Intrinsic formulae for the Casimir operators of semidirect products of the exceptional Lie algebra G_2 and a Heisenberg Lie algebra

Rutwig Campoamor-Stursberg†

† Laboratoire de Mathématiques et Applications
Faculté de Sciences et Techniques
Université de Haute Alsace
4, rue des Frères Lumière
F-68093 Mulhouse (France)

E-mail: `R.Campoamor@uha.fr`

Abstract. We show that the Casimir operators of the semidirect products $G_2\overrightarrow{\oplus}_{2\Gamma_{(a,b)}\oplus\Gamma_{(0,0)}}\mathfrak{h}$ of the exceptional Lie algebra G_2 and a Heisenberg algebra $\mathfrak{h}$ can be constructed explicitly from the Casimir operators of G_2.

1. Introduction

Casimir operators of Lie algebras constitute an important tool in representation theory and physical applications, since their eigenvalues can be used to label irreducible representations, and, in the case of physical symmetry groups, provide quantum numbers. For the classical algebras, explicit formulae to obtain the Casimir operators and their eigenvalues were succesively developed by Racah, Perelomov and Popov or Gruber and O'Raifeartaigh, among others [1, 2, 3, 4, 5]. For nonsemisimple algebras, the problem is far from being solved, although various results have been obtained [6, 7]. In [8, 9] the Casimir operators of various semidirect products $\mathfrak{g}$ of classical Lie algebras and Heisenberg algebras were obtained by application of the Perelomov-Popov formulae. The main idea was to introduce new variables that span a semisimple algebra isomorphic to the Levi part of $\mathfrak{g}$, and then to apply the well known formulae for invariants of the classical algebras. Such products hae been shown to be relevant for various physical problems, like the nuclear collective motions [10, 11].
In this work we show that this argument can be enlarged to cover the exceptional (complex) Lie algebra G_2. The same procedure can be applied to the higher rank exceptional algebras, but the low dimension of G_2 and its particular properties make it the adecuate frame for computations. The interest of a synthetic procedure for these algebras is out of discussion, for the various physical applications of exceptional algebras, as the labelling of internal states, since they entered elementary particle physics in the beginning sixties ([12, 13] and references therein), and which constitute nowadays an important tool in high energy physics.

The functional method to determine the (generalized) Casimir invariants of a Lie algebra $\mathfrak{g}$ has become the most extended in the literature, and is more practical than the traditional method of analyzing the centre of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. For the study of completely integrable Hamiltonian systems, where the existence of Casimir operators is not ensured, it allows to determine those solutions which are not interpretable in terms of $\mathcal{U}(\mathfrak{g})$.
Recall that if $\{X_1, .., X_n\}$ is a basis of $\mathfrak{g}$ and $\left\{C_{ij}^k\right\}$ the structure constants over this basis, we can represent $\mathfrak{g}$ in the space $C^\infty(\mathfrak{g}^*)$ by means of differential operators:

$$\widehat{X}_i = -C_{ij}^k x_k \frac{\partial}{\partial x_j}, \qquad (1)$$

where $[X_i, X_j] = C_{ij}^k X_k \quad (1 \leq i < j \leq n)$. In this context, an analytic function $F \in C^\infty(\mathfrak{g}^*)$ is called an invariant of $\mathfrak{g}$ if and only if it is a solution of the system:

$$\left\{\widehat{X}_i F = 0,\ 1 \leq i \leq n\right\}. \qquad (2)$$

If F is a polynomial, then it provides a classical Casimir operator, after symmetrization, while nonpolynomial solutions of system (2) are usually called "generalized Casimir invariants". The cardinal $\mathcal{N}(\mathfrak{g})$ of a maximal set of functionally independent solutions (in terms of the brackets of the algebra $\mathfrak{g}$ over a given basis) is easily obtained from the classical criteria for PDEs:

$$\mathcal{N}(\mathfrak{g}) := \dim \mathfrak{g} - \operatorname{rank} \left(C_{ij}^{k} x_k\right)_{1\leq i<j\leq \dim \mathfrak{g}}, \tag{3}$$

where $A(\mathfrak{g}) := \left(C_{ij}^{k} x_k\right)$ is the matrix which represents the commutator table of $\mathfrak{g}$ over the basis $\{X_1, .., X_n\}$. Evidently this quantity constitutes an invariant of $\mathfrak{g}$ and does not depend on the choice of basis. In particular, if the Lie algebra $\mathfrak{g}$ is supposed to be perfect, i.e., such that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ holds, then we can always find a maximal set of algebraically independent Casimir operators [14]. For semisimple Lie algebras $\mathfrak{g}$, Gruber and O'Raifeartaigh [2] developed an effective procedure to determine the Casimir invariants as trace operators. Given the matrix $g_{\mu\nu}$ associated to the Killing form over a basis $\{X_\mu\}$, they constructed the matrices

$$Q = (g_{\mu\nu})^{-1} X_\mu \otimes x_\nu \tag{4}$$

The trace operator TrQ^p has the property of commuting with the basis elements of $\mathfrak{g}$:

$$[TrQ^p, X_\mu] = 0, \tag{5}$$

that is, TrQ^p is a Casimir operator of degree p. This method was applied in [15] to determine the sixth order Casimir operator of the exceptional algebra G_2 in dependence of a A_2-basis.

2. Generators of G_2 in a A_2 basis

The motivation to determine the invariants of exceptional Lie algebras lies in their applications to particle physics, which require explicit formulae that can be manipulated in effective manner. The Casimir operators of the exceptional algebra G_2 have been determined by various authors, using different bases, like $A_1 + A_1$, A_2 or B_3-bases [15, 16, 17]. For the semidirect products of G_2 with Heisenberg algebras analyzed in this work, it is convenient to use a A_2-basis. Recall that the adjoint representation $\Gamma_{(1,0)}$ of G_2 decomposes like follows with respect to the A_2-subalgebra spanned by the long roots:

$$\Gamma_{(1,0)} = 8 + 3 + \overline{3} \tag{6}$$

According to this decomposition, we label the generators as E_{ij}, a_k, b^l $(i, j, k, l = 1, 2, 3)$ (with the constraint $E_{11} + E_{22} + E_{33} = 0$). We have the brackets:

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj} \tag{7}$$

$$[E_{ij}, a_k] = \delta_{jk} a_i \tag{8}$$

$$\left[E_{ij}, b^k\right] = -\delta_{ik} b^j \tag{9}$$

$$[a_i, a_j] = -2\varepsilon_{ijk} b^k \tag{10}$$

$$\left[b^i, b^j\right] = 2\varepsilon_{ijk} a_k \tag{11}$$

$$\left[a_i, b^j\right] = 3E_{ij} \tag{12}$$

Table 1. Eigenvalues of $\mathfrak{H}$ over the basis (7)-(12)

X	E_{12}	E_{23}	E_{13}	E_{21}	E_{32}	E_{31}	a_1	a_2	a_3	b^1	b^2	b^3
$\lambda_1(X)$	3	3	0	−3	−3	0	1	−2	1	−1	2	−1
$\lambda_2(X)$	−1	2	1	1	−2	−1	0	1	−1	0	−1	1

Table 2. The fundamental 7-dimensional representation of G_2 in the A_2-basis

	V_1	V_2	V_3	V_4	V_5	V_6	V_7
H_1	V_1	$2V_2$	V_3	$-V_4$	$-2V_5$	$-V_6$	0
H_2	0	$-V_2$	$-V_3$	V_4	V_5	0	0
E_{12}	0	0	0	0	V_1	V_2	0
E_{21}	V_5	V_6	0	0	0	0	0
E_{23}	0	$-V_4$	V_5	0	0	0	0
E_{32}	0	0	0	$-V_2$	V_3	0	0
E_{13}	0	0	V_1	0	0	V_4	0
E_{31}	V_3	0	0	V_6	0	0	0
a_1	0	0	V_2	0	$-V_4$	$-V_7$	$2V_1$
a_2	V_4	V_7	V_6	0	0	0	$2V_5$
a_3	$-V_2$	0	0	V_7	$-V_6$	0	$2V_3$
b^1	V_7	V_3	0	$-V_5$	0	0	$-2V_6$
b^2	0	0	0	V_1	V_7	V_3	$2V_2$
b^3	0	$-V_2$	V_7	0	0	$-V_5$	$2V_4$

The A_2 subalgebra is clearly spanned by the operators E_{ij}. For our purpose it is convenient to choose the Cartan subalgebra generated by the operators $H_1 = E_{11} - 2E_{22} + E_{33}$ and $H_2 = E_{22} - E_{33}$. The operators $\{a_1, a_2, a_3\}$ correspond to the fundamental quark representation 3, while $\{b^1, b^2, b^3\}$ corresponds to the antiquark representation $\overline{3}$ (the action is given in table 1)

With respect to this basis, the quadratic invariant of G_2, obtained using the Q-matrix (4), is given by

$$C_2 = H_1^2 + 3H_1H_2 + 3H_2^2 + 3\left(E_{12}E_{21} + E_{23}E_{32} + E_{13}E_{31}\right) + a_i b^i \tag{13}$$

The corresponding symmetrization gives the quadratic Casimir operator. The operator of order six can be computed by the same method (see [15] for the explicit expression in a slightly different basis from that used here).

Here we are interested on the semidirect products $\mathfrak{g} = G_2 \overrightarrow{\oplus}_R \mathfrak{h}$ of the exceptional algebra G_2 with a Heisenberg Lie algebra $\mathfrak{h}_m$. As usual, we use the characterizing property of this algebra, namely being 2-step nilpotent of dimension $(2m+1)$ $(m \geq 1)$ and having a one dimensional centre with coincides with the derived subalgebra. This implies that we can always find a basis $\{V_1, .., V_{2m}, V_0\}$ of $\mathfrak{h}_m$ such that the only brackets are

$$[V_i, V_{2m+1-i}] = \lambda_i V_0 \tag{14}$$

with $\lambda_i \neq 0$. Since the centre of such an algebra is nontrivial [18], the representation R is of the form $R = \alpha_i \Gamma_{(a_i,b_i)} \oplus \Gamma_{(0,0)}$. Without loss of generality we can suppose that for any index i we have $(a_i, b_i) \neq (0,0)$. In what follows, we will consider representations R of the form $R = 2\Gamma_{(a,b)} \oplus \Gamma_{(0,0)}$. If $\{V_1, .., V_p\}$ is a basis of the (irreducible) representation $\Gamma_{(a,b)}$, we label the basis of R as $\left\{V_1^1, .., V_p^1, V_1^2, .., V_p^2, V_0\right\}$, where V_0 is a basis of the trivial representation of G_2.
The lowest possible dimension for such a representation is 15, $R = 2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}$, where $\Gamma_{(0,1)}$ denotes the fundamental seven dimensional irreducible representation of G_2 (the action is given in table 2). With the preceding labelling of the basis, the only nontrivial brackets of the radical are given by:

$$\left[V_1^1, V_6^2\right] = \left[V_5^2, V_2^1\right] = \left[V_4^2, V_3^1\right] = V_0 \tag{15}$$
$$\left[V_6^1, V_1^2\right] = \left[V_3^2, V_4^1\right] = \left[V_2^2, V_5^1\right] = V_0 \tag{16}$$
$$\left[V_7^1, V_7^2\right] = 2V_0. \tag{17}$$

The number of invariants of such algebras is easily obtained.

Lemma 1 *For any* $(a,b) \in \mathbb{N} \times \mathbb{N}$ *following identity holds:*

$$\mathcal{N}\left(G_2 \overrightarrow{\oplus}_{2\Gamma_{(a,b)} \oplus \Gamma_{(0,0)}} \mathfrak{h}_{7q}\right) = 3$$

The proof follows at once either directly from the commutator matrix or using the reformulation of the Beltrametti-Blasi in terms of differential forms developed in [19]. As expected, one of the invariants of $\mathfrak{g} = G_2 \overrightarrow{\oplus}_{2\Gamma_{(a,b)} \oplus \Gamma_{(0,0)}} \mathfrak{h}_m$ is given by the generator V_0 of the centre. The other two invariants of $\mathfrak{g}$ depend on all its variables, the generator of the centre comprised [18].

3. An auxiliary result concerning $\mathfrak{sl}(2)$-representations

Let $l \in \mathbb{N}$ and $\delta = 0, 1$. Let $\{2l - \delta\}$ be the irreducible representation of $\mathfrak{sl}(2) = \left\{X_1, X_2, X_3 \mid [X_2, X_3] = X_1,\ [X_1, X_i] = 2(-1)^i X_i, i = 2, 3\right\}$ of highest weight $\lambda = 2l - \delta$‡. Then, for any $l \geq 1$ we can obtain a Lie algebra $\mathfrak{g} = \mathfrak{sl}(2) \overrightarrow{\oplus}_R \mathfrak{r}$ with radical isomorphic to $\mathfrak{h}_{2l+1-\delta}$, the $(4l + 3 - 2\delta)$-dimensional Heisenberg algebra, and representation $R = 2\{2l - \delta\} \oplus \{0\}$. Over a basis $\left\{V_1^1, .., V_{2l+1-\delta}^1, V_1^2, .., V_{2l+1-\delta}^2, V_0\right\}$, the action of $\mathfrak{sl}(2)$ is given, for $i = 1, 2$, by

$$\left[X_1, V_k^i\right] = (2l + 2 - 2k - \delta) V_k^i,\ k = 1..2l + 1 - \delta \tag{18}$$
$$\left[X_2, V_k^i\right] = (2l + 2 - k - \delta) V_{k-1}^i,\ k = 2..2l + 1 - \delta \tag{19}$$
$$\left[X_3, V_k^i\right] = k V_{k+1}^i,\ k = 1..2l - \delta \tag{20}$$

Suppose moreover that the brackets of the radical are given by

$$\left[V_k^1, V_{2l+2-k-\delta}^2\right] = \lambda_k V_0,\ 1 \leq k \leq 2l + 1 - \delta, \tag{21}$$

‡ With this notation, the case $\delta = 0$ corresponds to integer spin representations, while $\delta = 1$ corresponds to half-integer spin representations. This notation has been chosen to simplify the formulae.

where $\lambda_k \neq 0$ for all k. Using the determinantal formulae developed in [18], their application to this particular case allows to express the noncentral Casimir invariant in dependence of the quadratic operator of $\mathfrak{sl}(2)$

Lemma 2 *The noncentral Casimir operator of* $\mathfrak{g} = \mathfrak{sl}(2)\overrightarrow{\oplus}_R \mathfrak{r}_{2l+1-\delta}$ *is given by* $C = P_1^2 + 4P_2P_3$*, where*

$$P_1 = x_1 v_0 + \sum_{k=1}^{2l+1-\delta} \frac{(2l+2-2k-\delta)}{\lambda_k} v_k^1 v_{2l+2-k-\delta}^2 \tag{22}$$

$$P_2 = x_2 v_0 - \sum_{k=1}^{2l-\delta} \frac{k}{\lambda_k} v_k^1 v_{2l+1-k-\delta}^2 \tag{23}$$

$$P_3 = x_3 v_0 + \sum_{k=1}^{2l-\delta} \frac{k}{\lambda_k} v_k^1 v_{2l+3-k-\delta}^2 \tag{24}$$

Obviously the formula remains valid for any even multiple of the irreducible representation $\{2l+1-\delta\}$ and their direct sums [20]. Therefore we conclude that the Casimir operators of Lie algebras $\mathfrak{sl}(2)\overrightarrow{\oplus}_{2W\oplus\{0\}}\mathfrak{h}$, where W is an arbitrary representation of $\mathfrak{sl}(2)$, can be written as function of the quadratic Casimir operator of the Levi part, and that the coefficients of the new variables P_i are completely determined by W (the signs of $v_i^1 v_j^2$ are however dependent on the basis chosen).

As an example, let us consider the representation $R = 4\{1\} \oplus \{0\}$, where the action of $\mathfrak{sl}(2)$ over the basis $\{V_1^1, .., V_4^1, V_1^2, .., V_2^2, V_0\}$ is given by the following table:

	V_1^1	V_2^1	V_3^1	V_4^1	V_1^2	V_2^2	V_3^2	V_4^2	V_0
X_1	V_1^1	$-V_2^1$	V_3^1	$-V_4^1$	V_1^2	$-V_2^2$	V_3^2	$-V_4^2$	0
X_2	0	$-V_1^1$	0	V_3^1	0	$-V_1^2$	0	V_3^2	0
X_3	$-V_2^1$	0	V_4^1	0	$-V_2^2$	0	V_4^2	0	0

(25)

Observe that the signs differ from those considered in (18)-(20). The brackets of the radical are

$$\left[V_k^1, V_{5-k}^2\right] = -V_0, \ k = 1..4 \tag{26}$$

The noncentral Casimir operator I can be computed using the determinantal formula of [18]. It can be verified that this invariant can be written as $I = (x_1')^2 + 4x_2'x_3'$, where

$$x_1' = x_1 v_0 + \left(-v_1^1 v_4^2 + v_4^1 v_1^2\right) + \left(v_2^1 v_3^2 - v_3^1 v_2^2\right) \tag{27}$$

$$x_2' = x_2 v_0 + v_1^1 v_3^2 - v_3^1 v_1^2 \tag{28}$$

$$x_3' = x_3 v_0 + v_2^1 v_4^2 - v_4^1 v_2^2. \tag{29}$$

In particular, we observe that the coefficients of $v_i^1 v_j^2$ and $v_j^1 v_i^2$ have opposite sign. This will always be the case when the $\mathfrak{sl}_2$-representation is obtained by restriction of a representation R of a simple algebra $\mathfrak{s}$ (with $\mathfrak{sl}_2 \hookrightarrow \mathfrak{s}$) which involves irreducible representations of $\mathfrak{s}$ along with their dual representations.

4. The invariants of $\mathfrak{g}$ as functions of the Casimir operators of G_2

In this section we present a procedure to determine the Casimir operators of algebras $\mathfrak{g} = G_2 \overrightarrow{\oplus}_{2\Gamma_{(a,b)}\oplus\Gamma_{(0,0)}} \mathfrak{h}_m$ using the invariants of G_2. The idea to obtain the Casimir operators is similar to the argument employed in [8] for the $wsu(n)$ and $wsp(n)$ Lie algebras, among others. That is, to find new variables E'_{ij}, a'_i, b'^j which transform like the generators (7)-(12) of G_2 and such that $C_2\left(E'_{ij}, a'_i, b'^j\right)$ and $C_6\left(E'_{ij}, a'_i, b'^j\right)$ are solutions of the system (2) corresponding to the Lie algebra $G_2 \overrightarrow{\oplus}_{2\Gamma_{(a,b)}\oplus\Gamma_{(0,0)}} \mathfrak{h}_m$, where $C_2\left(E_{ij}, a_i, b^j\right)$ and $C_6\left(E_{ij}, a_i, b^j\right)$ are the Casimir operators of G_2 of degrees two and six, respectively.
To this extent, we can suppose that the new variables X' associated to the basis (7)-(12) must be of the form

$$X' = Xv_0 + \alpha_X^{ij} v_i^1 v_j^2, \quad i,j > 0, \alpha_X^{ij} \in \mathbb{R}, \tag{30}$$

where $\left\{V_1^1, .., V_p^1, V_1^2, .., V_p^2, V_0\right\}$ is a basis of R. In other words, we consider X' as quadratic polynomials in the variables of $\mathfrak{g}$. This fact implies the following conditions on the products $v_i^1 v_j^2$:

Lemma 3 *Let $\mathfrak{H}$ be the Cartan subalgebra of G_2 with respect to the basis* $(7)-(12)$*. Then for any $H \in \mathfrak{H}$ and X' in (30) following identity holds:*

$$\lambda_H(X) = \lambda_H\left(v_i^1\right) + \lambda_H\left(v_j^2\right) \tag{31}$$

In particular, if $X' \in \mathfrak{H}$, then

$$\lambda_H(X) = \lambda_H\left(v_k^1\right) + \lambda_H\left(v_l^2\right) = 0 \tag{32}$$

The proof follows from the following facts: First, for any element $H \in \mathfrak{H}$ and any $X, Y \in \mathfrak{g}$ we have

$$[H, XY] = HXYH - XYH = X[H,Y] + [H,X]Y = (\lambda_H(X) + \lambda_H(Y))XY. \tag{33}$$

Since the radical of $\mathfrak{g}$ is a Heisenberg Lie algebra, it is an algebra spanned by m creation and annihilation operators $\left\{B_i, B_i^\dagger\right\}$, as well as a unit operator I corresponding to the centre generator V_0. Therefore there exists an isomorphism:

$$\sigma : \mathfrak{h}_m \rightarrow \text{span}\left\{B_i, B_i^\dagger, I\right\}_{1\leq i\leq m} \tag{34}$$

In view of equations (33), this implies that the operators

$$\widetilde{E}_{ij} = E_{ij} + \mu^{\kappa l}\sigma\left(v_k^1\right)\sigma\left(v_l^2\right) \tag{35}$$

$$\widetilde{a}_i = a_i + \nu^{jk}\sigma\left(v_j^1\right)\sigma\left(v_k^2\right) \tag{36}$$

$$\widetilde{b}^j = b^j + \chi^{\kappa l}\sigma\left(v_k^1\right)\sigma\left(v_l^2\right) \tag{37}$$

transform in the say way as the generators of G_2, i.e., they satisfy the relations $(7)-(12)$. Thus, in particular

$$\left[\widetilde{H}, \widetilde{X}\right] = \lambda_{\widetilde{H}}\left(\widetilde{X}\right)\widetilde{X} = \lambda_H(X)\widetilde{X}, \tag{38}$$

since the eigenvalues are preserved by (35) – (37). Thus the identities (31) – (32) are satisfied.
In particular, it follows that the coupling of the variables $v_i^1 v_j^2$ associated to the radical does not depend on the choice of a Cartan subalgebra of G_2, and can easily be obtained from the weight diagram of the representation.
In order to determine the coefficients α_X^{ij} for X', we will consider the restriction of the representation R to the different $\mathfrak{sl}_2$-triples of G_2. This gives Lie algebras with Heisenberg radical, whose noncentral Casimir operator can be computed easily. Once this operator has been obtained, the result of the preceding section shows that we can rewrite it as a polynomial function of the quadratic Casimir operator of $\mathfrak{sl}(2)$, from which the coefficients will be deduced rapidly.

Proposition 1 *For any $X \in G_2$, the coefficients α_X^{ij} of $X' = Xv_0 + \alpha_X^{ij} v_i^1 v_j^2$ are completely determined by the $\mathfrak{sl}_2$-triples generated by $\{E_{ij}, E_{ji}, [E_{ij}, E_{ji}]\}_{i\neq j}$ and $\{a_i, b^i, E_i\}$ and the restriction of R to them.*

Proof. For any of the $\mathfrak{sl}_2$-triples $\{H_2, E_{23}, E_{32}\}$, $\{H_1 + H_2, E_{12}, E_{21}\}$, $\{H_1 + 2H_2, E_{13}, E_{31}\}$, $\{2H_1 + H_2, a_1, b^1\}$, $\{H_1, a_2, b^2\}$, and $\{H_1 + 3H_2, a_3, b^3\}$, the restriction of $2\Gamma_{(a,b)} \oplus \Gamma_{(0,0)}$ gives a Lie algebra with Levi part $\mathfrak{sl}(2)$ and Heisenberg radical. By lemma 2, the noncentral Casimir operator of these algebras can be written in dependence of the quadratic Casimir operator of $\mathfrak{sl}(2)$, which provides the expressions of the new variables E'_{ij}, a'_i, b'^j. These expressions are completely determined by the restriction of R to the different $\mathfrak{sl}_2$-triples, therefore by R. It is straightforward to verify that these satisfy the requirements of lemma 3 and equations (31)-(32), thus transform in the same way as the generators of G_2. ■

Proposition 2 *Let $\mathfrak{g} = G_2 \overrightarrow{\oplus}_{2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}} \mathfrak{h}_7$. A fundamental set of invariants of $\mathfrak{g}$ is given by $C_2\left(E'_{ij}, a'_i, b'^j\right)$, $C_6\left(E'_{ij}, a'_i, b'^j\right)$ and $C_1 = v_0$.*

The proof follows directly by insertion of the functions $C_2\left(E'_{ij}, a'_i, b'^j\right)$ and $C_6\left(E'_{ij}, a'_i, b'^j\right)$ in the system (2), C_1 being obviously an invariant for being a generator of the centre. In particular, $C_2\left(E'_{ij}, a'_i, b'^j\right)$ is given by

$$(H'_1)^2 + 3H'_1H'_2 + 3(H'_2)^2 + 3\left(E'_{12}E'_{21} + E'_{23}E'_{32} + E'_{13}E'_{31}\right) + \left(a'_1 b'^1 + a'_2 b'^2 + a'_3 b'^3\right). \quad (39)$$

We obtain an invariant of degree four, and of degree twelve for $C_6\left(E'_{ij}, a'_i, b'^j\right)$.

These results enable us to propose a procedure to compute the invariants of $\mathfrak{g} = G_2 \overrightarrow{\oplus}_{2\Gamma_{(a,b)} \oplus \Gamma_{(0,0)}} \mathfrak{h}_m$ in dependence of the Casimir operators $C_2\left(E'_{ij}, a'_i, b'^j\right)$ and $C_6\left(E'_{ij}, a'_i, b'^j\right)$ of G_2:

(i) Determine the noncentral Casimir operator $C(\mathfrak{s}_i)$ of $\mathfrak{s}_i \overrightarrow{\oplus}_{(2\Gamma_{(a,b)} \oplus \Gamma_{(0,0)})|_{\mathfrak{s}_i}} \mathfrak{h}_m$ for the different $\mathfrak{sl}_2$-triples $\mathfrak{s}_i$ generated by $\{H_2, E_{23}, E_{32}\}$, $\{H_1 + H_2, E_{12}, E_{21}\}$, $\{H_1 + 2H_2, E_{13}, E_{31}\}$, $\{2H_1 + H_2, a_1, b^1\}$, $\{H_1, a_2, b^2\}$, and $\{H_1 + 3H_2, a_3, b^3\}$.

(ii) Express $C(\mathfrak{s}_i)$ as function of the quadratic invariant of $\mathfrak{sl}_2$ using lemma 2.

(iii) Substitute E_{ij}, a_i, b^i by $E'_{ij}, a'_i, (b^i)'$ in the operators $C_2\left(E'_{ij}, a'_i, b'^j\right)$ and $C_6\left(E'_{ij}, a'_i, b'^j\right)$ of G_2.

To illustrate this method, we consider the two lowest dimensional cases, corresponding to the representations $R = 2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}$ and $R = 2\Gamma_{(1,0)} \oplus \Gamma_{(0,0)}$ of G_2.

4.1. $R = 2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}$

In this case, the highest weight is given by $\lambda = (0,1)$. The action of G_2 is given by table 2, while the brackets of the radical are given by equations (15)-(17). From table 2 and lemma 2, it follows that, for example, the new variable H_2' takes the form

$$H_2' = H_2 v_0 + \alpha_2^{25} v_2^1 v_5^2 + \alpha_2^{34} v_3^1 v_4^2 + \alpha_2^{43} v_4^1 v_3^2 + \alpha_2^{52} v_5^1 v_2^2, \tag{40}$$

where $\{V_1^1, .., V_7^1, V_1^2, .., V_7^2, V_0\}$ is a basis of R. Now the generators $\{H_2, E_{23}, E_{32}\}$ of G_2 span a subalgebra $\mathfrak{s}_1$ of G_2 isomorphic to $\mathfrak{sl}(2)$ (whose quadratic Casimir operators over the given basis is $H_2^2 + 4E_{23}E_{32}$). The restriction of R to $\mathfrak{s}_1$ gives:

$$R|_{\mathfrak{s}_1} = 4\{1\} \oplus 7\{0\}, \tag{41}$$

(compare with the example in section 3). The noncentral Casimir operator of $\mathfrak{s}_1 \overrightarrow{\oplus}_R \mathfrak{h}_7$ follows from the determinantal formulae developed in [18], and equals

$$\begin{aligned} C = &\left(v_4^1 v_3^2\right)^2 + \left(v_3^1 v_4^2\right)^2 + \left(v_5^1 v_2^2\right)^2 + \left(v_2^1 v_5^2\right)^2 + 4v_0 \left(E_{23}\left(v_2^1 v_3^2 - v_3^1 v_2^2\right) + E_{32}\left(v_4^1 v_5^2 - v_5^1 v_4^2\right)\right) \\ &+2\left(v_2^1 v_3^1 v_4^2 v_5^2 + v_3^1 v_5^1 v_2^2 v_4^2 + v_4^1 v_5^1 v_2^2 v_3^2 + v_2^1 v_4^1 v_3^2 v_5^2 - 2v_2^1 v_5^1 v_3^2 v_4^2 - 2v_3^1 v_4^1 v_2^2 v_5^2\right) + 4v_0^2 E_{23}E_{32} \\ &-2\left(v_2^1 v_5^1 v_2^2 v_5^2 + v_3^1 v_4^1 v_3^2 v_4^2\right) + 2H_2 v_0 \left(v_2^1 v_5^2 - v_5^1 v_2^2 + v_3^1 v_4^2 - v_4^1 v_3^2\right) + \left(H_2 v_0\right)^2. \end{aligned} \tag{42}$$

We observe that C does not depend on the variables $\{v_1^i, v_6^i, v_7^i\}_{i=1,2}$, since the action of $\mathfrak{s}_1$ over these elements is trivial [18]. A short calculation shows that (42) can be written as

$$C = (H_2')^2 + 4(E_{23}' E_{32}'), \tag{43}$$

where

$$H_2' = H_2 v_0 + (v_2^1 v_5^2 - v_5^1 v_2^2) + (v_3^1 v_4^2 - v_4^1 v_3^2) \tag{44}$$

$$E_{23}' = E_{23} v_0 + (v_4^1 v_5^2 - v_5^1 v_4^2) \tag{45}$$

$$E_{32}' = E_{32} v_0 + (v_2^1 v_3^2 - v_3^1 v_2^2). \tag{46}$$

Repeating the argument for the triples generated respectively by $\{H_1 + H_2, E_{12}, E_{21}\}$, $\{H_1 + 2H_2, E_{13}, E_{31}\}$, $\{2H_1 + H_2, a_1, b^1\}$, $\{H_1, a_2, b^2\}$, and $\{H_1 + 3H_2, a_3, b^3\}$, we obtain the expression for the remaining generators of G_2:

$$H_1' = H_1 v_0 + v_1^1 v_6^2 - 2v_2^1 v_5^2 - v_3^1 v_4^2 + v_4^1 v_3^2 + 2v_5^1 v_2^2 - v_6^1 v_1^2 \tag{47}$$

$$E_{12}' = E_{12} v_0 - v_1^1 v_2^2 + v_2^1 v_1^2 \tag{48}$$

$$E_{21}' = E_{21} v_0 + v_5^1 v_6^2 - v_6^1 v_5^2 \tag{49}$$

$$E_{13}' = E_{13} v_0 - v_1^1 v_4^2 + v_4^1 v_1^2 \tag{50}$$

$$E_{31}' = E_{31} v_0 + v_3^1 v_6^2 - v_6^1 v_3^2 \tag{51}$$

Table 3. The adjoint representation of G_2 in the A_2-basis

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9	V_{10}	V_{11}	V_{12}	V_{13}	V_{14}
H_1	0	0	$2V_3$	$-2V_4$	$-3V_5$	$3V_6$	$-V_7$	V_8	V_9	$-V_{10}$	$3V_{11}$	$-3V_{12}$	0	0
H_2	0	0	$-V_3$	V_4	$2V_5$	$-2V_6$	V_7	$-V_8$	0	0	$-V_{11}$	V_{12}	V_{13}	$-V_{14}$
E_{12}	$-3V_{11}$	V_{11}	0	V_9	V_{13}	0	0	0	0	$-V_3$	0	W_1	0	$-V_6$
E_{21}	$3V_{12}$	$-V_{12}$	$-V_{10}$	0	0	$-V_{14}$	0	0	V_4	0	$-W_1$	0	V_5	0
E_{23}	$3V_5$	$-2V_5$	$-V_7$	0	0	V_2	0	V_4	0	0	$-V_{13}$	0	0	V_{12}
E_{32}	$-3V_6$	$2V_6$	0	V_8	$-V_2$	0	$-V_3$	0	0	0	0	V_{14}	$-V_{11}$	0
E_{13}	0	$-V_{13}$	0	0	0	V_{11}	0	V_9	0	$-V_7$	0	$-V_5$	0	W_2
E_{31}	0	V_{14}	0	0	$-V_{12}$	0	$-V_{10}$	0	V_8	0	V_6	0	$-W_2$	0
a_1	$-3V_9$	0	$3V_{11}$	$-2V_7$	0	0	$3V_{13}$	$2V_3$	0	W_3	0	$-V_4$	0	$-V_8$
a_2	$2V_4$	$-V_4$	$-V_1$	0	0	$-V_8$	$3V_5$	$-2V_{10}$	$2V_7$	$3V_{12}$	$-V_9$	0	0	0
a_3	$-3V_8$	V_8	$3V_6$	$2V_{10}$	$-V_4$	0	$-W_4$	0	$-2V_3$	$3V_{14}$	0	0	$-V_9$	0
b^1	V_{10}	0	$2V_8$	$-3V_{12}$	0	0	$-2V_4$	$-3V_{14}$	$-W_3$	0	V_3	0	V_7	0
b^2	$-2V_3$	V_3	0	V_1	V_7	0	$2V_9$	$-3V_6$	$-3V_{11}$	$-2V_8$	0	V_{10}	0	0
b^3	V_7	$-V_7$	$-2V_9$	$-3V_5$	0	V_3	0	W_4	$-3V_{13}$	V_4	0	0	0	V_{10}

$W_1 = V_1 + V_2, W_2 = V_1 + 2V_2, W_3 = 2V_1 + 3V_2, W_4 = V_1 + 3V_2$

$$a'_1 = a_1 v_0 + v_1^1 v_7^2 - v_2^1 v_4^2 + v_4^1 v_2^2 - v_7^1 v_1^2 \tag{52}$$
$$a'_2 = a_2 v_0 + v_4^1 v_6^2 + v_5^1 v_7^2 - v_6^1 v_4^2 - v_7^1 v_5^2 \tag{53}$$
$$a'_3 = a_3 v_0 - v_2^1 v_6^2 + v_3^1 v_7^2 + v_6^1 v_2^2 - v_7^1 v_3^2 \tag{54}$$
$$b'^1 = b^1 v_0 - v_3^1 v_5^2 + v_5^1 v_3^2 - v_6^1 v_7^2 + v_7^1 v_6^2 \tag{55}$$
$$b'^2 = b^2 v_0 - v_1^1 v_3^2 - v_7^1 v_2^2 + v_3^1 v_1^2 + v_2^1 v_7^2 \tag{56}$$
$$b'^3 = b^3 v_0 + v_1^1 v_5^2 + v_4^1 v_7^2 - v_5^1 v_1^2 - v_7^1 v_4^2 \tag{57}$$

The substitution of (44)-(57) into the corresponding system (2)§ provides a Casimir operator of the algebra $\mathfrak{g} = G_2 \overrightarrow{\oplus}_{2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}} \mathfrak{h}_7$ as function of the quadratic Casimir operator of G_2. $C_2\left(E'_{ij}, a'_i, b'^j\right)$ has 95 terms before symmetrization, while $C_6\left(E'_{ij}, a'_i, b'^j\right)$‖ involves some thousand terms, for which reason we omit its explicit description here.

4.2. $R = 2\Gamma_{(1,0)} \oplus \Gamma_{(0,0)}$

The Lie algebra $\mathfrak{g} = G_2 \overrightarrow{\oplus}_{2\Gamma_{(1,0)} \oplus \Gamma_{(0,0)}} \mathfrak{h}_{14}$ is 43-dimensional, where $\Gamma_{(1,0)}$ is the adjoint representation of G_2 (the action is given in table 3). Over the basis $\{V_1^1, .., V_{14}^1; V_1^2, .., V_{14}^2; V_0\}$ of the radical, the brackets are given by

$$\left[V_1^1, V_1^2\right] = 6V_0; \quad \left[V_2^1, V_2^2\right] = 2V_0 \tag{58}$$

§ This system has been presented explicitly in Appendix A.
‖ The sixth order Casimir operator C_6 of G_2 obtained in [15] has 416 terms.

$$\left[V_3^1, V_4^2\right] = \left[V_4^1, V_3^2\right] = -\left[V_1^1, V_2^2\right] = -\left[V_2^1, V_1^2\right] = 3V_0 \quad (59)$$

$$\left[V_7^1, V_8^1\right] = \left[V_8^1, V_7^2\right] = \left[V_9^1, V_{10}^2\right] = \left[V_{10}^1, V_9^2\right] = 3V_0 \quad (60)$$

$$\left[V_{11}^1, V_{12}^2\right] = \left[V_{12}^1, V_{11}^2\right] = \left[V_{13}^1, V_{14}^2\right] = \left[V_{14}^1, V_{13}^2\right] = \left[V_5^1, V_6^2\right] = \left[V_6^1, V_5^2\right] = V_0 \quad (61)$$

We observe that the element V_1^1 has nontrivial bracket with the elements V_1^2 and V_2^2. This fact is only a consequence of having chosen the basis of the radical such that it transforms according to the brackets given in table 3. In fact, taking the change of variables

$$\widetilde{V}_1^1 = V_1^1;\ \widetilde{V}_2^1 = V_2^1;\ \widetilde{V}_1^2 = V_1^2 + 2V_2^2;\ \widetilde{V}_2^2 = V_2^2 + \frac{2}{3}V_1^2 \quad (62)$$

(the rest of the basis remaining unchanged) we obtain that

$$\left[\widetilde{V}_1^1, \widetilde{V}_1^2\right] = \left[\widetilde{V}_2^1, \widetilde{V}_2^2\right] = 0 \quad (63)$$

$$\left[\widetilde{V}_1^1, \widetilde{V}_2^2\right] = \left[\widetilde{V}_2^1, \widetilde{V}_1^2\right] = V_0, \quad (64)$$

which shows that the radical is effectively a Heisenberg Lie algebra. We now take the $\mathfrak{sl}_2$-triple of G_2 generated by $\{H_2, E_{23}, E_{32}\}$ of G_2. The corresponding algebra has dimension 32. Using the determinantal formulae and lemma 2, we get the noncentral Casimir operator $C = (H_2')^2 + 4(E_{23}'E_{32}')$, where

$$H_2' = H_2v_0 + \frac{1}{3}(-v_3^1v_4^2 + v_4^1v_3^2 + v_7^1v_8^2 - v_8^1v_7^2) + 2(v_5^1v_6^2 - v_6^1v_5^2) - v_{11}^1v_{12}^2 + v_{12}^1v_{11}^2 + \quad (65)$$
$$+v_{13}^1v_{14}^2 - v_{14}^1v_{13}^2$$

$$E_{23}' = E_{23}v_0 + \frac{1}{3}(v_4^1v_7^2 - v_7^1v_4^2) + v_{12}^1v_{13}^2 - v_{13}^1v_{12}^2 - v_5^1v_2^2 + v_5^2v_2^1 \quad (66)$$

$$E_{32}' = E_{32}v_0 + \frac{1}{3}(-v_3^1v_8^2 + v_8^1v_3^2) - v_{11}^1v_{14}^2 + v_{14}^1v_{11}^2 + v_6^1v_2^2 - v_6^2v_2^1 \quad (67)$$

This gives us the expressions for three of the G_2-generators. Repeating the same process with the other triples, we obtain the new variables for the remaining elements of the A_2-basis:

$$H_1' = H_1v_0 + \frac{2}{3}(v_3^1v_4^2 - v_4^1v_3^2) + 3(-v_5^1v_6^2 + v_6^1v_5^2 + v_{11}^1v_{12}^2 - v_{12}^1v_{11}^2) + \quad (68)$$
$$\frac{1}{3}(-v_7^1v_8^2 + v_8^1v_7^2 + v_9^1v_{10}^2 - v_{10}^1v_9^2)$$

$$E_{12}' = E_{12}v_0 + \frac{1}{3}(-v_3^1v_9^2 + v_9^1v_3^2) - v_6^1v_{13}^2 + v_{13}^1v_6^2 - v_{11}^1(v_1^2 + v_2^2) + v_{11}^2(v_1^1 + v_2^1) \quad (69)$$

$$E_{21}' = E_{12}v_0 + \frac{1}{3}(v_4^1v_{10}^2 - v_{10}^1v_4^2) + v_5^1v_{14}^2 - v_{14}^1v_5^2 + v_{12}^1(v_1^2 + v_2^2) - v_{12}^2(v_1^1 + v_2^1) \quad (70)$$

$$E_{13}' = E_{13}v_0 + \frac{1}{3}(-v_7^1v_9^2 + v_9^1v_7^2) - v_5^1v_{11}^2 + v_{11}^1v_5^2 - v_{13}^1(v_1^2 + 2v_2^2) + v_{13}^2(v_1^1 + 2v_2^1) \quad (71)$$

$$E_{31}' = E_{31}v_0 + v_6^1v_{12}^2 - v_{12}^1v_6^2 + \frac{1}{3}(v_8^1v_{10}^2 - v_{10}^1v_8^2) + v_{14}^1(v_1^2 + 2v_2^2) - v_{14}^2(v_1^1 + 2v_2^1) \quad (72)$$

$$a_1' = a_1v_0 - v_4^1v_{11}^2 + v_{11}^1v_4^2 - v_8^1v_{13}^2 + v_{13}^1v_8^2 + \frac{2}{3}(v_3^1v_7^2 - v_7^1v_3^2) + \frac{2}{3}(-v_9^1v_1^2 + v_9^2v_1^1) + \quad (73)$$

$+v_9^2v_2^1 - v_9^1v_2^2$

$$a_2' = a_2v_0 + v_5^1v_8^2 - v_5^2v_8^1 + \frac{2}{3}(v_7^1v_{10}^2 - v_{10}^1v_7^2) - v_9^1v_{12}^2 + v_{12}^1v_9^2 + \frac{1}{3}(v_4^1v_1^2 - v_4^2v_1^1) \quad (74)$$

$$a_3' = a_3v_0 + \frac{2}{3}(v_3^1v_{10}^2 + v_{10}^1v_3^2) - v_4^1v_6^2 + v_6^1v_4^2 - v_9^1v_{14}^2 + v_{14}^1v_9^2 + \frac{1}{3}(v_8^1v_1^2 - v_8^1v_1^1) + \quad (75)$$

$+v_8^1v_2^2 - v_2^1v_8^2$

$$(b^1)' = b^1v_0 + v_3^1v_{12}^2 - v_{12}^1v_3^2 + \frac{2}{3}(-v_4^1v_8^2 + v_8^1v_4^2 + v_{10}^1v_1^2 - v_{10}^2v_1^1) + v_7^1v_{14}^2 - v_{14}^1v_7^2 + \quad (76)$$

$+v_{10}^1v_2^2 - v_{10}^2v_2^1$

$$(b^2)' = b^2v_0 - v_6^1v_7^2 + v_7^1v_6^2 + \frac{2}{3}(-v_8^1v_9^2 + v_9^1v_8^2) + v_{10}^1v_{11}^2 - v_{11}^1v_{10}^2 + \frac{1}{3}(-v_3^1v_1^2 + v_3^2v_1^1) \quad (77)$$

$$(b^3)' = b^3v_0 + v_3^1v_5^2 - v_5^1v_3^2 + \frac{2}{3}(v_4^1v_9^2 - v_9^1v_4^2) + v_{10}^1v_{13}^2 - v_{13}^1v_{10}^2 + \frac{1}{3}(v_7^1v_1^2 - v_7^2v_1^1) - \quad (78)$$

$-v_7^1v_2^2 + v_7^2v_2^1$

The substitution of (65)-(78) into (13) gives a degree four Casimir operator of the algebra.

5. Reduction to the A_2-subalgebra

Since the operators H_i, E_{ij} $(i \neq j)$ generate a copy of A_2, the restriction of R to this subalgebra gives a Lie algebra $\mathfrak{sl}(3)\overrightarrow{\oplus}_{R|}\mathfrak{h}_p$ with Heisenberg radical. Although this algebra is in general no more perfect, we can always find a fundamental set of invariants generated by Casimir operators. In fact, let $R = 2\Gamma_{(a,b)} \oplus \Gamma_{(0,0)}$ be a representation of G_2 and $\left\{V_1^1, .., V_p^1, V_1^2, .., V_p^2, V_0\right\}$ be a basis of R (where $p = \frac{\dim R-1}{2}$). Further let

$$\left[V_i^1, V_{p+1-i}^2\right] = \lambda_i V_0 \quad (79)$$

be the brackets of the Heisenberg radical. Suppose that the restriction of R to A_2 equals

$$R| = \sum W_i \oplus kW_0 \quad (80)$$

where W_i is a nontrivial A_2-representation and W_0 is the trivial representation. First of all, k must be odd. In fact, if V_j^1 is a noncentral element of $\mathfrak{sl}(3)\overrightarrow{\oplus}_{R|}\mathfrak{h}_p$ such that $\left[X, V_j^1\right] = 0$ for all $X \in A_2$, then the Jacobi identity applied to the triple $\left\{H, V_j^1, V_{p+1-j}^2\right\}$ (H an element of the Cartan subalgebra of A_2) shows that $\left[X, V_{p+1-j}^2\right] = 0$, $\forall X \in A_2$ as well. Thus $k = 2q+1$. Further, the corresponding equations $\widehat{V}_j^1$ and $\widehat{V}_{p+1-j}^2$ are:

$$\widehat{V}_j^1 = -\lambda_j v_0 \frac{\partial F}{\partial v_{p+1-j}^2} = 0 \quad (81)$$

$$\widehat{V}_{p+1-j}^2 = \lambda_j v_0 \frac{\partial F}{\partial v_j^1} = 0, \quad (82)$$

hence $\frac{\partial F}{\partial v_{p+1-j}^2} = \frac{\partial F}{\partial v_j^1} = 0$. That is, the invariants of $\mathfrak{sl}(3)\overrightarrow{\oplus}_{R|}\mathfrak{h}_p$ do not depend on these variables. As a consequence, the invariants of $\mathfrak{sl}(3)\overrightarrow{\oplus}_{R|}\mathfrak{h}_p$ are the same as those of the Lie algebra $\mathfrak{sl}(3)\overrightarrow{\oplus}_{\sum W_i \oplus W_0} h_t$ where $t = \frac{\dim R-1-2q}{2}$. Since this algebra is perfect, we can find a fundamental set of invariants formed by Casimir operators.

The preceding method shows that the Casimir invariants of $\mathfrak{sl}(3)\overrightarrow{\oplus}_{R|}\mathfrak{h}_p$ can also be determined using the expressions of the Casimir invariants of A_2. Over the basis (7)-(12), these are given by

$$I_2 = 3\left(H_1H_2 + E_{12}E_{21} + E_{13}E_{31} + E_{23}E_{32} + H_2^2\right) + H_1^2 \quad (83)$$

$$I_3 = 27\left(E_{21}E_{31}E_{23} + E_{12}E_{13}E_{32} + H_2E_{12}E_{21} - H_2E_{13}E_{31}\right) + 9H_1\left(E_{23}E_{32} + E_{12}E_{21} - 2E_{13}E_{31} + H_1H_2 + H_2^2\right) + 2H_1^3 \quad (84)$$

The substitution of the E'_{ij} into (83) and (84) provides the Casimir operators of the corresponding semidirect products.
For the two examples exhibited before, the restrictions of R give:

$$\left(2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}\right)|_{A_2} = 3 \oplus \overline{3} \oplus 1 \oplus 3 \oplus \overline{3} \oplus 1 \oplus 1 \quad (85)$$

$$\left(2\Gamma_{(1,0)} \oplus \Gamma_{(0,0)}\right)|_{A_2} = 8 \oplus 3 \oplus \overline{3} \oplus 8 \oplus 3 \oplus \overline{3} \oplus 1. \quad (86)$$

In the first case, the Casimir operators will be independent of V_7^1 and V_7^2, while in the second case they depend on all variables (including the generator of the centre).

6. Conclusions

We have seen that the Casimir operators of semidirect products $G_2\overrightarrow{\oplus}_{2\Gamma_{(a,b)}\oplus\Gamma_{(0,0)}}\mathfrak{h}$ of the exceptional Lie algebra G_2 of rank two and Heisenberg Lie algebras $\mathfrak{h}_m$ can be computed from the Casimir operators of G_2 by substituting the generators by new variables. These new variables are determined by the restriction of R to the different $\mathfrak{sl}_2$-triples, and can easily be deduced from the determinantal formulae of [18] and lemma 2. Indeed the method remains valid on different bases of G_2, but as shown in [17], the A_2 is highly convenient. This basis also allows us to determine the invariants of algebras with Heisenberg radical and Levi part isomorphic to A_2, which coincides with the results obtained in [8]. We could have also deduced the results using the reduction to the A_2-subalgebra of G_2, but the A_1-reduction turns out to be more practical, since the Casimir operators of the induced algebras can be computed without effort and presented in explicit form. For the case of G_2, the determination of the two noncentral Casimir operators of the semidirect product $G_2\overrightarrow{\oplus}_{2\Gamma_{(a,b)}\oplus\Gamma_{(0,0)}}\mathfrak{h}$ reduces to the computation of six determinants, corresponding to the different $\mathfrak{sl}_2$-triples, and which can be rewritten in terms of the $\mathfrak{sl}_2$-invariant. The advantage of this method in comparison with a direct integration of system (2) or other less direct reduction methods is considerable. With the method proposed here for G_2, the corresponding eigenvalues of the Casimir operators can also be deduced in closed form.
The same method should work for the remaining (complex) exceptional Lie algebras F_4, E_6, E_7 and E_8. Indeed lemma 3 remains valid for these algebras, and the reduction to the different $\mathfrak{sl}_2$-triples follows at once from their structure and the root theory.

The important question that arises at once from this method is how to generalize this to the real forms of the (exceptional) simple Lie algebras. For the corresponding normal real forms the procedure is formally the same, but for the remaining real forms,

Table 4. Commutator table for the real Lie algebra $\mathfrak{so}(3) \overrightarrow{\oplus}_{\Gamma^{\left(0,\frac{1}{2}\right),II}|\oplus\Gamma_{(0,0)}} \mathfrak{h}_2$

$[.]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}
X_1	0	$-X_3$	X_2	0	$-X_6$	X_5	$\frac{X_{10}}{2}$	$\frac{X_9}{2}$	$\frac{-X_8}{2}$	$\frac{-X_7}{2}$	0
X_2	X_3	0	$-X_1$	X_6	0	$-X_4$	$\frac{-X_8}{2}$	$\frac{X_7}{2}$	$\frac{-X_{10}}{2}$	$\frac{X_9}{2}$	0
X_3	$-X_2$	X_1	0	$-X_5$	X_4	0	$\frac{X_9}{2}$	$\frac{-X_{10}}{2}$	$\frac{-X_7}{2}$	$\frac{X_8}{2}$	0
X_4	0	$-X_6$	X_5	0	X_3	$-X_2$	$\frac{-X_8}{2}$	$\frac{-X_7}{2}$	$\frac{-X_{10}}{2}$	$\frac{-X_9}{2}$	0
X_5	X_6	0	$-X_4$	$-X_3$	0	X_1	$\frac{-X_{10}}{2}$	$\frac{X_9}{2}$	$\frac{X_8}{2}$	$\frac{-X_7}{2}$	0
X_6	$-X_5$	X_4	0	X_2	$-X_1$	0	$\frac{-X_7}{2}$	$\frac{X_8}{2}$	$\frac{-X_9}{2}$	$\frac{X_{10}}{2}$	0
X_7	$-\frac{X_{10}}{2}$	$\frac{X_8}{2}$	$\frac{-X_9}{2}$	$\frac{X_8}{2}$	$\frac{X_{10}}{2}$	$\frac{X_7}{2}$	0	X_{11}	0	0	0
X_8	$\frac{-X_9}{2}$	$\frac{-X_7}{2}$	$\frac{X_{10}}{2}$	$\frac{X_7}{2}$	$\frac{-X_9}{2}$	$\frac{-X_8}{2}$	$-X_{11}$	0	0	0	0
X_9	$\frac{X_8}{2}$	$\frac{X_{10}}{2}$	$\frac{X_7}{2}$	$\frac{X_{10}}{2}$	$\frac{-X_8}{2}$	$\frac{X_9}{2}$	0	0	0	$-X_{11}$	0
X_{10}	$\frac{X_7}{2}$	$\frac{-X_9}{2}$	$\frac{-X_8}{2}$	$\frac{X_9}{2}$	$\frac{X_7}{2}$	$\frac{-X_{10}}{2}$	0	0	X_{11}	0	0
X_{11}	0	0	0	0	0	0	0	0	0	0	0

lemma 3 will not be applicable in general, since it requires a diagonal action of the Cartan subalgebra. In this case, the $\mathfrak{sl}_2$-triples used for the complex case must be replaced either by its compact form $\mathfrak{so}(3)$ or some other semisimple algebra of higher rank. In this frame, the detailed description of subalgebras of semisimple Lie algebras are of great importance [21, 22]. As an example where the reduction to three dimensional simple subalgebras still works, take the eleven dimensional real Lie algebra $\mathfrak{so}(3,1) \overrightarrow{\oplus}_{\Gamma^{\left(0,\frac{1}{2}\right),II}\oplus\Gamma(0,0)} \mathfrak{h}_2$, where $\Gamma^{\left(0,\frac{1}{2}\right),II}$ is the "realification" of the two dimensional representation $\Gamma^{\left(0,\frac{1}{2}\right)}$¶ of the Lorentz algebra $\mathfrak{so}(3,1)$. Over a basis $\{X_1,..,X_{11}\}$, the brackets are given in table 4.

It is easily seen that this algebra satisfies $\mathcal{N}(\mathfrak{g}) = 3$, where $I = x_{11}$ is one of the invariants for generating the centre. To obtain the other two Casimir operators, we want to make use of the invariants of the Levi subalgebra, which are, over the given basis, given by

$$C_1 = x_1^2 + x_2^2 + x_3^3 - x_4^2 - x_5^2 - x_6^2 \tag{87}$$

$$C_2 = x_1x_4 + x_2x_5 + x_3x_6. \tag{88}$$

If we take the $\mathfrak{so}(3)$-subalgebra $\mathfrak{s}_1$ spanned by $\{X_1, X_2, X_3\}$, the induced algebra $\mathfrak{so}(3) \overrightarrow{\oplus}_{\Gamma^{\left(0,\frac{1}{2}\right),II|\mathfrak{s}_1}} \mathfrak{h}_2$ has a (noncentral) Casimir operator C which can be determined with the determinantal formulae of [18]. This determinant can be rewritten as

$$C = Y_1^2 + Y_2^2 + Y_3^2, \tag{89}$$

where

$$Y_1 = x_1x_{11} - \frac{1}{2}x_7x_9 + \frac{1}{2}x_8x_{10} \tag{90}$$

¶ Since the representation $\Gamma^{\left(0,\frac{1}{2}\right)}$ of $\mathfrak{so}(3,1)$ is complex, the corresponding real representation has double size, i.e., dimension four.

$$Y_2 = x_2x_{11} + \frac{1}{4}\left(x_9^2 + x_{10}^2 - x_7^2 - x_8^2\right) \tag{91}$$

$$Y_3 = x_3x_{11} + \frac{1}{2}x_8x_9 + \frac{1}{2}x_7x_{10}. \tag{92}$$

If we further consider the $\mathfrak{so}(3)$-subalgebras generated respectively by $\{X_1, X_5, X_6\}$, $\{X_2, X_4, X_6\}$ and $\{X_3, X_4, X_5\}$, the corresponding Casimir invariants of the induced algebras can also be written in terms of the $\mathfrak{so}(3)$-quadratic invariant. The computation provides the expressions

$$Y_4 = x_4x_{11} + \frac{1}{4}\left(x_7^2 - x_8^2 - x_9^2 + x_{10}^2\right) \tag{93}$$

$$Y_5 = x_5x_{11} - \frac{1}{2}x_7x_9 - \frac{1}{2}x_8x_{10} \tag{94}$$

$$Y_6 = x_6x_{11} - \frac{1}{2}x_7x_8 + \frac{1}{2}x_9x_{10}. \tag{95}$$

A straightforward computation shows that $Y_1^2 + Y_2^2 + Y_3^3 - Y_4^2 - Y_5^2 - Y_6^2$ and $Y_1Y_4 + Y_2Y_5 + Y_3Y_6$ are invariants of $\mathfrak{so}(3) \overrightarrow{\oplus}_{\Gamma\left(0,\frac{1}{2}\right),III|\oplus\Gamma_{(0,0)}} \mathfrak{h}_2$.

Therefore, the reduction method can also be used for real Lie algebras. The difficulty for the general case of real forms $\mathfrak{s}$ of a simple Lie algebra is to find the appropiate subalgebra to make the reduction and obtain the new variables corresponding to the generators of $\mathfrak{s}$.

Finally, we can ask whether the semidirect products of simple and Heisenberg Lie algebras is the only case where the Casimir operators can be described using the classical formulae for these invariants, or if other radicals are possible [6, 23] . Even if Heisenberg algebras occupy a privileged position within the possible candidates for radicals of semidirect products, due to the deep relation between their compatibility and the quaternionic, complex or real character of representations of simple Lie algebras, examples where the method is still applicable for solvable non-nilpotent radicals exist. To this extent consider the seven dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{\{1\}\oplus 2\{0\}} A_{4,9}^1$ given by the structure constants

$$C_{12}^2 = -C_{13}^3 = C_{67}^7 = 2 \tag{96}$$

$$C_{23}^1 = C_{14}^4 = C_{25}^4 = C_{34}^5 = C_{45}^6 = C_{47}^4 = C_{57}^5 = -C_{15}^5 = 1 \tag{97}$$

over a basis $\{X_1, .., X_7\}$. It is straighforward to verify that $\mathcal{N}\left(\mathfrak{sl}(2,\mathbb{R}) \overrightarrow{\oplus}_{\{1\}\oplus 2\{0\}} A_{4,9}^1\right) = 1$, and that this algebra does not admit a fundamental set of invariants generated by a Casimir operator, but a harmonic invariant. Introducing the "rational" variables

$$Y_1 := \frac{x_1x_6 + x_4x_5}{x_6} \tag{98}$$

$$Y_2 := \frac{2x_2x_6 - x_4^2}{2x_6} \tag{99}$$

$$Y_3 := \frac{2x_3x_6 + x_5^2}{2x_6}, \tag{100}$$

the function $Y_1^2 + 4Y_2Y_3$ turns out to be an invariant of the algebra. Observe that $x_1^2 + 4x_2x_3$ is the invariant of the Levi part.

Having in mind this example, one could be tempted to analyze the possibility of extending these methods to contractions of semidirect products $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_{R\oplus D_0}\mathfrak{h}$, since the contraction of the Casimir operators of $\mathfrak{g}$ provides invariants of the contraction [24]. However, in general these Casimir operators will not be expressible as a function of the invariants of the contracted algebra. With the notation of [25], the Lie algebra $L_{6,2}$ and its contraction $L_{5,1} \oplus L_0$ is the lowest dimensional example of this impossibility.

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Appendix A. The system for $R = 2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}$

In this appendix we present explicitely the system (2) for the Lie algebra $G_2 \overrightarrow{\oplus}_R \mathfrak{h}_7$, where $R = 2\Gamma_{(0,1)} \oplus \Gamma_{(0,0)}$. Since there is no ambiguity, we denote the variables associated to the generators E_{ij}, b^i, a_i of G_2 with the same symbol, while $\{V_1^1, .., V_7^1, V_1^2, .., V_7^2, V_0\}$ is a basis of R. Further, the symbol F_z denotes $\frac{\partial F}{\partial z}$.

$$2b^2F_{b^2} - 2a_2F_{a_2} - 3E_{23}F_{E_{23}} + 3E_{32}F_{E_{32}} - b^3F_{b^3} + a_3F_{a_3} + a_1F_{a_1} - b^1F_{b^1} + \tag{A.1}$$
$$+3E_{12}F_{E_{12}} - 3E_{21}F_{E_{21}} + v_1^1F_{v_1^1} + 2v_2^1F_{v_2^1} + v_3^1F_{H_17} - v_4^1F_{v_4^1} - 2v_5^1F_{v_5^1} - v_6^1F_{v_6^1} +$$
$$+v_1^2F_{v_1^2} + 2v_2^2F_{v_2^2} + v_3^2F_{v_3^2} - v_4^2F_{v_4^2} - 2v_5^2F_{v_5^2} - v_6^2F_{v_6^2} = 0$$
$$-b^2F_{b^2} + a_2F_{a_2} + 2E_{23}F_{E_{23}} - 2E_{32}F_{E_{32}} + b^3F_{b^3} - a_3F_{a_3} - E_{12}F_{E_{12}} + E_{21}F_{E_{21}} + \tag{A.2}$$
$$+E_{13}F_{E_{13}} - E_{31}F_{E_{31}} - v_2^1F_{v_2^1} - v_3^1F_{v_3^1} + v_4^1F_{v_4^1} + v_5^1F_{v_5^1} - v_2^2F_{v_2^2} - v_3^2F_{v_3^2} +$$
$$+v_4^2F_{v_4^2} + v_5^2F_{v_5^2} = 0$$
$$-2b^2F_{H_1} + b^2F_{H_2} + H_1F_{a_2} + b^3F_{E_{23}} + 2a_1F_{b^3} - 3E_{32}F_{a_3} - 3E_{12}F_{a_1} - 2a_3F_{b^1} + \tag{A.3}$$
$$+b^1F_{E_{21}} + v_1^1F_{v_4^1} + v_7^1F_{v_5^1} + v_3^1F_{v_6^1} + 2v_2^1F_{v_7^1} + v_1^2F_{v_4^2} + v_7^2F_{v_5^2} + v_3^2F_{v_6^2} + 2v_2^2F_{v_7^2} = 0$$
$$2a_2F_{H_1} - a_2F_{H_2} - H_1F_{b^2} - a_3F_{E_{32}} + 3E_{23}F_{b^3} - 2b^1F_{a_3} + 2b^3F_{a_1} + 3E_{21}F_{b^1} + \tag{A.4}$$
$$-a_1F_{E_{12}} + v_4^1F_{v_1^1} + v_7^1F_{v_2^1} + v_6^1F_{v_3^1} + 2v_5^1F_{v_7^1} + v_4^2F_{v_1^2} + v_7^2F_{v_2^2} + v_6^2F_{v_3^2} + 2v_5^2F_{v_7^2} = 0$$
$$3E_{23}F_{H_1} - 2E_{23}F_{H_2} - b^3F_{b^2} + H_2F_{E_{32}} + a_2F_{a_3} - E_{13}F_{E_{12}} + E_{21}F_{E_{31}} - v_4^1F_{v_2^1} + \tag{A.5}$$
$$+v_5^1F_{v_3^1} - v_4^2F_{v_2^2} + v_5^2F_{v_3^2} = 0$$
$$-3E_{32}F_{H_1} + 2E_{32}F_{H_2} + a_3F_{a_2} - H_2F_{E_{23}} - b^2F_{b^3} + E_{31}F_{E_{21}} - E_{12}F_{E_{13}} - v_2^1F_{v_4^1} + \tag{A.6}$$
$$+v_3^1F_{v_5^1} - v_2^2F_{v_4^2} + v_3^2F_{v_5^2} = 0$$
$$b^3F_{H_1} - b^3F_{H_2} - 2a_1F_{b^2} - 3E_{23}F_{a_2} + b^2F_{E_{32}} + (H_1 + 3H_2)F_{a_3} - 3H_13F_{a_1} + 2a_2F_{b^1} + \tag{A.7}$$
$$+b^1F_{E_{31}} - v_1^1F_{v_2^1} + v_7^1F_{v_3^1} - v_5^1F_{v_6^1} + 2v_4^1F_{v_7^1} - v_1^2F_{v_2^2} + v_7^2F_{v_3^2} - v_5^2F_{v_6^2} + 2v_4^2F_{v_7^2} = 0$$
$$-a_3F_{H_1} + a_3F_{H_2} + 3E_{32}F_{b^2} + 2b^1F_{a_2} - a_2F_{E_{23}} - (H_1 + 3H_2)F_{b^3} - 2b^2F_{a_1} + 3E_{31}F_{b^1} + \tag{A.8}$$
$$-a_1F_{E_{13}} - v_2^1F_{v_1^1} + v_7^1F_{v_4^1} - v_6^1F_{v_5^1} + 2v_3^1F_{v_7^1} - v_2^2F_{v_1^2} + v_7^2F_{v_4^2} - v_6^2F_{v_5^2} + 2v_3^2F_{v_7^2} = 0$$
$$-a_1F_{H_1} + 3E_{12}F_{b^2} - 2b^3F_{a_2} + 3E_{13}F_{b^3} + 2b^2F_{a_3} + (2H_1 + 3H_2)F_{b^1} - a_2F_{E_{21}} + \tag{A.9}$$
$$-a_3F_{E_{31}} + v_2^1F_{v_3^1} - v_4^1F_{v_5^1} - v_7^1F_{v_6^1} + 2v_1^1Fv_7^1 + v_2^2F_{v_3^2} - v_4^2F_{v_5^2} - v_7^2F_{v_6^2} + 2v_1^2F_{v_7^2} = 0$$
$$b^1F_{H_1} + 2a_3F_{b^2} - 3E_{21}F_{a_2} - 2a_2F_{b^3} - 3E_{31}F_{a_3} - (2H_1 + 3H_2)F_{a_1} + b^2F_{E_{12}} + \tag{A.10}$$
$$b^3F_{E_{13}} + v_7^1F_{v_1^1} + v_3^1F_{v_2^1} - v_5^1F_{v_4^1} - 2v_6^1F_{v_7^1} + v_7^2F_{v_1^2} + v_3^2F_{v_2^2} - v_5^2F_{v_4^2} - 2v_6^2F_{v_7^2} = 0$$
$$-3E_{12}F_{H_1} + E_{12}F_{H_2} + a_1F_{a_2} + E_{13}F_{E_{23}} - b^2F_{b^1} + (H_1 + H_2)F_{E_{21}} - E_{32}F_{E_{31}} + \tag{A.11}$$
$$+v_1^1F_{v_5^1} + v_2^1F_{v_6^1} + v_1^2F_{v_5^2} + v_2^2F_{v_6^2} = 0$$
$$3E_{21}F_{H_1} - E_{21}F_{H_2} - b^1F_{b^2} - E_{31}F_{E_{32}} + a_2F_{a_1} - (H_1 + H_2)F_{E_{12}} + E_{23}F_{E_{13}} + \tag{A.12}$$
$$+v_5^1F_{v_1^1} + v_6^1F_{v_2^1} + v_5^2F_{v_1^2} + v_6^2F_{v_2^2} = 0$$
$$-E_{13}F_{H_2} + E_{12}F_{E_{32}} + a_1F_{a_3} - b^3F_{b^1} - E_{23}F_{E_{21}} + (H_1 + 2H_2)F_{E_{31}} + v_1^1F_{v_3^1} + \tag{A.13}$$
$$+v_4^1F_{v_6^1} + v_1^2F_{v_3^2} + v_4^2F_{v_6^2} = 0$$
$$E_{31}F_{H_2} - E_{21}F_{E_{23}} - b^1F_{b^3} + a_3F_{a_1} + E_{32}F_{E_{12}} - (H_1 + 2H_2)F_{E_{13}} + v_3^1F_{v_1^1} + \tag{A.14}$$

$$+v_6^1 F_{v_4^1} + v_3^2 F_{v_1^2} + v_6^2 F_{v_4^2} = 0$$

$$-v_1^1 F_{H_1} - v_4^1 F_{a_2} + v_2^1 F_{a_3} - v_7^1 F_{b^1} - v_5^1 F_{E_{21}} - v_3^1 F_{E_{31}} + v_0 F_{v_6^2} = 0 \tag{A.15}$$

$$-2v_2^1 F_{H_1} + v_2^1 F_{H_2} - v_7^1 F_{a_2} + v_4^1 F_{E_{23}} + v_1^1 F_{b^3} - v_3^1 F_{b^1} - v_6^1 F_{E_{21}} - v_0 F_{v_5^2} = 0 \tag{A.16}$$

$$-v_3^1 F_{H_1} + v_3^1 F_{H_2} - v_6^1 F_{a_2} - v_5^1 F_{E_{23}} - v_7^1 F_{b^3} - v_2^1 F_{a_1} - v_1^1 F_{E_{13}} - v_0 F_{v_4^2} = 0 \tag{A.17}$$

$$v_4^1 F_{H_1} - v_4^1 F_{H_2} - v_1^1 F_{b^2} + v_2^1 F_{E_{32}} - v_7^1 F_{a_3} + v_5^1 F_{b^1} - v_6^1 F_{E_{31}} - v_0 F_{v_3^2} = 0 \tag{A.18}$$

$$2v_5^1 F_{H_1} - v_5^1 F_{H_2} - v_7^1 F_{b^2} - v_3^1 F_{E_{32}} + v_6^1 F_{a_3} + v_4^1 F_{a_1} - v_1^1 F_{E_{12}} - v_0 F_{v_2^2} = 0 \tag{A.19}$$

$$v_6^1 F_{H_1} - v_3^1 F_{b^2} + v_5^1 F_{b^3} + v_7^1 F_{a_1} - v_2^1 F_{E_{12}} - v_4^1 F_{E_{13}} + v_0 F_{v_1^2} = 0 \tag{A.20}$$

$$-2v_2^1 F_{b^2} - 2v_5^1 F_{a_2} - 2v_4^1 F_{b^3} - 2v_3^1 F_{a_3} - 2v_1^1 F_{a_1} + 2v_6^1 F_{b^1} + 2v_0 F_{v_7^2} = 0 \tag{A.21}$$

$$-v_1^2 F_{H_1} - v_4^2 F_{a_2} + v_2^2 F_{a_3} - v_7^2 F_{b^1} - v_5^2 F_{E_{21}} - v_3^2 F_{E_{31}} - v_0 F_{v_6^1} = 0 \tag{A.22}$$

$$-2v_2^2 F_{H_1} + v_2^2 F_{H_2} - v_7^2 F_{a_2} + v_4^2 F_{E_{23}} + v_1^2 F_{b^3} - v_3^2 F_{b^1} - v_6^2 F_{E_{21}} + v_0 F_{v_5^1} = 0 \tag{A.23}$$

$$-v_3^2 F_{H_1} + v_3^2 F_{H_2} - v_6^2 F_{a_2} - v_5^2 F_{E_{23}} - v_7^2 F_{b^3} - v_2^2 F_{a_1} - v_1^2 F_{E_{13}} + v_0 F_{v_4^1} = 0 \tag{A.24}$$

$$v_4^2 F_{H_1} - v_4^2 F_{H_2} - v_1^2 F_{b^2} + v_2^2 F_{E_{32}} - v_7^2 F_{a_3} + v_5^2 F_{b^1} - v_6^2 F_{E_{31}} + v_0 F_{v_3^1} = 0 \tag{A.25}$$

$$2v_5^2 F_{H_1} - v_5^2 F_{H_2} - v_7^2 F_{b^2} - v_3^2 F_{E_{32}} + v_6^2 F_{a_3} + v_4^2 F_{a_1} - v_1^2 F_{E_{12}} + v_0 F_{v_2^1} = 0 \tag{A.26}$$

$$v_6^2 F_{H_1} - v_3^2 F_{b^2} + v_5^2 F_{b^3} + v_7^2 F_{a_1} - v_2^2 F_{E_{12}} - v_4^2 F_{E_{13}} - v_0 F_{v_1^1} = 0 \tag{A.27}$$

$$-2v_2^2 F_{b^2} - 2v_5^2 F_{a_2} - 2v_4^2 F_{b^3} - 2v_3^2 F_{a_3} - 2v_1^2 F_{a_1} + 2v_6^2 F_{b^1} - 2v_0 F_{v_7^1} = 0 \tag{A.28}$$