Upper and lower bounds on Mathieu characteristic numbers of integer orders

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Abstract

For each Mathieu characteristic number of integer order (MCN) we construct sequences of upper and lower bounds both converging to the MCN. The bounds arise as zeros of polynomials in sequences generated by recursion. This result is based on a constructive proof of convergence for Ince's continued fractions. An important role is also played by the fact that the continued fractions define meromorphic functions.

1 Introduction

Consider Mathieu's equation in the standard form

$$\frac{d^2y}{dt^2} + (a - 2q\cos 2t)\, y = 0 \,, \tag{1}$$

where a and q are real parameters and t is a real variable. The problem of studying vibrations of an elliptic membrane, which originally led Mathieu [6] to study this equation, requires finding periodic solutions of period 2π for it. Stating it better, for any given $q \in \mathbb{R}$ the problem is to show existence of values for a such that (1) has periodic solutions with period 2π. Applications which lead to the same problem include also other boundary value problems in regions with elliptic symmetry [7]

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and linear stability regions of upside-down pendula with periodic vertical driving [1].

By using a method based on continued fractions, Ince [5] showed that for any $q \neq 0$ there exist infinite sets $A(q) = \{a_0(q), a_1(q), a_2(q), \ldots\}$, $a_i(q) < a_{i+1}(q)$, $i = 0, 1, 2, \ldots$, and $B(q) = \{b_1(q), b_2(q), \ldots\}$, $b_i(q) < b_{i+1}(q)$, $i = 1, 2, \ldots$ such that for all $a \in A(q)$ Mathieu's equation has an even periodic solution with period 2π and for all $a \in B(q)$ it has an odd periodic solution with period 2π. Furthermore, for the given value of q these are the only values for a such that (1) has a solution of period 2π. The elements of $A(q) \cup B(q)$ are known as Mathieu characteristic numbers of integer orders (MCNs).

Nowadays, long after the appearance of classical books on the subject, such as [7] and [3], the computation of periodic solutions of period 2π to Mathieu's equation still deserves attention. A recent review [2] claims that the major difficulty is the computation of MCNs. Most available methods for that require an initial estimate of the MCN. Methods for obtaining such estimates are hardly discussed. One exception is [9], which discusses the problem in the context of eigenvalues for infinite matrices and uses the bisection method to find upper and lower bounds for MCNs.

Our purpose in this paper is to provide sequences of upper and lower bounds to any MCN of integer order, for any $q \in \mathbb{R}$, without recourse to any initial estimates. Furthermore, each sequence converges to the desired MCN. Elements of these sequences may thus provide rigorous estimates for MCNs to be used in conjunction with currently employed methods.

We shall work in the framework of Ince's continued fractions [5] and bounds will arise as roots of polynomial equations. To achieve our goal, we shall constructively prove convergence of Ince's continued fractions and meromorphism of the limiting function. The convergence issue is only superficially treated in [7] and [3], as we shall explain in section 2. Both books also claim that Ince's continued fractions converge but do not specify the values for parameter a in (1) in which this happens. We shall prove the continued fractions converge at all $a \in \mathbb{C}$ except for a countable infinite set of real values. We shall also see that some of these values are themselves MCNs.

MCNs will appear as solutions to transcendent equations (13) and (14) below, in which continued fractions appear at the right-hand side. Meromorphism of the function defined by such continued fractions will be important in proving that those equations do have solutions. It will also be used in proving that the sequence of lower bounds to MCNs converge to the MCNs themselves and not to lower bounds. As far as we know, such meromorphism result is also new.

We now give some notations and conventions necessary for stating our main result.

Without loss of generality we may consider $q > 0$ for the rest of the paper. In fact, for $n = 0, 1, 2, \ldots$ it is easy to see that $a_{2n}(q) = a_{2n}(-q)$, $b_{2n}(q) = b_{2n}(-q)$ and

$a_{2n+1}(q) = b_{2n+1}(-q)$.

Define then

$$\alpha_n^{(e)}(a) = \frac{(-1)^{n+1}}{q}\,(a - 4n^2) \tag{2}$$

and

$$\alpha_n^{(o)}(a) = \frac{(-1)^{n+1}}{q}\,(a - (2n+1)^2)\ , \tag{3}$$

where labels e and o stand respectively for even and odd as will be seen later. With p meaning either e or o, we recursively construct polynomials $R_n^{(p)}(a)$ and $S_n^{(p)}(a)$ respectively of degrees $n-1$ and n by

$$\begin{aligned} R_0^{(p)}(a) &= 0 \\ R_1^{(p)}(a) &= 1 \\ R_{n+1}^{(p)}(a) &= \alpha_{n+1}^{(p)}(a)R_n^{(p)}(a) + R_{n-1}^{(p)}(a)\ , n = 1, 2, 3, \dots \end{aligned} \tag{4}$$

and

$$\begin{aligned} S_0^{(p)}(a) &= 1 \\ S_1^{(p)}(a) &= \alpha_1^{(p)}(a) \\ S_{n+1}^{(p)}(a) &= \alpha_{n+1}^{(p)}(a)S_n^{(p)}(a) + S_{n-1}^{(p)}(a)\ , n = 1, 2, 3, \dots\ . \end{aligned} \tag{5}$$

Define also polynomials

$$T_n(a) = a\,S_n^{(e)}(a) - 2q\,R_n^{(e)}(a)\ , \tag{6}$$

$$U_n(a) = (a - q - 1)\,S_n^{(o)}(a) - q\,R_n^{(o)}(a)\ , \tag{7}$$

and

$$V_n(a) = (a + q - 1)\,S_n^{(o)}(a) - qR_n^{(o)}(a)\ . \tag{8}$$

Finally, let

$$X_n^-(a) = X_n(a) - (-1)^n X_{n-1}(a)\ , \tag{9}$$

where the letter X may stand for $S^{(e)}$, T, U or V.

We shall prove in section 3 that for any n all of the zeros of all the above defined polynomials, *i.e.* $R_n^{(e,o)}$, $S_n^{(e,o)}$, T_n, U_n, V_n, $S_n^{(e)-}$, T_n^-, U_n^- and V_n^-, are real-valued and have multiplicity 1. For each n, the (real) zeros of each of the above polynomials will be labeled as $x_{n,j}$, where x stands for the lower case letter naming the polynomial X, n is its index and j numbers the zeros in increasing order. For example, the zeros of $S_4^{(e)-}$ will be labeled $s_{4,1}^{(e)-} < s_{4,2}^{(e)-} < s_{4,3}^{(e)-} < s_{4,4}^{(e)-}$.

We can now state our main result:

Theorem 1.1 *Let $q > 0$ be fixed, $a_k(q)$ and $b_k(q)$ denote the MCNs as usual, and polynomials $R_n^{(e,o)}, S_n^{(e,o)}, T_n, U_n, V_n, S_n^{(e)-}, T_n^-, U_n^-, V_n^-$ be defined by (4)-(9).*

(i) *For each fixed $i \in \mathbb{N}$, sequences $(s^{(e)}_{n,i})$, $(t_{n,i})$, $(u_{n,i})$ and $(v_{n,i})$ are decreasing with*

$$\lim_{n\to\infty} s^{(e)}_{n,i} = b_{2i}(q) ,$$

$$\lim_{n\to\infty} t_{n,i} = a_{2(i-1)}(q) ,$$

$$\lim_{n\to\infty} u_{n,i} = a_{2i-1}(q)$$

and

$$\lim_{n\to\infty} v_{n,i} = b_{2i-1}(q) .$$

In particular, each element in the sequences is an upper bound to the corresponding MCN.

(ii) *Let X stand for any of the polynomial sequences mentioned in part (i) and $N_i(q)$ be defined either as the smallest $n \in \mathbb{N}$ such that*

$$n > \frac{1}{2}\sqrt{2q + x_{n,i}} ,$$

or as $N_i(q) = 1$ in the case $x_{n,i} < -2q$ for all values of n such that $x_{n,i}$ exists. Then sequences $(x^-_{n,i})$ are increasing for $n > N_i(q)$ and converge to the same MCNs as the $(x_{n,i})$. In particular, each $x^-_{n,i}$ with $n > N_i(q)$ is a lower bound to the corresponding MCN.

The plan for the rest of this paper is as follows. In the next section, we briefly derive the equations involving continued fractions to be satisfied by a and q in order that (1) has solutions with period 2π and proceed to some initial results concerning convergence of these continued fractions. In section 3 we introduce the concept of sequences of polynomials with interlaced zeros (SPIZ). It will follow that the sequences $R^{(e,o)}_n$, $S^{(e,o)}_n$, T_n, U_n, V_n are all SPIZs. This will unify the question of existence of real zeros for all kinds of polynomials studied, as well as prove the convergence of the sequences of their zeros appearing in Theorem 1.1. In section 4 we use tools from Complex Analysis to finish the proof of the continued fractions' convergence and also prove that limit functions are meromorphic. As a consequence, it will follow that the limits of the sequences in Theorem 1.1 are indeed the MCNs.

2 The continued fractions of Ince

Let us search for solutions of period 2π for (1) as Fourier series

$$y(t) = \sum_{n=-\infty}^{\infty} A_n e^{int} . \tag{10}$$

Differentiating the series and substituting in (1), we find that the Fourier coefficients of the solutions must satisfy the recurrence relation

$$(a-n^2)A_n \;-\; q(A_{n+2}+A_{n-2}) \;=\; 0\;,$$

for which we have 2 classes of solutions: one in which

$$A_{\pm 1} \;=\; A_{\pm 3} \;=\; A_{\pm 5} \;=\; \ldots \;=\; 0$$

and the other such that

$$A_0 \;=\; A_{\pm 2} \;=\; A_{\pm 4} \;=\; \ldots \;=\; 0\;.$$

The solutions in the first class will be termed of *even order*, because they involve only Fourier coefficients of even order. Accordingly, solutions in the second class will be termed of *odd order*.

Defining

$$v_k \;=\; \frac{A_{k+2}}{A_k}\;,$$

the recurrence relation above may be rewritten either as

$$v_{n-2} \;=\; \frac{1}{\frac{a-n^2}{q} - v_n} \tag{11}$$

or

$$v_n \;=\; \frac{a-n^2}{q} - \frac{1}{v_{n-2}}\;. \tag{12}$$

By using (11) repeatedly with even values of n, beginning with $n=2$ we obtain

$$\begin{aligned} v_0 \;&=\; \frac{1}{\frac{a-4}{q} - v_2} \;=\; \frac{1}{\frac{a-4}{q} + \frac{1}{-\frac{a-16}{q}+v_4}} \;=\; \ldots \;= \\ &=\; \frac{1}{\alpha_1^{(e)} + \frac{1}{\alpha_2^{(e)} + \frac{1}{\alpha_3^{(e)} + \ddots}}}\;, \end{aligned}$$

where $\alpha_n^{(e)}$ was defined in (2).

Similarly, iterating (12), beginning with $n=0$, we get

$$v_0 \;=\; \frac{a}{q} \;-\; \frac{1}{\alpha_1^{(e)} + \frac{1}{\alpha_2^{(e)} + \frac{1}{\alpha_3^{(e)} + \ddots}}}\;.$$

If we equate the last two expressions for v_0, we finally get

$$\frac{a}{2q} = \cfrac{1}{\alpha_1^{(e)} + \cfrac{1}{\alpha_2^{(e)} + \cfrac{1}{\alpha_3^{(e)} + \ddots}}} \; . \tag{13}$$

The elements in the continued fraction at the right-hand side of the last equation depend on a and q; provided its convergence, this equation should be interpreted as the relation between a and q to be satisfied so that Mathieu's equation has an even order solution with period 2π.

We will show later that the continued fraction in (13) converges for any complex a, except for a countable infinite set of real numbers $\{s_1^{(e)}, s_2^{(e)}, \ldots\}$. We will also show that for any fixed $q > 0$, (13) has an infinite number of solutions for a, which will be the MCNs denoted as $a_0(q), a_2(q), a_4(q), \ldots$. In fact, the solutions of Mathieu's equation corresponding to these values have the form

$$\sum_{n=0}^{\infty} c_n^{(1)} \cos 2nt \; .$$

This can be seen first of all because $A_{2n+1} = 0$ for all integer n. Then using (13) in conjunction with (11) and (12) we show inductively that $A_{2n} = A_{-2n}$, $n = 1, 2, \ldots$.

Before proceeding, we make some comments on the convergence of the right-hand side in (13) as quoted in the literature we were able to trace. Analogous results are quoted also regarding (14) below.

In [7], page 29, (13) is written in the equivalent form

$$a = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} \; ,$$

with $a_1 = -1/2q^2$ and $a_n = -1/(16n^2(n-1)^2)$ for $n \geq 2$ and $b_n = 1 - a/(4n^2)$. Its convergence is said to be consequence of the fact that the nth denominators b_n tend to 1, whereas the numerators a_n tend to 0. No proof of such a result is mentioned. Also, as we shall prove that the continued fraction in (13) diverges for a countable infinite set of real values for a, such a "theorem" is false.

The same equation is written in [3] at page 16 as

$$\frac{a}{2q} = \cfrac{-q}{4 - a - \cfrac{q^2}{16 - a - \ddots}} \; .$$

Convergence of the right-hand side follows, according to the book, from the fact that numerators and denominators in the continued fraction are polynomials in a such that the degrees of the numerators are less than twice the degrees of denominators. The proof of such a fact is said to be found in an old book by Perron [8] we were not able to find. No reference is made to the divergence of the continued fraction for some a's.

Furthermore, neither [7], nor [3] discuss existence and number of roots for equations such as (13).

Conditions for the existence of odd order solutions can be found by a procedure similar to the one leading to (13). Start taking $n = 3$ in (11). Using it repeatedly, we get

$$v_1 = \frac{1}{\alpha_1^{(o)} + \frac{1}{\alpha_2^{(o)} + \frac{1}{\alpha_3^{(o)} + \ddots}}} ,$$

where $\alpha_n^{(o)}$ was defined in (3). By iterating (12) starting at $n = 1$, we get

$$v_1 = \frac{a-1}{q} + \frac{1}{-\frac{a-1}{q} + \frac{1}{\alpha_1^{(o)} + \frac{1}{\alpha_2^{(o)} + \ddots}}} = \frac{a-1}{q} + \frac{1}{-\frac{a-1}{q} + v_1} ,$$

where the last equality results from the first expression for v_1. Solving the last equation for $\frac{a-1}{q}$ we finally get to

$$\frac{a-1}{q} \pm 1 = \frac{1}{\alpha_1^{(o)} + \frac{1}{\alpha_2^{(o)} + \frac{1}{\alpha_3^{(o)} + \ddots}}} . \tag{14}$$

This equation is analogous to (13). We will show that the continued fraction converges except for some special values of a and that for either sign in the left-hand side, it has a countable infinite set of solutions. The solutions for the minus sign will be denoted, as usual, $a_1(q), a_3(q), \ldots$ because it is easy to prove that $A_{2n-1} = A_{-(2n-1)}$, $n = 1, 2, \ldots$, thus the Fourier expansion of the corresponding periodic solution for Mathieu's equation will be in the form

$$\sum_{n=0}^{\infty} c_n^{(2)} \cos{(2n+1)t} .$$

Analogously, the solutions for (14) with the plus sign in the left-hand side will be denoted $b_1(q), b_3(q), \ldots$ because the Fourier series for the solution of Mathieu's equation will have the form

$$\sum_{n=0}^{\infty} c_n^{(3)} \sin{(2n+1)t} .$$

The reader familiar with Mathieu functions will notice the lack, up to now, of solutions with period 2π with Fourier series of the form

$$\sum_{n=1}^{\infty} c_n^{(4)} \sin 2nt \ ,$$

which occur when a equals the MCNs usually denoted as $b_2(q), b_4(q), \ldots$. The fact is that these solutions belong to the even order class, but they also have $A_0 = 0$, so that our reasoning, based on equating two expressions for $v_0 = A_2/A_0$ should clearly fail. But it would not be a big surprise if these solutions would occur at the values for a such that v_0 is not well-defined, *i.e.* at the values $\{s_1^{(e)}, s_2^{(e)}, \ldots\}$ for a such that the continued fraction in (13) diverges. We will show later that this is indeed the case.

Let us now start our study of the continued fractions appearing in (13) and (14) by some general results. First of all, the nth approximant for a continued fraction of the form

$$\frac{1}{\beta_1 \ + \ \frac{1}{\beta_2 + \frac{1}{\ddots}}} \ , \tag{15}$$

where $\beta_1, \beta_2, \ldots \in \mathbb{C}$, is defined as

$$f_n \ = \ \frac{1}{\beta_1 \ + \ \frac{1}{\beta_2 + \frac{1}{\ddots + \frac{1}{\beta_n}}}} \ .$$

We shall need also numbers P_n and Q_n inductively defined by

$$\begin{aligned} P_0 \ &= \ 0 \\ P_1 \ &= \ 1 \\ P_{n+1} \ &= \ \beta_{n+1} \, P_n \ + \ P_{n-1} \quad , n = 1, 2, \ldots \end{aligned} \tag{16}$$

and

$$\begin{aligned} Q_0 \ &= \ 1 \\ Q_1 \ &= \ \beta_1 \\ Q_{n+1} \ &= \ \beta_{n+1} \, Q_n \ + \ Q_{n-1} \quad , n = 1, 2, \ldots \ . \end{aligned} \tag{17}$$

Proposition 2.1 *Consider continued fractions of the form (15) and let*

$$F_n(w) \ = \ \frac{1}{\beta_1 \ + \ \frac{1}{\beta_2 + \frac{1}{\ddots + \frac{1}{\beta_n + w}}}} \ . \tag{18}$$

Then, for each $n = 1, 2, \ldots$ we have

(i)

$$F_n(w) \;=\; \frac{P_{n-1}w \;+\; P_n}{Q_{n-1}w \;+\; Q_n} \; .$$

(ii)

$$f_n \;=\; \frac{P_n}{Q_n} \; .$$

(iii)

$$P_{n+1}\,Q_n \;-\; P_n\,Q_{n+1} \;=\; (-1)^n \; .$$

(iv)

$$f_{n+1} - f_n \;=\; \frac{(-1)^n}{Q_n\,Q_{n+1}} \; .$$

Proof:

(i) By induction. A slick proof using the matrix representation of the group of Möbius transformations is given as Theorem 12.1a in volume II of [4].

(ii) Just put $w = 0$ in the expression for $F_n(w)$.

(iii) By induction.

(iv) Just use (ii) and (iii). ∎

A simple result we shall use quite often is the following

Lemma 2.2 *If for some $n_0 \in \mathbb{N}$ we have $|\beta_{n_0}| > 2$ and $|Q_{n_0-1}| \geq |Q_{n_0-2}|$, then $|Q_{n_0}| > |Q_{n_0-1}|$. Analogously for P replacing Q.*

Proof: By using the recurrence relation in (17), we have

$$\begin{aligned} |Q_{n_0}| \;&=\; |\beta_{n_0}\,Q_{n_0-1} \;+\; Q_{n_0-2}| \;\geq\; |\beta_{n_0}|\,|Q_{n_0-1}| \;-\; |Q_{n_0-2}| \\ &>\; |Q_{n_0-1}| \; . \end{aligned}$$ ∎

A first result concerning convergence of the continued fractions is the following

Proposition 2.3 *Suppose there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $|\beta_n| > 2 + \delta \;\forall n > n_0$ and $|Q_{n_0}| > |Q_{n_0-1}|$. Then, continued fraction (15) converges.*

Proof: Mimicking the proof of Lemma 2.2, it can be seen that for $n > n_0$ one has $|Q_n| > (1+\delta)|Q_{n-1}|$. Thus, if $n \geq n_0$ and $k \in \mathbb{N}$,

$$|Q_{n+k}| \ > \ (1+\delta)^k \, |Q_n| \ .$$

By using (ii) and (iv) in Proposition 2.1 it follows that, for $n \geq n_0$,

$$\begin{aligned} |f_{n+k} \ - \ f_n| \ &= \ |\frac{P_{n+k}}{Q_{n+k}} \ - \ \frac{P_n}{Q_n}| \ \leq \ \sum_{l=0}^{k-1} |\frac{P_{n+l+1}}{Q_{n+l+1}} \ - \ \frac{P_{n+l}}{Q_{n+l}}| \ = \ \sum_{l=0}^{k-1} \frac{1}{|Q_{n+l+1}| \, |Q_{n+l}|} \\ &< \ \frac{1}{(1+\delta) \, |Q_n|^2} \left(1 \ + \ \frac{1}{(1+\delta)^2} \ + \ \frac{1}{(1+\delta)^4} \ + \ \ldots \right) \ . \end{aligned}$$

As the geometric series above converges and $|Q_n|$ may be taken arbitrarily large if n is taken sufficiently larger than n_0, then (f_n) is a Cauchy sequence of complex numbers. ∎

3 Sequences of Polynomials with Interlaced Zeros

We may now start relating the sequences of polynomials referred to in Theorem 1.1 with continued fractions of the form (15). The $R_n^{(e,o)}$ and $S_n^{(e,o)}$ defined in (4) and (5) are such that the nth approximant to the continued fraction in (13) is $\frac{R_n^{(e)}}{S_n^{(e)}}$ and and the nth approximant to the continued fraction appearing in (14) is $\frac{R_n^{(o)}}{S_n^{(o)}}$.

Also, polynomial T_n is such that its zeros are the roots of the equation obtained when one substitutes the continued fraction appearing at the right-hand side of (13) by its nth approximant. Similarly, U_n and V_n are such that their zeros are the roots of the equation obtained approximating the continued fraction in (14).

We shall see that all these sequences of polynomials belong to an interesting class with remarkable properties. These properties are the unifying feature of all statements in Theorem 1.1.

Before giving the definition, let us state some terminology. Let $\Pi_1, \Pi_2, \ldots$ be a sequence of polynomials with real-valued coefficients. In case they exist, we shall denote the real zeros of Π_n as $r_{n,1} \ \leq r_{n,2} \ \leq \ \ldots \ \leq \ r_{n,k}$, where k is the number of real roots for Π_n. For notational simplicity, it will be useful to define $r_{n,0} = -\infty$ and $r_{n,k+1} = +\infty$.

Definition 3.1 *We shall say $\Pi_1, \Pi_2, \ldots$ is a sequence of polynomials with interlaced zeros (SPIZ for short) if*

(i) Π_1 is not the null polynomial, has degree $d \geq 0$ and all its zeros are real-valued with multiplicity 1;

(ii) Π_2 *has degree* $d+1$*, all its zeros are real-valued with multiplicity 1 and each zero of* Π_2 *is located between two consecutive zeros of* Π_1*, i.e.*

$$r_{1,i-1} \;<\; r_{2,i} \;<\; r_{1,i} \;\; i = 1, 2, \ldots, d+1 \; .$$

In case Π_1 *has degree* $d = 0$*, this item applies automatically.*

(iii) There exist polynomials β_n *of degree 1 such that*

$$\Pi_{n+1} \;=\; \beta_{n+1}\, \Pi_n \;+\; \Pi_{n-1} \; , \;\;\; n = 2, 3, \ldots \; ,$$

(iv) $\Pi_n(+\infty)$ *and* $\Pi_{n+2}(+\infty)$ *have opposite signs for all* $n \in \mathbb{N}$*.*

The name given is justified by the following

Theorem 3.2 *If* $\Pi_1, \Pi_2, \ldots$ *is a SPIZ, then all zeros of each* Π_n *are real-valued with multiplicity 1. Furthermore, for every* $n \geq 2$ *the zeros of* Π_n *and* Π_{n-1} *are interlaced, i.e.*

$$r_{n-1,i-1} \;<\; r_{n,i} \;<\; r_{n-1,i} \;\; i = 1, 2, \ldots, d+n-1 \; .$$

Proof: By (iii) and (iv) in Definition 3.1, β_n and β_{n+1} must have opposite signs at $+\infty$. Combining the two possibilities for the sign of β_1 at $+\infty$ with the possible signs at $+\infty$ of Π_1 and Π_2, one can see that there exist 4 different sign attributions for the β_n and Π_n compatible with the definition of SPIZ. Let us take one of them for definiteness, the proof being analogous for the other ones. Consider then

$$\beta_n(+\infty) \;=\; \begin{cases} +\infty, & n \text{ is odd} \\ -\infty, & n \text{ is even} \end{cases} \tag{19}$$

and

$$\Pi_n(+\infty) \;=\; \begin{cases} +\infty, & n = 0 \bmod 4 \;\text{ or }\; n = 1 \bmod 4 \\ -\infty, & n = 2 \bmod 4 \;\text{ or }\; n = 3 \bmod 4 \end{cases} \tag{20}$$

The proof proceeds by induction. The thesis is certainly true in cases $n = 1$ and $n = 2$ by (i) and (ii) in Definition 3.1. Suppose now it holds for $n = 1, 2, \ldots, p$.

If d is the degree of Π_1, then Π_p has degree $d+p-1$ and the same number of real and distinct zeros. We evaluate Π_{p+1} at the largest zero of Π_p:

$$\Pi_{p+1}(r_{p,d+p-1}) \;=\; \beta_{p+1}(r_{p,d+p-1})\, \Pi_p(r_{p,d+p-1}) \;+\; \Pi_{p-1}(r_{p,d+p-1}) \;=\; \Pi_{p-1}(r_{p,d+p-1}) \; .$$

By the induction hypothesis, $r_{p,d+p-1}$ is larger than the largest zero of Π_{p-1}, and so $\Pi_{p+1}(r_{p,d+p-1})$ has the same sign as $\Pi_{p-1}(+\infty)$. But, by (iv) in Definition 3.1, $\Pi_{p-1}(+\infty)$ and $\Pi_{p+1}(+\infty)$ have opposite signs. This argument shows that Π_{p+1} has different signs at $+\infty$ and $r_{p,d+p-1}$. By the Intermediate Value Theorem (IVT), it must have at least one zero larger than $r_{p,d+p-1}$.

We proceed by evaluating Π_{p+1} at each zero of Π_p. By the recursion relation in (ii) of Definition 3.1,

$$\Pi_{p+1}(r_{p,i}) \;=\; \Pi_{p-1}(r_{p,i}) \;.$$

By the induction hypothesis, $r_{p,i} \in (r_{p-1,i-1}, r_{p-1,i})$ and also, as the zeros of Π_{p-1} have multiplicity 1, it changes sign as it passes through each of them. This proves that the signs of $\Pi_{p+1}(r_{p,i})$ and $\Pi_{p+1}(r_{p,i+1})$ are opposite. By the IVT, there must exist at least one zero of Π_{p+1} between each pair of consecutive zeros of Π_p. By the same reasoning with the IVT, Π_{p+1} must have at least 1 zero smaller than the smallest zero of Π_p. Up to now we have counted at least $d+p$ zeros for Π_{p+1}, which has degree $d+p$. Then, at every place we stated that Π_{p+1} should have at least one zero, it must have exactly one zero. This ends the proof. ∎

Theorem 3.2 above is a particular case of and can be used to prove a result in the Sturm theory for difference equations, see propositions 2.1 and 2.2 in [10]. According to the author of that paper, although Sturm theory for differential equations is quite standard, its discrete version is not so. This is why we decided to keep the proof here.

The preceding result was concerned with existence of the zeros appearing in Theorem 1.1 as upper bounds to MCNs. The existence of zeros related to the lower bounds in the same theorem will arise due to the next result, when applied to the polynomials defined by (9):

Theorem 3.3 *Let $\Pi_1, \Pi_2, \ldots$ be a SPIZ and define $\Theta_n = \Pi_n - \Pi_{n-1}$ and $\Psi_n = \Pi_n + \Pi_{n-1}$. Then every zero of each Θ_n or Ψ_n is real-valued with multiplicity 1.*

Furthermore:

(i) If $\Pi_n(+\infty)$ and $\Pi_{n-1}(+\infty)$ have opposite signs, then the zeros $\rho_{n,1} < \rho_{n,2} < \ldots < \rho_{n,d+n-1}$ of Θ_n are such that

$$r_{n-1,i-1} \;<\; \rho_{n,i} \;<\; r_{n,i}$$

and the zeros $\rho'_{n,1} < \rho'_{n,2} < \ldots < \rho'_{n,d+n-1}$ of Ψ_n are such that

$$r_{n,i} \;<\; \rho'_{n,i} \;<\; r_{n-1,i} \;.$$

(ii) If $\Pi_n(+\infty)$ and $\Pi_{n-1}(+\infty)$ have the same sign, then positions of the zeros of Θ_n and Ψ_n are interchanged with respect to (i), i.e.

$$r_{n-1,i-1} \;<\; \rho'_{n,i} \;<\; r_{n,i}$$

and

$$r_{n,i} \;<\; \rho_{n,i} \;<\; r_{n-1,i} \;.$$

Proof: The proof is similar to that of Theorem 3.2, evaluating $\Pi_n \pm \Pi_{n-1}$ at $\pm\infty$ and at the zeros of Π_n and Π_{n-1} and applying the IVT.■

For easiness of reference to the zeros of polynomials Θ_n and Ψ_n, let us make one more notational definition:

Definition 3.4 *Let $\rho_{n,i}$ and $\rho'_{n,i}$ be defined as in Theorem 3.3. Then*

(i)

$$\rho^-_{n,i} = \min\{\rho_{n,i}, \rho'_{n,i}\}$$

(ii)

$$\rho^+_{n,i} = \max\{\rho_{n,i}, \rho'_{n,i}\}$$

With this definition, it follows that for all $n \in \mathbb{N}$ and $i = 1, 2, \ldots, d + n - 1$,

$$\rho^-_{n,i} \in (r_{n-1,i-1}\,, r_{n,i}) \quad \text{and} \quad \rho^+_{n,i} \in (r_{n,i}\,, r_{n-1,i})\ .$$

The next result, which proof is a simple verification, states that all polynomials appearing in Theorem 1.1 are either members of some SPIZ, or of the kind appearing in Theorem 3.3:

Proposition 3.5 *The following are all SPIZ:*

(i) $R_1^{(e,o)}, R_2^{(e,o)}, \ldots$

(ii) $S_0^{(e,o)}, S_1^{(e,o)}, S_2^{(e,o)} \ldots$

(iii) $T_0, T_1, T_2 \ldots$

(iv) $U_0, U_1, U_2 \ldots$

(v) $V_0, V_1, V_2, \ldots$

Notice that Theorem 3.2 and Proposition 3.5 prove that the sequences appearing in part (i) of Theorem 1.1 are decreasing. The next result proves that they are convergent by exhibiting to each of them the lower bounds appearing in part (ii) of the same theorem.

Before stating the theorem, we shall impose one more constraint on the kind of SPIZ appearing in what follows. Notice that all SPIZ in the statement of Proposition 3.5 either have $\beta_n = \alpha_n^{(e)}$ or $\beta_n = \alpha_n^{(o)}$. In both cases the β_n satisfy the following

Definition 3.6 *Writing the β_n appearing in Definition 3.1 as $\beta_n(a) = \lambda_n(a - B_n)$, we shall say a SPIZ is admissible if there exist $\lambda > 0$ and $N_0 \in \mathbb{N}$ such that the slopes λ_n satisfy $|\lambda_n| \geq \lambda$ for all n and the zeros B_n are such that $B_{n+1} - B_n > \frac{2}{\lambda}$ if $n \geq N_0$.*

It is an easy matter to see that if the β_n satisfy the above admissibility condition, then for any fixed $x \in \mathbb{R}$, there exists $n_1(x)$ such that

$$|\beta_n(a)| > 2 \ \ \forall a < x, \ \ \forall n \geq n_1(x) \ . \tag{21}$$

In the proof of the next theorem we will need to make sure that for each fixed i, $|\beta_n(a)| > 2 \ \ \forall a < r_{n,i}$, where $r_{n,i}$ is the ith root of Π_n. Of course, as $(r_{n,i})_n$ is decreasing, this condition will hold true for large enough n. In cases $\beta_n = \alpha_n^{(e)}$ or $\beta_n = \alpha_n^{(o)}$, it can be seen that it holds for $n \geq N_i(q)$, where $N_i(q)$ is defined at part (ii) of Theorem 1.1.

Theorem 3.7 *Let $\Pi_1, \Pi_2, \ldots$ be any admissible SPIZ and let $r_{n,i}$ be the zeros of its elements denoted as in Definition 3.1. Given any $i \in \mathbb{N}$, let n_i be such that for $n \geq n_i$ one has $|\beta_n(a)| > 2 \ \ \forall a < r_{n,i}$. Then for $n \geq n_i$ one has*

$$r_{n+1,i} \in [\rho^-_{n,i}, r_{n,i})$$

and

$$\rho^-_{n+1,i} > \rho^-_{n,i} \ .$$

As a consequence, for each i, the sequences $(r_{n,i})_{n=1}^{\infty}$ converge and the $\rho^-_{n,i}$ with $n \geq n_i$ are lower bounds to $\lim_{n\to\infty} r_{n,i}$.

Proof: Suppose that β_n and Π_n have the same signs as in (19) and (20) and n is odd. That implies $\beta_{n+1}(a) > 2$ if $n \geq n_i$ and $a < r_{n,i}$ and $\Pi_n(+\infty)$ and $\Pi_{n-1}(+\infty)$ have the same sign. In this case, $\rho^-_{n,i}$ is a zero of $\Pi_n + \Pi_{n-1}$ and $\rho^+_{n,i-1}$ is a zero of $\Pi_n - \Pi_{n-1}$. If n is even, or if β_n and Π_n have other possible signs, the proof is similar.

As $r_{n,i-1} < \rho^+_{n,i-1} < r_{n-1,i-1} < \rho^-_{n,i} < r_{n,i}$, we may partition $(r_{n,i-1}, r_{n,i})$ in 3 intervals: $I_1 = (r_{n,i-1}, \rho^+_{n,i-1})$, $I_2 = [\rho^+_{n,i-1}, \rho^-_{n,i})$ and $I_3 = [\rho^-_{n,i}, r_{n,i})$. To prove the first assertion of the theorem, we shall show that $r_{n+1,i}$ is neither in I_1, nor in I_2. By Theorem 3.2 it must lie in I_3.

First of all, $r_{n+1,i}$ is not in I_1 because, as $\Pi_n = \Pi_{n-1}$ at $\rho^+_{n,i-1}$ and neither Π_n, nor Π_{n-1} has any zero in I_1, then they must have the same sign in I_1. But, as $\beta_{n+1} > 0$, property (iii) in the definition of SPIZ implies Π_{n+1} will have the same sign as Π_n and Π_{n-1} in I_1, hence no zero there.

Neither is $r_{n+1,i}$ in I_2. In fact, the only roots of $|\Pi_n| = |\Pi_{n-1}|$ in $(r_{n,i-1}, r_{n,i})$ are $\rho^-_{n,i}$ and $\rho^+_{n,i-1}$ and, as Π_n vanishes at the end points of that interval, one must have $|\Pi_n| \geq |\Pi_{n-1}|$ in I_2. As $|\beta_{n+1}| > 2$ in I_2, then, using Lemma 2.2 in I_2 we have

$$|\Pi_{n+1}| \ > \ |\Pi_n| \ > \ 0 \ ,$$

the last inequality holding because Π_n has no zero in I_2. That leads us to conclude that Π_{n+1} has no zero in I_2, either.

To show that $\rho^-_{n+1,i} > \rho^-_{n,i}$, notice that $|\Pi_n| = |\Pi_{n-1}|$ at $\rho^-_{n,i}$. Together with $|\beta_{n+1}(\rho^-_{n,i})| > 2$, Lemma 2.2 implies $|\Pi_{n+1}(\rho^-_{n,i})| > |\Pi_n(\rho^-_{n,i})|$. As $|\Pi_{n+1}| < |\Pi_n|$ in $(\rho^-_{n+1,i}, \rho^+_{n+1,i})$, then $\rho^-_{n,i}$ cannot be in that interval. But $r_{n+1,i}$ does lie in that interval and is larger than $\rho^-_{n,i}$ by the first assertion in this theorem. Then, the second one follows. ∎

To resume this section, we prove now a simple result, which will be very useful in the next section:

Lemma 3.8 *Let $\Pi_1, \Pi_2, \ldots$ be any admissible SPIZ, $r_i = \lim_{n\to\infty} r_{n,i}$ and $l_i = \lim_{n\to\infty} \rho^-_{n,i}$. Then:*

(i) For any fixed $z \in \mathbb{C}$, we either have $|\Pi_n(z)| \overset{n\to\infty}{\to} \infty$, or $|\Pi_n(z)| \overset{n\to\infty}{\to} 0$.

(ii) $\lim_{n\to\infty} \Pi_n(a) = 0$ for any $a \in [l_i, r_i]$, $i = 1, 2, \ldots$.

Proof: If $(|\Pi_n(z)|)_{n>N}$ is a decreasing sequence for some N, then there must exist $L = \lim_{n\to\infty} |\Pi_n(z)|$. But if there is no N with such property, then, by property (21) and by the arguments in the proof of Proposition 2.3, we must have $|\Pi_n(z)| \overset{n\to\infty}{\to} \infty$.

Suppose then, $|\Pi_n(z)| \overset{n\to\infty}{\to} L$. As, by Definition 3.6, for any fixed z, $|\beta_n(z)| \overset{n\to\infty}{\to} \infty$, then taking the limit as $n \to \infty$ in (iii) of Definition 3.1 we get $L = 0$.

To prove the second part, let n_i be as in Theorem 3.7. By that theorem,

$$\bigcap_{n\geq n_i} [\rho^-_{n,i}, r_{n,i}] = [l_i, r_i] .$$

Because $|\Pi_n(a)| \leq |\Pi_{n-1}(a)|$ if $a \in [\rho^-_{n,i}, r_{n,i}]$ and, for $n \geq n_i$, $[\rho^-_{n,i}, r_{n,i}] \supset [l_i, r_i]$, then $|\Pi_n(a)| \leq |\Pi_{n-1}(a)|$ for all $a \in [l_i, r_i]$ if $n \geq n_i$. By the first part in this lemma, this implies $\Pi_n(a) \overset{n\to\infty}{\to} 0$ for $a \in [l_i, r_i]$. ∎

4 Convergence of the continued fractions and holomorphism

Let us summarize what we have succeeded to prove up to now in Theorem 1.1.

First of all, the theorem refers in parts (i) and (ii) to real zeros of some polynomials. We have already proved that all zeros of these polynomials are indeed real-valued. It also states that certain sequences of zeros are decreasing and convergent. That was also proved. According to the theorem, some other sequences are increasing and converge to the same limit as the corresponding decreasing sequences. We have shown that those sequences are indeed increasing and convergent, but their limits may be in principle smaller than the limits of the corresponding decreasing sequences. In this section, we shall prove that this is not the case.

Most important of all, the polynomial sequences in Theorem 1.1 appear when we substitute continued fractions in (13) and (14) by their approximants. We have not yet proved that those continued fractions converge. And if they do, it might happen that limits of solutions to the equations obtained by replacing in (13) and (14) the continued fractions by their approximants are not solutions of (13) and (14) themselves. We shall also prove that the continued fractions converge at all complex values for a with the exception of the set of limits of the zeros of the denominators of the approximants. And it will also result, by showing that the continued fractions in (13) and (14) define meromorphic functions, that the limits of the zeros of the polynomials in Theorem 1.1 are indeed solutions to those equations.

We begin with a simple result extending to the complex domain something already well-known for real numbers:

Lemma 4.1 *Let $\Pi_1, \Pi_2, \ldots$ be any admissible SPIZ. Then, for all $z \in \mathbb{C}$ with $\operatorname{Re} z < \rho^-_{n,1}$, we have*

$$|\Pi_n(z)| > |\Pi_{n-1}(z)| .$$

Proof: Writing Π_k in factored form, we have

$$\left|\frac{\Pi_n(z)}{\Pi_{n-1}(z)}\right| = |\lambda_n| \left(\prod_{i=1}^{d_n-1} \left|\frac{z - r_{n,i}}{z - r_{n-1,i}}\right|\right) |z - r_{n,d_n}| ,$$

where d_n is the degree of Π_n and λ_n is the slope of β_n . Let $z = a + i\,b$ with $a < \rho^-_{n,1} < r_{n,1}$. As $r_{n,i} < r_{n-1,i}$ and $\operatorname{Re} z < \rho^-_{n,1} \le r_{n,i}$, it is straightforward to verify that for all i, $|\frac{z-r_{n,i}}{z-r_{n-1,i}}|$ is an increasing function of $|b|$. Also $|z - r_{n,d_n}|$ is increasing in $|b|$. Thus $|\frac{\Pi_n(z)}{\Pi_{n-1}(z)}| \ge |\frac{\Pi_n(a)}{\Pi_{n-1}(a)}|$.

Because the zeros $r_{n,i}$ of Π_n are located at intervals $(\rho^-_{n,i}, \rho^+_{n,i})$, at which endpoints we have $|\Pi_n| = |\Pi_{n-1}|$, then for $a \in \mathbb{R}$ we have $|\Pi_n(a)| > |\Pi_{n-1}(a)|$, unless $a \in [\rho^-_{n,i}, \rho^+_{n,i}]$ for some i. Condition $a < \rho^-_{n,1}$ implies then $|\frac{\Pi_n(a)}{\Pi_{n-1}(a)}| > 1$, proving the lemma. ∎

An immediate consequence of using this lemma together with Proposition 2.3 is the convergence of continued fractions of the form

$$\cfrac{1}{\beta_1(z) + \cfrac{1}{\beta_2(z) + \cfrac{1}{\beta_3(z) + \ddots}}} ,$$

if $\operatorname{Re} z$ is small enough and $\beta_n(z)$ satisfies the admissibility condition in Definition 3.6.

In fact, there exists x_1 such that $|\beta_n(a)| > 2$ for all $a < x_1$ with $n \ge 2$. By taking $x = \min\{x_1, \rho^-_{1,1}\}$, we have for all $z \in \mathbb{C}$ with $\operatorname{Re} z < x$ both conditions of Proposition 2.3 holding with $n_0 = 1$.

This convergence result may be further generalized on using the same idea with some more labor, working not with the whole continued fractions, but rather with their "remainders".

Define the remainder Ω_m as the continued fraction

$$\Omega_m(z) \;=\; \cfrac{1}{\beta_{m+1}(z) + \cfrac{1}{\beta_{m+2}(z) + \cfrac{1}{\beta_{m+3}(z) + \ddots}}} \tag{22}$$

and the truncated remainder $\Omega_{m,k}$ at order k as the kth approximant to Ω_m, *i.e.*,

$$\Omega_{m,k}(z) \;=\; \cfrac{1}{\beta_{m+1}(z) \;+\; \cfrac{1}{\beta_{m+2}(z) + \cfrac{1}{\ddots + \cfrac{1}{\beta_{m+k}(z)}}}} \;. \tag{23}$$

By applying Proposition 2.1 to Ω_m, we have

$$\Omega_{m,k}(z) \;=\; \frac{P_k^{(m)}(z)}{Q_k^{(m)}(z)} \;,$$

where

$$\begin{aligned} Q_0^{(m)} &= 1 \\ Q_1^{(m)} &= \gamma_1^{(m)} \\ Q_{n+1}^{(m)} &= \gamma_{n+1}^{(m)}\, Q_n^{(m)} \;+\; Q_{n-1}^{(m)}, \quad n = 1, 2, \ldots \;, \end{aligned} \tag{24}$$

$P_k^{(m)}$ is defined by an analogous recurrence relation and

$$\gamma_n^{(m)} \;=\; \beta_{m+n} \;.$$

Observe that for all m, $P_1^{(m)}, P_2^{(m)}, \ldots$ and $Q_0^{(m)}, Q_1^{(m)}, \ldots$ are admissible SPIZ.

We may apply Theorems 3.3 and 3.7 to the sequence $Q_0^{(m)}, Q_1^{(m)}, \ldots$. In this context, define $r_{k,i}^{(m)}$ as the ith zero of $Q_k^{(m)}$ and $\rho_{k,i}^{(m)-}$ as the smallest between the ith zeros of $Q_k^{(m)} \pm Q_{k-1}^{(m)}$.

We may prove now a very important intermediate result:

Proposition 4.2 *Let $a \in \mathbb{R}$ be given and choose $m_0 \in \mathbb{N}$ with $B_{m_0+1} > a + \frac{1}{\lambda}$ and $m_0 \geq N_0$, where N_0 is defined at Definition 3.6. Then, for all $m \geq m_0$, $\Omega_m(z)$ is holomorphic in $C_a = \{z \in \mathbb{C}\,; \mathrm{Re}\, z < a\}$.*

Proof: Let $m \geq m_0$; as $r_{1,1}^{(m)} = B_{m+1}$, then condition $B_{k+1} - B_k > 2/\lambda$ in Definition 3.6 implies $|\gamma_n^{(m)}(x)| > 2$ for all $x \leq r_{1,1}^{(m)}$ if $n \geq 1$. By Theorem 3.7, sequence $(\rho_{n,1}^{(m)-})_{n\geq 1}$ is increasing.

As $\rho_{1,1}^{(m)-} = B_{m+1} - \frac{1}{|\lambda_{m+1}|}$, then $B_{m_0+1} > a + \frac{1}{\lambda}$ implies $\rho_{1,1}^{(m)-} > a$ for all $m \geq m_0$, implying that all zeros of the $Q_n^{(m)}$ are larger than a. So for any $m \geq m_0$ and $n \in \mathbb{N}$ we have $\rho_{n,1}^{(m)-} > a$. It is also true that $|\gamma_n^{(m)}(z)| > 2$ for all $n \geq 2$, $z \in C_a$.

From the conclusions in the previous paragraph, we can now show that Ω_m converges if $z \in C_a$ and $m \geq m_0$. In fact, $|\gamma_n^{(m)}(z)| > 2$ for all $n \geq 2$, $z \in C_a$ and $\rho_{1,1}^{(m)-} > a$. By Lemma 4.1 we can then show validity of the hypothesis of Proposition 2.3, thus concluding convergence of the continued fraction Ω_m if $z \in C_a$.

By the second assertion in Theorem 3.7, the $\rho_{n,1}^{(m)-}$ form an increasing sequence beginning from $n = 1$. This shows that there are no zeros of the $Q_k^{(m)}$ in C_a and thus the approximants $\frac{P_k^{(m)}}{Q_k^{(m)}}$ are holomorphic in C_a. In order to show that Ω_m is holomorphic, it suffices to show that the convergence of the approximants to Ω_m is uniform in compact subsets of C_a.

Let then $\Lambda \subset C_a$ be compact and $\delta = |\gamma_2^{(m)}(a)| - 2 > 0$. We shall proceed as in the proof of Proposition 2.3, now seeking uniform bounds. If $M_k(\Lambda) = \min_{z\in\Lambda} |Q_k^{(m)}(z)|$, then, as $M_{k+1}(\Lambda) = |Q_{k+1}^{(m)}(z_0)|$ for some $z_0 \in \Lambda$ and $|\gamma_{k+1}^{(m)}(z_0)| > 2 + \delta$, it follows that

$$M_{k+1}(\Lambda) \;>\; (1+\delta)\,|Q_k^{(m)}(z_0)| \;\geq\; (1+\delta)\,M_k(\Lambda)\,.$$

We conclude that $M_k(\Lambda) \overset{k\to\infty}{\to} \infty$. Take now $n > k$ and repeat an estimate such as in the proof of Proposition 2.3: for all $z \in \Lambda$,

$$|\Omega_{m,n}(z) - \Omega_{m,k}(z)| \;<\; \frac{1}{(1+\delta)\,M_k(\Lambda)^2}\left(1 \;+\; \frac{1}{(1+\delta)^2} \;+\; \frac{1}{(1+\delta)^4} \;+\; \ldots\right)\,.$$

This proves uniform convergence and terminates the proof. ∎

Having proved convergence and holomorphism for Ω_m, we can now tackle the problem of convergence of the continued fractions in the right-hand sides of (13) and (14) with the bonus of proving holomorphism of the limiting function.

For the next few pages we will be considering

$$f(z) \;=\; \cfrac{1}{\beta_1(z) + \cfrac{1}{\beta_2(z) + \cfrac{1}{\beta_3(z) + \ddots}}}\,, \tag{25}$$

where either $\beta_n(z) = \alpha_n^{(e)}(z)$, or $\beta_n(z) = \alpha_n^{(o)}(z)$. Of course the left-hand side function $f(z)$ will be defined if the continued fraction converges, or if it tends to the point at infinity, in the Riemann sphere sense. Polynomials $P_k(z)$ and $Q_k(z)$ will be defined as in (16) and (17), such that the kth approximant to f will be

$$f_k(z) \;=\; \frac{P_k(z)}{Q_k(z)}\,.$$

By Proposition 3.5, in both cases for β_n, both sequences $P_1, P_2, \ldots$ and $Q_0, Q_1, Q_2, \ldots$ will be SPIZ.

We begin with the following

Proposition 4.3 *Function f defined by (25) is meromorphic in all of $\mathbb{C}$.*

Proof: We are going to show that for any $a \in \mathbb{R}$ given, f is meromorphic in $C_a = \{z \in \mathbb{C}\,; \operatorname{Re} z < a\}$.

By (i) in Proposition 2.1, if we define

$$F_m(z,w) \;=\; \cfrac{1}{\beta_1(z) \;+\; \cfrac{1}{\beta_2(z) + \cfrac{1}{\ddots + \cfrac{1}{\beta_m(z)+w}}}} \;,$$

then

$$F_m(z,w) \;=\; \frac{P_{m-1}(z)\,w \;+\; P_m(z)}{Q_{m-1}(z)\,w \;+\; Q_m(z)} \;.$$

If $k > m$, then

$$f_k(z) = F_m(z, \Omega_{m,k-m}(z)) \;. \tag{26}$$

Given any $a \in \mathbb{R}$, by Proposition 4.2 we may choose m large enough so that for $z \in C_a$ there exists $\lim_{k\to\infty} \Omega_{m,k-m}(z) = \Omega_m(z)$, Ω_m being holomorphic in C_a. So,

$$f(z) \;=\; \lim_{k\to\infty} F_m(z, \Omega_{m,k-m}(z)) \;=\; \frac{P_{m-1}(z)\,\Omega_m(z) \;+\; P_m(z)}{Q_{m-1}(z)\,\Omega_m(z) \;+\; Q_m(z)} \;,$$

where we allow for the possibility that $f(z) = \infty$ in the case

$$\Omega_m(z) \;=\; -\frac{Q_m(z)}{Q_{m-1}(z)} \;. \tag{27}$$

This shows that f is meromorphic in C_a, because it is the quotient of holomorphic functions. Furthermore, the possible poles of f in C_a must be the solutions to (27) in C_a for m large enough. ∎

We now know that the continued fractions we are interested in do converge at all $z \in \mathbb{C}$, except for a possible set of isolated poles. But, do poles exist? If the answer is yes, where do they lie? To answer these questions, we first characterize the poles of $f(z)$:

Proposition 4.4 *Let f be defined by (25). Then it has a pole in z if $|Q_n(z)| \overset{n\to\infty}{\longrightarrow} 0$.*

Proof: Suppose $|Q_n(z)| \overset{n\to\infty}{\longrightarrow} 0$. Then, by (iii) in Proposition 2.1, $P_n(z)$ does not tend to 0 and of course $f_n(z) = P_n(z)/Q_n(z)$ must tend to infinity. Then f has a pole in z. ∎

In order to prove Proposition 4.7 below, which includes the converse of Proposition 4.4, let us first quote the following corollary to Rouché's theorem, see [4], vol. I, Corollary 4.10e:

Theorem 4.5 *Let ϕ_n, $n = 0, 1, 2, \ldots$ be holomorphic in some fixed region R and let (ϕ_n) converge uniformly on every compact subset of R. If the limit function ϕ does not vanish identically, then each zero of ϕ is a limit of zeros of the functions ϕ_n.*

In order to be able to use the theorem above, we must first show

Proposition 4.6 *Let $f(z)$ be defined by (25) and $f_n(z) = P_n(z)/Q_n(z)$ be its nth approximant. If $\Lambda \subset \mathbb{C}$ is a compact set which does not contain any pole of f, then $f_n \to f$ uniformly in Λ.*

Proof: By the proof of Proposition 4.2, there exists $m \in \mathbb{N}$ such that $\Omega_{m,k} \stackrel{k\to\infty}{\to} \Omega_m$ uniformly in Λ. By using (26), (18) and (iii) in Proposition 2.1, we get for $k, l > m$

$$|f_k(z) - f_l(z)| = \frac{|\Omega_{m,k-m}(z) - \Omega_{m,l-m}(z)|}{|Q_{m-1}(z)\Omega_{m,k-m}(z) + Q_m(z)|\,|Q_{m-1}(z)\Omega_{m,l-m}(z) + Q_m(z)|} . \tag{28}$$

As the poles of f satisfy (27) and Λ is compact and does not contain any poles of f, then there exists $M > 0$ such that for all $z \in \Lambda$ we have $|Q_{m-1}(z)\Omega_m(z) + Q_m(z)| \geq M$. In fact, if there were not such an M, then for any $j \in \mathbb{N}$ we would find $z_j \in \Lambda$ such that $|Q_{m-1}(z_j)\Omega_m(z_j) + Q_m(z_j)| < 1/j$. By taking a convergent subsequence of the $(z_j)_{j\in\mathbb{N}}$ we would have $z^* \in \Lambda$ with $|Q_{m-1}(z^*)\Omega_m(z^*) + Q_m(z^*)| = 0$, which is in contradiction with the non-existence of poles of f in Λ.

We continue by writing

$$Q_{m-1}(z)\Omega_{m,k-m}(z) + Q_m(z) = Q_{m-1}(z)\Omega_m(z) + Q_m(z) - Q_{m-1}(z)\left(\Omega_m(z) - \Omega_{m,k-m}(z)\right),$$

which implies for large enough k and all $z \in \Lambda$

$$|Q_{m-1}(z)\Omega_{m,k-m}(z) + Q_m(z)| > \frac{M}{2} , \tag{29}$$

because for large enough k we have

$$|\Omega_m(z) - \Omega_{m,k-m}(z)| < \frac{M}{2\max_{z\in\Lambda}|Q_{m-1}(z)|}$$

for all $z \in \Lambda$.

Using (29) in (28) we finally prove the proposition. ∎

We now can now complete the characterization of the poles of f:

Proposition 4.7 *Consider Q_n as a SPIZ and let $r_{n,i}$ and $\rho^-_{n,i}$ be as in section 3. For each $i \in \mathbb{N}$, let $r_i = \lim_{n\to\infty} r_{n,i}$. Then, for each $i = 1, 2, \ldots$:*

(i) f has a pole in r_i.

(ii)

$$\lim_{n\to\infty} \rho^-_{n,i} \;=\; r_i \;.$$

(iii) The only poles of f are the r_i, $i = 1, 2, \ldots$.

Proof: Let l_i be defined as in Lemma 3.8. Then this lemma shows that $Q_n(a) \overset{n\to\infty}{\to} 0$ for $a \in [l_i, r_i]$. By Proposition 4.4, f has poles at every point of $[l_i, r_i]$. As poles must be isolated, $l_i = r_i$.

To prove (iii), notice that the poles of f are the zeros of $1/f$. As the roles of P_n and Q_n can be interchanged in Proposition 4.6, we may apply Theorem 4.5 to conclude that the poles of f are the r_i. ∎

Proposition 4.8 *Considering the notations introduced before Theorem 1.1, we have*

$$\lim_{n\to\infty} x_{n,i} \;=\; \lim_{n\to\infty} x^-_{n,i} \;,$$

where X may stand for $S^{(e)}$, T, U or V.

Proof: As $Q_n = S^{(e)}_n$ if we take $\beta_n = \alpha^{(e)}_n$, then, by (ii) in Proposition 4.7, we have just proved our claim for $X = S^{(e)}$.

To prove it for $X = T$, notice that by (6) we have

$$\frac{T_n(a)}{S^{(e)}_n(a)} \;=\; a \;-\; 2q\,\frac{R^{(e)}_n(a)}{S^{(e)}_n(a)} \;.$$

As, by Proposition 4.3, $R^{(e)}_n(a)/S^{(e)}_n(a)$ converges to a meromorphic function, then $\phi(a) = \lim_{n\to\infty} T_n(a)/S^{(e)}_n(a)$ is also meromorphic.

Define $l_i = \lim_{n\to\infty} t^-_{n,i}$ and $t_i = \lim_{n\to\infty} t_{n,i}$. By Lemma 3.8, $\lim_{n\to\infty} T_n(a) = 0$ for all $a \in [l_i, t_i]$, $i = 1, 2, \ldots$. $S^{(e)}_n$ does not tend to 0 at these points, because if it did so, (6) and Proposition 2.1(iii) would make T_n not tend to 0. Then ϕ has zeros at all $a \in [l_i, t_i]$. As zeros of meromorphic functions are isolated, then $l_i = t_i$ for $i = 1, 2, \ldots$.

An analogous proof works in proving the claim for $X = U$ or $X = V$. ∎

By now, the only facts remaining without proof in Theorem 1.1 are the identity between the limits of sequences of polynomial zeros and MCNs. These are again consequences of Theorem 4.5:

Corollary 4.9 *Using the same notations as in Theorem 1.1, we have, for $i = 1, 2, \ldots$:*

(i) $\lim_{n\to\infty} t_{n,i} \;=\; a_{2(i-1)}(q)$

(ii) $\lim_{n\to\infty} u_{n,i} \;=\; a_{2i-1}(q)$

(iii) $\lim_{n\to\infty} v_{n,i} \;=\; b_{2i-1}(q)$

Proof: To prove (i), just notice that if $\beta_n = \alpha_n^{(e)}$, then, by Proposition 4.6, $\frac{a}{2q} - f_n(z)$ converges uniformly to a holomorphic function at all compacts not containing any pole of f. By Theorem 4.5, solutions to (13) are then the limits of the $t_{n,i}$. As already noticed in the comments after (13), these are the MCNs $a_{2(i-1)}(q)$.

Proofs for (ii) and (iii) follow an analogous path. ∎

As in deriving (13) we have divided by the A_0 coefficient in the complex Fourier series (10), we may be missing some odd solutions of even order to Mathieu's equation. By proceeding in a way similar to the derivation of (13), we may start with a Fourier sine series $y(t) = \sum_{n=1}^{\infty} C_n \sin 2nt$ series instead of a complex Fourier series and derive an equation involving continued fractions for the a values at which these solutions occur. It is not difficult to see that the whole theory in this paper applies as well and proves that that these solutions are the ones corresponding to the $b_{2i}(q)$ MCNs, $i = 1, 2, \ldots$, with $b_{2i}(q) = \lim_{n\to\infty} s_{n,i}^{(e)}$.

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