-C_2' N^\alpha} \le Ce^{-cN^\alpha}$.

The lower bound on diffusion exponents now follows from (the whole-line version) of Corollary 2.2. For each $\alpha > 0$, $P(\Omega_N(\alpha))$ is summable over N. Thus, by Borel-Cantelli, the assumption of Corollary 2.2 is satisfied for almost every ω and $\theta = \frac{1}{2} + \alpha$. We get that almost surely $\beta_f^-(p) \ge p - \frac{1}{2} - \alpha$ for every compactly supported f that is not orthogonal to all solutions of $-u'' + V_\omega u = E_0 u$. We finally take $\alpha = \frac{1}{n} \to 0$, using a countable intersection of full measure sets. □

5. The Bernoulli-Anderson model: A concrete example

The existence of critical energies for a given pair g_0 and g_1 is not a generic property. In fact, as (21) immediately implies vanishing of the Lyapunov exponent, the set of critical energies must be discrete by the results of [4]. However, it is easy to give examples where critical energies exist, see [5]. The most simple one is given by $g_0 = 0$ and $g_1 = \lambda\chi_{[0,1]}$, $\lambda > 0$, that is, V_ω consists of constant steps of height 0 or λ. This example will be discussed in more detail in this section.

In this case, all energies $E > \lambda$ satisfy (20)(ii). For such energies, the transfer matrices are

$$T^{(0)}(E) = \begin{pmatrix} \cos k & \frac{1}{k}\sin k \\ -k\sin k & \cos k \end{pmatrix} \tag{27}$$

and

$$T^{(1)}(E) = \begin{pmatrix} \cos\alpha & \frac{1}{\alpha}\sin\alpha \\ -\alpha\sin\alpha & \cos\alpha \end{pmatrix} \tag{28}$$

where $k = \sqrt{E}$ and $\alpha = \sqrt{E-\lambda}$. If $E = n^2\pi^2$, $n \in \mathbb{N}$, then $T^{(0)}(E) = \pm I$. On the other hand, if $E = n^2\pi^2 + \lambda$, then $T^{(1)}(E) = \pm I$. In both cases, $T^{(0)}(E)$ and $T^{(1)}(E)$ commute. Thus, when $\lambda \in (0, \pi^2)$, we have the following critical energies:

$$\{n^2\pi^2 : n \in \mathbb{N}\} \cup \{n^2\pi^2 + \lambda : n \in \mathbb{N}\}. \tag{29}$$

The richness of the set of critical energies gives us considerable flexibility in choosing the initial state f in Theorem 3 and Theorem 4, respectively. In fact, every non-zero f with support in $[0,1]$ satisfies the required non-orthogonality condition for at least one of the critical energies and we get from Theorem 3:

Corollary 5.1. *Let $\lambda \in (0, \pi^2)$, $g_0 = 0$, $g_1 = \lambda\chi_{[0,1]}$, and H_ω be given by* (17) *and* (18)*. Then*

$$\beta_f^-(p) \geq p - 1 \tag{30}$$

for any $f \in L^2(0,1)$, $f \neq 0$ and any ω.

Proof. There is at least one $n \in \mathbb{N}$ such that

$$\int_0^1 f(x)\sin(\pi n x)\,dx \neq 0 \quad \text{or} \quad \int_0^1 f(x)\cos(\pi n x)\,dx \neq 0. \tag{31}$$

If $f \neq \text{const}$ on $[0,1]$, then this follows (with even n) as $\{1\} \cup \{\sin(2\pi kx), \cos(2\pi kx); k \in \mathbb{N}\}$ span $L^2(0,1)$. For $0 \neq f = \text{const}$, we may choose $n = 1$.

Now consider two cases: If $\omega_0 = 0$, that is, $V_\omega(x) = 0$ on $[0,1]$, then by (31), f is not orthogonal in $L^2(0,1)$ to the space of solutions of $-u'' = n^2\pi^2 u$. Thus, (30) follows from Theorem 3 applied to the critical energy $E_0 = n^2\pi^2$. If, on the other hand, $\omega_0 = 1$, then we conclude in the same way, now based on the critical energy $E_0 = n^2\pi^2 + \lambda$. □

In the case where the ω_n are i.i.d. random variables, we can say even more almost surely by invoking Theorem 4. The condition $\eta_0 \neq \eta_1 \mod \pi$ of Theorem 4 is fulfilled for all critical energies (with η_0, η_1 now given by k, α) throughout the λ-interval $(0, \pi^2)$ under consideration.

Corollary 5.2. *If λ, g_0, g_1 and H_ω are as above, and the ω_n are i.i.d. random variables, then*

$$\beta_f^-(p) \geq p - 1/2 \tag{32}$$

for almost every ω and any $f \in L^2(0,1)$, $f \neq 0$.

The previous corollary is particularly interesting as in this case it was proven in [4] that the operator H_ω given by (17), (18) almost surely exhibits pure point spectrum with exponentially decaying eigenfunctions, assuming only that $g_0 \neq g_1$. Thus the case $g_0 = 0$, $g_1 = \lambda\chi_{[0,1]}$ gives an example of a continuum random Schrödinger operator with coexistence of spectral localization and super-diffusive transport ($\beta_f^-(2) \geq 3/2$).

Also, [4] establishes dynamical localization for H_ω in the following sense: If $g_0 \neq g_1$, then there is a discrete set $M \subset \mathbb{R}$ such that for every compact interval $I \subset \mathbb{R} \setminus M$, every compact set $K \subset \mathbb{R}$, and every $p > 0$,

$$\mathbb{E}\left\{\sup_{t\in\mathbb{R}} \||X|^{p/2} e^{-itH_\omega} P_I(H_\omega)\chi_K\|\right\} < \infty, \tag{33}$$

where P_I is the spectral projection onto I. Corollary 5.2 shows that the insertion of $P_I(H_\omega)$ is crucial here: $\mathbb{E}\{\sup_t \||X|^{p/2}e^{-itH_\omega}\chi_{[0,1]}\|\} < \infty$ would imply that $\beta_f^-(p) = 0$ for almost every ω and every f supported in $[0,1]$, contradicting (32) if $p > 1/2$. Thus, dynamical localization holds for the model (17), (18) in general only away from a discrete set of critical energies.

6. Self-Similar Potentials

In this section, we discuss operators whose potentials are generated by means of a substitution rule. The inherent self-similar structure of such potentials is expressed by the existence of a renormalization scheme that gives rise to a dynamical system, the so-called trace map, which governs the evolution of transfer matrix traces along the different levels of the hierarchy. Results on the dynamics of the trace map can often be used to establish power-law bounds for the norms of transfer matrices associated with suitable energies. In the discrete case, three prominent models were studied in [7], namely, the Fibonacci model, the period doubling model, and the Thue-Morse model. The strongest dynamical bound was obtained for the Thue-Morse model. We shall carry out an explicit analysis for continuum operators with Thue-Morse and period doubling symmetry, obtaining the same quantitative bounds, and then discuss the Fibonacci case briefly.

The Thue-Morse substitution on the alphabet $\{a, b\}$ is given by $S(a) = ab$, $S(b) = ba$. This mapping extends to words over this alphabet by concatenation. Thus, for example, $S^2(a) = abba$, $S^3(a) = abbabaab$. Let Ω_{TM} be the associated (two-sided) subshift, that is,

$$\Omega_{\text{TM}} = \{\omega \in \{a,b\}^{\mathbb{Z}} : \text{ every subword of } \omega \text{ is contained in } S^n(a) \text{ for some } n \in \mathbb{Z}_+\}.$$

Now choose two numbers $l_a, l_b > 0$ and two local potentials $V_a \in L^1(0, l_a)$ and $V_b \in L^1(0, l_b)$. Each sequence $\omega \in \Omega_{\text{TM}}$ generates a potential on $\mathbb{R}$ by

$$V_\omega(x) = V_{\omega_0}(x) \text{ on } (0, l_{\omega_0}),\;\; V(x) = V_{\omega_1}(x - l_{\omega_0}) \text{ on } (l_{\omega_0}, l_{\omega_0} + l_{\omega_1}), \text{ etc.},$$

and similarly on the left half-line, using $\{\omega_j\}_{-\infty<j\leq -1}$.

Theorem 5. *For every pair (V_a, V_b), there are $E_0 \in \mathbb{R}$ and $C > 0$ such that for every $\omega \in \Omega_{\text{TM}}$,*

$$\|M_\omega(x, y, E_0)\| \leq C \text{ for all } x, y \in \mathbb{R}. \tag{34}$$

Proof. If $x_1 \dots x_n$ is a word over the alphabet $\{a, b\}$ and $E \in \mathbb{R}$, we denote by $M(x_1 \dots x_n, E)$ the transfer matrix over an interval of length $l_{x_1} + \cdots + l_{x_n}$ with potential given by $V_{x_1} \cdots V_{x_n}$ and energy E. Define

$$M_k^{(0)}(E) = M(S^k(0), E), \;\; M_k^{(1)}(E) = M(S^k(1), E),$$

and

$$x_k(E) = \operatorname{tr} M_k^{(0)}(E), \; y_k(E) = \operatorname{tr} M_k^{(1)}(E).$$

It is clear that $x_k = y_k$ for $k \geq 1$ and it follows from the substitution rule that

$$M_k^{(0)}(E) = M_{k-1}^{(1)}(E) M_{k-1}^{(0)}(E), \; M_k^{(1)}(E) = M_{k-1}^{(0)}(E) M_{k-1}^{(1)}(E)$$

and

$$x_k(E) = x_{k-2}(E)^2 (x_{k-1}(E) - 2) + 2 \;\text{ for } k \geq 3. \tag{35}$$

The relation (35) is called the Thue-Morse trace map.

Fix some $k \geq 3$ and let $\mathcal{E}_k = \{E : x_{k-2}(E) = 0\}$. It follows from Floquet theory that $\mathcal{E}_k$ is countably infinite. We claim that for every $E \in \mathcal{E}_k$,

$$M_k^{(0)}(E) = M_k^{(1)}(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{36}$$

This is a consequence of the Cayley-Hamilton theorem:

$$\begin{aligned} M_k^{(0)}(E) &= M_{k-2}^{(0)}(E) \, M_{k-2}^{(1)}(E) \, M_{k-2}^{(1)}(E) \, M_{k-2}^{(0)}(E) \\ &= M_{k-2}^{(0)}(E) \left(x_{k-2}(E) M_{k-2}^{(1)}(E) - I \right) M_{k-2}^{(0)}(E) \\ &= -M_{k-2}^{(0)}(E) M_{k-2}^{(0)}(E) \\ &= -\left(x_{k-2}(E) M_{k-2}^{(0)}(E) - I \right) \\ &= I \end{aligned}$$

and, similarly, $M_k^{(1)}(E) = I$. This yields (36). From this, (34) follows readily. □

Thus, we can apply Corollary 2.1 with $\alpha = 0$ and obtain $\beta^-_{\omega,f}(p) \geq p - 1$ for suitable compactly supported f. Note that we can consider operators either on the half-line or on the whole line; the respective version of Corollary 2.1 then tells us what is required from f.

Next we consider the period doubling substitution on the alphabet $\{a, b\}$, which is given by $S(a) = ab$, $S(b) = aa$. Again, we define the associated (two-sided) subshift Ω_{PD} and choose two local potentials $V_a \in L^1(0, l_a)$ and $V_b \in L^1(0, l_b)$, generating potentials V_ω as before.

Theorem 6. *For every pair (V_a, V_b), there are $E_0 \in \mathbb{R}$ and $C > 0$ such that for every $\omega \in \Omega_{\text{PD}}$,*

$$\|M_\omega(x, y, E_0)\| \leq C(1 + |x - y|) \text{ for all } x, y \in \mathbb{R}. \tag{37}$$

Proof. Define

$$M_k^{(0)}(E) = M(S^k(0), E), \;\; M_k^{(1)}(E) = M(S^k(1), E),$$

and

$$x_k(E) = \operatorname{tr} M_k^{(0)}(E), \; y_k(E) = \operatorname{tr} M_k^{(1)}(E).$$

It follows from the substitution rule that

$$M_k^{(0)}(E) = M_{k-1}^{(1)}(E) M_{k-1}^{(0)}(E), \; M_k^{(1)}(E) = M_{k-1}^{(0)}(E) M_{k-1}^{(0)}(E)$$

and

$$x_k(E) = x_{k-1}(E)y_{k-1}(E) - 2, \;\; y_k(E) = (x_{k-1}(E))^2 - 2. \tag{38}$$

The relation (38) is called the period doubling trace map.

Fix some $k \geq 2$ and let $\mathcal{E}_k = \{E : x_{k-1}(E) = 0\}$. It follows again from Floquet theory that $\mathcal{E}_k$ is countably infinite. For $E \in \mathcal{E}_k$, (38) yields $x_k(E) = -2$. Thus, there is a constant γ such that

$$M_k^{(0)}(E) \text{ is conjugate to } \begin{pmatrix} -1 & \gamma \\ 0 & -1 \end{pmatrix}.$$

Moreover, it follows from the Cayley-Hamilton theorem that

$$M_k^{(1)}(E) = M_{k-1}^{(0)}(E)M_{k-1}^{(0)}(E) = x_{k-1}(E)M_{k-1}^{(0)}(E) - \mathrm{Id} = -\mathrm{Id}.$$

The bound (37) is now an immediate consequence of these two observations. □

Thus, we can apply Corollary 2.1 with $\alpha = 1$ and obtain $\beta^-_{\omega,f}(p) \geq (p-5)/2$ for suitable compactly supported f.

We conclude this section with a brief discussion of the Fibonacci case. The Fibonacci substitution on the alphabet $\{a, b\}$ is given by $S(a) = ab$, $S(b) = a$. As before, we may define the subshift Ω_F generated by the substitution and, given two local potentials, a family of Schrödinger operators. It is possible to prove power-law upper bounds for the associated transfer matrices for suitable energies. This is technically much more involved than in the Thue-Morse or period doubling case, but it can be accomplished using ideas from [7, 8, 15, 19]. The analysis in those papers is to a large extent independent of the explicit form of the transfer matrices and is mainly based on the renormalization scheme that arises naturally from the substitution rule.

The Fibonacci case is different from Thue-Morse and period doubling on a conceptual level as there are no "exceptional" enegies. In fact, one can prove power-law bounds for all energies in the spectrum. In the discrete case, one can even choose the power uniformly on the spectrum. Thus, the methods in [7] and this paper should not be expected to give the best dynamical results in the Fibonacci case. Indeed, the best known dynamical results for the discrete version of the Fibonacci potential are contained in [8]. The latter paper combines ideas from [7] and [16, 19] and gives quite strong dynamical bounds in cases where one has quite good solution estimates.

7. Further remarks

In this section we address a number of issues that are suggested by our work. Most importantly, we extend the main result in the discrete case from [7] to more general finitely supported initial states.

7.1. Dynamical Bounds for Discrete Schrödinger Operators.

Given a bounded $V : \mathbb{Z} \to \mathbb{R}$, we may consider the discrete Schrödinger operator

$$[H\varphi](n) = \varphi(n+1) + \varphi(n-1) + V(n)\varphi(n)$$

on $\mathcal{H} = \ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{1, 2, \dots\}$, (with a suitable boundary condition at the origin; e.g., Dirichlet) and the associated difference equation

$$u(n+1) + u(n-1) + V(n)u(n) = E_0 u(n). \tag{39}$$

The particular solution of (39) satisfying $u(0) = 0$, $u(1) = 1$ will be denoted by u_{0,E_0}.

The position operator acts as $[X\varphi](n) = n\varphi(n)$ and we can define the quantities $M_f(T,p)$, $M_{f,D}(T,p)$, $\beta_f^-(p)$, and $\beta_{f,D}^-(p)$ as before; compare (3) and (4).

Transfer matrices and the sets $P(\alpha, C, N)$ are also defined in a completely analogous way. With the standard scalar product on $\mathcal{H}$, we may now state the following pair of results, which are the discrete analogs of Theorems 1 and 2.

Theorem 7. *Suppose $E_0 \in \mathbb{R}$ is such that there exist $C > 0$ and $\alpha > 0$ with $E_0 \in P(\alpha, C, N)$ for all sufficiently large N. Let $A(N)$ be a subset of $P(\alpha, C, N)$ containing E_0 such that* $\mathrm{diam}(A(N)) \longrightarrow 0$ *as $N \to \infty$. Then, for every finitely supported $f \in \ell^2(\mathbb{Z}_+)$ with $\langle u_{0,E_0}, f\rangle \neq 0$, there exists $\widetilde{C} > 0$ such that for T large enough, $M_{f,D}(T,p) \geq \widetilde{C}|B(T)|T^{\frac{p-3\alpha}{1+\alpha}}$, where $B(T)$ is the $1/T$ neighborhood of $A(T^{\frac{1}{1+\alpha}})$.*

Theorem 8. *Suppose $E_0 \in \mathbb{R}$ is such that there exist $C > 0$ and $\alpha > 0$ with $E_0 \in P(\alpha, C, N)$ for sufficiently large N. Let $A(N)$ be a subset of $P(\alpha, C, N)$ containing E_0 such that* $\mathrm{diam}(A(N)) \longrightarrow 0$ *as $N \to \infty$. Let $f \in \ell^2(\mathbb{Z})$ be finitely supported and satisfy $\langle u, f\rangle \neq 0$ for at least one solution u of* (39)*. Then, for T large enough, $M_f(T,p) \geq \widetilde{C}|B(T)|T^{\frac{p-3\alpha}{1+\alpha}}$, where $B(T)$ is the $1/T$ neighborhood of $A(T^{\frac{1}{1+\alpha}})$.*

These results are proved in the exact same way as their continuum counterparts. When specializing Theorems 7 and 8 to the case $f = \delta_1$, we recover the results of Damanik and Tcheremchantsev from [7] (for isolated critical energies). Notice that the assumption $\langle u_{0,E_0}, f\rangle \neq 0$ (resp., $\langle u, f\rangle \neq 0$ for at least one solution u of (39)) is trivially satisfied in this case and hence was not an issue in [7]. This is also the reason why the results of [7] do not immediately suggest the correct formulation of an extension to more general initial states.

Let us discuss the example of a (random) dimer model in more detail. On the one hand, this will generalize results of [18] and, on the other hand, this will provide the discrete analog of our discussion in Section 5; particularly, Corollaries 5.1 and 5.2. A dimer model is a discrete Schrödinger operator on the whole line whose potential V takes values in the set $\{\lambda, -\lambda\}$, $\lambda > 0$, and satisfies $V(2n) = V(2n-1)$ for all n. A random dimer model is a family of dimer models $\{H_\omega\}$, where $\omega \in \{\lambda, -\lambda\}^{\mathbb{Z}}$, $V_\omega(2n) = \omega_n$, and the ω_n's are i.i.d. random variables. Notice that the energies $E_0 = \pm\lambda$ are critical if $0 < \lambda < 1$. It is straightforward to see that Theorem 8 above, combined with [18, Theorem 7], implies the following:

Corollary 7.1. *Let $\lambda \in (0,1)$ and $f \neq 0$ be supported in $\{1,2\}$. Then, for every $p > 0$,*

(i) $\beta_{f,\omega}^-(p) \geq p - 1$ *for every ω,*
(ii) $\beta_{f,\omega}^-(p) \geq p - 1/2$ *for almost every ω.*

A straightforward calculation shows that the condition $\eta_0 - \eta_1 \neq 0 \mod \pi$, required in [18, Theorem 7], holds at $E_0 = \pm\lambda$ for every $\lambda \in (0,1)$.

For $f = \delta_1$, the above was shown in [18]. Here we see that, by translation invariance, we may in fact take all non-trivial initial states f that have their support in one of the random blocks. In other words, this is the precise analog of Corollaries 5.1 and 5.2 from Section 5.

7.2. **Open Problems.** We conclude this paper with a discussion of open problems that are suggested by our work and previous ones.

Our dynamical results require a certain non-orthogonality condition from the initial state f. Such a condition can certainly not be dropped in general as the Bernoulli-Anderson example shows: If f has no energy near a critical one, the results of [4] show that no non-trivial dynamical lower bound exists. This suggests studying cases where $\langle u_{0,E_0}, f\rangle = 0$, E_0 critical, but $\langle u_{0,E}, f\rangle \neq 0$ for energies E close to E_0. For example, is it possible to prove a non-trivial dynamical lower bound if the function $E \mapsto \langle u_{0,E}, f\rangle$ vanishes to a finite order at E_0? (Note that all roots of $\langle u_{0,E}, f\rangle$ are of finite order if f is compactly supported and non-zero: In this case $\{u_{0,E} : E \in \mathbb{R}\}$ is total in L^2 over the support of f and thus the analytic function $\langle u_{0,E}, f\rangle$ doesn't vanish identically.)

Our results for the Bernoulli-Anderson model once again motivate the following question: Is the bound (6) optimal? For the random dimer model, it was conjectured in [9] that indeed $\beta^-_{\delta_1,\omega}(2) = 3/2$ for almost every ω. That 3/2 is a lower bound was shown in [18], and here we proved an analogous result for the Bernoulli-Anderson model with a critical energy. Proving dynamical upper bounds, especially on moments of the position operator, is a hard problem. The only existing results in this direction for random Schrödinger operators[1] establish complete dynamical localization in the sense that all diffusion exponents vanish. It is not clear how to deal with critical energies in terms of proving dynamical upper bounds, and we consider this an interesting open problem.

Appendix A. Existence of analytic eigenvectors

The following fact from Floquet theory has been used in the proof of Lemma 4.1 above. This is probably well known. We include a proof mainly for the reason that we use it not only for energies E_c in the interior of stability intervals (where $|D_0(E_c)| < 2$), but also at degenerate band edges (E_c in the interior of the spectrum, but $|D_0(E_c)| = 2$). For general background on Floquet theory see [10].

Lemma A.1. *Let $H^{(0)}$ be a periodic Schrödinger operator as in Section 3 and E_c in the interior of $\sigma(H^{(0)})$. Then there exists an open neighborhood I of E_c and analytic functions $\rho_\pm : I \to \mathbb{C}$ and $v_\pm : I \to \mathbb{C}^2$ such that for each $E \in I$, $T_0(E)v_\pm(E) = \rho_\pm(E)v_\pm(E)$, the $v_\pm(E)$ are linearly independent, and $\rho_-(E) = \overline{\rho_+(E)}$, $v_-(E) = \overline{v_+(E)}$.*

Proof. Define $D_0(E) := \operatorname{tr} T_0(E)$. As E_c belongs to the spectrum of $H^{(0)}$, we have

$$-2 \leq D_0(E) \leq 2.$$

We consider two cases:

Case 1: $-2 < D_0(E_c) < 2$.

Then, $-2 < D_0(E) < 2$ for all E in an interval I around E_c. In this interval $T_0(E)$ has the different complex conjugate eigenvalues

$$\rho_\pm(E) = \frac{1}{2}(D_0(E) \pm i\sqrt{4 - D_0(E)^2}) \tag{40}$$

[1]For a class of sparse potentials, Tcheremchantsev has explicitly determined the diffusion exponents [25].

with corresponding linearly independent complex conjugate eigenvectors

$$v_\pm(E) = \begin{pmatrix} 1 \\ c_\pm(E) \end{pmatrix}. \tag{41}$$

Here, $\sqrt{\ }: [0,\infty) \longrightarrow [0,\infty)$ is the usual square root and

$$c_\pm(E) = \frac{\rho_\pm(E) - u_N(1,E)}{u_D(1,E)} \tag{42}$$

where u_N and u_D are the solutions of $-u'' + V_0 u = Eu$ with initial conditions $u_N(0) = u_D'(0) = 1$, $u_N'(0) = u_D(0) = 0$. This is well known and easily checked. In particular, $u_D(1,E) \neq 0$ as E is not an eigenvalue of the Dirichlet problem on $[0,1]$.

Case 2: $|D_0(E_c)| = 2$.

As E_c is in the interior of the spectrum of $H^{(0)}$, we are at a degenerate band edge. Thus, $T_0(E_c)$ is equal to id or $-$id.

Assume w.l.o.g. $D_0(E_c) = 2$. As E_c belongs to the interior of the spectrum of $H^{(0)}$, D_0 has a local maximum at E_c, $D_0'(E_c) = 0$. As local extreme values of D_0 are non-degenerate (see, e.g., Section 2.3 of [10], in particular page 29), $D_0''(E_c) < 0$. This implies that $\sqrt{2 - D_0(E)}$ and therefore $\sqrt{4 - D_0(E)^2}$ have a branch which is analytic in a neighborhood I of E_c. We now use this branch in the definition of $\rho_\pm(E)$ via (40). Thus $\rho_\pm(E)$ are analytic and complex conjugate eigenvalues of $T_0(E)$ near E_c. One checks that

$$\rho_\pm'(E_c) = \pm i\sqrt{c} \neq 0, \tag{43}$$

where $c = |D_0''(E_c)|/2$. We again define $c_\pm$ and $v_\pm$ by (42) and (41), which makes them analytic up to a possible singularity at E_c. However, both $\rho_\pm(E) - u_N(1,E)$ and $u_D(1,E)$ have first order zeros at E_c. For the former, this follows from (43), noting that $u_N(1,E)$ is real. For the latter, this can be seen by using that the Prüfer phase

$$\theta(1,E) := \arctan \frac{u_D(1,E)}{u_D'(1,E)} \tag{44}$$

has positive E-derivative (one way to prove this may be found in [24, Section 4]). Using that $u_D(1,E_c) = 0$ and $u_D'(1,E_c) = 1$ it is now easily verified from (44) that $(\partial_E u_D)(1,E_c) = (\partial_E \theta)(1,E_c) > 0$.

We conclude that the singularity of $c_\pm$ at $E = E_c$ is movable. We also see, from l'Hospital's rule, that $c_\pm(E_c)$ has non-vanishing imaginary part. We conclude that for all E near E_c the $v_\pm(E)$ are eigenvectors of $T_0(E)$ (this is trivial for $E = E_c$), which are complex conjugate and linearly independent. □

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