ABSOLUTELY CONTINUOUS SPECTRUM FOR THE ISOTROPIC MAXWELL OPERATOR WITH COEFFICIENTS THAT ARE PERIODIC IN SOME DIRECTIONS AND DECAY IN OTHERS

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Abstract. The purpose of this paper is to prove that the spectrum of an isotropic Maxwell operator with electric permittivity and magnetic permeability that are periodic along certain directions and tending to a constant super-exponentially fast in the remaining directions is purely absolutely continuous. The basic technical tools is a new "operatorial" identity relating the Maxwell operator to a vector-valued Schrödinger operator. The analysis of the spectrum of that operator is then handled as in [4, 5].

0. The main result

In $\mathbb{R}^3$, we study the Maxwell operator

$$M = i\begin{pmatrix} 0 & \varepsilon^{-1}\nabla\times\cdot \\ -\mu^{-1}\nabla\times\cdot & 0 \end{pmatrix} \tag{0.1}$$

acting on the space $\mathcal{H}(\varepsilon)\oplus\mathcal{H}(\mu)$. Here, ∇ denotes the gradient of a function, div the divergence of a vector field, $\times$ the standard cross-product in $\mathbb{R}^3$, and we defined

$$\mathcal{H}(\varepsilon) := \{u \in L^2(\mathbb{R}^3, \varepsilon(x)dx)\otimes\mathbb{C}^3;\ \mathrm{div}(\varepsilon x) = 0\}.$$

$\mathcal{H}(\varepsilon)$ is endowed with its natural scalar product

$$\langle f, g\rangle_\varepsilon = \int_{\mathbb{R}^3}\langle f(x), g(x)\rangle_{\mathbb{C}}\,\varepsilon(x)dx$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denotes the usual scalar product in $\mathbb{C}^3$.
Pick $d\in\{1,2\}$. Let (x,y) denote the points of the space $\mathbb{R}^3$. Define $\Omega = \mathbb{R}^{3-d}\times(0,2\pi)^d$.

We assume that the scalar functions ε and μ satisfy

(H1): $\forall l\in\mathbb{Z}^d$, $\forall(x,y)\in\mathbb{R}^3$,

$$\varepsilon(x, y+2\pi l) = \varepsilon(x,y),\quad \mu(x,y+2\pi l) = \mu(x,y);$$

(H2): the functions ε and μ are twice continuously differentiable in Ω;
(H3): there exist $\varepsilon_0>0$ and $\mu_0>0$ such that, for any $a>0$, one has

$$\sup_{0\le|\alpha|\le 2}\ \sup_{(x,y)\in\Omega} e^{a|x|}(|\partial^\alpha(\varepsilon-\varepsilon_0)(x,y)| + |\partial^\alpha(\mu-\mu_0)(x,y)|) < +\infty;$$

(H4): there exists $c_0>0$ such that $\forall(x,y)\in\mathbb{R}^3$, $\varepsilon(x,y)\ge c_0$ and $\mu(x,y)\ge c_0$.

Then, our main result is

Theorem 0.1. *Under assumptions (H1)–(H4), the spectrum of M is purely absolutely continuous.*

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In [9], A. Morame proved that the spectrum of the Maxwell operator (0.1) is absolutely continuous when the electric permittivity ε and the magnetic permeability μ are periodic with respect to a non-degenerate lattice in $\mathbb{R}^3$. In [11], T. Suslina proved the absolute continuity of the spectrum of the Maxwell operator (0.1) in a strip when the electric permittivity ε and the magnetic permeability μ are periodic along the strip (with perfect conductivity conditions imposed on the boundary of the strip).
In both papers, the authors first apply a standard idea in the spectral theory of the Maxwell operator to circumvent one of the first technical difficulties one encounters when dealing with the Maxwell system: the fact that the domain of the Maxwell operator, $\mathcal{H}(\varepsilon)\oplus\mathcal{H}(\mu)$, consists of only the divergence free vectors (up to multiplication by ε or μ). To resolve that difficulty, the standard idea [1] is to extend the Maxwell operator to an operator acting on $L^2(\mathbb{R}^3)\otimes\mathbb{C}^8$. We introduce such an extension that slightly differs from the one considered in [1, 8, 11, 9] as we require some additional properties.

Consider the matrix of first order linear differential expressions

$$\mathcal{M} = i\begin{pmatrix} 0 & \varepsilon^{-1}\nabla\times\cdot & 0 & \nabla(\varepsilon^{-1}\cdot) \\ -\mu^{-1}\nabla\times\cdot & 0 & \nabla(\mu^{-1}\cdot) & 0 \\ 0 & (\varepsilon\mu)^{-1}\operatorname{div}(\mu\,\cdot) & 0 & 0 \\ (\varepsilon\mu)^{-1}\operatorname{div}(\varepsilon\,\cdot) & 0 & 0 & 0 \end{pmatrix}. \tag{0.2}$$

It naturally defines an elliptic self-adjoint operator on

$$\mathcal{H}_{\mathrm{tot}} := L^2(\mathbb{R}^3,\varepsilon(x)dx;\mathbb{C}^3)\oplus L^2(\mathbb{R}^3,\mu(x)dx;\mathbb{C}^3)\oplus L^2(\mathbb{R}^3,\varepsilon(x)dx)\oplus L^2(\mathbb{R}^3,\mu(x)dx)$$

with domain

$$H^1(\mathbb{R}^3;\mathbb{C}^3)\oplus H^1(\mathbb{R}^3;\mathbb{C}^3)\oplus H^1(\mathbb{R}^3)\oplus H^1(\mathbb{R}^3).$$

Let Π be the orthogonal projector on $\mathcal{H}(\varepsilon)\oplus\mathcal{H}(\mu)\oplus\{0\}\oplus\{0\}$ in $\mathcal{H}_{\mathrm{tot}}$. One checks that

$$[\Pi,\mathcal{M}] = 0. \tag{0.3}$$

This is a consequence of the well known facts that gradient fields are orthogonal (for the standard scalar product) to divergence free fields, and that curl fields are divergence free.
Moreover, one computes

$$\Pi\mathcal{M}\Pi = \Pi\begin{pmatrix} M & 0 \\ 0 & 0\end{pmatrix}\Pi. \tag{0.4}$$

This and equation (0.3) imply that Theorem 0.1 is an immediate consequence of

Theorem 0.2. *Under assumptions(H1)–(H4), the spectrum of $\mathcal{M}$ is purely absolutely continuous.*

In the cases dealt with in [9, 11], to prove the absolute continuity of the spectrum of $\mathcal{M}$ (or rather said their analogue of $\mathcal{M}$), the authors perform the Bloch-Floquet-Gelfand reduction that brings them back to studying an operator with compact resolvent. Because of this, they only need to show that $\mathcal{M}$ has no eigenvalue. To prove this, they show that the fact that $\mathcal{M}$ has an eigenvalue implies that some Schrödinger operator with a potential having the same symmetry properties as ε and μ has an eigenvalue. The well known argument showing that this is impossible relies on the fact that the reduced operator has compact resolvent.

In our case, by assumption (H1), the Bloch-Floquet-Gelfand reduction can only be done in the y-variable; hence, the resolvent of the reduced operator is not compact. So, the standard argument does not apply. To analyze the reduced $\mathcal{M}$, we first show an "operatorial" identity that brings us back to analyzing a Schrödinger operator; then, to analyze this Schrödinger operator, we apply the method developed in [4].

Consider the following differential matrices acting on twice differentiable functions valued in $\mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C} \oplus \mathbb{C}$

$$\Delta_8 := \begin{pmatrix} \Delta_3 & 0 & 0 & 0 \\ 0 & \Delta_3 & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix} \quad \text{where} \quad \Delta_3 := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix}, \tag{0.5}$$

$$\mathcal{A} = i \begin{pmatrix} 0 & -\mu z \times \cdot & 0 & -\mu z \cdot \\ \varepsilon z \times \cdot & 0 & -\varepsilon z \cdot & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{0.6}$$

$$\mathcal{J} = \begin{pmatrix} \varepsilon^{-1/2} \cdot & 0 & 0 & 0 \\ 0 & \mu^{-1/2} \cdot & 0 & 0 \\ 0 & 0 & \mu^{1/2} \cdot & 0 \\ 0 & 0 & 0 & \varepsilon^{1/2} \cdot \end{pmatrix} \tag{0.7}$$

where Δ is the standard Laplace operator in $\mathbb{R}^3$ and

$$z = \nabla((\varepsilon\mu)^{-1}). \tag{0.8}$$

We prove

Theorem 0.3. *One computes*

$$\varepsilon\mu \mathcal{J}^{-1}(\mathcal{M} + \mathcal{A})\mathcal{M}\mathcal{J} = -\Delta_8 + \mathcal{V} + \mathcal{F} \tag{0.9}$$

where

- Δ_8 *is the diagonal Laplace operator defined in* (0.5),
- $\mathcal{V}$ *is the zeroth-order matrix and* $\mathcal{F}$ *the first-order matrix defined by*

$$\mathcal{V} = \begin{pmatrix} V(\varepsilon)\cdot & 0 & 0 & 0 \\ 0 & V(\mu)\cdot & 0 & 0 \\ 0 & 0 & v(\mu)\cdot & 0 \\ 0 & 0 & 0 & v(\varepsilon)\cdot \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 & 0 & -F(\varepsilon, \mu, \cdot) & 0 \\ 0 & 0 & 0 & F(\mu, \varepsilon, \cdot) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{0.10}$$

and, for $\{f, g\} = \{\mu, \varepsilon\}$*, we have defined*

$$V(f) = v(f)Id - 2Jac(s(f)), \quad v(f) = s^2(f) + \operatorname{div} s(f) \quad \text{and} \quad s(f) = f^{-1/2}\nabla(f^{1/2}), \tag{0.11}$$

$$F(f, g, \cdot) = f^{-1/2}\nabla(\varepsilon\mu) \times \nabla(g^{-1/2} \cdot), \tag{0.12}$$

and $Jac(g)$ *denotes the Jacobian of a differentiable function* $g: \mathbb{R}^3 \to \mathbb{R}^3$.

Remark 0.1. If the functions ε, μ are such that the product $\varepsilon\mu$ is constant then $\mathcal{A} = 0$ and $\mathcal{F} = 0$. This idea was used in [2].

Remark 0.2. Though computations analogous to those leading to Theorem 0.3 have been done in [9, 11], to our knowledge, the "operatorial" identity (0.9) is new. We hope it will also prove useful beyond the present study [3].

Remark 0.3. As a consequence of (0.9), for $\lambda \in \mathbb{C}$, we obviously obtain

$$\varepsilon\mu\mathcal{J}^{-1}(\mathcal{M} + \mathcal{A} + \lambda)(\mathcal{M} - \lambda)\mathcal{J} = -\Delta_8 + \mathcal{V} - \varepsilon\mu\mathcal{J}^{-1}(\lambda\mathcal{A} + \lambda^2)\mathcal{J} + \mathcal{F}. \tag{0.13}$$

These equalities being written between differential matrices can be complemented with boundary conditions to yield equalities between operators. Among the boundary conditions we will need are the quasi-periodic Floquet boundary conditions described in section 2.

Remark 0.4. One can consider another extension of the initial operator (0.1),

$$\mathcal{M} = i \begin{pmatrix} 0 & \varepsilon^{-1}\nabla\times\cdot & 0 & \nabla(\alpha_2\beta_2\,\cdot) \\ -\mu^{-1}\nabla\times\cdot & 0 & \nabla(\alpha_1\beta_1\,\cdot) & 0 \\ 0 & \beta_1\operatorname{div}(\mu\,\cdot) & 0 & 0 \\ \beta_2\operatorname{div}(\varepsilon\,\cdot) & 0 & 0 & 0 \end{pmatrix}$$

with positive functions $\alpha_1, \alpha_2, \beta_1, \beta_2$. This operator is self-adjoint in the space

$$L^2(\mathbb{R}^3, \varepsilon(x)dx; \mathbb{C}^3) \oplus L^2(\mathbb{R}^3, \mu(x)dx; \mathbb{C}^3) \oplus L^2(\mathbb{R}^3, \alpha_1(x)dx) \oplus L^2(\mathbb{R}^3, \alpha_2(x)dx)$$

and (0.4) holds. If $\alpha_1\beta_1^2 = \varepsilon^{-1}\mu^{-2}$ and $\alpha_2\beta_2^2 = \varepsilon^{-2}\mu^{-1}$ and we take

$$\mathcal{A} = i \begin{pmatrix} 0 & -\mu z\times\cdot & 0 & -\beta_2^{-1}\varepsilon^{-1}z\,\cdot \\ \varepsilon z\times\cdot & 0 & -\beta_1^{-1}\mu^{-1}z\,\cdot & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathcal{J} = diag(\varepsilon^{-1/2}, \mu^{-1/2}, \alpha_1^{-1}\beta_1^{-1}\mu^{-1/2}, \alpha_2^{-1}\beta_2^{-1}\varepsilon^{-1/2})$ then formulae (0.9), (0.13) still hold (our choice in this paper is $\alpha_1 = \varepsilon$, $\alpha_2 = \mu$, $\beta_1 = \beta_2 = \varepsilon^{-1}\mu^{-1}$).

1. A useful formula: the proof of Theorem 0.3

The computations leading to Theorem 0.3 are quite similar to those done in [11]. We first compute

$$\mathcal{M}\mathcal{J} = i \begin{pmatrix} 0 & \varepsilon^{-1}\nabla\times(\mu^{-1/2}\,\cdot) & 0 & \nabla(\varepsilon^{-1/2}\,\cdot) \\ -\mu^{-1}\nabla\times(\varepsilon^{-1/2}\,\cdot) & 0 & \nabla(\mu^{-1/2}\,\cdot) & 0 \\ 0 & (\varepsilon\mu)^{-1}\operatorname{div}(\mu^{1/2}\cdot) & 0 & 0 \\ (\varepsilon\mu)^{-1}\operatorname{div}(\varepsilon^{1/2}\,\cdot) & 0 & 0 & 0 \end{pmatrix}.$$

Hence, as $\operatorname{div}(\nabla\times\cdot) = 0$ and $\nabla\times\nabla\cdot = 0$, we obtain

$$\varepsilon\mu\mathcal{J}^{-1}\mathcal{M}^2\mathcal{J} = -\begin{pmatrix} a(\varepsilon,\mu) & 0 & 0 & 0 \\ 0 & a(\mu,\varepsilon) & 0 & 0 \\ 0 & 0 & b(\mu) & 0 \\ 0 & 0 & 0 & b(\varepsilon) \end{pmatrix} \tag{1.1}$$

where, for $\{f, g\} = \{\varepsilon, \mu\}$, we have defined

$$a(f,g) = -f^{1/2}g\nabla\times(g^{-1}\nabla\times(f^{-1/2}\,\cdot)) + (fg)f^{1/2}\nabla(f^{-1}(fg)^{-1}\operatorname{div}(f^{1/2}\cdot)), \tag{1.2}$$

$$b(f) = f^{-1/2}\operatorname{div}(f\nabla(f^{-1/2}\,\cdot)). \tag{1.3}$$

On the other hand,

$$\varepsilon\mu\mathcal{J}^{-1}\mathcal{A}\mathcal{M}\mathcal{J} = -\begin{pmatrix} c(\varepsilon) & 0 & -d(\mu) & 0 \\ 0 & c(\mu) & 0 & d(\varepsilon) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

where, for $f \in \{\varepsilon, \mu\}$, we have defined

$$c(f) = \varepsilon\mu(f^{1/2}z\times\nabla\times(f^{-1/2}\,\cdot) - zf^{-1/2}\operatorname{div}(f^{1/2}\cdot)), \tag{1.5}$$

$$d(f) = (\varepsilon\mu)^{3/2}f^{1/2}z\times\nabla(f^{-1/2}\,\cdot). \tag{1.6}$$

For $\{f, g\} = \{\varepsilon, \mu\}$, using (1.6) and

$$f\nabla(f^{-1}) = -f^{-1}\nabla f, \tag{1.7}$$

we compute

$$d(f) = -(\varepsilon\mu)^{-1/2}f^{1/2}\nabla(\varepsilon\mu)\times\nabla(f^{-1/2}\,\cdot) = -g^{-1/2}\nabla(\varepsilon\mu)\times\nabla(f^{-1/2}\,\cdot) = -F(g,f,\cdot) \tag{1.8}$$

which gives formula (0.12) for the coefficient of the matrix $\mathcal{F}$ in Theorem 0.3.
Recall that, for $u: \mathbb{R}^3 \to \mathbb{R}$ and $v: \mathbb{R}^3 \to \mathbb{R}^3$ both once differentiable, one has

$$\nabla \times (uv) = u\,(\nabla \times v) + (\nabla u) \times v. \tag{1.9}$$

Using this, (1.2), (1.5) and (0.8), we compute

$$-\varepsilon^{1/2}\mu\nabla \times ((\varepsilon\mu)^{-1}\varepsilon\nabla \times (\varepsilon^{-1/2}\,\cdot)) + \varepsilon^{3/2}\mu\nabla((\varepsilon\mu)^{-1}) \times \nabla \times (\varepsilon^{-1/2}\,\cdot) = -\varepsilon^{-1/2}\nabla \times (\varepsilon\nabla \times (\varepsilon^{-1/2}\,\cdot)),$$

and

$$\varepsilon^{3/2}\mu\nabla((\varepsilon\mu)^{-1}\varepsilon^{-1}\operatorname{div}(\varepsilon^{1/2}\,\cdot)) - \varepsilon^{1/2}\mu\nabla((\varepsilon\mu)^{-1})\operatorname{div}(\varepsilon^{1/2}\,\cdot) = \varepsilon^{1/2}\nabla(\varepsilon^{-1}\operatorname{div}(\varepsilon^{1/2}\,\cdot)),$$

so

$$c(\varepsilon) + a(\varepsilon,\mu) = -\varepsilon^{-1/2}(\nabla \times (\varepsilon\nabla \times (\varepsilon^{-1/2}\,\cdot))) + \varepsilon^{1/2}\nabla(\varepsilon^{-1}\operatorname{div}(\varepsilon^{1/2}\,\cdot)). \tag{1.10}$$

To complete the proof of Theorem 0.3, taking (1.1), (1.3), (1.4) and (1.10) into account, we are only left with proving the following

Lemma 1.1. *One has*

$$-\varepsilon^{-1/2}(\nabla \times (\varepsilon\nabla \times (\varepsilon^{-1/2}\,\cdot))) + \varepsilon^{1/2}\nabla(\varepsilon^{-1}\operatorname{div}(\varepsilon^{1/2}\,\cdot)) = \Delta_3 - V(\varepsilon) \tag{1.11}$$

and

$$\varepsilon^{-1/2}\operatorname{div}(\varepsilon\nabla(\varepsilon^{-1/2}\,\cdot)) = \Delta - v(\varepsilon) \tag{1.12}$$

where V and v are defined in Theorem 0.3.

Proof. We start with the proof of (1.12). Using (1.7) and (0.11), we compute

$$\begin{aligned}\varepsilon^{-1/2}\operatorname{div}(\varepsilon^{1/2}(\nabla\,\cdot) - s(\varepsilon)\varepsilon^{1/2}\,\cdot) &= \Delta + \langle s(\varepsilon), \nabla\cdot\rangle - \operatorname{div} s(\varepsilon) - \langle s(\varepsilon), \varepsilon^{-1/2}\nabla(\varepsilon^{1/2}\,\cdot)\rangle \\ &= \Delta - (\operatorname{div} s(\varepsilon) + s(\varepsilon)^2),\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^3$.
Let us now prove (1.11). Using (1.9) we compute

$$\begin{aligned}&\varepsilon^{-1/2}(\nabla\times(\varepsilon\nabla \times (\varepsilon^{-1/2}\,\cdot))) \\ &\quad= \varepsilon^{-1/2}\nabla \times (\varepsilon^{1/2}\nabla \times \cdot - \varepsilon^{1/2}s(\varepsilon) \times \cdot) \\ &\quad= (\nabla \times \cdot)^2 + s(\varepsilon) \times (\nabla \times \cdot) - s(\varepsilon) \times (s(\varepsilon) \times \cdot) - \nabla \times (s(\varepsilon) \times \cdot).\end{aligned} \tag{1.13}$$

The classical formula gives

$$\nabla \times (s(\varepsilon) \times \cdot) = s(\varepsilon)\operatorname{div}\,\cdot - \cdot\,\operatorname{div} s(\varepsilon) - \langle s(\varepsilon), \nabla\rangle\,\cdot + \langle\cdot,\nabla\rangle s(\varepsilon). \tag{1.14}$$

For the second term in (1.11) we have

$$\begin{aligned}\varepsilon^{1/2}\nabla(\varepsilon^{-1}\operatorname{div}(\varepsilon^{1/2}\,\cdot)) &= \varepsilon^{1/2}\nabla(\varepsilon^{-1/2}\operatorname{div}(\cdot) + \varepsilon^{-1/2}\langle s(\varepsilon),\cdot\rangle) \\ &= \nabla(\operatorname{div}\cdot) - s(\varepsilon)\operatorname{div}(\cdot) - s(\varepsilon)\langle s(\varepsilon),\cdot\rangle + \nabla\langle s(\varepsilon),\cdot\rangle.\end{aligned} \tag{1.15}$$

Summarizing (1.13), (1.14) and (1.15) we obtain

$$\begin{aligned}&-\varepsilon^{-1/2}(\nabla \times (\varepsilon\nabla \times (\varepsilon^{-1/2}\,\cdot))) + \varepsilon^{1/2}\nabla(\varepsilon^{-1}\operatorname{div}(\varepsilon^{1/2}\,\cdot)) \\ &\qquad= \Delta_3 - (|s(\varepsilon)|^2 + \operatorname{div} s(\varepsilon))\,\cdot \\ &\qquad\qquad - s(\varepsilon) \times (\nabla \times \cdot) - \langle s(\varepsilon), \nabla\rangle\,\cdot + \langle\,\cdot,\nabla\rangle s(\varepsilon) + \nabla\langle s(\varepsilon),\cdot\rangle,\end{aligned}$$

where the well-known formulas

$$\nabla\operatorname{div} - (\nabla\times)^2 = \Delta_3$$

and

$$s(\varepsilon)\langle s(\varepsilon),\cdot\rangle - s(\varepsilon) \times (s(\varepsilon) \times \cdot) = |s(\varepsilon)|^2\cdot$$

are used. Now the simple calculations

$$s(\varepsilon)\times(\nabla\times\cdot)+\langle s(\varepsilon),\nabla\rangle\cdot=\mathrm{Jac}(\,\cdot\,)s(\varepsilon),$$
$$\langle\,\cdot\,,\nabla\rangle s(\varepsilon)=\mathrm{Jac}(s(\varepsilon))^t\,\cdot,\quad \nabla\langle s(\varepsilon),\cdot\rangle=\mathrm{Jac}(\,\cdot\,)s(\varepsilon)+\mathrm{Jac}(s(\varepsilon))\,\cdot$$

complete the proof of Lemma 1.1. □

2. Proof of Theorem 0.2

In our previous work [4, 5], we proved the absolute continuity of the spectrum of the Schrödinger operator where the properties of the potential were similar to those imposed on permittivity ε and the permeability μ in Theorem 0.1 and 0.2. The scheme of the proof of Theorem 0.2 is globally the same as that of Theorem 1.1 in [4, 5]; so, we will omit some details.

First, basing on the relation (0.13), we construct a convenient representation of the resolvent $(\mathcal{M}-\lambda)^{-1}$ (see Lemma 2.3 below).

First of all we need to define some notations. Let $\langle x\rangle=\sqrt{x^2+1}$. For $a\in\mathbb{R}$, introduce the spaces

$$L_{p,a}=\{f:e^{a\langle x\rangle}f\in L_p(\Omega)\},\qquad H^l_a=\{f:e^{a\langle x\rangle}f\in H^l(\Omega)\},$$

where $1\le p\le\infty$ and $H^l(\Omega)$ is the standard Sobolev space. Introduce the function spaces in Ω with quasi-periodic boundary conditions

$$H^l_a(k):=\left\{f\in H^l_a:(D^\alpha f)\mid_{y_j=2\pi}=e^{2\pi ik_j}(D^\alpha f)\mid_{y_j=0},\quad|\alpha|\le l-1\right\}\text{ and }H^l(k):=H^l_0(k).$$

Finally, for X and Y Banach spaces, $B(X,Y)$ is the space of all bounded operators from X to Y, and $B(X)=B(X,X)$, both endowed with their natural norm topology.

Due to the Bloch-Floquet-Gelfand transformation, the Maxwell operator $\mathcal{M}$ is unitary equivalent to the direct integral $\int^{\oplus}_{[0,1)^d}\mathcal{M}(k)dk$, where $\mathcal{M}$ is the operator given by the differential expression (0.2) on the domain $\mathrm{Dom}\,\mathcal{M}(k)=H^1(k)$. The Laplace operator on the domain $H^2(k)$ will be denoted by $\Delta(k)$.
In [4, 5], we essentially proved the following result

Lemma 2.1. *Assume that the pair $(k_0,\lambda_0)\in\mathbb{R}^{d+1}$ satisfies*

$$(k_0+n)^2\neq\varepsilon_0\mu_0\lambda_0,\qquad\forall n\in\mathbb{Z}^d.\tag{2.1}$$

Then, there exist numbers $\delta>0$, $a>0$, an open set $\Xi_0\subset\mathbb{C}^{d+1}$ such that

$$(B_\delta(k_0)\cup\{k(\tau)\}_{\tau\in\mathbb{R}})\times B_\delta(\lambda_0)\subset\Xi_0,$$

where $B_\delta(k_0)$ is a ball in real space

$$B_\delta(k_0)=\{k\in\mathbb{R}^d:|k-k_0|<\delta\},$$

and $k(\tau)=(\tilde k_1+i\tau,\tilde k')$ with fixed $\tilde k\in B_\delta(k_0)$, and there exists an analytic $B(L_{2,a},H^2_{-a})$-valued function R_0, defined in Ξ_0, having the properties

- *for $(k,\lambda)\in\Xi_0$, $k\in\mathbb{R}^d$, $\mathrm{Im}\,\lambda>0$, $U\in L_{2,a}$, one has*
$$R_0(k,\lambda)U=(-\Delta(k)-\varepsilon_0\mu_0\lambda)^{-1}U;$$
- $$\|R_0(k(\tau),\lambda)\|_{B(H^2_a,H^2_{-a})}\le C|\tau|^{-1};\tag{2.2}$$
- *$R_0(k,\lambda)L_{2,a}\subset H^2_{-a}(k)$.*

This lemma is proved in [4] (see Theorem 3.1) except for the fact that estimate (2.2) is replaced with

(2.3) $$\|R_0(k(\tau),\lambda)\|_{B(L_{2,a},L_{2,-a})} \le C|\tau|^{-1}.$$

The proof of estimate (2.2) is exactly the same as that of (2.3).

Clearly, in Lemma 2.1, we can replace $\Delta(k)$ with $\Delta_8(k)$ (defined in (0.5)) at the expense of changing the constants; the resolvent of $\Delta_8(k)$ (and its analytic extension) will henceforth be denoted by $R^0_{\mathcal{M}}(k,\lambda)$. So

$$R^0_{\mathcal{M}}(k,\lambda) = R_0(k,\lambda)\,\mathrm{Id}_{\mathbb{C}^8}.$$

To deal with the potential, we prove

Lemma 2.2. *Let ε, μ satisfy hypothesis (H1)-(H4), $\mathcal{A}$ be defined by (0.6), and (k_0,λ_0) satisfy (2.1). Then, there exist $\delta>0$, $a>0$, an open set $\Xi\subset\mathbb{C}^{d+1}$ with $B_\delta(k_0)\times B_\delta(\lambda_0)\subset\Xi$, a function $h:\ \Xi\to\mathbb{C}$ analytic in Ξ with the property*

(2.4) $$\forall\lambda\in B_\delta(\lambda_0),\quad \exists k\in B_\delta(k_0)\quad \text{such that}\quad h(k,\lambda)\neq 0,$$

and there exists an analytic $B(L_{2,a},H^2_{-a})$-valued function Z, defined in

$$\Xi_1 := \{(k,\lambda)\in\Xi : h(k,\lambda)\neq 0\},$$

such that, for $(k,\lambda)\in\Xi_1$, $k\in\mathbb{R}^d$, $\operatorname{Im}\lambda^2>0$, $U\in H^2_a(k)$, one has

(2.5) $$Z(k,\lambda)\left(-\Delta_8+\mathcal{V}-\varepsilon\mu\mathcal{J}^{-1}(\lambda\mathcal{A}+\lambda^2)\mathcal{J}\right)U = U$$

and

$$Z(k,\lambda)L_{2,a}\subset H^2_{-a}(k).$$

Proof. Note that

$$\mathcal{V}-\varepsilon\mu\mathcal{J}^{-1}(\lambda\mathcal{A}+\lambda^2)\mathcal{J} = -\varepsilon_0\mu_0\lambda^2+\mathcal{W}(\lambda),$$

where, by assumptions (H2)-(H3), $\lambda\mapsto\mathcal{W}(\lambda)$ is an entire function valued in $L_{\infty,b}$ for any $b\in\mathbb{R}$. Set

$$Z(k,\lambda) = \left(I+R^0_{\mathcal{M}}(k,\lambda^2)\mathcal{W}(\lambda)\right)^{-1}R^0_{\mathcal{M}}(k,\lambda^2).$$

The operator of multiplication by $\mathcal{W}$ is bounded as an operator from H^2_{-a} to H^2_a, and is compact as an operator from H^2_{-a} to $L_{2,a}$. It remains to use the estimation (2.2) and the analytic Fredholm alternative in the Hilbert space H^2_{-a} (see e.g. [7, 10]) to complete the proof of Lemma 2.2. □

In the following lemma, we construct an analytic extension of the resolvent of Maxwell operator to the non-physical sheet. Set

$$Q(\lambda) = \varepsilon\mu\mathcal{J}^{-1}(\mathcal{M}+\mathcal{A}+\lambda).$$

Then, for any $b\in\mathbb{R}$, Q is an entire function with values in $B(H^1_b,L_{2,b})$. The next result we need is

Lemma 2.3. *Under the assumptions of Lemma 2.2, on the set Ξ_1, we define the operator-function*

$$(k,\lambda)\mapsto R_{\mathcal{M}}(k,\lambda) := \mathcal{J}Z(k,\lambda)(I-\mathcal{F}Z(k,\lambda))Q(\lambda).$$

Then, one has

1. *$(k,\lambda)\mapsto R_{\mathcal{M}}(k,\lambda)$ is analytic in Ξ_1 with values in $B(H^1_a,H^2_{-a})$);*
2. *for $(k,\lambda)\in\Xi_1$, $k\in\mathbb{R}^d$, $\operatorname{Im}\lambda^2>0$, there exists $\mathcal{H}(k)\subset H^1_a(k)$ such that $\overline{\mathcal{H}(k)}=L_2(\Omega)$ and for $U\in\mathcal{H}(k)$,*

$$R_{\mathcal{M}}(k,\lambda)U = (\mathcal{M}(k)-\lambda)^{-1}U$$

Proof. The first property is true because $\mathcal{F}$ is a bounded operator from H^1_{-a} to $L_{2,a}$.
To prove the second one, pick $(k,\lambda) \in \Xi_1$ such that $k \in \mathbb{R}^d$ and $\operatorname{Im}\lambda^2 > 0$; define

$$\mathcal{H}(k) = (\mathcal{M}(k) - \lambda) H^2_a(k).$$

That $\mathcal{H}(k)$ is dense in $L^2(\Omega)$ is a consequence of the self-adjointness of $\mathcal{M}$ and the fact that $\lambda \notin \mathbb{R}$.
Let $W \in H^2_a(k)$ and $U = (\mathcal{M} - \lambda)\mathcal{J} W$. Then, one computes

$$\begin{aligned}
R_{\mathcal{M}}(k,\lambda)U &= \mathcal{J} Z(k,\lambda)(I - \mathcal{F} Z(k,\lambda)) Q(\lambda)(\mathcal{M} - \lambda)\mathcal{J} W \\
&= \mathcal{J}(Z(k,\lambda) - Z(k,\lambda)\mathcal{F} Z(k,\lambda)) \left(-\Delta_8 + \mathcal{V} - \varepsilon\mu \mathcal{J}^{-1}(\lambda\mathcal{A} + \lambda^2)\mathcal{J} + \mathcal{F}\right) W \\
&= \mathcal{J}\,[W + Z(k,\lambda)\mathcal{F} W \\
&\qquad - Z(k,\lambda)\mathcal{F} Z(k,\lambda) \left(-\Delta_8 + \mathcal{V} - \varepsilon\mu \mathcal{J}^{-1}(\lambda\mathcal{A} + \lambda^2)\mathcal{J} + \mathcal{F}\right) W] \\
&= \mathcal{J}(W - Z(k,\lambda)\mathcal{F} Z(k,\lambda)\mathcal{F} W)
\end{aligned} \tag{2.6}$$

where we used (0.13) and (2.5). Furthermore, one can check that

$$\mathcal{F} Z(k,\lambda)\mathcal{F} = 0.$$

Plugging this into (2.6), we obtain

$$R_{\mathcal{M}}(k,\lambda)U = \mathcal{J} W = (\mathcal{M}(k) - \lambda)^{-1} U.$$

This completes the proof of Lemma 2.3. □

Remark 2.1. One presumably has $\mathcal{H}(k) = H^1_a(k)$.

Lemma 2.4. *Let G_0 and G be two Hilbert spaces, $G_0 \subset G$, and G_0^* be a dual space to G_0 with respect to the scalar product in G. Let B be a self-adjoint operator in G. Suppose that R_B is an analytic function defined in a complex neighborhood of an interval $[\alpha,\beta]$ except at a finite number of points $\{\mu_1,\dots,\mu_N\}$, that the values of R_B are in $B(G_0,G_0^*)$ and that*

$$R_B(\lambda)\varphi = (B - \lambda)^{-1}\varphi \quad \text{if } \operatorname{Im}\lambda > 0, \varphi \in \mathcal{H}$$

where $\mathcal{H} \subset G_0$ is dense in G. Then, the spectrum of B in the set $[\alpha,\beta] \setminus \{\mu_1,\dots,\mu_N\}$ is absolutely continuous. If $\Lambda \subset [\alpha,\beta]$, $\operatorname{mes}\Lambda = 0$ and $\mu_j \notin \Lambda$, $j = 1,\dots,N$, then $E_B(\Lambda) = 0$, where E_B is the spectral projector of B.

This lemma is an immediate consequence of Proposition 2 and equation (18) in section 1.4.5 of [12].
Now, let G be a Hilbert space, and let $(H(k))_{k \in \mathbb{C}^d}$ be an analytic family of self-adjoint operators on G. On $\mathcal{G} = L^2([0,1)^d, G)$, following [10], one defines the self-adjoint operator

$$H = \int^{\oplus}_{[0,1)^d} H(k)\,dk.$$

The following abstract theorem on the spectrum of the fibered operator H is based on the Lemma 2.4. Its proof repeats the proof of Theorem 1.1 in [4] although this explicit formulation is not given there.

Theorem 2.1. *Suppose that there exists a sequence of analytic functions $f_m : \mathbb{C}^{d+1} \to \mathbb{C}$ such that*

$$\forall\lambda \quad \exists k \quad \text{such that} \quad f_m(k,\lambda) \neq 0,$$

and the set of real points (k,λ) where $f_m(k,\lambda) \neq 0$ for all m can be represented as

$$\mathbb{R}^{d+1} \setminus \bigcup_{m=1}^{\infty} \{(k,\lambda) : f_m(k,\lambda) = 0\} = \bigcup_{j=1}^{\infty} B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j).$$

Suppose moreover that, for every j, there exist

- *an analytic scalar function h_j defined in a complex neighborhood of $\overline{B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j)}$ satisfying property (2.4);*
- *a Hilbert space $G_j(k) \subset G$, its dual $G_j^*(k)$ with respect to the scalar product in G, and a set $\mathcal{H}_j(k)$ such that*
$$\mathcal{H}_j(k) \subset G_j(k) \subset G, \quad \overline{\mathcal{H}_j(k)} = G;$$
- *an analytic $B(G_j, G_j^*)$-valued function R_j defined on the set $\{(k,\lambda);\ h_j(k,\lambda) \neq 0\}$ such that for $k \in \mathbb{R}^d$, $\operatorname{Im}\lambda > 0$, $f \in \mathcal{H}_j(k)$,*
$$R_j(k,\lambda)f = (H(k) - \lambda)^{-1} f.$$

Then, the spectrum of H is purely absolutely continuous.

The spectral theory of a class of analytically fibered operators has been studied in [6]; their definition of an analytically fibered operator cannot be used in the present case as they require the resolvent of the fiber operators to be compact.

Theorem 2.1 completes the proof of Theorem 0.2 if we take
$$G = L_2(\Omega), \quad H(k) = \mathcal{M}(k), \quad H = \mathcal{M}, \quad f_n(k,\lambda) = (k+n)^2 - \varepsilon_0\mu_0\lambda^2,$$
use Lemma 2.3 in a neighborhood of each pair (k,λ) for which f_n does not vanish, and set
$$\mathcal{H}_j(k) = (\mathcal{M} - \lambda)H^2_{a_j}(k), \quad G_j(k) = H^1_{a_j}(k), \quad H^2_{-a_j}(k) \subset G_j^*(k) = H^{-1}_{-a_j}(k), \quad R_j = R_{\mathcal{M}}.$$

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