Effective Dynamics of Magnetic Vortices

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Abstract

We study solutions of Ginzburg-Landau-type evolution equations (both dissipative and Hamiltonian) with initial data representing collections of widely-spaced vortices. We show that for long times, the solutions continue to describe collections of vortices, and we identify (to leading order in the vortex separation) the dynamical system describing the motion of the vortex centres (*effective dynamics*).

Contents

1 Introduction

In this paper we study effective dynamics of magnetic (Abrikosov) vortices in a macroscopic model of superconductivity, and of Nielsen-Olesen or Nambu strings in the Abelian Higgs model of particle physics. In both cases the equilibrium configurations are described by the Ginzburg-Landau equations:

$$\begin{aligned} -\Delta_A\psi &= \lambda(1-|\psi|^2)\psi \\ curl^2 A &= Im(\bar{\psi}\nabla_A\psi) \end{aligned} \tag{1}$$

where $(\psi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$, $\nabla_A = \nabla - iA$, and $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively. Equations (1) are the Euler-Lagrange equations for the Ginzburg-Landau energy functional

$$\mathcal{E}_{GL}(\psi, A) := \frac{1}{2}\int_{\mathbb{R}^2}\left\{|\nabla_A\psi|^2 + (curl A)^2 + \frac{\lambda}{2}(|\psi|^2-1)^2\right\}. \tag{2}$$

In the case of superconductivity, the function $\psi : \mathbb{R}^2 \to \mathbb{C}$ is called the *order parameter*; $|\psi|^2$ gives the density of superconducting electrons. The vector field $A : \mathbb{R}^2 \to \mathbb{R}^2$ is the magnetic potential. The r.h.s. of the equation for A is the superconducting current. In the case of particle physics, ψ and A are the

Higgs and Abelian gauge (electro-magnetic) fields, respectively. (See [R] for reviews, and [No] for historical and physics background.)

In addition to being translationally and rotationally invariant, equations (1) are invariant under gauge transformations:

$$(\psi, A) \mapsto (e^{i\chi}\psi, A + \nabla\chi)$$

for any $\chi : \mathbb{R}^2 \to \mathbb{R}$ (solutions are mapped to solutions under this transformation).

We consider various time-dependent versions of the Ginzburg-Landau equations (1). The first example is the gradient-flow equations

$$\begin{aligned} \partial_t\psi &= \Delta_A\psi + \lambda(1 - |\psi|^2)\psi \\ \partial_t A &= -curl^2 A + Im(\bar{\psi}\nabla_A\psi), \end{aligned} \tag{3}$$

a model in superconductivity theory ([GE, T]). We will refer to equations (3) as the *superconductor model* (they are sometimes called the *Gorkov-Eliashberg equations* or *time-dependent Ginzburg-Landau equations*).

The second example is

$$\begin{aligned} \partial_t^2\psi &= \Delta_A\psi + \lambda(1 - |\psi|^2)\psi \\ \partial_t^2 A &= -curl^2 A + Im(\bar{\psi}\nabla_A\psi), \end{aligned} \tag{4}$$

coupled (covariant) wave equations describing the $U(1)$-gauge Higgs model of elementary particle physics ([JT]) (written here in the *temporal gauge*). We will refer to equations (4) as the *Higgs model* (they are sometimes also called the *Maxwell-Higgs equations*).

The general framework we develop in this paper also applies to coupled (complex) Schrödinger and Maxwell equations

$$\begin{aligned} \gamma\partial_t\psi &= \Delta_A\psi + \lambda(1 - |\psi|^2)\psi \\ \partial_t^2 A &= -curl^2 A + Im(\bar{\psi}\nabla_A\psi) \end{aligned} \tag{5}$$

with $Re\gamma \geq 0$, or the Chern-Simons variant of these equations, though the implementation for $Re\gamma = 0$ requires some additional technical steps.

Finite energy states (ψ, A) are classified by the topological degree

$$deg(\psi) := deg\left(\left.\frac{\psi}{|\psi|}\right|_{|x|=R} \right),$$

where R is sufficiently large (the winding number of ψ at infinity). For each such state we have the quantization of magnetic flux:

$$\int_{\mathbb{R}^2} B = 2\pi deg(\psi) \in 2\pi\mathbb{Z},$$

where $B := curl A$ is the magnetic field associated with the vector potential A.

In each case the equations have "radially symmetric" (more precisely *equivariant*) solutions of the form

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a_n(r)\nabla(n\theta) \ , \tag{6}$$

where n is an integer and (r, θ) are the polar coordinates of $x \in \mathbb{R}^2$. As $r \to \infty$, $a_n(r)$ and $f_n(r)$ converge to 1 exponentially fast with the rates 1 and $m_\lambda := \min(\sqrt{2\lambda}, 2)$, respectively:

$$f_n(r) = 1 + O(e^{-m_\lambda r}) \quad \text{and} \quad a_n(r) = 1 + O(e^{-r}) \ .$$

At the origin, $f_n(r)$ vanishes like r^n and $a_n(r)$ like r^2. Hence $1 - f_n(r)$ and $1 - a_n(r)$ are well localized near the origin.

The pair $(\psi^{(n)}, A^{(n)})$ is called the *n-vortex* (*magnetic* or *Abrikosov* ([A, No]) in the case of superconductors, and *Nielsen-Olesen* or *Nambu string* in the particle physics case). Note that $deg(\psi^{(n)}) = n$. No other static solutions of the Ginzburg-Landau equations are rigorously known, though there is a physical argument and experimental evidence for the existence of vortex lattices – the Abrikosov lattices.

Observe that (in the present scaling) the length scale for the magnetic field (the *penetration depth*) is 1, and the length scale for the order parameter (the *coherence length*) is $\frac{1}{m_\lambda}$, where $m_\lambda = \min(\sqrt{2\lambda}, 2)$. More precisely, the following asymptotics for the field components of the n-vortex were established in [P] (see also [JT]): as $r := |x| \to \infty$,

$$\begin{array}{c} j^{(n)}(x) = n\beta_n K_1(|x|)[1 + o(e^{-m_\lambda r})]J\hat{x} \\ B^{(n)}(r) = n\beta_n K_1(r)[1 - \frac{1}{2r} + O(1/r^2)] \\ |1 - f_n(r)| \leq ce^{-m_\lambda r} \\ |f_n'(r)| \leq ce^{-m_\lambda r}. \end{array} \tag{7}$$

Here $j^{(n)} := Im(\overline{\psi^{(n)}}\nabla_{A^{(n)}}\psi^{(n)})$ is the n-vortex supercurrent, and $\beta_n > 0$ is a constant. K_1 is the modified Bessel function of order 1 of the second kind. Since $K_1(r)$ behaves like $ce^{-r}/\sqrt{r}$ for large r, we see that the length scale for

j and B is 1. Note that the two length length scales $1/m_\lambda$ and 1 coincide at $\lambda = 1/2$. Superconductors are referred to as *Type I* if $\lambda < 1/2$, and *Type II* if $\lambda > 1/2$.

Consider test functions describing several vortices, with the centers at points $z_1,\ z_2, \dots$ and with the degrees $n_1,\ n_2,\ \dots$, glued together. An example of such a function can be easily constructed as $v_{\underline{z},\chi} = (\psi_{\underline{z},\chi}, A_{\underline{z},\chi})$ with

$$\psi_{\underline{z},\chi}(x) = e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - z_j) \tag{8}$$

and

$$A_{\underline{z},\chi}(x) = \sum_{j=1}^{m} A^{(n_j)}(x - z_j) + \nabla\chi(x)\ , \tag{9}$$

where $\underline{z} = (z_1, z_2, \dots)$ and χ is an arbitrary real function yielding the gauge transformation (the integer degrees of the vortices, $\underline{n} = (n_1, \dots, n_m)$, are suppressed in the notation). Define the inter-vortex separation

$$R(\underline{z}) := \min_{j \neq k} |a_j - a_k|.$$

Since vortices are exponentially localized, for large separation $R(\underline{z})$ (compared with $[\min(m_\lambda, 1)]^{-1}$) such test functions are approximate – but not exact – solutions of the stationary Ginzburg-Landau equations.

When $\lambda > 1/2$, we take $n_j = \pm 1$, since vortices with $|n| \geq 2$ are known to be unstable ([GS]).

Now consider a time-dependent Ginzburg-Landau equation with an initial condition $v_{\underline{z}_0,\chi_0}$ and ask the following questions: does the solution at time t describe well-localized vortices at some locations $\underline{z} = \underline{z}(t)$ (and with a gauge transformation $\chi = \chi(t)$) and, if it does, what is the dynamic law of the vortex centers $\underline{z}(t)$ (and of $\chi(t)$)?

We describe here answers to these questions for the superconductor model (3) and Higgs model (4). Precise statements (Theorems 1 and 2) are given in Section 2.3.

Consider the superconductor model (3) with initial data (ψ_0, A_0) close to some $v_{\underline{z}_0,\chi_0}$ with $e^{-R(\underline{z}_0)}/\sqrt{R(\underline{z}_0)} < \epsilon$. We show that the solution can be written as

$$(\psi(t), A(t)) = v_{\underline{z}(t),\chi(t)} + O(\epsilon \log^{1/4}(1/\epsilon)) \tag{10}$$

and that the vortex dynamics is governed by the system

$$\gamma_{n_j}\dot{z}_j = -\nabla_{z_j} W(\underline{z}) + O(\epsilon^2 \log^{3/4}(1/\epsilon)). \tag{11}$$

Here $\dot{z}_j$ denotes dz_j/dt, $W(\underline{z}) := \mathcal{E}_{GL}(v_{\underline{z},\chi}) - \sum_{j=1}^m E^{(n_j)}$, where $E^{(n)} := \mathcal{E}_{GL}(\psi^{(n)}, A^{(n)})$, is the effective energy, and γ_n are the numbers given by

$$\gamma_n := \frac{1}{2}\|\nabla_{A^{(n)}}\psi^{(n)}\|_2^2 + \|curl A^{(n)}\|_2^2. \tag{12}$$

In general, these statements hold only as long as the path $\underline{z}(t)$ does not violate a condition of large separation: $R(\underline{z}(t)) > \log(1/\epsilon) + c$. In the *repulsive* case, when $\lambda > 1/2$ and $n_j = +1$ (or $n_j = -1$) for all j, the above statements hold for all time t. A precise statement is given in Theorem 1.

The leading-order term in the r.h.s of (11) is of order ϵ (see Lemma 11 and Remark 5). For $\lambda > 1/2$, the leading order of $W(\underline{z})$ for large $R(\underline{z})$ is:

$$W(\underline{z}) \sim \sum_{j \neq k} (const) n_j n_k \frac{e^{-|z_j - z_k|}}{\sqrt{|z_j - z_k|}}$$

(see Section 4.2).

For the Higgs model equations with initial data (ψ_0, A_0) close to some $v_{\underline{z}_0,\chi_0}$ (and with appropriate initial momenta), we show that

$$\|(\psi(t), A(t)) - v_{\underline{z}(t),\chi(t)}\|_{H^1} + \|(\partial_t\psi(t), \partial_t A(t)) - \partial_t v_{\underline{z}(t),\chi(t)}\|_{L^2} = o(\sqrt{\epsilon}) \tag{13}$$

with

$$\gamma_{n_j}\ddot{z}_j = -\nabla_{z_j} W(\underline{z}(t)) + o(\epsilon) \tag{14}$$

for times up to (approximately) order $\frac{1}{\sqrt{\epsilon}} \log\left(\frac{1}{\epsilon}\right)$. Here $\ddot{z}_j(t)$ denotes $d^2 z_j(t)/dt^2$. This result is stated precisely in Section 2.3 (see Theorem 2).

The resulting dynamics of vortices induced by the field dynamics of (ψ, A) is called the *effective dynamics*.

We now outline some previous works on vortex dynamics, including related works on the Gross-Pitaevski (or nonlinear Schrödinger) equation

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + \frac{1}{\epsilon^2}(|\psi|^2 - 1)\psi \tag{15}$$

in a bounded domain, used in the theory of superfluids (see [TT]). It is obtained from (5) by setting $\gamma = i$ and $A = 0$. The landmark previous developments are summarized in the table below

Type of Eqns / Type of Results	Superfluid	Superconductor	Higgs
Nonrigorous	Onsager ‘49	Perez-Rubinstein ’83 ($\lambda \gg \frac{1}{2}$) E ’84 ($\lambda \gg \frac{1}{2}$)	Manton ‘82 ($\lambda \approx \frac{1}{2}$)
Rigorous	Colliander-Jerrard ’00 F.-H. Lin-Xin ‘00	Demoulini-Stuart ’97 ($\lambda \approx \frac{1}{2}$)	Stuart ‘94 ($\lambda \approx \frac{1}{2}$)

In more detail, non-rigorous results for the Ginzburg-Landau equation (15) without the magnetic component, were obtained by L. Onsager ([O]), A. Fetter ([F]), R. Creswick and M. Morrison ([CM]), J. Neu ([N]), L.M. Pismen and D. Rodriguez ([PiR]), D. Rodriguez, L.M. Pismen and L. Sirovich ([PRS]), L.M. Pismen and J. Rubinstein ([PiR]), N. Ercolani and R. Montgomery ([EM]), W. E ([E]), Yu. Ovchinnikov and I.M. Sigal ([OS]).

Rigorous results are contained in J.E. Colliander and R.L. Jerrard ([CJ]), F.-H. Lin and J. Xin ([LX]), based on Bethuel, Brézis and Hélein ([BBH]). Let ψ^ϵ be the solution of Eqn (15) with a "low energy" initial condition. Then these papers show that as $\epsilon \to 0$, the "renormalized" energy density

$$\frac{1}{|\log \epsilon|}\left(\frac{1}{2}|\nabla\psi^\epsilon|^2 + \frac{1}{4\epsilon^2}(|\psi^\epsilon|^2-1)^2\right)$$

converges weakly to a sum of δ-functions located at points $\underline{z}(t) := \big(z_1(t), \ldots, z_k(t)\big)$ which solve the Hamiltonian equation $\dot{z} = J\nabla H(z)$ with appropriate initial conditions and Hamiltonian H. Also [CJ] prove the Bethuel-Brézis-Hélein type result $\forall \rho > 0$, as $\epsilon \to 0$

$$\min_{\alpha\in[0,2\pi]} ||\psi^\epsilon - e^{i\alpha} H_{\underline{Z}(t)}||_{H^1(T^2_\rho)} \to 0$$

where $H_{\underline{Z}}$ is the Bethuel-Brézis-Hélein canonical harmonic map with singularities at $z_1, \ldots, z_N$ and $T^2_\rho = T^2/\cup_i B_\rho(z_i)$, and [LX] show that the rescaled linear momentum $Im(\bar{\psi}_\epsilon \nabla\psi_\epsilon)$ converges (on the time-scale $O(1)$) to a solution of an incompressible Euler equation. The results above describe the dynamics of the vortex centers, but say nothing about the vortex structure of the solutions.

In the magnetic case non-rigorous results were obtained in N. Manton ([M]) ($\lambda \approx \frac{1}{2}$), M. Atiyah and N. Hitchin ([AH]) ($\lambda \approx \frac{1}{2}$), L. Perez and J. Rubinstein ([PR]), and W.E ([E]).

Rigorous results were obtained in D. Stuart ([S]) ($\lambda \approx \frac{1}{2}$), and S. Demoulini and D. Stuart ([DS]) ($\lambda \approx \frac{1}{2}$).

Finally, we mention the recent results [EW, IWW, ABF, AF, BF, CC, DeS, Pe, RSK, SW1, SW2, SW3, BP, BS, BJ, FTY, TY1, TY2, TY3, FGJS] on interface, bubble, spike, and soliton dynamics.

The rest of the paper is organized as follows. Ginzburg-Landau preliminaries are given in Sections 2.1 and 2.2. The effective dynamics results described above (Theorems 1 and 2) are stated precisely in Section 2.3. Theorem 2 is proved in Section 3.1, and Theorem 1 in Section 3.2. The key technical estimates used in the proofs are themselves proved in Section 4. Technical complications are relegated to appendices (Sections 5.1- 5.3).

Notation. Here, and in what follows, H^s denotes the Sobolev space $H^s(\mathbb{R}^2;\mathbb{C}\times\mathbb{R}^2)$ (same for L^2, etc.). For $\mu = (\phi,\alpha), \nu = (\chi,\beta) \in L^2$, $\langle \mu,\nu\rangle$ denotes the real L^2-inner product

$$\langle \mu,\nu\rangle := \int_{\mathbb{R}^2} \{Re(\overline{\mu}\nu) + \alpha\cdot\beta\}. \tag{16}$$

Moreover, we will use the same symbol to denote the real inner-product in $L^2\times L^2$: for $\xi = (\xi_1,\xi_2)$, $\eta = (\eta_1,\eta_2)$, we write

$$\langle \xi,\eta\rangle := \langle \xi_1,\eta_1\rangle + \langle \xi_2,\eta_2\rangle. \tag{17}$$

L^p-norms are denoted with a subscript p: $\|\cdot\|_p = \|\cdot\|_{L^p}$. The letter c will denote a generic constant, independent of any small parameters present, which may change from line to line.

2 Ginzburg-Landau preliminaries and results

2.1 Ginzburg-Landau equations

The Ginzburg-Landau energy functional $\mathcal{E}_{GL}$ (see (2)) is a smooth functional on the following affine space of configurations of degree n:

$$X^{(n)} := \{(\psi,A) : \mathbb{R}^2 \to \mathbb{C}\times\mathbb{R}^2 \ \mid\ (\psi,A) - (\psi^{(n)},A^{(n)}) \in H^1\}$$

where $(\psi^{(n)}, A^{(n)})$ is the exact n-vortex solution of the Ginzburg-Landau equations (see (2)). The variational derivative $\mathcal{E}'_{GL}(\psi, A)$ is the (negative of the) right hand side of the Ginzburg-Landau evolution equations (3) (or (4)).

With the notation $u = (\psi, A)$, the superconductor model equations (3) can be written as

$$\partial_t u(t) = -\mathcal{E}'_{GL}(u(t)).$$

We consider solutions of (3) satisfying $u = (\psi, A) \in C^1(\mathbb{R}^+; X^{(n)})$ (see [DS] for existence theory).

It is convenient to write the Higgs model equations (4) as a first-order Hamiltonian system. Introduce the momenta

$$(\pi(t), E(t)) := (-\partial_t \psi(t), -\partial_t A(t))$$

($E(t)$ is the electric field). The Hamiltonian is

$$\mathcal{H}(\psi, A, \pi, E) := \mathcal{E}_{GL}(\psi, A) + \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\pi|^2 + |E|^2 \right\}, \tag{18}$$

a smooth functional on the space $X^{(n)} \times L^2$. The space $X^{(n)} \times L^2$, viewed as a real space, admits the non-degenerate symplectic form

$$\omega(\xi, \eta) = \langle \xi, \mathbb{J}^{-1} \eta \rangle \tag{19}$$

where $\langle \cdot, \cdot \rangle$ is the real inner product on the tangent space to $X^{(n)} \times L^2$ defined in (17), and $\mathbb{J}$ is the symplectic operator

$$\mathbb{J} := \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

(in block notation). Setting $w := (\psi, A, \pi, E)$, the Higgs model (4) is equivalent to the equation

$$\partial_t w(t) = \mathbb{J}\mathcal{H}'(w(t)). \tag{20}$$

We consider solutions in the space $w \in C^1(\mathbb{R}; X^{(n)} \times L^2)$ which conserve the Hamiltonian functional $\mathcal{H}$ (see [BM] for existence theory).

2.2 Multi-vortex configurations

We begin by constructing a manifold of multi-vortex configurations, made up of collections of widely-spaced vortices "glued" together. Such a collection is determined by $m \in \mathbb{Z}^+$ vortex locations, $\underline{z} = (z_1, \ldots, z_m) \in \mathbb{R}^{2m}$ and m vortex degrees, $\underline{n} = (n_1, \ldots, n_m) \in \mathbb{Z}^m$, associated with these locations (the latter will often be suppressed in the notation), together with a gauge transformation. So the manifold we construct may be parameterized by a subset of $\mathbb{R}^{2m} \times$ {gauge transformations}.

Recall $(\psi^{(n)}, A^{(n)})$ denotes the equivariant, n-vortex static solution of the Ginzburg-Landau equations (see (6)). To a triple $\underline{z} \in \mathbb{R}^{2m}$, $\underline{n} \in \mathbb{Z}^m$, $\chi : \mathbb{R}^2 \to \mathbb{R}$, we associate the function

$$v_{\underline{z},\chi} := (\psi_{\underline{z},\chi}, A_{\underline{z},\chi}), \tag{21}$$

where

$$\psi_{\underline{z},\chi}(x) = e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - z_j)$$

and

$$A_{\underline{z},\chi}(x) = \sum_{j=1}^{m} A^{(n_j)}(x - z_j) + \nabla\chi(x).$$

Here $(n_1, \ldots, n_m)$ are the fixed topological degrees of the vortices, $n_j \in \mathbb{Z} \backslash \{0\}$. For given $\underline{z} \in \mathbb{R}^{2m}$, the gauge transformations will be of the form

$$\chi(x) = \sum_{j=1}^{m} z_j \cdot A^{(n_j)}(x - z_j) + \tilde{\chi}(x)$$

with $\tilde{\chi} \in H^2(\mathbb{R}^2; \mathbb{R})$. The gauge transformation is taken to be of this form to ensure that $v_{\underline{z},\chi}$ lies in $X^{(n)}$.

Given a vortex configuration $\underline{z} = (z_1, \ldots, z_m)$, the inter-vortex distance is defined to be

$$R(\underline{z}) := \min_{1 \leq j < k \leq m} |z_j - z_k|.$$

To ensure that our multi-vortex configurations are approximate solutions of the Ginzburg-Landau equations, the inter-vortex separation will be taken large.

In the Higgs model case, momenta must be included. To do this, we first introduce the "almost zero-modes". Define the gauge "almost zero-modes"

$$G_\gamma^{(\underline{z},\chi)} := \langle \gamma, \partial_\chi \rangle v_{\underline{z},\chi} \tag{22}$$

for $\gamma : \mathbb{R}^2 \to \mathbb{R}$, and the gauge-invariant translational "almost zero-modes"

$$T_{jk}^{(\underline{z},\chi)} := (\partial_{z_{jk}} + \langle A_k^{(n_j)}(\cdot - z_j), \partial_\chi \rangle) v_{\underline{z},\chi}. \tag{23}$$

From explicit expressions for $G_\gamma^{(\underline{z},\chi)}$ and $T_{jk}^{(\underline{z},\chi)}$ (see (44) and (45)), one can deduce that $G_\gamma^{(\underline{z},\chi)}, T_{jk}^{(\underline{z},\chi)} \in H^s$, provided $\gamma \in H^{s+1}$. Then for momentum parameters $\underline{p} = (p_1, \ldots, p_m) \in \mathbb{R}^{2m}$ and $\zeta \in H^1(\mathbb{R}^2; \mathbb{R})$, we define the (π, E) (momentum) component to be

$$\phi_{\underline{z},\chi,\underline{p},\zeta} := \sum_{j=1}^m p_j \cdot T_j^{(\underline{z},\chi)} + G_\zeta^{(\underline{z},\chi)} \in L^2. \tag{24}$$

We will often denote the full set of parameters by $\sigma := (\underline{z}, \chi, \underline{p}, \zeta)$ and $\phi_{\underline{z},\chi,\underline{p},\zeta}$ by ϕ_σ.

An important role will be played by the interaction energy of a multi-vortex configuration (see Section 4.2):

$$W(\underline{z}) := \mathcal{E}_{GL}(v_{\underline{z},\chi}) - \sum_{j=1}^m E^{(n_j)} \tag{25}$$

where, recall, $E^{(n)} := \sum_{j=1}^m \mathcal{E}_{GL}(\psi^{(n)}, A^{(n)})$. Due to the gauge invariance of $\mathcal{E}_{GL}$, this interaction energy is independent of the gauge transformation χ.

2.3 Main results

The main result in the superconductor model case is as follows:

Theorem 1 *Suppose $\lambda > 1/2$ and $n_j = +1$ (or $n_j = -1$) for $j = 1, \ldots, m$. There are $d_0, d, \epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ the following holds: let $(\psi(t), A(t))$ solve (3) with initial data satisfying*

$$\|(\psi_0, A_0) - v_{\underline{z}_0,\chi_0}\|_{H^1} < d_0 \epsilon \log^{1/4}(1/\epsilon)$$

with $e^{-R(\underline{z}_0)}/\sqrt{R(\underline{z}_0)} < d_0\epsilon$. Then for $t \geq 0$,

$$\|(\psi(t), A(t)) - v_{\underline{z}(t),\chi(t)}\|_{H^1} < d\epsilon \log^{1/4}(1/\epsilon)$$

for a path $v_{\underline{z}(t),\chi(t)} \in M_{as}$ satisfying

$$|\gamma_{n_j} \dot{z}_j(t) + \nabla_{z_j} W(\underline{z})| < d\epsilon^2 \log^{3/4}(1/\epsilon), \tag{26}$$

$$\|\partial_t \chi(t) - \sum_{j=1}^m \dot{z}_j(t) \cdot A^{(n_j)}(\cdot - z_j(t))\|_{H^1} < d\epsilon^2 \log^{3/4}(1/\epsilon).$$

Here γ_n is a positive constant, given explicitly in (12).

For the Higgs model equations, we have the following result:

Theorem 2 *Suppose $\lambda > 1/2$ and $n_j = +1$ (or $n_j = -1$) for $j = 1, \ldots, m$. Let $\alpha(\epsilon)$ be a function satisfying $\sqrt{\epsilon} < \alpha(\epsilon) << \log^{-1/4}(1/\epsilon)$. There are $d_0, d, \tau, \epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following holds: let $w(t) = (\psi(t), A(t), \pi(t), E(t))$ solve (20), with initial data satisfying*

$$\|(\psi_0, A_0) - v_{\underline{z}_0,\chi_0}\|_{H^1} + \|(\pi_0, E_0) - \phi_{\underline{z}_0,\chi_0,\underline{p}_0,\zeta_0}\|_2 < d_0\epsilon \log^{1/2}(1/\epsilon)$$

with $e^{-R(\underline{z}_0)}/\sqrt{R(\underline{z}_0)} + |\underline{p}_0|^2 + \|\zeta_0\|_2^2 < d_0\epsilon$. Then for $0 \le t \le \frac{\tau}{\sqrt{\epsilon}} \log\left(\frac{\alpha(\epsilon)}{\sqrt{\epsilon}}\right)$,

$$\|(\psi(t), A(t)) - v_{\underline{z}(t),\chi(t)}\|_{H^1} + \|(\pi(t), E(t)) - \phi_{\sigma(t)}\|_2 < d\alpha(\epsilon)\sqrt{\epsilon}\log^{1/2}(1/\epsilon) \quad (27)$$

for a path $\sigma(t) = (\underline{z}(t), \chi(t), \underline{p}(t), \zeta(t))$ satisfying, for all j,

$$|\dot{z}_j - p_j| + |\gamma_{n_j}\dot{p}_j + \nabla_{z_j}W(\underline{z})| < \epsilon\alpha(\epsilon)\log^{1/2}(1/\epsilon) = o(\epsilon)$$

$$\|\partial_t\chi - \sum_{j=1}^{m} \dot{z}_j \cdot A^{(n_j)}(x - z_j) - \zeta\|_{H^1} + \|\partial_t\zeta\|_{H^{1-s}} < \epsilon\alpha(\epsilon)\log^{1/2}(1/\epsilon) = o(\epsilon) \quad (28)$$

for any $s > 0$.

Remark 1 *The inter-vortex force is of size ϵ: $\nabla W(\underline{z}) = O(\epsilon)$ (see Lemma 11 and Remark 5).*

Remark 2 *The condition $\lambda > 1/2$ and $n_j = +1$ in Theorems 1 and 2 ensures that the inter-vortex interaction is repulsive, and therefore that the inter-vortex separation does not become too small in the given time interval. In fact the theorems apply, without these restrictions, for any initial vortex configuration whose evolution (namely (28) or (26)) preserves an appropriate large-separation condition. In the Type II case ($\lambda > 1/2$), this condition is $e^{-R(\underline{z}(t))}/\sqrt{R(\underline{z}(t)} < \epsilon$ (and $|\underline{p}(t)| + \|\zeta(t)\|_{L^2} < \epsilon$ in the Higgs model case). In the Type-I case ($\lambda < 1/2$), this condition must be appropriately modified (see Remark 5 of Section 4.2).*

Remark 3 *In Theorem 2, since $|\dot{z}_j| \le c\sqrt{\epsilon}$ over the time interval $0 \le t \le T = \frac{\tau}{\sqrt{\epsilon}} \log\left(\frac{\alpha}{\sqrt{\epsilon}}\right)$, vortices can move a distance*

$$\sqrt{\epsilon}T = \tau \log \frac{\alpha(\epsilon)}{\sqrt{\epsilon}} \sim R(\underline{z}(0)) >> 1.$$

3 Proofs

We start by proving Theorem 2 for the Higgs model in Section 3.1. The proof is considerably more involved than that of Theorem 1 for the superconductor model. The latter proof is sketched in Section 3.2.

The proof of Theorem 2 given in the following section is based on a series of propositions and lemmas. Propositions 1-3 summarize our geometric construction, and Lemmas 1-7, whose proofs are left to Section 4, provide the (elementary) analytic building blocks. Several technical lemmas are relegated to appendices.

3.1 Effective dynamics of vortices: Higgs model

In this section we prove Theorem 2. Let $w(t)$ solve (20) with $w \in C^1(\mathbb{R}; X^{(n)} \times L^2)$. In what follows, we denote

$$X := H^1(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \times L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2).$$

3.1.1 Manifold of multi-vortex configurations

We begin by defining the manifold of multi-vortex configurations. Let

$$\Sigma := \{(\underline{z}, \chi, \underline{p}, \zeta) \mid \underline{z} \in \mathbb{R}^{2m}, \chi - \underline{z} \cdot A_{\underline{z}} \in H^2(\mathbb{R}^2; \mathbb{R}), \underline{p} \in \mathbb{R}^{2m}, \zeta \in H^1(\mathbb{R}^2; \mathbb{R})\},$$

where $\underline{z} \cdot A_{\underline{z}} := \sum_{j=1}^m z_j \cdot A_j$ with $A_j(x) := A^{(n_j)}(x - z_j)$. This set is a manifold under the explicit parametrization map $\delta : Y_{2,1} \to \Sigma$ defined by

$$\delta : (\underline{z}, \tilde{\chi}, \underline{p}, \zeta) \mapsto (\underline{z}, \tilde{\chi} + \underline{z} \cdot A_{\underline{z}}, \underline{p}, \zeta). \tag{29}$$

Here

$$Y_{r,s} := \mathbb{R}^{2m} \times H^r(\mathbb{R}^2; \mathbb{R}) \times \mathbb{R}^{2m} \times H^s(\mathbb{R}^2; \mathbb{R}).$$

We define an open domain in Σ by

$$\Sigma_\epsilon := \{(\underline{z}, \chi, \underline{p}, \zeta) \in \Sigma \mid e^{-R(\underline{z})}/\sqrt{R(\underline{z})} < \epsilon,\ |\underline{p}| + \|\zeta\|_{H^1} < \sqrt{\epsilon}\}.$$

For each $\sigma := (\underline{z}, \chi, \underline{p}, \zeta) \in \Sigma$, introduce the multi-vortex configuration

$$w_\sigma := (v_{\underline{z},\chi}, \phi_\sigma) \in X^{(n)} \times L^2 \tag{30}$$

(recall $v_{\underline{Z},\chi}$ and ϕ_σ are defined in (21) and (24)). Finally, we define the space

$$M_{mv} := \{w_\sigma \mid \sigma \in \Sigma_\epsilon\} \subset X^{(n)} \times L^2.$$

The map $\gamma : \Sigma_\epsilon \to X^{(n)} \times L^2$ given by $\gamma : \sigma \to w_\sigma$, parameterizes M_{mv}, so that $M_{mv} = \gamma(\Sigma_\epsilon)$. It is easy to check that γ is C^1. It is shown in Section 5.1 that for all $\sigma \in \Sigma_\epsilon$, its Fréchet derivative $D\gamma(\sigma) : T_\sigma\Sigma_\epsilon \to X$ is one-to-one. Hence M_{mv} is a manifold.

For each $\sigma \in \Sigma_\epsilon$, the tangent space to M_{mv} at w_σ will be denoted by $T_{w_\sigma}M_{mv}$. It can be identified with a subspace of X; specifically, $T_{w_\sigma}M_{mv} = D\gamma(\sigma)(T_\sigma\Sigma_\epsilon)$.

For use in computations and estimates below, we introduce convenient bases in $T_\sigma\Sigma_\epsilon$ and $T_{w_\sigma}M_{mv}$. In terms of the coordinates in (29), the basis in $T_\sigma\Sigma_\epsilon$ is given by

$$\{\partial_{z_{ij}} + \langle z_i \cdot \partial_{x_j} A_i, \partial_{\tilde{\chi}}\rangle, \partial_{\tilde{\chi}(x)}, \partial_{p_{ij}}, \partial_{\zeta(x)}\}. \tag{31}$$

We denote the coordinates of $\sigma' \in T_\sigma\Sigma_\epsilon$ in this basis by $\sigma'_{coord} \in Y_{2,1}$. Define the map $\Gamma_\sigma : Y_{2,1} \to X$ by

$$\Gamma_\sigma \sigma'_{coord} := D\gamma(\sigma)\sigma'. \tag{32}$$

For $\sigma(t)$ a path in Σ_ϵ, this definition implies

$$\partial_t w_{\sigma(t)} = \Gamma_{\sigma(t)}\dot{\sigma}(t),$$

where $\dot{\sigma}(t)$ is the coordinate representation of the vector $\partial_t\sigma(t) \in T_{\sigma(t)}\Sigma_\epsilon$:

$$\dot{\sigma}(t) := (\dot{\underline{z}}(t), \partial_t^{\underline{Z}(t)}\chi(t), \dot{\underline{p}}(t), \partial_t\zeta(t)),$$

with $\dot{\underline{z}}(t) := d\underline{z}(t)/dt$, $\dot{\underline{p}}(t) := d\underline{p}(t)/dt$, and

$$\partial_t^{\underline{Z}(t)}\chi(x,t) := \partial_t\chi(x,t) - \sum_{j=1}^{m} \dot{z}_j(t) \cdot A^{(n_j)}(x - z_j(t)).$$

Let $\partial^A_{z_{ij}} := \partial_{z_{ij}} + \langle A_{ij}, \partial_\chi\rangle$. The basis for $T_{w_\sigma}M_{mv}$ (which is the image of the basis (31) under $D\gamma(\sigma)$) is given by:

$$\tau^z_{ij} := \partial^A_{z_{ij}} w_\sigma, \quad \tau^p_{ij} := \partial_{p_{ij}} w_\sigma, \quad \tau^\chi_x := \partial_{\chi(x)} w_\sigma, \quad \tau^\zeta_x := \partial_{\zeta(x)} w_\sigma. \tag{33}$$

Note that the tangent vector τ_{ij}^z is defined by differentiating w_σ *covariantly.* The point here is that $(\partial_z^A)^m w_\sigma$ lies in $H^1 \times L^2$ for any m, while $\partial_z w_\sigma$ does not. Explicit expressions for these tangent vectors are given in (105)- (108).

For a vector $\alpha \in \mathbb{R}^{2m}$, we will set $\alpha \cdot \tau^\# := \sum_{ij} \alpha_{ij} \tau_{ij}^\#$ for $\tau_{ij}^\# = \tau_{ij}^z, \tau_{ij}^p$, and for a function γ, set $\langle \gamma, \tau^\# \rangle := \int \gamma(x) \tau_x^\# dx$ for $\tau_x^\# = \tau_x^\chi, \tau_x^\zeta$. As a result of these definitions, and the relation

$$\begin{aligned} \partial_t &= \dot{\underline{z}} \cdot \partial_{\underline{z}} + \langle \partial_t \chi, \partial_\chi \rangle + \dot{\underline{p}} \cdot \partial_{\underline{p}} + \langle \partial_t \zeta, \partial_\zeta \rangle \\ &= \dot{\underline{z}} \cdot \partial_{\underline{z}}^A + < \partial_t^{\underline{z}(t)} \chi, \partial_\chi > + \dot{\underline{p}} \cdot \partial_{\underline{p}} + \langle \partial_t \zeta, \partial_\zeta \rangle, \end{aligned} \tag{34}$$

we have

$$\Gamma_\sigma \sigma'_{coord} = \underline{z}' \cdot \tau^z + \langle \chi', \tau^\chi \rangle + \underline{p}' \cdot \tau^p + \langle \zeta', \tau^\zeta \rangle, \tag{35}$$

where $\sigma'_{coord} = (\underline{z}', \chi', \underline{p}', \zeta') \in Y_{2,1}$.

In what follows, all of our computations are done in these bases, and we omit the subscript "coord" from the coordinate representation σ'_{coord} of a vector $\sigma' \in T_\sigma \Sigma_\epsilon$.

3.1.2 Reduced (vortex) Hamiltonian system

As was discussed above, the Maxwell-Higgs equations constitute a Hamiltonian system on the phase-space $X^{(n)} \times L^2$ with Hamiltonian (18). Our goal below is to project this Hamiltonian system onto the manifold M_{mv} (more precisely, onto TM_{mv}) with the smallest error possible. Below we describe an equivalent Hamiltonian structure on the parameter space $Y_{2,1}$ which is used in our analysis. We begin by setting

$$X_{r,s} := H^r(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \times H^s(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2)$$

(note that $X = X_{1,0}$). The operator Γ_σ has adjoint Λ_σ (with respect to the $\mathbb{R}^{2m} \times L^2 \times \mathbb{R}^{2m} \times L^2$ inner-product on $Y_{r,s}$, and the real $L^2 \times L^2$ inner-product on $X_{r,s}$) given by

$$\Lambda_\sigma : \xi \mapsto \langle D_\sigma w_\sigma, \xi \rangle \tag{36}$$

or, in our coordinates in $T_\sigma \Sigma_\epsilon$,

$$\Lambda_\sigma : \xi \mapsto (\langle \tau_{ij}^z, \xi \rangle, \langle \tau_x^\chi, \xi \rangle, \langle \tau_{ij}^p, \xi \rangle, \langle \tau_x^\zeta, \xi \rangle). \tag{37}$$

It is shown in Section 5.1 that Γ_σ and Λ_σ are bounded uniformly in $\sigma \in \Sigma_\epsilon$ between the following spaces:

$$\Gamma_\sigma : Y_{r,s} \to X_{r-1,s-1}, \tag{38}$$

for any r and s satisfying $min(r, 1) > s - 1$, and

$$\Lambda_\sigma : X_{r,s} \to Y_{r-1,s-1}, \tag{39}$$

for any r and s satisfying $min(s, 1) > r - 1$. (In the Physics literature the operators Γ_σ and Λ_σ are called bra and ket vectors, with notation $\Gamma_\sigma = |Dw_\sigma\rangle$ and $\Lambda_\sigma = \langle Dw_\sigma|.$)

Define the operators

$$V_\sigma := \Lambda_\sigma \mathbb{J}^{-1} \Gamma_\sigma : Y_{r,s} \to Y_{s-2,r-2}, \tag{40}$$

where $s - 1 < min(r, 2)$.

Relation (40) shows that $V_\sigma^* = -V_\sigma$ in the sense of the L^2 inner product. The operators V_σ define a symplectic form on $Y_{2,1}$ by

$$\omega_{red}(\sigma', \sigma'')(\sigma) := \langle \sigma', V_\sigma \sigma'' \rangle.$$

The non-degeneracy of this symplectic form follows from:

Proposition 1 (non-degeneracy of reduction) *For ϵ sufficiently small, and $\sigma \in \Sigma_\epsilon$, the operator V_σ is invertible.*

Proof: The invertibility of the operator V_σ for sufficiently small ϵ follows from the following expression, shown in Section 5.1:

$$V_\sigma = \begin{pmatrix} R_1 & -B \\ B^* & R_2 \end{pmatrix} \tag{41}$$

where

$$B = \begin{pmatrix} D & O(\epsilon \log^{1/2}(1/\epsilon)) \\ O(\epsilon \log^{1/2}(1/\epsilon)) & K \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & O(\sqrt{\epsilon}) \\ -O(\sqrt{\epsilon})^* & 0 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 0 & O(\epsilon \log^{1/2}(1/\epsilon)) \\ -O(\epsilon \log^{1/2}(1/\epsilon))^* & 0 \end{pmatrix},$$

$O(\sqrt{\epsilon})$ stands for an operator whose norm is bounded by $c\sqrt{\epsilon}$, D is a matrix of the form $D_{jk,lm} = \gamma_{n_j} \delta_{jl} \delta_{km} + O(\epsilon \log^{1/2}(1/\epsilon))$, and K is the operator $K := -\Delta + |\psi_{\underline{Z},\chi}|^2$. Since D and K are invertible, the operators U_σ and V_σ are obviously invertible if ϵ is sufficiently small. □

The symplectic form $\omega_{red}(\sigma', \sigma'')$ and the reduced (vortex) Hamiltonian $h(\sigma) := \mathcal{H}(w_\sigma)$ give a reduced Hamiltonian system on $Y_{2,1}$. The corresponding Hamiltonian equation is

$$\dot{\sigma} = V_\sigma^{-1} Dh(\sigma).$$

This equation will turn out to be the leading-order equation for the dynamics of the parameters σ. The next proposition computes the Hamiltonian $h(\sigma)$ explicitly.

Proposition 2 *If $e^{-R(\underline{z})}/\sqrt{R(\underline{z})} < \epsilon$, then*

$$\begin{aligned} h(\sigma) := \mathcal{H}(w_\sigma) = \sum_{j=1}^m E^{(n_j)} + W(\underline{z}) + \frac{1}{2}\sum_{j=1}^m \gamma_{n_j}|p_j|^2 + \frac{1}{2}\langle \zeta, K\zeta\rangle \\ + O(\epsilon \log^{1/2}(1/\epsilon)(|p|^2 + \|\zeta\|_2^2)) \end{aligned} \tag{42}$$

where, recall, $E^{(n)} := \mathcal{E}_{GL}(\psi^{(n)}, A^{(n)})$, $\gamma_n = \frac{1}{2}\|\nabla_{A^{(n)}}\psi^{(n)}\|_2^2 + \|curl A^{(n)}\|_2^2$, and $W(\underline{z})$ is defined in (25) (it is computed to leading order in Section 4.2). Further, we have

$$Dh(\sigma) := D_\sigma \mathcal{H}(w_\sigma) = (\nabla W(\underline{z}), 0, \underline{\gamma}\cdot \underline{p}, K\zeta) + O(\epsilon \log^{1/2}(1/\epsilon)(|p| + \|\zeta\|_2)), \tag{43}$$

where $\underline{\gamma}\cdot\underline{p}$ denotes $(\gamma_1 p_1, \ldots, \gamma_m p_m)$.

Proof. We begin with auxiliary computations establishing the approximate orthogonality of the tangent vectors introduced above. To this end, we record explicit expressions for $T_{jk}^{(\underline{Z},\chi)}$ and $G_\gamma^{(\underline{Z},\chi)}$, which follow readily from definitions (22) and (23):

$$T_{jk}^{(\underline{Z},\chi)} = -(e^{i\chi}[\prod_{l\neq j}\psi^{(n_l)}(x - z_l)](\nabla_{A_k}\psi)^{(n_j)}(x - z_j),\ B^{(n_j)}(x - z_j)e_k^\perp) \tag{44}$$

and

$$G_\gamma^{(\underline{Z},\chi)} = (i\gamma\psi_{\underline{Z},\chi},\ \nabla\gamma). \tag{45}$$

Here $B^{(n)} = \nabla \times A^{(n)}$ is the n-vortex magnetic field, $e_1^\perp := (0,1)$ and $e_2^\perp := (-1,0)$.

By the above explicit expressions, the exponential decay estimates (7), and Lemma 12, we see

$$|\langle T_{jr}^{(\underline{Z},\chi)}, T_{ks}^{(\underline{Z},\chi)}\rangle| \leq c\epsilon \log^{1/2}(1/\epsilon)$$

when $j \neq k$. When $j = k$, we compute

$$\langle T_{jr}^{(\underline{Z},\chi)}, T_{js}^{(\underline{Z},\chi)} \rangle = \langle (\nabla_{Ar}\psi)_j, (\nabla_{As}\psi)_j \rangle + \langle B_j J\hat{e}_r, B_j J\hat{e}_s \rangle + O(\epsilon),$$

and the leading term is easily computed to be $\gamma_{n_j}\delta_{rs}$ where γ_{n_j} is given in (12). Thus we have the approximate orthogonality relation

$$D_{jk,lm} := \langle T_{jk}^{(\underline{Z},\chi)}, T_{lm}^{(\underline{Z},\chi)} \rangle = \gamma_{n_j}\delta_{jl}\delta_{km} + O(\epsilon \log^{1/2}(1/\epsilon)). \tag{46}$$

A similar computation yields

$$|\langle T_{jk}^{(\underline{Z},\chi)}, G_{\gamma}^{(\underline{Z},\chi)} \rangle| \leq c\epsilon \log^{1/2}(1/\epsilon)\|\gamma\|_2. \tag{47}$$

Finally, the corresponding relation for the approximate gauge modes (see (22)) is

$$\langle G_{\gamma}^{(\underline{Z},\chi)}, G_{\zeta}^{(\underline{Z},\chi)} \rangle = \langle \gamma, (-\Delta + |\psi_{\underline{Z},\chi}|^2)\zeta \rangle, \tag{48}$$

a straightforward calculation.

Now using $w_\sigma = (v_{\underline{Z},\chi}, \phi_\sigma)$ (with ϕ_σ defined in (24)), and

$$\|\phi_\sigma\|_2^2 = p_{jk}p_{rs}\langle T_{jk}^{(\underline{Z},\chi)}, T_{rs}^{(\underline{Z},\chi)} \rangle + \langle G_{\zeta}^{(\underline{Z},\chi)}, G_{\zeta}^{(\underline{Z},\chi)} \rangle - 2p_{jk}\langle T_{jk}^{(\underline{Z},\chi)}, G_{\zeta}^{(\underline{Z},\chi)} \rangle$$

together with (46)-(48), we obtain (42) and (43). $\square$

3.1.3 Projections Q_σ

Here we construct operators Q_σ used to engineer a convenient splitting of (20). We define the operator $Q_\sigma : X \to T_{w_\sigma}M_{mv}$ as

$$Q_\sigma := \Gamma_\sigma V_\sigma^{-1}\Lambda_\sigma \mathbb{J}^{-1}. \tag{49}$$

Due to the expression for V_σ in (40), we see that Q_σ is a projection, $Q_\sigma^2 = Q_\sigma$, and it satisfies

$$KerQ_\sigma = (\mathbb{J}T_{w_\sigma}M_{mv})^\perp \tag{50}$$

and

$$Q_\sigma^* = -\mathbb{J}Q_\sigma\mathbb{J}. \tag{51}$$

Finally, we list two estimates which follow readily from the definitions above:

$$\|Q_\sigma\|_{X_{r,s}\to X_{r,s}} \leq c \tag{52}$$

for any r and s satisfying $s < min(r+1, 1)$, and for $\sigma = \sigma(t)$ a path in Σ_σ,

$$\|[Q_\sigma, \partial_t]\|_{X\to X} \leq c\|\dot{\sigma}\|_{Y_{1,0}} \tag{53}$$

where, recall, $\dot{\sigma} = (\dot{\underline{z}}, \partial_t^{\underline{Z}}\chi, \dot{\underline{p}}, \partial_t\zeta)$. To obtain this estimate one uses relation (34).

3.1.4 Splitting

The next proposition establishes a coordinate system (adapted to the projection Q_σ) on a tubular neighbourhood of M_{mv}. Let, for $0 < d_0 < 1$,

$$\Sigma_\epsilon^0 := \{(\underline{z}, \chi, \underline{p}, \zeta) \in \Sigma \mid e^{-R(\underline{Z})}/\sqrt{R(\underline{z})} < d_0\epsilon, |\underline{p}| + \|\zeta\|_{H^1} < d_0\sqrt{\epsilon}\}$$

(which parameterizes a manifold somewhat smaller than M_{mv}). Set

$$U_\delta := \{w \in X^{(n)} \times L^2 \mid \|w - w_\sigma\|_X < \delta, \text{ for some } \sigma \in \Sigma_\epsilon^0\}.$$

Proposition 3 (coordinates) *For ϵ sufficiently small, there is $\delta >> \epsilon$, and a C^1 map*

$$S : U_\delta \to \Sigma_\epsilon$$

satisfying $Q_{S(w)}(w - w_{S(w)}) = 0$ for $w \in U_\delta$. Moreover, $DS(w)$ is bounded uniformly in $w \in U_\delta$.

Proof: The proof is an application of the implicit function theorem. Define

$$g : U_\delta \times \Sigma_\epsilon^0 \to Y_{-1,0}$$

by

$$g(w, \sigma) := \Lambda_\sigma \mathbb{J}^{-1}(w - w_\sigma).$$

One can check that this is a C^1 map. Obviously, $g(w_\sigma, \sigma) = 0$. Note that, due to (32), $D_\sigma g(w_\sigma, \sigma) : Y_{2,1} \to Y_{-1,0}$ is given by

$$D_\sigma g(w_\sigma, \sigma) = -\Lambda_\sigma \mathbb{J}^{-1}\Gamma_\sigma = -V_\sigma.$$

which is invertible for $\sigma \in \Sigma_\epsilon$ with ϵ sufficiently small. So the implicit function theorem applies to provide a C^1 map $w \mapsto S(w)$ from an H^1-ball of size δ of a given $w_\sigma \in M_{mv}$ into Σ, satisfying $g(w, S(w)) = 0$. Allowing σ to vary in Σ_ϵ^0, we can construct such a ball about any such w_σ.

Using the definitions (32) and (36) of the operators Γ_σ and Λ_σ, the explicit expressions (105)- (110) for the basis $\{\tau^z, \tau^\chi, \tau^p, \tau^\zeta\}$, and expression (41), one can check the following: there is δ_0 independent of $\sigma \in \Sigma_\epsilon$ such that for all $w \in B_X(w_\sigma; \delta_0)$, the norms

$$\|V_\sigma^{-1}\|_{Y_{-1,0}\to Y_{2,1}}, \quad \|\Gamma_\sigma\|_{Y_{2,1}\to X_{1,0}}, \quad \|\Lambda_\sigma\|_{X_{0,1}\to Y_{-1,0}},$$
$$\|D_\sigma\Lambda_\sigma\|_{X_{0,1}\times Y_{2,1}\to Y_{-1,0}}, \quad \|D_\sigma\Gamma_\sigma\|_{Y_{2,1}\times Y_{2,1}\to X_{1,0}}, \quad \|D_\sigma^2\Lambda_\sigma\|_{Y_{2,1}\times Y_{2,1}\times X_{0,1}\to Y_{-1,0}}$$

are bounded uniformly in σ. This fact implies that the balls on which the maps S are defined can be taken to be of uniform size $\delta << \sqrt{\epsilon}$, which implies $S(w) \in \Sigma_\epsilon$. Thus we obtain a well-defined C^1 map $S : w \mapsto S(w)$ from the tubular neighborhood U_δ into Σ_ϵ, with $g(w, w_{S(w)}) = 0$. This map obviously satisfies also $Q_{S(w)}(w - w_{S(w)}) = 0$. The uniform boundedness of $DS(w)$ follows readily from the formula $DS(w) = -[D_\sigma g(w, S(w))]^{-1} D_w g(w, S(w))$ and the uniform estimates mentioned above. $\square$

Now suppose $w(t)$ solves the Higgs model equations (20) with initial data $w(0) = w_0$ as specified in Theorem 2. In particular, we have $w(0) \in U_\delta$. Let $0 < T_1 \le \infty$ be the time of first exit of $w(t)$ from U_δ. For $0 \le t < T_1$ we may write

$$w(t) = w_{\sigma(t)} + \xi(t) \tag{54}$$

with $w_{\sigma(t)} \in M_{mv}$, and $Q_{\sigma(t)}\xi(t) \equiv 0$ (by choosing $\sigma(t) = S(w(t))$). By our choice of initial data,

$$\|\xi(0)\|_X < c\|\xi_0\|_X < cd_0\epsilon \log^{1/4}(\epsilon), \tag{55}$$

where $\xi_0 := w(0) - w_{\sigma_0}$. Indeed, using (54) and the equation $w(0) = w_{\sigma_0} + \xi_0$, we find

$$\xi(0) = w_{\sigma_0} - w_{\sigma(0)} + \xi_0. \tag{56}$$

Next, since $\sigma(0) = S(w(0))$ and $\sigma_0 = S(w_{\sigma_0})$ (see Proposition 3), and since $w(0) - w_{\sigma_0} = \xi_0$, Proposition 3 gives

$$\|\sigma_0 - \sigma(0)\|_{Y_{2,1}} \le c\|\xi_0\|_X.$$

The last estimate, together with the estimate $\|D_\sigma w_\sigma\|_{Y_{2,1}\to X} \le c$ implies that

$$\|w_{\sigma_0} - w_{\sigma(0)}\|_X \le c\|\xi_0\|_X,$$

which, together with (56), yields (55).

3.1.5 Effective dynamics

Insert the decomposition (54) into the equations (20) and expand in a Taylor series to obtain

$$\partial_t w_\sigma + \partial_t \xi = \mathbb{J}[\mathcal{H}'(w_\sigma) + L_\sigma \xi + N_\sigma(\xi)] \tag{57}$$

where $L_\sigma := \mathcal{H}''(w_\sigma)$ is the Hessian of $\mathcal{H}$ at w_σ, and

$$N_\sigma(\xi) := \mathcal{H}'(w_\sigma + \xi) - \mathcal{H}'(w_\sigma) - L_\sigma \xi$$

consists of the terms nonlinear in ξ. Apply the projection Q_σ to (57) and use $Q_\sigma \partial_t w_\sigma = \partial_t w_\sigma$ (since $\partial_t w_\sigma \in T_{w_\sigma} M_{mv}$) to obtain

$$\partial_t w_\sigma - Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma) = Q_\sigma[\mathbb{J} L_\sigma \xi - \partial_t \xi + \mathbb{J} N_\sigma(\xi)]. \tag{58}$$

This equation governs the effective dynamics of the parameters $\sigma(t)$. The terms of leading order are on the left hand side. We now show, starting with the nonlinear term, that the right hand side is of lower order.

Lemma 1 (nonlinear estimate 1) *For $\sigma \in \Sigma_\epsilon$, and $\xi := (\xi_1, \xi_2) \in H^1 \times L^2$,*

$$N_\sigma(\xi) = \begin{pmatrix} (N_\sigma)_1(\xi_1) \\ 0 \end{pmatrix}$$

with

$$\|(N_\sigma)_1(\xi_1)\|_{H^{-s}} \leq c_s(\|\xi_1\|_{H^1}^2 + \|\xi_1\|_{H^1}^3)$$

for any $s > 0$.

This lemma is proved in Section 4.5. From now on, we fix $s > 0$ ($s = 1/2$, say). Thus by (52), we have

$$\|Q_\sigma \mathbb{J} N_\sigma(\xi)\|_{L^2 \times H^{-s}} \leq c(\|\xi\|_X^2 + \|\xi\|_X^3).$$

To minimize writing in the rest of this section, we make the additional assumption

$$\|\xi\|_X < 1, \tag{59}$$

which we shall justify later. Then the above estimate becomes

$$\|Q_\sigma \mathbb{J} N_\sigma(\xi)\|_{L^2 \times H^{-s}} \leq c\|\xi\|_X^2. \tag{60}$$

Using the fact that $Q_\sigma \xi \equiv 0$ and the bound (53), we have

$$\|Q_\sigma \partial_t \xi\|_X \leq c\|\dot{\sigma}\|_{Y_{1,0}} \|\xi\|_X. \tag{61}$$

To bound the remaining term on the right hand side of (58), we need the following lemma whose proof is given in Section 4.3.

Lemma 2 (approximate zero-modes) *For $\sigma \in \Sigma_\epsilon$ and any $\beta \in L^2 \times L^2$, we have*

$$\|L_\sigma Q_\sigma \beta\|_{L^2\times L^2} \le c\sqrt{\epsilon}\|\beta\|_{L^2\times L^2}. \tag{62}$$

Fix $\eta \in L^2 \times L^2$. Using the symmetry of L_σ, and (62), we have

$$\begin{aligned} |\langle \eta, Q_\sigma \mathbb{J} L_\sigma \xi\rangle| &= |\langle L_\sigma Q_\sigma \mathbb{J}\eta, \xi\rangle| \le \|\xi\|_{H^1\times L^2}\|L_\sigma Q_\sigma \mathbb{J}\eta\|_{H^{-1}\times L^2} \\ &\le c\sqrt{\epsilon}\|\xi\|_X\|\eta\|_{L^2\times L^2} \end{aligned}$$

and hence

$$\|Q_\sigma \mathbb{J} L_\sigma \xi\|_{L^2\times L^2} \le c\sqrt{\epsilon}\|\xi\|_X. \tag{63}$$

Collecting (60), (61), and (63), we obtain a bound on the right hand side of the effective dynamics law (58):

$$\|\partial_t w_\sigma - Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma)\|_{L^2\times H^{-s}} \le c(\sqrt{\epsilon} + \|\xi\|_X + \|\dot\sigma\|_{Y_{1,0}})\|\xi\|_X. \tag{64}$$

Finally, we translate (64) into parametric form, in order to remove $\dot\sigma$ from the r.h.s., and to see that it yields (28) in the leading order. For $\sigma \in C^1(\mathbb{R};\Sigma_\epsilon)$, we recall

$$\partial_t w_\sigma = \Gamma_\sigma \dot\sigma \tag{65}$$

where $\dot\sigma = (\dot z, \dot p, \partial_t^z \chi, \partial_t \zeta)$. Next, using (49), we find

$$Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma) = \Gamma_\sigma V_\sigma^{-1}\Lambda_\sigma \mathcal{H}'(w_\sigma). \tag{66}$$

Now the definition of Λ_σ, (36), implies that

$$\Lambda_\sigma \mathcal{H}'(w_\sigma) = D_\sigma \mathcal{H}(w_\sigma) = \partial_\sigma h(\sigma). \tag{67}$$

The last two equations yield

$$Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma) = \Gamma_\sigma V_\sigma^{-1} D_\sigma \mathcal{H}(w_\sigma). \tag{68}$$

Comparing (65) with (68), we obtain

$$\partial_t w_\sigma - Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma) = \Gamma_\sigma(\dot\sigma - V_\sigma^{-1}\partial_\sigma h(\sigma)).$$

Now the mapping properties (38) and (39), and the fact that $\Lambda_\sigma\Gamma_\sigma : Y_{1,1-s} \to Y_{-1,-s-1}$ is invertible, imply that

$$\|\dot\sigma - V_\sigma^{-1}\partial_\sigma h(\sigma)\|_{Y_{1,1-s}} \le c\|\partial_t w_\sigma - Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma)\|_{L^2\times H^{-s}}.$$

Since $Dh(\sigma) = \langle \mathcal{H}'(w_\sigma), D_\sigma w_\sigma \rangle = O(\epsilon^{1/2})$ (this estimate is part of Lemma 5 below), (64) implies that

$$\|\dot\sigma - V_\sigma^{-1} D_\sigma h(\sigma)\|_{Y_{1,1-s}} \le c(\sqrt{\epsilon} + \|\xi\|_X)\|\xi\|_X.$$

Now using (43) and (41), we arrive at

$$V_\sigma^{-1} D_\sigma h(\sigma) = (\underline{\mathrm{p}}, \zeta, -\underline{\gamma}^{-1} \cdot \partial_{\underline{\mathrm{z}}} W, 0) + O(\epsilon \log^{1/2}(1/\epsilon)(|\underline{\mathrm{p}}| + \|\zeta\|_2)),$$

where we have used the notation $\underline{\gamma}^{-1} \cdot \partial_{\underline{\mathrm{z}}} W = (\gamma_{n_1}^{-1}\partial_{z_1} W, \dots, \gamma_{n_m}^{-1}\partial_{z_m} W)$. Combine this relation with the estimate above to obtain finally

$$\sum_{j=1}^m |\dot z_j(t) - p_j(t)| + \sum_{j=1}^m |\dot p_j(t) + \gamma_{n_j}^{-1} \nabla_{z_j} W(\underline{\mathrm{z}}(t))| + \|\partial_t^{\mathrm{Z}} \chi(t) - \zeta(t)\|_{H^1} + \|\partial_t \zeta(t)\|_{H^{1-s}} \le c[(\sqrt{\epsilon} + \|\xi\|_X)\|\xi\|_X + \epsilon^{3/2}\log^{1/2}(1/\epsilon)]. \tag{69}$$

3.1.6 Energy estimates

Our remaining task is to control the remainder $\xi(t)$ for long times. The idea is similar to techniques used to prove orbital stability of solitary waves in Hamiltonian systems (see, eg, [W, GSS]): exploit conservation of energy – in this case *both* for the PDE (20) *and* for the leading order effective dynamics (28) – in order to control the fluctuations. We begin with a Taylor expansion of the Hamiltonian:

$$\mathcal{H}(w_\sigma + \xi) = \mathcal{H}(w_\sigma) + \langle \mathcal{H}'(w_\sigma), \xi \rangle + \frac{1}{2}\langle \xi, L_\sigma \xi \rangle + R_\sigma(\xi) \tag{70}$$

(this equation defines $R_\sigma(\xi)$). The following lemma, proved in Section 4.4, allows us to control ξ by the Hamiltonian.

Lemma 3 (coercivity) *For ϵ sufficiently small, $\sigma \in \Sigma_\epsilon$, and $\xi \in kerQ_\sigma$,*

$$\frac{1}{c}\|\xi\|_X \le \langle \xi, L_\sigma \xi \rangle \le c\|\xi\|_X^2.$$

Using this lemma, together with conservation of the Hamiltonian, in (70), we obtain

$$\begin{aligned} \|\xi\|_X^2 &\le c[\mathcal{H}(w(0)) - \mathcal{H}(w_\sigma) - \langle \mathcal{H}'(w_\sigma), \xi \rangle - R_\sigma(\xi)] \\ &= c[\mathcal{H}(w_{\sigma(0)}) - \mathcal{H}(w_\sigma) + \langle \mathcal{H}'(w_{\sigma(0)}), \xi(0) \rangle - \langle \mathcal{H}'(w_\sigma), \xi \rangle \\ &\quad + \frac{1}{2}\langle \xi(0), L_{\sigma(0)}\xi(0) \rangle + R_{\sigma(0)}(\xi(0)) - R_\sigma(\xi)]. \end{aligned} \tag{71}$$

The following lemma bounds the super-quadratic terms on the right-hand side of (71).

Lemma 4 (nonlinear estimate 2) *For* $\sigma \in \Sigma$,

$$|R_\sigma(\xi)| \le c(\|\xi\|_X^3 + \|\xi\|_X^4).$$

This is proved is Section 4.5. To control the terms linear in ξ, we need another key lemma:

Lemma 5 (approximate solution properties) *For* $\sigma \in \Sigma_\epsilon$, *we have*

1. $\|\mathcal{H}'(w_\sigma)\|_{H^1 \times L^2} \le c\sqrt{\epsilon}$
2. $\|[\mathcal{H}'(w_\sigma)]_1\|_{H^1} = \|\mathcal{E}'_{GL}(v_{\underline{Z},X})\|_{H^1} \le c\epsilon \log^{1/4}(1/\epsilon)$
3. $\|\bar{Q}_\sigma \mathbb{J}\mathcal{H}'(w_\sigma)\|_{L^2 \times L^2} \le c\epsilon \log^{1/4}(1/\epsilon)$, *where* $\bar{Q}_\sigma := \mathbf{1} - Q_\sigma$.

This lemma is proved in Section 4.1. Using the third statement of the lemma and (51), we find

$$\begin{aligned} |\langle \mathcal{H}'(w_\sigma), \xi\rangle| &= |\langle \mathcal{H}'(w_\sigma), \bar{Q}_\sigma \xi\rangle| \\ &= |\langle \mathbb{J}\bar{Q}_\sigma \mathbb{J}\mathcal{H}'(w_\sigma), \xi\rangle| \\ &\le \|\bar{Q}_\sigma \mathbb{J}\mathcal{H}'(w_\sigma)\|_{L^2 \times L^2} \|\xi\|_X \\ &\le c\epsilon \log^{1/4}(1/\epsilon)\|\xi\|_X. \end{aligned} \tag{72}$$

Collecting Lemma 3, Lemma 4, and (72) in (71) (and remembering the intermediate assumption $\|\xi\|_X < 1$), we obtain

$$\begin{aligned} \|\xi\|_X^2 \le c[&\mathcal{H}(w_{\sigma(0)}) - \mathcal{H}(w_\sigma) + (\epsilon \log^{1/4}(1/\epsilon) + \|\xi\|_X^2)\|\xi\|_X \\ &+ (\epsilon \log^{1/4}(1/\epsilon) + \|\xi(0)\|_X)\|\xi(0)\|_X]. \end{aligned} \tag{73}$$

3.1.7 Approximate conservation of the reduced energy, $\mathcal{H}(w_\sigma)$

It remains to control $\mathcal{H}(w_{\sigma(0)}) - \mathcal{H}(w_\sigma)$. The estimate below involves a delicate estimate of the contribution of the nonlinear terms.

Proposition 4 *Let* $M(t) := \sup_{0 \le s \le t} \|\xi(s)\|_X$. *Then*

$$|\mathcal{H}(v_{\sigma(0)}) - \mathcal{H}(v_{\sigma(t)})| \le ct\sqrt{\epsilon}M(t)(\epsilon \log^{1/2}(1/\epsilon) + M(t)) + \sqrt{\epsilon}M^2(t). \tag{74}$$

Proof: First, we differentiate in time and use the effective dynamics law (58):

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(w_\sigma) &= \langle \mathcal{H}'(w_\sigma), \partial_t w_\sigma\rangle \\ &= \langle \mathcal{H}'(w_\sigma), Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma) + Q_\sigma[\mathbb{J}L_\sigma\xi - \partial_t\xi] + Q_\sigma\mathbb{J}N_\sigma(\xi)\rangle \\ &= \langle \mathcal{H}'(w_\sigma), Q_\sigma[\mathbb{J}L_\sigma\xi - \partial_t\xi]\rangle + \langle \mathcal{H}'(w_\sigma), Q_\sigma\mathbb{J}N_\sigma(\xi)\rangle\end{aligned} \tag{75}$$

where we have used the fact that $(Q_\sigma\mathbb{J})^* = -Q_\sigma\mathbb{J}$.

We start by estimating the first inner-product on the right-hand side. First we exploit the fact – Lemma 5 part 2 – that the first component of $\mathcal{H}'(w_\sigma)$ is smaller than the second: using $Q_\sigma\partial_t\xi = [Q_\sigma, \partial_t]\xi$, (53) and (63), we have

$$|\langle[\mathcal{H}'(w_\sigma)]_1, [Q_\sigma(\mathbb{J}L_\sigma\xi - \partial_t\xi)]_1\rangle| \le c\epsilon\log^{1/4}(1/\epsilon)\|\xi\|_X(\sqrt{\epsilon} + \|\dot\sigma\|_{Y_{1,0}}).$$

Combined with (69), this yields

$$|\langle[\mathcal{H}'(w_\sigma)]_1, [Q_\sigma(\mathbb{J}L_\sigma\xi - \partial_t\xi)]_1\rangle| \le c\epsilon^{3/2}\log^{1/4}(1/\epsilon)\|\xi\|_X.$$

To deal with the second component, we have to exploit a key cancellation. This is expressed in the following lemma, which can be considered a refinement of both (61) and of (63).

Lemma 6 *For $\sigma \in C^1(\mathbb{R}; \Sigma_\epsilon)$ and $Q_\sigma\xi \equiv 0$, we have*

$$\begin{aligned}|\langle[\mathcal{H}'(w_\sigma)]_2, &[Q_\sigma(\mathbb{J}L_\sigma\xi - \partial_t\xi)]_2\rangle| \le c\sqrt{\epsilon}[\epsilon\log^{1/2}(1/\epsilon) + |\dot{\underline{p}} + \nabla_z W(\underline{z})| \\ &+ \|\partial_t\zeta\|_2 + \sqrt{\epsilon}(|\dot{\underline{z}} - \underline{p}| + \|\partial_t^{\underline{z}}\chi - \zeta\|_{H^1})]\|\xi\|_X.\end{aligned}$$

This lemma is proved in Section 5.2. Combining the above estimates with (69) yields

$$|\langle\mathcal{H}'(w_\sigma), Q_\sigma[\mathbb{J}L_\sigma\xi - \partial_t\xi]\rangle| \le c[\epsilon^{3/2}\log^{1/2}(1/\epsilon) + \epsilon\|\xi\|_X]\|\xi\|_X. \tag{76}$$

Finally, we must control the second inner-product on the right hand side of (75). This is problematic since, so far, we have control over (the second component of) the first factor only in L^2 (see Lemma 5), and over the second factor only in H^{-s} for $s > 0$ (see Lemma 1). The solution is to isolate the worst term and to use the detailed structure of the equations to deal with it.

First we claim that $\mathcal{H}'(w_\sigma)$ is of the form

$$\mathcal{H}'(w_\sigma) = \mathbb{J}H_\zeta^\sigma + \mathcal{H}'_{rest} \tag{77}$$

with $H^\sigma_\zeta := \langle \zeta, \partial_\chi w_\sigma \rangle$, and $\mathcal{H}'_{rest}$ satisfying the estimate

$$\|\mathcal{H}'_{rest}\|_{H^{1-s}\times H^1} \le c\sqrt{\epsilon}. \tag{78}$$

Indeed, (77)- (78) is easily obtained from the explicit expression

$$\mathcal{H}'(w_\sigma) = (\mathcal{E}'_{GL}(v_{\underline{\mathrm{Z}},\chi}), \phi_\sigma), \tag{79}$$

where $\phi_\sigma = p_{jk} T^{(\underline{\mathrm{Z}},\chi)}_{jk} + G^{(\underline{\mathrm{Z}},\chi)}_\zeta$,, and estimates $\|\mathcal{E}'_{GL}(v_{\underline{\mathrm{Z}},\chi})\|_{H^1} \le c\epsilon \log^{1/4}(1/\epsilon)$ (Lemma 5, part 2) and $|p| + \|\zeta\|_{H^1} \le \sqrt{\epsilon}$. Thus

$$\begin{aligned}\langle \mathcal{H}'(w_\sigma), Q_\sigma \mathbb{J} N_\sigma(\xi)\rangle &= -\langle Q_\sigma[-H^\sigma_\zeta + \mathbb{J}\mathcal{H}'_{rest}], N_\sigma(\xi)\rangle \\ &= \langle G^{(\underline{\mathrm{Z}},\chi)}_\zeta, (N_\sigma)_1(\xi_1)\rangle - \langle Q_\sigma \mathbb{J}\mathcal{H}'_{rest}, N_\sigma(\xi)\rangle.\end{aligned}$$

So by (78), (60), and (51), we have

$$|\langle \mathcal{H}'(w_\sigma), Q_\sigma \mathbb{J} N_\sigma(\xi)\rangle - \langle G^{(\underline{\mathrm{Z}},\chi)}_\zeta, (N_\sigma)_1(\xi_1)\rangle| \le c\sqrt{\epsilon}\|\xi\|^2_X. \tag{80}$$

Next, we single out the worst term in the nonlinearity:

$$(N_\sigma)_1(\xi_1) = (0, Im(\bar{\xi}_\psi \nabla_{A_{\underline{\mathrm{Z}},\chi}} \xi_\psi)) + N_{rest}$$

where

$$\|N_{rest}\|_{H^{-s}\times L^2} \le c\|\xi\|^2_X.$$

Recall here that we are writing $\xi = (\xi_1, \xi_2)$, $\xi_1 = (\xi_\psi, \xi_A)$, and $\xi_2 = (\xi_\pi, \xi_E)$. Hence

$$|\langle G_\zeta, N_{rest}\rangle| \le c\|\zeta\|_{H^1}\|\xi\|^2_X.$$

Then in light of (80), we have

$$|\langle \mathcal{H}'(w_\sigma), Q_\sigma \mathbb{J} N_\sigma(\xi)\rangle - \langle \nabla\zeta, Im(\bar{\xi}_\psi \nabla_{A_{\underline{\mathrm{Z}},\chi}} \xi_\psi)\rangle| \le c\sqrt{\epsilon}\|\xi\|^2_X. \tag{81}$$

It remains to estimate $\langle \nabla\cdot\zeta, Im(\bar{\xi}_\psi \nabla_{A_{\underline{\mathrm{Z}},\chi}} \xi_\psi)\rangle$ (note that the estimates available to us so far control the first factor in L^2 (and no better) and just fail to control the second in L^2). The key is to recognize this quantity as (essentially) a time derivative. Using the basic equation (57) and (79) compute

$$\begin{aligned}\frac{d}{dt}\langle \zeta, Im(\bar{\xi}_\psi \xi_\pi)\rangle = \langle \partial_t\zeta, Im(\bar{\xi}_\psi \xi_\pi)\rangle + \langle \zeta, Im(\bar{\xi}_\psi[-\partial_t[\phi_\sigma]_\pi + [\mathcal{E}'_{GL}(v_{\underline{\mathrm{Z}},\chi})]_\psi \\ + [\mathcal{E}''_{GL}(v_{\underline{\mathrm{Z}},\chi})\xi_1]_\psi + [(N_\sigma)_1]_\psi] + \bar{\xi}_\pi[\partial_t\psi_{\underline{\mathrm{Z}},\chi} - [\phi_\sigma]_\psi])\rangle.\end{aligned}$$

We estimate each term on the RHS as follows:

$$|\langle\partial_t\zeta, Im(\bar{\xi}_\psi\xi_\pi)\rangle| \le c\|\partial_t\zeta\|_{H^{1-s}}\|\xi\|_X^2$$

$$|\langle\zeta, Im(\bar{\xi}_\psi[(N_\sigma)_1]_\psi)\rangle| \le c\|\zeta\|_{H^1}(\|\xi\|_X^3 + \|\xi\|_X^4)$$

$$|\langle\zeta, Im(\bar{\xi}_\psi[\mathcal{E}'_{GL}(v_{\underline{\mathrm{Z}},\chi})]_\psi)\rangle| \le c\epsilon\log^{1/4}(1/\epsilon)\|\zeta\|_{H^1}\|\xi\|_X$$

$$\begin{aligned}|\langle\zeta, Im(\bar{\xi}_\psi\partial_t[\phi_\sigma]_\pi)\rangle| \le{}& c(|\underline{\dot{\mathrm{p}}}| + (|\underline{\mathrm{p}}| + \|\partial_t\zeta\|_2)(|\underline{\dot{\mathrm{z}}}| + \|\partial_t^{\underline{\mathrm{Z}}}\chi\|_2)\\ &+ \|\partial_t\zeta\|_2)\|\zeta\|_{H^1}\|\xi\|_X\end{aligned}$$

and

$$\begin{aligned}|\langle\zeta,&Im(\bar{\xi}_\pi[\partial_t\psi_{\underline{\mathrm{Z}},\chi} - [\phi_\sigma]_\psi])\rangle|\\ &= |\langle\zeta, Im(\bar{\xi}_\pi[(\underline{\mathrm{p}} - \underline{\dot{\mathrm{z}}})_{jk}[T_{jk}^{(\underline{\mathrm{Z}},\chi)}]_\psi + [G^{(\underline{\mathrm{Z}},\chi)}_{\partial_t^{\underline{\mathrm{Z}}}\chi-\zeta}]_\psi])\rangle|\\ &\le c(|\underline{\mathrm{p}} - \underline{\dot{\mathrm{z}}}| + \|\partial_t^{\underline{\mathrm{Z}}}\chi - \zeta\|_{H^1})\|\zeta\|_{H^1}\|\xi\|_X.\end{aligned}$$

We write

$$[\mathcal{E}''_{GL}(v_{\underline{\mathrm{Z}},\chi})\xi_1]_\psi = -\Delta_{A_{\underline{\mathrm{Z}},\chi}}\xi_\psi + \mathcal{E}''_{rest}$$

with

$$|\langle\zeta, Im(\bar{\xi}_\psi\mathcal{E}''_{rest})\rangle| \le c\|\zeta\|_{H^1}\|\xi\|_X^2.$$

Collecting these estimates yields

$$\begin{aligned}|\frac{d}{dt}&\langle\zeta, Im(\bar{\xi}_\psi\xi_\pi)\rangle + \langle\zeta, Im(\bar{\xi}_\psi\Delta_{A_{\underline{\mathrm{Z}},\chi}}\xi_\psi)\rangle|\\ &\le c\|\xi\|_X(\|\partial_t\zeta\|_{H^{1-s}}\|\xi\|_X + \sqrt{\epsilon}[\|\xi\|_X + \epsilon\log^{1/4}(1/\epsilon) + |\underline{\dot{\mathrm{p}}}|\\ &\quad + \sqrt{\epsilon}(|\underline{\dot{\mathrm{z}}}| + \|\partial_t^{\underline{\mathrm{Z}}}\chi\|_2) + |\underline{\mathrm{p}} - \underline{\dot{\mathrm{z}}}| + \|\partial_t^{\underline{\mathrm{Z}}}\chi - \zeta\|_{H^1}])\\ &\le c\sqrt{\epsilon}\|\xi\|_X(\|\xi\|_X + \epsilon\log^{1/4}(1/\epsilon)),\end{aligned} \tag{82}$$

using (69). Noting that

$$-\langle\zeta, Im(\bar{\xi}_\psi\Delta_{A_{\underline{\mathrm{Z}},\chi}}\xi_\psi)\rangle = \langle\nabla\zeta, Im(\bar{\xi}_\psi\nabla_{A_{\underline{\mathrm{Z}},\chi}}\xi_\psi)\rangle,$$

and combining (75), (76), (81), and (82), we have

$$|\frac{d}{dt}[\mathcal{H}(w_\sigma) + \langle\zeta, Im(\bar{\xi}_\psi\xi_\pi)\rangle]| \le c\sqrt{\epsilon}\|\xi\|_X(\|\xi\|_X + \epsilon\log^{1/4}(1/\epsilon)).$$

Integrating this in time, and defining $M(t) := \sup_{0\le s\le t}\|\xi(s)\|_X$, leads to (74).
□

Returning to (73), we obtain

$$\|\xi(t)\|_X^2 \le c[t\sqrt{\epsilon}M(t)(M(t)+\epsilon\log^{1/4}(1/\epsilon)) + M(t)(\epsilon\log^{1/2}(1/\epsilon)+M^2(t)) \\ + \sqrt{\epsilon}M^2(t)M(0)(\epsilon\log^{1/4}(1/\epsilon)+M(0))].$$

It now follows that there is a constant $\tau' > 0$, such that for $0 \le t \le \min\left(\frac{\tau'}{\sqrt{\epsilon}}, T_1\right)$ (recall T_1 is the time of first exit of $w(t)$ from U_δ), we have

$$\|\xi(t)\|_X < c(\epsilon\log^{1/2}(1/\epsilon) + \|\xi(0)\|_X). \tag{83}$$

In particular, the intermediate assumption (59) is justified.

3.1.8 A priori momentum bound

We wish to iterate the above argument to extend the time interval. The problem is that the vortex velocities can, in principle, grow to size $\epsilon t >> \sqrt{\epsilon}$ if $t >> 1/\sqrt{\epsilon}$ (this would mean we leave the manifold M_{mv}, and many of the above estimates fail). We show here that this does not happen. To this end we use the approximate conservation of the reduced (vortex) energy $H(w_\sigma)$, together with the repulsivity of the interaction energy. Indeed, since we are in the "repulsive" case ($\lambda > 1/2$ and $n_1 = \cdots = n_m = \pm 1$), we have the following lemma, which is proved in Section 4.2.

Lemma 7 (interaction energy) *For $R(\underline{z})$ large,*

$$W(\underline{z}) = \sum_{j\neq k} n_j n_k c_{jk} \frac{e^{-|z_j - z_k|}}{\sqrt{|z_j - z_k|}} + o(e^{-R(\underline{z})}/\sqrt{R(\underline{z})}). \tag{84}$$

Here $c_{jk} > 0$ are constants.

Remark 4 *One can see from this expression that like-signed vortices repel, while opposite-signed vortices attract.*

Conservation of energy for the PDE, $\mathcal{H}(w(t)) = \mathcal{H}(w(0))$, together with the decomposition (54) and a Taylor expansion yields

$$\mathcal{H}(w_\sigma) - \mathcal{H}(w_{\sigma(0)}) = \langle \mathcal{H}'(w_\sigma), \xi\rangle - \langle \mathcal{H}'(w_{\sigma(0)}), \xi(0)\rangle \\ + O(\|\xi\|_X^2 + \|\xi(0)\|_X^2).$$

We saw above (in (72)) that $|\langle\mathcal{H}'(w_\sigma),\xi\rangle| \le c\epsilon\log^{1/4}(1/\epsilon)\|\xi\|_X$, which gives

$$\mathcal{H}(w_\sigma) - \mathcal{H}(w_{\sigma(0)}) \le c((\|\xi\|_X + \|\xi(0)\|_X)(\epsilon\log^{1/4}(1/\epsilon) + \|\xi\|_X + \|\xi(0)\|_X)).$$

By estimates (42) and $K \ge c$ for some $c > 0$, we have, for ϵ sufficiently small,

$$W(\underline{z}) + |\underline{p}|^2 + \|\zeta\|_{H^1}^2 < \alpha[\mathcal{H}(w_\sigma) - \sum_{j=1}^m E^{(n_j)}]$$

for some constant $\alpha > 0$. In light of Lemma 7, the assumptions on the initial conditions in Theorem 2 imply $\mathcal{H}(w_{\sigma(0)}) - \sum_{j=1}^m E^{(n_j)} < c'd_0\epsilon$, for some c'. We choose $d_0 < \frac{\alpha'}{2c'\alpha}$, where α' is a constant to be chosen below. So provided

$$c((\|\xi(t)\|_X + \|\xi(0)\|_X)(\epsilon\log^{1/4}(1/\epsilon) + \|\xi(t)\|_X + \|\xi(0)\|_X)) < \frac{\alpha'}{2\alpha}\epsilon, \tag{85}$$

we have $\alpha[\mathcal{H}(w_\sigma) - \sum_{j=1}^m E^{(n_j)}] < \alpha'\epsilon$, and therefore

$$W(\underline{z}) + |\underline{p}|^2 + \|\zeta\|_{H^1}^2 < \alpha'\epsilon. \tag{86}$$

So by (84), if $\alpha' < \min(c_{jk})$, then as long as condition (85) holds, $|p|^2 + \|\zeta\|_{H^1}^2 < \epsilon$, and $R(\underline{z})e^{-R(\underline{z})} < \epsilon$. Hence $\sigma \in \Sigma_\epsilon$, and $w_\sigma \in M_{mv}$.

In particular, this estimate shows that $T_1 > \tau'/\sqrt{\epsilon}$. Hence, we have shown:

Lemma 8 *There are $\tau' > 0$ and $d > 0$, such that inequality (83) holds for $0 \le t \le \tau'/\sqrt{\epsilon}$, provided*

$$\|\xi(0)\|_X + \|\xi(t)\|_X < d\sqrt{\epsilon}. \tag{87}$$

3.1.9 Iteration

We may iterate Lemma 8 for as long as the conditions $\sigma \in \Sigma_\epsilon$, and $\|\xi(t)\|_X < d\sqrt{\epsilon}$ hold. Iterating N times starting with $\xi(0)$ and satisfying $\|\xi(0)\|_X \le d_0\sqrt{\epsilon}$ yields

$$\|\xi(t)\|_X \le Cc^N\epsilon\log^{1/2}(1/\epsilon) \quad \text{for} \quad 0 \le t \le \tau'N/\sqrt{\epsilon},$$

where C is another constant. The condition (87) limiting the number of iterations, ensures both that (85) holds (so that $\sigma \in \Sigma_\epsilon$ remains true), and that the remainder in the effective dynamics law is sub-leading order. Thus we can take $c^N = \alpha(\epsilon)/\sqrt{\epsilon}$ for any $\alpha_0\sqrt{\epsilon} < \alpha(\epsilon) << \log^{-1/2}(1/\epsilon)$ (with $\alpha_0 > 1$). This gives a total time interval of length

$$T = \frac{\tau}{\sqrt{\epsilon}} \log\left(\frac{\alpha(\epsilon)}{\sqrt{\epsilon}}\right),$$

where $\tau = \tau'/\log c$, over which we have the bound

$$\|\xi(t)\|_X < C\alpha(\epsilon)\sqrt{\epsilon}\log^{1/2}(1/\epsilon) \tag{88}$$

for $0 \le t \le T$.

Finally, equation (69) implies

$$\begin{aligned}|\underline{\dot{z}}(t) - \underline{p}(t)| + |\underline{\dot{p}}(t) + \underline{\gamma}^{-1} \cdot \nabla_{\underline{Z}} W(\underline{z})| + \|\partial_t^{\underline{Z}}\chi(t) - \zeta(t)\|_{H^{1-s}} + \|\partial_t\zeta(t)\|_{H^1} \\ \le c\sqrt{\epsilon}\|\xi\|_X \le c\epsilon\alpha(\epsilon)\log^{1/2}(1/\epsilon) = o(\epsilon).\end{aligned}$$

This completes the proof of Theorem 2. □

3.2 Effective dynamics of vortices: superconductor model

Here we just sketch the proof of Theorem 1 since it proceeds as above. The important difference is that we can control the remainder for all times.

The set-up is as follows. For the superconductor model our manifold of multi-vortex configurations is taken to be

$$M_{mv} = \{v_{\underline{Z},\chi} \mid e^{-R(\underline{Z})}/\sqrt{R(\underline{z})} < \epsilon,\ \chi \in H^2_{\underline{Z}}(\mathbb{R}^2;\mathbb{R})\}.$$

A solution $u(t) = (\psi(t), A(t))$ of (3) is decomposed as

$$u(t) = v_{\underline{Z}(t),\chi(t)} + \xi(t)$$

with $P_{\underline{Z},\chi}\xi \equiv 0$, where $P_{\underline{Z},\chi}$ denotes the orthogonal projection from H^1 onto the tangent space $T_{v_{\underline{Z},\chi}}M_{mv}$. Substituting this into (3) yields

$$\partial_t v_{\underline{Z},\chi} + \partial_t\xi = -[\mathcal{E}'_{GL}(v_{\underline{Z},\chi}) + L_{\underline{Z},\chi}\xi - N_{v_{\underline{Z},\chi}}(\xi)]$$

where $L_{\underline{Z},\chi} := \mathcal{E}''_{GL}(v_{\underline{Z},\chi})$. The equation governing the effective dynamics of $\underline{z}(t)$ and $\chi(t)$ is derived by applying the projection $P_{\underline{Z},\chi}$ to this:

$$\partial_t v_{\underline{Z},\chi} + P_{\underline{Z},\chi}\mathcal{E}'_{GL}(v_{\underline{Z},\chi}) = -P_{\underline{Z},\chi}[L_{\underline{Z},\chi}\xi - N_{v_{\underline{Z},\chi}}(\xi) + \partial_t\xi].$$

To estimate the RHS, we have the following properties

$$\|P_{\underline{Z},\chi} L_{\underline{Z},\chi}\xi\|_{L^2} \leq c\epsilon \log^{1/2}(1/\epsilon)\|\xi\|_{H^1}$$

$$\|P_{\underline{Z},\chi} N_{v_{\underline{Z},\chi}}(\xi)\|_{H^{-s}} \leq c(\|\xi\|_{H^1}^2 + \|\xi\|_{H^1}^3)$$

$$\|P_{\underline{Z},\chi}\partial_t\xi\|_{H^1} \leq c(|\underline{\dot{z}}| + \|\partial_t^{\underline{Z}}\chi\|_{L^2})\|\xi\|_{H^1}.$$

Combining these yields

$$\|\partial_t v_{\underline{Z},\chi} + P_{\underline{Z},\chi}\mathcal{E}'_{GL}(v_{\underline{Z},\chi})\|_{H^{-s}} \leq c(\epsilon\log^{1/2}(1/\epsilon) + |\underline{\dot{z}}| + \|\partial_t^{\underline{Z}}\chi\|_2 + \|\xi\|_{H^1})\|\xi\|_{H^1}$$

which in parametric form reads (recall the notation $\gamma\underline{\dot{z}} := (\gamma_{n_1}\dot{z}_1, \ldots, \gamma_{n_m}\dot{z}_m)$)

$$|\gamma\underline{\dot{z}} + \nabla_z W(\underline{z})| + \|\partial_t^{\underline{Z}}\chi\|_{H^{1-s}} \leq c(\epsilon\log^{1/2}(1/\epsilon) + \|\xi\|_{H^1})\|\xi\|_{H^1}. \tag{89}$$

In order to control $\|\xi\|_{H^1}$ for *all* time we need

Lemma 9 *There is $\delta > 0$ such that for $R(\underline{z})$ sufficiently large,*

$$\langle L_{\underline{Z},\chi}\xi, L_{\underline{Z},\chi}\xi\rangle > \delta\|\xi\|_{H^2}^2.$$

This lemma is proved in Section 4.4.

We use the fact that the main part of the energy difference, $\mathcal{E}(v_{\underline{Z},\chi} + \xi) - \mathcal{E}(v_{\underline{Z},\chi})$, namely $\frac{1}{2}\langle\xi, L_{\underline{Z},\chi}\xi\rangle$, is a decaying quantity. Compute

$$\begin{aligned}\frac{d}{dt}\langle\xi, L_{\underline{Z},\chi}\xi\rangle &= 2\langle\partial_t\xi, L_{\underline{Z},\chi}\xi\rangle + \langle[\partial_t, L_{\underline{Z},\chi}]\xi, \xi\rangle \\ &= \langle -\mathcal{E}'(v_{\underline{Z},\chi}) - L_{\underline{Z},\chi}\xi - N_{v_{\underline{Z},\chi}}(\xi), L_{\underline{Z},\chi}\xi\rangle \\ &\quad + O((|\underline{\dot{z}}| + \|\partial_t\chi\|_2)\|\xi\|_{H^1}^2).\end{aligned}$$

Now we use

$$|\langle\mathcal{E}'(v_{\underline{Z},\chi}), L_{\underline{Z},\chi}\xi\rangle| < c\epsilon\log^{1/4}(1/\epsilon)\|\xi\|_{H^1},$$

$$|\langle N_{v_{\underline{Z},\chi}}(\xi), L_{\underline{Z},\chi}\xi\rangle| < c\|\xi\|_{H^2}^2(\|\xi\|_{H^1} + \|\xi\|_{H^1}^2),$$

and

$$\langle\xi, L_{\underline{Z},\chi}\xi\rangle \leq \beta\|\xi\|_{H^1}^2,$$

together with Lemma 9, to obtain

$$\begin{aligned}(\frac{d}{dt} + \frac{\delta}{2\beta})\langle\xi, L_{\underline{Z},\chi}\xi\rangle \leq\ & \|\xi\|_{H^2}^2[c(\|\xi\|_{H^1} + \|\xi\|_{H^1}^2 \\ & + |\underline{\dot{z}}| + \|\partial_t\chi\|_2) - \delta/2] + c\epsilon\log^{1/4}(1/\epsilon)\|\xi\|_{H^1}.\end{aligned}$$

So as long as

$$c(\|\xi\|_{H^1} + \|\xi\|_{H^1}^2 + |\dot{\underline{z}}| + \|\partial_t \chi\|_2) \leq \delta/2, \tag{90}$$

we have

$$\frac{d}{dt}(e^{(\delta/2\beta)t}\langle \xi, L_{\underline{Z},\chi}\xi\rangle) \leq c\epsilon \log^{1/4}(1/\epsilon) e^{(\delta/\beta)t}\|\xi\|_{H^1}.$$

Setting $M(t) := \sup_{0\leq s\leq t}\|\xi(s)\|_{H^1}$ and integrating in time leads to

$$\langle \xi, L_{\underline{Z},\chi}\xi\rangle \leq e^{-(\delta/2\beta)t}\langle \xi(0), L_{\underline{Z}_0,\chi_0}\xi(0)\rangle + c\epsilon \log^{1/4}(1/\epsilon)M(t).$$

Finally using

$$c\langle \xi, L_{\underline{Z},\chi}\xi\rangle < \|\xi\|_{H^1}^2 < \frac{1}{\gamma}\langle \xi, L_{\underline{Z},\chi}\xi\rangle,$$

we find

$$M^2(t) \leq c[e^{-(\delta/2\beta)t}M^2(0) + \epsilon \log^{1/4}(1/\epsilon)M(t)]$$

and so if $M(0) = O(\epsilon \log^{1/4}(1/\epsilon))$ we have

$$\|\xi(t)\|_{H^1} < c\epsilon \log^{1/4}(1/\epsilon)$$

for all t, as long as (90) and $e^{-R(\underline{z})}/\sqrt{R(\underline{z})} < \epsilon$ hold. By (89), we see

$$|\dot{\underline{z}} + \nabla_{\underline{Z}}W(\underline{z})| + \|\partial_t^{\underline{Z}}\chi\|_{H^{1-s}} \leq c\epsilon^2 \log^{3/4}(1/\epsilon)$$

so the intermediate assumption (90) is justified. Finally, in the repulsive case, $\mathcal{E}(u) - \sum_{j=1}^m E^{(n_j)} \leq c\epsilon$ implies $e^{-R(\epsilon)}/\sqrt{R(\underline{z})} < \epsilon$ holds for all t. □

4 Key properties

In this section we prove the lemmas used in the proofs of Theorems 1 and 2.

4.1 Approximate static solution property

Proof of Lemma 5: The main fact we use here is that since we consider the Type-II regime ($\lambda > 1/2$), the effects of the magnetic field and current dominate those of the order parameter at large distances.

In what follows, a subindex k will denote an equivariant field component, of degree n_k, centred at z_k: eg, $\psi_k := \psi^{(n_k)}(\cdot - z_k)$, $(\nabla_A\psi)_k = \nabla_{A^{(n_k)}(\cdot - z_k)}\psi^{(n_k)}(\cdot - z_k)$, etc.

We first prove

Lemma 10

$$\|\mathcal{E}'_{GL}(v_{\underline{z},\chi})\|_2 \le ce^{-R(\underline{z})}/R^{1/4}(\underline{z}). \tag{91}$$

Proof: The proof is a computation using the fact that $u^{(n_j)} = (\psi^{(n_j)}, A^{(n_j)})$ satisfies the Ginzburg-Landau equations, together with the exponential decay (7). We start with

$$[\mathcal{E}'_{GL}(v_{\underline{Z},\chi})]_\psi = -\Delta_{A_{\underline{Z},\chi}}\psi_{\underline{Z},\chi} + \lambda(|\psi_{\underline{Z},\chi}|^2 - 1)\psi_{\underline{Z},\chi}.$$

Using gauge covariance and the covariant product rule, we find

$$\Delta_{A_{\underline{Z},\chi}}\psi_{\underline{Z},\chi} = e^{i\chi}\left[\sum_j(\prod_{k\neq j}\psi_k)(\Delta_A\psi)_j + \sum_{j\neq k}(\prod_{l\neq j,k}\psi_l)(\nabla_A\psi)_j\cdot(\nabla_A\psi)_k\right]. \tag{92}$$

A little computation plus (7) yields

$$|(\prod_{j=1}^m f_j^2) - 1 - \sum_{j=1}^m (f_j^2 - 1)| \le c\sum_{j\neq k} e^{-m_\lambda(|x-z_j|+|x-z_k|)}. \tag{93}$$

Using (92) and (93), together with the fact that $u^{(n_j)}$ solves the Ginzburg-Landau equations, we arrive at

$$|[\mathcal{E}'_{GL}(v_{\underline{Z},\chi})]_\psi(x) - [E^{(\underline{Z},\chi)}]_\psi(x)| \le c\sum_{j\neq k} e^{-m_\lambda(|x-z_j|+|x-z_k|)}$$

where

$$E_\psi^{(\underline{Z},\chi)} := -e^{i\chi}\sum_{j\neq k}(\prod_{l\neq j,k}\psi_l)(\nabla_A\psi)_j\cdot(\nabla_A\psi)_k.$$

Using Lemma 12 (in Appendix 3, Section 5.3) with $\alpha = \beta = 2m_\lambda > 2$, $\gamma = \delta = 0$, we obtain

$$\|[\mathcal{E}'_{GL}(v_{\underline{Z},\chi})]_\psi - [E^{(\underline{Z},\chi)}]_\psi\|_2 \le ce^{-m_\lambda R(\underline{Z})}R(\underline{z})^{3/2} << e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}. \tag{94}$$

We turn now to

$$[\mathcal{E}'_{GL}(v_{\underline{Z},\chi})]_A = curl B_{\underline{Z},\chi} - j_{\underline{Z},\chi}.$$

Observing that $curl B_{\underline{Z},\chi} = \sum_{j=1}^m curl B_j$, and

$$j_{\underline{Z},\chi} = \sum_{j=1}^m j_j + \sum_{j=1}^m(\prod_{k\neq j} f_k^2 - 1)j_j,$$

using the Ginzburg-Landau equation $curl B_j - j_j = 0$, invoking an equation similar to (93) for $\prod_{k\neq j} f_k^2 - 1$, and using (7), we arrive at

$$|[\mathcal{E}'_{GL}(v_{\underline{Z},\chi})]_A(x) - E_A^{(\underline{Z},\chi)}(x)| \leq c \sum_{j,k,l \text{ distinct}} e^{-m_\lambda(|x-z_j|+|x-z_k|)-|x-z_l|}$$

where

$$E_A^{(\underline{Z},\chi)} := \sum_{j\neq k}(1-f_j^2)j_k.$$

Estimating as above gives

$$\|[\mathcal{E}'_{GL}(v_{\underline{Z},\chi})]_A - E_A^{(\underline{Z},\chi)}\|_2 \leq ce^{-m_\lambda R(\underline{Z})}R(\underline{z})^{3/2} << e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}. \tag{95}$$

Using (7) again, we obtain the following pointwise estimate for $E^{(\underline{Z},\chi)} = (E_\psi^{(\underline{Z},\chi)}, E_A^{(\underline{Z},\chi)})$:

$$|E^{(\underline{Z},\chi)}| \leq c\sum_{k\neq j} \frac{e^{-|x-z_j|}}{(1+|x-z_j|)^{1/2}} \frac{e^{-|x-z_k|}}{(1+|x-z_k|)^{1/2}}.$$

Applying Lemma 12 with $\alpha = \beta = 2$ and $\gamma = \delta = 1$ yields

$$\|E^{(\underline{Z},\chi)}\|_2 \leq ce^{-R(\underline{Z})}/R(\underline{z})^{1/4}. \tag{96}$$

Then (94)-(96) yield (91). □

Now we consider the manifold of approximate solutions for the Higgs model equations. Parts 1 and 2 of Lemma 5 follow immediately from the expression (cf. (79))

$$\mathcal{H}'(w_\sigma) = (\mathcal{E}'_{GL}(v_{\underline{Z},\chi}), p_{jk}T_{jk}^{(\underline{Z},\chi)} + G_\zeta^{(\underline{Z},\chi)}),$$

together with Lemma 10, and the fact that $\sigma \in \Sigma_\epsilon$ implies $|\underline{p}| + \|\zeta\|_{H^1} < \epsilon$ and $e^{-R(\underline{Z})}/\sqrt{R(\underline{z})} < \epsilon$, which implies $e^{-R(\underline{Z})}/R(\underline{z})^{1/4} < c\epsilon \log^{1/4}(1/\epsilon)$. The refined statement, part 3 of Lemma 5, follows from the fact that for $\sigma \in \Sigma_\epsilon$,

$$Q_\sigma = \begin{pmatrix} P_{\underline{Z},\chi} & 0 \\ 0 & P_{\underline{Z},\chi} \end{pmatrix} + O(\sqrt{\epsilon})$$

where $P_{\underline{Z},\chi}$ denotes the orthogonal projection onto the span of $T_{jk}^{(\underline{Z},\chi)}$ and $G_\gamma^{(\underline{Z},\chi)}$ (see Eqn. (115) of Appendix 2) and so, since $\bar{P}_{\underline{Z},\chi}[p_{jk}T_{jk}^{(\underline{Z},\chi)} + G_\zeta^{(\underline{Z},\chi)}] = 0$,

$$\|\bar{Q}_\sigma \mathbb{J}\mathcal{H}'(w_\sigma)\|_{L^2\times L^2} \leq c\epsilon \log^{1/4}(1/\epsilon)$$

as required. This completes the proof of Lemma 5. □

4.2 Inter-vortex interaction

The reduced energy is a function of the vortex positions alone:

$$W(\underline{z}) := \mathcal{E}_{GL}(v_{\underline{Z},\chi}) - \sum_{j=1}^{m} E^{(n_j)}.$$

In this section, we compute – to leading order in the vortex separation – $W(\underline{z})$, and $\nabla W(\underline{z})$, the inter-vortex force entering the effective vortex dynamic laws.

Proof of Lemma 7: Noting that $B = curl A$, we re-write the Ginzburg-Landau energy as

$$\mathcal{E}_{GL}(A,\psi) = \frac{1}{2}\int_{\mathbb{R}^2}\{|\nabla_A\psi|^2 + B^2 + \frac{\lambda}{2}(|\psi|^2-1)^2\}.$$

For $(\psi, A) = (\psi_{\underline{Z},\chi}, A_{\underline{Z},\chi})$, we have

$$\nabla_A\psi = e^{i\chi}\sum_{j=1}^{m}(\prod_{k\neq j}\psi_k)(\nabla_A\psi)_j,$$

and $B = B_1 + \cdots + B_m$. So plugging into $\mathcal{E}_{GL}$ and using the notation $j_l := Im(\bar{\psi}_l\nabla_{A_l}\psi_l)$ and $f_l := |\psi_l|$, we find

$$\mathcal{E}_{GL}(\psi_{\underline{Z},\chi}, A_{\underline{Z},\chi}) = \sum_{j=1}^{m} E^{(n_j)} + LO + Rem$$

where

$$LO := \frac{1}{2}\sum_{l\neq k}\int_{\mathbb{R}^2}[j_l\cdot j_k + B_lB_k]$$

and

$$\begin{aligned}Rem = &\frac{1}{2}\sum_{j=1}^{m}\int(\prod_{k\neq j}f_k^2-1)|(\nabla_A\psi)_j|^2 + \frac{1}{2}\sum_{j\neq l}\int(\prod_{k\neq j,l}f_k^2)[Re(\bar{\psi}\nabla_A\psi)]_j[Re(\bar{\psi}\nabla_A\psi)]_l\\ &+\frac{1}{2}\sum_{j\neq l}\int(\prod_{k\neq j,l}f_k^2-1)j_k\cdot j_l\\ &+\frac{\lambda}{4}\int[\sum_{j\neq l}(f_j^2-1)(f_l^2-1) + \sum_{j\neq l\neq k}(f_j^2-1)(f_l^2-1)(f_k^2-1)+\cdots].\end{aligned}$$

For each term in Rem, the integrand is bounded by $e^{-(\min(m_\lambda,2)|x-z_j|+m_\lambda|x-z_k|)}$ or $e^{-(m_\lambda|x-z_k|+|x-z_j|+|x-z_l|)}$, and so, after integration, is $<< e^{-R(\underline{z})}/\sqrt{R(\underline{z})}$ (using Lemma 12 and $m_\lambda > 1$). Using the Ginzburg-Landau equation $curl B = j$, we can re-write the leading-order term as

$$LO = \frac{1}{2}\sum_{l\neq k}\int_{\mathbb{R}^2}[B_l(-\Delta+1)B_k].$$

A computation gives $(-\Delta+1)B = n(2(1-a)ff' + a'(1-f^2))/r > 0$. By (7), $|(-\Delta+1)B(x)| < ce^{-m_\lambda|x|}$, and $B_n(x) = c_n n e^{-|x|}/\sqrt{|x|}[1+O(1/|x|)]$, $c_n > 0$. Applying Lemma 13 yields

$$LO = \frac{1}{2}\sum_{l\neq k} c_l n_l n_k \frac{e^{-|z_l-z_k|}}{\sqrt{|z_l-z_k|}}\int_{\mathbb{R}^2} e^{x\cdot(z_l-z_k)/|z_l-z_k|}(2(1-a_k)f_k f_k' + a_k'(1-f_k^2))/r dx$$
$$+ o\left(\frac{e^{-R(\underline{z})}}{\sqrt{R(\underline{z})}}\right).$$

Lemma 7 follows. $\Box$.

Now we turn to the estimate of the force:

Lemma 11 *We have*

$$\nabla_{z_l} W(\underline{z}) = \sum_{j\neq l} n_j n_l C_{jl}\frac{e^{-|z_j-z_l|}}{\sqrt{|z_j-z_l|}}\frac{z_j-z_l}{|z_j-z_l|} + o(e^{-R(\underline{z})}/\sqrt{R(\underline{z})}) \qquad (97)$$

as $R(\underline{z}) \to \infty$. Here $C_{jl} > 0$ are constants.

Proof: By the definition of $W(\underline{z})$, $\nabla_{z_{lm}} W(\underline{z}) = \langle \mathcal{E}_{GL}'(v_{\underline{z},\chi}), T_{lm}^{(\underline{z},\chi)}\rangle$. Equations (94) and (95) imply that

$$\nabla_{z_{lm}} W(\underline{z}) = \langle E^{(\underline{z},\chi)}, T_{lm}^{(\underline{z},\chi)}\rangle + o(e^{-R(\underline{z})}/\sqrt{R(\underline{z})}), \qquad (98)$$

where $E^{(\underline{z},\chi)}$ is defined in the proof of Lemma 10. We first compute

$$\langle E_\psi^{(\underline{z},\chi)}, [T_{lm}^{(\underline{z},\chi)}]_\psi\rangle = \sum_{j\neq k}\alpha_{lm}^{jk}$$

where (recall the notation $\psi_k(x) := \psi^{(n_k)}(x-z_k)$, etc.)

$$\alpha_{lm}^{jk} = \langle (\prod_{r\neq j,k}\psi_r)(\nabla_A\psi)_j\cdot(\nabla_A\psi)_k, (\prod_{t\neq l}\psi_t)([\nabla_A]_m\psi)_l\rangle.$$

First, we note that $\alpha_{lm}^{jk} = \alpha_{lm}^{kj}$. Second, we use (7) to conclude that if $l \neq j$ and $l \neq k$, then $|\alpha_{lm}^{jk}| << e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}$. It remains to compute α_{jm}^{jk}. We rewrite to get

$$\alpha_{jm}^{jk} = \sum_s \int (\prod_{r \neq j,k} f_r^2) Re[\overline{([\nabla_A]_m \psi)_j}([\nabla_A]_s \psi)_j (\overline{\psi}[\nabla_A]_s \psi)_k],$$

and use

$$|\prod_{r \neq j,k} f_r^2 - 1| \leq c e^{-m_\lambda \cdot \max(|x - z_j|, |x - z_k|)}$$

to conclude that $\alpha_{jm}^{jk} = \tilde{\alpha}_{jm}^{jk} + o(e^{-R(\underline{Z})}/\sqrt{R(\underline{z})})$, where

$$\tilde{\alpha}_{jm}^{jk} = \sum_s Re \int \overline{([\nabla_A]_m \psi)_j}([\nabla_A]_s \psi)_j (\overline{\psi}[\nabla_A]_s \psi)_k.$$

Writing everything out in terms of the vortex profiles f_j and a_j and taking the real part, we find (applying Lemma 12 again) that

$$\tilde{\alpha}_{lm}^{lk} = -\sum_s \int \left[\frac{n(1-a)}{r}(J\hat{x})_s\right]_k \left[\frac{n(1-a)ff'}{r}[\hat{x}_m (J\hat{x})_s - (J\hat{x})_m \hat{x}_s]\right]_j + o(e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}).$$

Now using the fact that $[\hat{x}_m (J\hat{x})_s - (J\hat{x})_m \hat{x}_s]$ equals 0 if $s = m$, -1 if $(s, m) = (1, 2)$, and 1 if $(s, m) = (2, 1)$, and summing over s, we arrive at

$$\tilde{\alpha}_{lm}^{lk} = -\int \left[\frac{n(1-a)}{r}\right]_k \left[\frac{n(1-a)ff'}{r}\right]_j (\widehat{x - z_k})_m + o(e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}).$$

Now we apply (a slight variant of) Lemma 13 to obtain

$$\begin{aligned}\tilde{\alpha}_{jm}^{jk} = &-n_j n_k \frac{e^{-|z_j - z_k|}}{\sqrt{|z_j - z_k|}} (\widehat{z_j - z_k})_m \int_{\mathbb{R}^2} e^{x \cdot (z_k - z_j)/|z_j - z_k|} (1-a) ff'/r dx \\ &+ o(e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}).\end{aligned}$$

Thus

$$\langle E_{\psi}^{\underline{Z},\chi}, [T_{lm}^{(\underline{Z},\chi)}]_{\psi} \rangle = n_l \sum_{k \neq l} c_{lk} n_k \frac{e^{-|z_l - z_k|}}{\sqrt{|z_l - z_k|}} (\widehat{z_l - z_k})_m + o(e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}). \quad (99)$$

The computation of $\langle E_A^{\underline{Z},\chi}, [T_{lm}^{(\underline{Z},\chi)}]_A\rangle$ is similar, but simpler. We just report the result:

$$\langle E_A^{\underline{Z},\chi}, [T_{lm}^{(\underline{Z},\chi)}]_A\rangle = n_l \sum_{k\neq l} c'_{lk} n_k \frac{e^{-|z_l - z_k|}}{\sqrt{|z_l - z_k|}} (\widehat{z_k - z_l})_m + o(e^{-R(\underline{Z})}/\sqrt{R(\underline{z})}). \quad (100)$$

Combining (99) and (100) with (98) yields (97). □

Remark 5 *Similar computations can be made for the Type-I case, $\lambda < 1/2$. In this case,*

$$W(\underline{z}) = O(e^{-m_\lambda R(\underline{z})})$$

as $R(\underline{z}) \to \infty$, and the inter-vortex forces are attractive.

4.3 Approximate zero-mode property

Proof of Lemma 2. Set $L_{\underline{Z},\chi} := \mathcal{E}''_{GL}(v_{\underline{Z},\chi})$. For any, j, we may write

$$L_{\underline{Z},\chi} = L_j + V_{(j)}$$

where $L_j := \mathcal{E}''_{GL}(g_{\chi_{(j)}} u^{(n_j)}(\cdot - z_j))$, $\chi_{(j)} := \chi + \sum_{k\neq j} \theta(\cdot - z_k)$, and $V_{(j)}$ is a multiplication operator satisfying

$$|V_{(j)}(x)| \leq c e^{-min_{k\neq j}|x - z_k|}.$$

The notation $g_\gamma u$ stands for the result of acting on u by a gauge transformation γ. Recall the translational modes $T_{jk}^{(\underline{Z},\chi)}$ are given in (44). Using the fact that

$$L_j(e^{i\chi_{(j)}}(\nabla_{A_k}\psi)_j, B_j \hat{e}_j^{\perp}) = 0,$$

we get the easy estimates $\|L_j T_{jk}^{(\underline{Z},\chi)}\|_2 \leq c e^{-R(\underline{Z})}$, and

$$\|V_{(j)} T_{jk}^{(\underline{Z},\chi)}\|_2 \leq c e^{-R(\underline{Z})}.$$

Thus

$$\|L_{\underline{Z},\chi} T_{jk}^{(\underline{Z},\chi)}\|_2 \leq c\epsilon \log^{1/2}(1/\epsilon). \quad (101)$$

To deal with the gauge modes, $G_\gamma^{(\underline{Z},\chi)} := \langle \gamma, \partial_\chi v_{\underline{Z},\chi}\rangle$, we use (45), which gives

$$L_{\underline{Z},\chi} G_\gamma^{(\underline{Z},\chi)} = (i\gamma[\mathcal{E}'_{GL}(v_{\underline{Z},\chi})]_\psi, 0)$$

and so

$$\|L_{\underline{Z},\chi}G_{\gamma}^{(\underline{Z},\chi)}\|_2 \le c\epsilon \log^{1/4}(1/\epsilon)\|\gamma\|_2. \tag{102}$$

Now by (32), and (105)- (110), $RanQ_\sigma = T_{w_\sigma}M_{mv}$ consists of vectors of the form $(\alpha \cdot T^{(\underline{Z},\chi)} + G_{\gamma}^{(\underline{Z},\chi)}, O_{L^2}(\sqrt{\epsilon}))$ with $\alpha \in \mathbb{R}^{2m}$ and $\gamma \in H^1$. This, together with

$$L_\sigma = \begin{pmatrix} L_{\underline{Z},\chi} & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

and (101) and (102), yields Lemma 2. □

4.4 Coercivity of the Hessian

Proof of Lemma 3. Suppose $\eta := (\eta_\psi, \eta_A)$ is orthogonal to each approximate translational zero-mode, $T_{jk}^{(\underline{Z},\chi)}$, and to the approximate gauge zero-modes, $G_{\gamma}^{(\underline{Z},\chi)}$ (which means $Im(\overline{\psi_{\underline{Z},\chi}}\eta_\psi) = \nabla \cdot \eta_A$ by an integration by parts). Set $L := L_{\underline{Z},\chi}$. Our first goal is to show

$$\langle \eta, L\eta \rangle \quad \ge \quad c_1\|\eta\|_{H^1}^2.$$

Let $\{\chi_j\}$ be a partition of unity associated to the vortex centres. That is, $\sum_{j=0}^m \chi_j^2 = 1$, χ_j is supported in a ball of fixed radius about z_j $(j = 1, \dots, m)$, and χ_0 is supported away from all the vortices. By the IMS formula ([CFKS]),

$$L = \sum \chi_j L \chi_j - 2\sum |\nabla \chi_j|^2.$$

We can choose $\{\chi_j\}$ such that $|\nabla\chi_j| \le cR^{-1}$, where $R := R(\underline{z})$. As in Section 4.3, set

$$L_j := \mathcal{E}_{GL}''(g_{\chi(j)}u^{(n_j)}(\cdot - z_j)),$$

and write, for each $1 \le j \le m$, $L = L_j + V_{(j)}$. Since

$$|V_{(j)}(x)| \le c\sum_{k\neq j} e^{-|x-z_k|},$$

we can choose $\{\chi_j\}$ so that $\|V_{(j)}\chi_j\|_\infty \le c\sqrt{\epsilon}$, and so

$$\langle \chi_j\eta, L\chi_j\eta\rangle \ge \langle \chi_j\eta, L_j\chi_j\eta\rangle - c\sqrt{\epsilon}\|\eta\|_2^2.$$

for $1 \le j \le m$. Also, since χ_0 is supported away from all the vortices,

$$\langle \chi_0\eta, L\chi_0\eta\rangle \ge c_2\|\chi_0\eta\|_{H^1}^2$$

for some $c_2 > 0$. Thus

$$\langle \eta, L\eta \rangle \quad \geq \quad \sum_{j=1}^{m} \langle \chi_j \eta, L_j \chi_j \eta \rangle + c_2 \|\chi_0 \eta\|_{H^1}^2 - c(\sqrt{\epsilon} + R^{-2}) \|\eta\|_{H^1}^2.$$

Now let $\{\tilde{T}_{jk}\}$ $(k = 1, 2)$ be the exact translational zero-eigenfunctions of L_j (see [GS] for a discussion). We have

$$|\langle \tilde{T}_{jk}, \chi_j \eta \rangle| \leq c\epsilon,$$

and

$$Im(e^{-i\chi_{(j)}} \bar{\psi}_j \chi_j \eta_\psi) - \nabla \cdot (\chi_j \eta_A) = O(R^{-1}).$$

So by the n-vortex stability result of [GS] (for $n_j = \pm 1$), we have

$$\langle \chi_j \eta, L_j \chi_j \eta \rangle \geq c_3 \|\chi_j \eta\|_{H^1}^2 - c\epsilon \|\eta\|_2^2,$$

and so

$$\langle \eta, L\eta \rangle \geq [c_4 - c(\sqrt{\epsilon} + R^{-2})] \|\eta\|_{H^1}^2 \geq c_1 \|\eta\|_{H^1}^2 \tag{103}$$

for ϵ sufficiently small.

For the Higgs model, the linearized operator acts as the identity on the momentum components:

$$L_\sigma := \mathcal{H}''(w_\sigma) = \begin{pmatrix} L_{\underline{Z},\chi} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Observing that $Q_\sigma \xi \equiv 0$ implies $P_{\underline{Z},\chi} \xi_1 = O(\sqrt{\epsilon})$ (see Eqn. (115) in Appendix 2), we have

$$\langle \xi, L_\sigma \xi \rangle \geq [\gamma - O(\sqrt{\epsilon})] \|\xi_1\|_{H^1}^2 + \|\xi_2\|_2^2.$$

This proves Lemma 3 (the upper bound is straightforward). □

Proof of Lemma 9: Set $L := L_{\underline{Z},\chi}$. First observe, using $LP_{\underline{Z},\chi} = o(1)$, that

$$\langle L\xi, L\xi \rangle = \langle L^{1/2}\xi, LL^{1/2}\xi \rangle = \langle P_{\underline{Z},\chi} L^{1/2}\xi, LL^{1/2}\xi \rangle \\ + \langle \bar{P}_{\underline{Z},\chi} L^{1/2}\xi, (P_{\underline{Z},\chi} + \bar{P}_{\underline{Z},\chi}) LL^{1/2}\xi \rangle \geq (c_1 - o(1)) \|\xi\|_{H_1}^2.$$

Now since $\|L + \Delta\|_{H^1 \to L^2} \leq c$, for any $0 < \delta < 1$ we have

$$\langle L\xi, L\xi \rangle = \delta \langle L\xi, L\xi \rangle + (1 - \delta) \langle L\xi, L\xi \rangle \\ \geq \delta \langle \Delta\xi, \Delta\xi \rangle - c\delta \|\xi\|_{H^1}^2 + (1 - \delta)(c_1 - o(1)) \|\xi\|_{H^1}^2 \geq \tilde{\gamma} \|\xi\|_{H^2}^2$$

for δ and ϵ sufficiently small. □

4.5 Remainder estimates for GL functional

Proof of Lemma 4. For $v = (\psi, A)$, set

$$R_v(\xi) := \mathcal{E}_{GL}(v + \xi) - \langle \mathcal{E}'_{GL}(v), \xi \rangle - \frac{1}{2}\langle \xi, \mathcal{E}''_{GL}\xi \rangle.$$

Here

$$\mathcal{E}'_{GL}(v) = \big(-\Delta_A\psi + \lambda(|\psi|^2 - 1)\psi, -curl^2 A - Im(\bar{\psi}\nabla_A\psi)\big)$$

and $\mathcal{E}''_{GL}(v)$ is the Hessian of $\mathcal{E}_{GL}$ at v (which we don't write out explicitly here). After some computation, we find, for $\xi = (\eta, \alpha)$,

$$\begin{aligned} R_v(\xi) = \int \Big\{ & |\alpha|^2 Re(\bar{\eta}\psi_{\underline{Z},\chi}) - \alpha \cdot Im(\bar{\eta}\nabla_{A_{\underline{Z},\chi}}\eta) \\ & + \lambda Re(\overline{\psi_{\underline{Z},\chi}}\eta)|\eta|^2 + \frac{1}{2}|\alpha|^2|\eta|^2 + \frac{\lambda}{4}|\eta|^4 \Big\} \end{aligned}$$

and so using Hölder's inequality, and the Sobolev embedding $\|g\|_p \leq c_p\|g\|_{H^1}$ in two dimensions, we obtain easily

$$|R_v(\xi)| \leq c(\|\xi\|^3_{H^1} + \|\xi\|^4_{H^1}). \tag{104}$$

□

Proof of Lemma 1. The most problematic term in $N_v(\xi)$ is of the form $\xi\nabla\xi$, so we will just bound this one (the rest are straightforward):

$$\begin{aligned} \|\xi\nabla\xi\|_{H^{-s}} = \sup_{\|\eta\|_{H^s}=1} |(\eta, \xi\nabla\xi)| &\leq \sup \|\eta\xi\|_2\|\nabla\xi\|_2 \\ &\leq c \sup \|\eta\|_p\|\xi\|_q\|\xi\|_{H^1} \leq c\|\xi\|^2_{H^1} \end{aligned}$$

where $1/p + 1/q = 1/2$ and q is taken large enough so that $H^s \subset L^p$. □

5 Appendices

5.1 Appendix 1: Operators V_σ

In this section we consider the key operators $V_\sigma := \Lambda_\sigma \mathbb{J}^{-1}\Gamma_\sigma$, where Γ_σ and Λ_σ are given in (35) and (37), and we show for them the relation (41), implying, in particular, the invertibility of V_σ. We also prove the auxiliary properties (38) and (39) of the operators Γ_σ and Λ_σ. To this end, we use the following explicit expressions for the basis vectors (33) for the tangent space $T_{w_\sigma} M_{mv}$:

$$\tau^z_{jk} = (T^{(\underline{z},\chi)}_{jk}, S^\sigma_{jk}), \tag{105}$$

$$\tau^p_{jk} = (0, T^{(\underline{z},\chi)}_{jk}), \tag{106}$$

$$\tau^\chi_x = (G^{(\underline{z},\chi)}_{\delta_x}, F^\sigma_{\delta_x}), \tag{107}$$

$$\tau^\zeta_x = (0, G^{(\underline{z},\chi)}_{\delta_x}) \tag{108}$$

where

$$S^\sigma_{jk} := ((\partial_{z_{jk}} + iA_{jk})[\phi_\sigma]_\pi, \partial_{z_{jk}}[\phi_\sigma]_E) \tag{109}$$

and

$$F^\sigma_{\delta_x} := \partial_{\chi(x)}\phi_\sigma = (i\delta_x[\phi_\sigma]_\pi, 0). \tag{110}$$

In what follows we omit the super- and sub-indices $(\underline{z}, \chi)$ and σ. Using Equations (105)- (110) it is not difficult to verify properties (38) and (39). For example, to show (39) we calculate using definitions (37) and (105)- (110),

$$\Lambda_\sigma \xi = (\alpha, \gamma, \beta, \eta),$$

where $\xi = (\xi_1, \xi_2)$

$$\alpha_{jk} := \langle T_{jk}, \xi_1\rangle + \langle S_{jk}, \xi_2\rangle,$$
$$\gamma(x) := \langle G_{\delta_x}, \xi_1\rangle + \langle F_{\delta_x}, \xi_2\rangle,$$
$$\beta_{jk} := \langle T_{jk}, \xi_2\rangle, \quad \eta(x) := \langle G_{\delta_x}, \xi_2\rangle.$$

Clearly $|\alpha| \le c\|\xi\|_{X_{r,s}}$ for any r, s, and similarly for β. Furthermore, due to (45),

$$\langle G_{\delta_x}, \xi_1\rangle = Im(\bar{\psi}\xi_\psi) - div\xi_A$$

and due to (110),

$$\langle F_{\delta_x}, \xi_2\rangle = Im(\bar{\phi}_\pi \xi_\pi).$$

Recall that $\phi := \sum_{j=1}^m p_j \cdot T^{(\underline{z},\chi)}_j + G^{(\underline{z},\chi)}_\zeta$ and that $\zeta \in H^1$. Using that $H^1(\mathbb{R}^2) \cdot H^s(\mathbb{R}^2) \subset H^{r'}(\mathbb{R}^2)$ for $r' < min(s, 1)$, we obtain

$$\|\gamma\|_{H^{r-1}} \le c(\|\xi_\psi\|_{H^{r-1}} + \|\xi_A\|_{H^r} + \|\xi_\pi\|_{H^s}) \le c\|\xi\|_{H^r \times H^s},$$

provided $r - 1 < min(s, 1)$. Similarly, we have

$$\|\eta\|_{H^{s-1}} \le c\|\xi_2\|_{H^s} \le c\|\xi\|_{H^{any} \times H^s}.$$

Summing this up, we conclude that

$$\|\Lambda_\sigma \xi\|_{Y_{r-1,s-1}} \leq c\|\xi\|_{X_{r,s}}$$

provided $r-1 < min(s,1)$, which implies (39). (38) is obtained similarly.

Now we use explicit expressions (105)- (108) for the basis (33) in order to establish equation (41). Equation (41) follows from the relations $\langle \tau, \mathbb{J}^{-1}\tau\rangle = 0$, where $\tau = \tau^z_{ij}$, τ^p_{ij}, τ^χ_x, or τ^ζ_x, and the relations

$$\langle \tau^z_{ij}, \mathbb{J}^{-1}\tau^p_{kl}\rangle = -\langle \tau^p_{ij}, \mathbb{J}^{-1}\tau^z_{kl}\rangle = \gamma_{n_i}\delta_{ik}\delta_{jl} + O(\epsilon \log^{1/2}(1/\epsilon)), \tag{111}$$

$$\langle \tau^z_{ij}, \mathbb{J}^{-1}\tau^\chi_y\rangle, \langle \tau^\chi_x, \mathbb{J}^{-1}\tau^z_{kl}\rangle = O(\sqrt{\epsilon}), \tag{112}$$

$$\begin{aligned} \langle \tau^z_{ij}, \mathbb{J}^{-1}\tau^\zeta_y\rangle &= \langle \tau^\zeta_x, \mathbb{J}^{-1}\tau^z_{kl}\rangle = \langle \tau^p_{ij}, \mathbb{J}^{-1}\tau^\chi_y\rangle \\ &= \langle \tau^\chi_x, \mathbb{J}^{-1}\tau^p_{kl}\rangle = \langle \tau^p_{ij}, \mathbb{J}^{-1}\tau^\zeta_y\rangle \\ &= \langle \tau^\zeta_x, \mathbb{J}^{-1}\tau^p_{kl}\rangle = O(\epsilon \log^{1/4}(1/\epsilon)), \end{aligned} \tag{113}$$

$$\langle \tau^\chi_x, \mathbb{J}^{-1}\tau^\zeta_y\rangle = -\langle \tau^\zeta_x, \mathbb{J}^{-1}\tau^\chi_y\rangle = -K_{xy} \tag{114}$$

where K_{xy} is the integral kernel of the operator $K = -\Delta + |\psi_{\underline{Z},\chi}|^2$.

We will not present here proofs of all the relations (111)- (114), but rather illustrate our arguments by establishing two of the relations, say $\langle \tau^z_{ij}, \mathbb{J}^{-1}\tau^\chi_y\rangle$ and $\langle \tau^z_{ij}, \mathbb{J}^{-1}\tau^\zeta_y\rangle$ (see (112) and (113)). In what follows, we omit the superscripts in $T^{(\underline{Z},\chi)}_{jk}$, $G^{(\underline{Z},\chi)}_{\delta_x}$, S^σ_{jk}, and F^σ_{jk}, and the subscripts in $\psi_{\underline{Z},\chi}$ and ϕ_σ. Using Equations (105) and (107), we obtain

$$\langle \tau^z_{jk}, \mathbb{J}^{-1}\tau^\chi_y\rangle = \langle T_{jk}, F_{\delta_y}\rangle - \langle S_{jk}, G_{\delta_y}\rangle.$$

Using the explicit expressions (44), (45), (109), and (110) for the vectors on the r.h.s, we compute

$$\langle \tau^z_{jk}, \mathbb{J}^{-1}\tau^\chi_y\rangle = -Im(e^{-i\chi}\bar{\psi}_{(jk)}\phi_\pi) + Im[\overline{(\partial_{z_{jk}} + iA_{jk})\phi_\pi}\psi] + div(\partial_{z_{jk}}\phi_A),$$

where $\psi_{(jk)}(x) = [\prod_{l\neq j}\psi^{(n_l)}(x-z_l)]([\nabla_A]_k\psi)^{(n_j)}(x-z_j)$. Recalling the definition (24) of ϕ, and using $|p| + \|\zeta\|_{H^1} < \epsilon$, we conclude that (112) is true.

To prove that $\langle \tau^z_{ij}, \mathbb{J}^{-1}\tau^\zeta_y\rangle = O(\epsilon)$, use Equations (105) and (107) to obtain

$$\langle \tau^z_{jk}, \mathbb{J}^{-1}\tau^\zeta_y\rangle = \langle T_{jk}, G_{\delta_y}\rangle.$$

Now using (44) and (45) and the equation $curl B^{(n)} = Im(\bar{\psi}^{(n)}\nabla_{A^{(n)}}\psi^{(n)})$ and estimate (7), we find

$$|\langle T_{jk}, G_{\delta_y}\rangle| = |Im(\prod_{l\neq j}\overline{\psi^{(n_l)}(x-z_l)([\nabla_A]_k\psi)^{(n_j)}(x-z_j)}\prod_m \psi^{(n_m)}(x-z_m))$$
$$- curl B^{(n_j)}(x-z_j)|$$
$$= |Im[(\prod_{l\neq j}|\psi^{(n_l)}(x-z_l)|^2 - 1)\overline{([\nabla_A]_k\psi)^{(n_j)}(x-z_j)}\psi^{(n_j)}(x-z_j)]|$$

and using Lemma 12, we see $|\langle T_{jk}, G_{\delta_y}\rangle| \leq c\epsilon\log^{1/4}(1/\epsilon)$.

5.2 Appendix 2: Proof of Lemma 6

We first note that using $Q_\sigma\xi = 0$ and therefore $Q_\sigma\partial_t\xi = [Q_\sigma, \partial_t]\xi$, and using (53) and (79), we obtain

$$\langle\mathcal{H}'(w_\sigma), Q_\sigma\partial_t\xi\rangle = \langle\phi_\sigma, ([Q_\sigma,\partial_t]\xi)_2\rangle + O(\epsilon\log^{1/4}(1/\epsilon)\|\xi\|_X\|\dot\sigma\|_{Y_{1,0}})$$

Using the above expressions for V_σ, Γ_σ, and Λ_σ, and differentiating Q_σ with respect to t leads to

$$([Q_\sigma,\partial_t]\xi)_2 = -D^{-1}_{jk,lm}\langle(\dot z_{qr}[\partial_{z_{qr}} + \begin{pmatrix} iA_{qr} & 0 \\ 0 & 0\end{pmatrix}] + \langle\partial_t^{\underline{Z}}\chi, \partial_\chi\rangle)T^{(\underline{Z},\chi)}_{jk}, \xi_2\rangle T^{(\underline{Z},\chi)}_{lm}$$
$$- G_{(-\Delta+|\psi_{\underline{Z},\chi}|^2)^{-1}\langle(\dot z_{qr}[\partial_{z_{qr}}+\begin{pmatrix} iA_{qr} & 0 \\ 0 & 0\end{pmatrix}]+\langle\partial_t^{\underline{Z}}\chi,\partial_\chi\rangle)G^{(\underline{Z},\chi)}_\delta,\xi_2\rangle}$$
$$+ O(\|\xi\|_X(|\underline{\dot p}| + \|\partial_t\zeta\|_2 + \sqrt{\epsilon}(|\underline{\dot z}| + \|\partial_t^{\underline{Z}}\chi\|_{H^1}))).$$

Using $\phi_\sigma = p_{st}T^{(\underline{Z},\chi)}_{st} + G^{(\underline{Z},\chi)}_\zeta$, we find

$$\langle\phi_\sigma, [Q_\sigma\partial_t\xi]_2\rangle = \langle\dot z_{qr}S^\sigma_{qr} - F^\sigma_{\partial_t^{\underline{Z}}\chi}, \xi_2\rangle + O(\sqrt{\epsilon}\|\xi\|_X(|\underline{\dot p}| + \|\partial_t\zeta\|_2 + \sqrt{\epsilon}(|\underline{\dot z}| + \|\partial_t^{\underline{Z}}\chi\|_{H^1})))$$

and so conclude

$$|\langle\mathcal{H}'(w_\sigma), Q_\sigma\partial_t\xi\rangle - \langle\dot z_{qr}S^\sigma_{qr} - F^\sigma_{\partial_t^{\underline{Z}}\chi}, \xi_2\rangle| \leq c(\sqrt{\epsilon}\|\xi\|_X(|\underline{\dot p}| + \|\partial_t\zeta\|_2 + \sqrt{\epsilon}\log^{1/4}(1/\epsilon)(|\underline{\dot z}| + \|\partial_t^{\underline{Z}}\chi\|_{H^1}))).$$

We turn now to computation of $\langle\mathcal{H}'(w_\sigma), Q_\sigma\mathbb{J}L_\sigma\xi\rangle$. To do this, we have to refine the above computations, and compute Q_σ up to $O(\epsilon)$. We find

$$Q_\sigma = \begin{pmatrix} P_{\underline{Z},\chi} & 0 \\ 0 & P_{\underline{Z},\chi}\end{pmatrix} + \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0\end{pmatrix} + O(\epsilon\log^{1/2}(1/\epsilon)) \qquad (115)$$

where $P_{\underline{Z},\chi}$ denotes the orthogonal projection onto the span of the vectors $T_{jk}^{(\underline{Z},\chi)}$ and $G_\gamma^{(\underline{Z},\chi)}$, and Q_{12} and Q_{21} are $O(\sqrt{\epsilon})$. We will need the explicit form of Q_{21}:

$$Q_{21} = D^{-1}\langle T^{(\underline{Z},\chi)}, \cdot\rangle S^\sigma + (-\Delta + |\psi_{\underline{Z},\chi}|^2)^{-1}\langle G^{(\underline{Z},\chi)}, \cdot\rangle F^\sigma \\ - D^{-1}\langle S^\sigma, \cdot\rangle T^{(\underline{Z},\chi)} - (-\Delta + |\psi_{\underline{Z},\chi}|^2)^{-1}\langle F^\sigma, \cdot\rangle G_\sigma.$$

We have

$$\begin{aligned}\langle [\mathcal{H}'(w_\sigma)]_2, [Q_\sigma \mathbb{J} L_\sigma \xi]_2\rangle &= -\langle Q_\sigma \mathbb{J}\mathcal{H}'(w_\sigma), L_\sigma\xi\rangle \\ &= \langle [Q_\sigma(\phi_\sigma, -\mathcal{E}'(v_{\underline{Z},\chi}))]_1, \mathcal{E}''(v_{\underline{Z},\chi})\xi_1\rangle + \langle [Q_\sigma\phi_\sigma, -\mathcal{E}'(v_{\underline{Z},\chi}))]_2, \xi_2\rangle \\ &= \langle P_{\underline{Z},\chi}\phi + O(\epsilon^{3/2}\log^{1/2}(1/\epsilon)), \mathcal{E}''(v_{\underline{Z},\chi})\xi_1\rangle \\ &\quad + \langle -P_{\underline{Z},\chi}\mathcal{E}'(v_{\underline{Z},\chi}) + Q_{21}\phi + O(\epsilon^{3/2}\log^{1/2}(1/\epsilon)), \xi_2\rangle \\ &= \langle \mathcal{E}''(v_{\underline{Z},\chi})P_{\underline{Z},\chi}\phi, \xi_1\rangle - \langle \mathcal{E}'(v_{\underline{Z},\chi}), P_{\underline{Z},\chi}\xi_2\rangle + \langle Q_{21}\phi, \xi_2\rangle \\ &\quad + O(\epsilon^{3/2}\log^{1/2}(1/\epsilon)\|\xi\|_X).\end{aligned}$$

Now use $\mathcal{E}''(v_{\underline{Z},\chi})P_{\underline{Z},\chi} = O(\epsilon\log^{1/2}(1/\epsilon))$ and the fact that

$$0 = Q_\sigma\xi = (P_{\underline{Z},\chi}\mathbf{1} + O(\sqrt{\epsilon}))\xi$$

which implies $P_{\underline{Z},\chi}\xi_j = O(\sqrt{\epsilon}\|\xi\|_X)$ to find

$$\langle \mathcal{H}'(w_\sigma), Q_\sigma\mathbb{J}L_\sigma\xi\rangle = \langle Q_{21}\phi, \xi_2\rangle + O(\epsilon^{3/2}\log^{1/2}(1/\epsilon)\|\xi\|_X).$$

From the form of Q_{21} given above, and the fact that $\langle T^{(\underline{Z},\chi)}, \xi_2\rangle$ and $\langle G^{(\underline{Z},\chi)}, \xi_2\rangle$ are $O(\sqrt{\epsilon}\|\xi\|_X)$, we see

$$\begin{aligned}\langle \mathcal{H}'(w_\sigma), Q_\sigma\mathbb{J}L_\sigma\xi\rangle &= \langle D^{-1}\langle T^{(\underline{Z},\chi)}, \phi_\sigma\rangle S^\sigma + (-\Delta + |\psi_{\underline{Z},\chi}|^2)^{-1}\langle G^{(\underline{Z},\chi)}, \phi_\sigma\rangle F^\sigma, \xi_2\rangle \\ &\quad + O(\epsilon^{3/2}\log^{1/2}(1/\epsilon)\|\xi\|_X) \\ &= \langle p_{jk}S^\sigma_{jk} - F_\zeta, \xi_2\rangle + O(\epsilon^{3/2}\log^{1/2}(1/\epsilon)\|\xi\|_X).\end{aligned}$$

Combining this with the above computation of $\langle \mathcal{H}'(w_\sigma), Q_\sigma\partial_t\xi\rangle$, with the facts that $|\dot{\underline{p}}| = |\dot{\underline{p}} + \nabla_z W(\underline{z})| + O(\epsilon)$ and $|\dot{\underline{z}}| + \|\partial_t^{\underline{Z}}\chi\|_{H^1} = |\dot{\underline{z}} - \underline{p}| + \|\partial_t^z\chi - \zeta\|_{H^1} + O(\sqrt{\epsilon})$ proves Lemma 6 □

5.3 Appendix 3: Two technical lemmas

Lemma 12 *Let $0 < \alpha \le \beta$ and $0 \le \delta, \gamma < 3/2$. Then*

$$\int_{\mathbb{R}^2} \frac{e^{-\alpha|x|}e^{-\beta|x-a|}}{|x|^\gamma|x-a|^\delta}dx \le c\frac{e^{-\alpha|a|}}{|a|^{\gamma+\delta-2}} \begin{cases} |a|^{-1/2} & \alpha = \beta \\ |a|^{\delta-2} & \alpha < \beta \end{cases}. \tag{116}$$

Proof: We prove only the case $\alpha = \beta$, since the remaining cases follow from Lemma 13 below. Define I by

$$
\begin{aligned}
2|a|^{2-\gamma-\delta} I &:= \int_{\mathbb{R}^2} \frac{e^{-\alpha|x|-\beta|x-a|}}{|x|^\gamma |x-a|^\delta} dx \\
&= \int_0^\infty \frac{dr}{r^{\gamma-1}} \int_0^{2\pi} d\theta \frac{e^{-\alpha r - \beta\sqrt{r^2+|a|^2-2r|a|\cos(\theta)}}}{(r^2+|a|^2-2r|a|\cos(\theta))^{\delta/2}}.
\end{aligned}
$$

Changing variables to $u = 1-\cos(\theta)$ and $t = r/|a|$, and using $\alpha = \beta$, we estimate

$$
\begin{aligned}
I &\leq \int_0^\infty \frac{dt}{t^{\gamma-1}} \int_0^1 \frac{du}{\sqrt{u}} \frac{e^{-\alpha|a|(t+\sqrt{(t-1)^2+2tu})}}{((t-1)^2+2tu)^{\delta/2}} \\
&= \int_0^{1/2}\int_0^1 + \int_{1/2}^1\int_0^1 + \int_1^\infty\int_0^1 =: I_1 + I_2 + I_3.
\end{aligned} \tag{117}
$$

Using $(t-1)^2 + 2tu \geq [1-t+tu]^2$ (for $0 \leq u \leq 1$), we have

$$
\begin{aligned}
I_1 &\leq c\int_0^{1/2} \frac{dt}{t^{\gamma-1}} \int_0^1 \frac{du}{\sqrt{u}} e^{-\alpha|a|(1+tu)} \\
&= ce^{-\alpha|a|}/\sqrt{|a|} \int_0^{1/2} \frac{dt}{t^{\gamma-1/2}} \int_0^{|a|t} e^{-\alpha v}/\sqrt{v}dv \\
&\leq ce^{-\alpha|a|}/\sqrt{|a|}.
\end{aligned} \tag{118}
$$

Now estimating $\sqrt{(t-1)^2+2tu} \geq \sqrt{(1-t)^2+u}$ for $t \geq 1/2$ and changing the variables of integration as $x = |a|(1-t)$ and $y = |a|\sqrt{u}$ we obtain

$$
\begin{aligned}
I_2 &\leq ce^{-\alpha|a|}/|a|^{2-\delta} \int_0^{|a|} dx \int_0^{|a|} dy \frac{e^{-\alpha(-x+\sqrt{x^2+y^2})}}{(x^2+y^2)^{\delta/2}} \\
&\leq ce^{-\alpha|a|}/|a|^{2-\delta} \int_0^{|a|} dx \int_0^{|a|} dy \frac{e^{-\alpha y^2/(2\sqrt{x^2+y^2})}}{(x^2+y^2)^{\delta/2}} \\
&\leq e^{-\alpha|a|}/|a|^{2-\delta} \int_0^{2|a|} \frac{dr}{r^{\delta-1}} \int_0^{2\pi} d\theta e^{-\alpha r\cos^2(\theta)/2}.
\end{aligned}
$$

We have

$$\begin{aligned}\int_0^{2\pi} d\theta e^{-\alpha r\cos^2(\theta)/2} &= c\int_{-1}^{1}\frac{ds}{\sqrt{1-s^2}}e^{-r(1+s)\alpha/4}\\ &\le c\int_0^1 \frac{ds}{\sqrt{1-s}}e^{-r(1-s)\alpha/4} + ce^{-r\alpha/4}\\ &= \frac{c}{\sqrt{r}}\int_0^r \frac{dv}{\sqrt{v}}e^{-\sqrt{\alpha}/4} + ce^{-r\alpha/4} \le c/\sqrt{r}.\end{aligned}$$

Hence

$$I_2 \le c\frac{e^{-\alpha|a|}}{\sqrt{|a|}}. \tag{119}$$

Finally,

$$\begin{aligned} I_3 &\le \int_0^\infty \frac{d\tau}{(\tau+1)^{\gamma-1}}\int_0^1 \frac{du}{\sqrt{u}}\frac{e^{-\alpha|a|(\tau+1+\sqrt{\tau^2+2u})}}{(\tau^2+2u)^{\delta/2}}\\ &= 2e^{-\alpha|a|}\int_0^\infty \frac{d\tau}{(\tau+1)^{\gamma-1}}\int_0^1 dv\frac{e^{-\alpha|a|(\tau+\sqrt{\tau^2+2v^2})}}{(\tau^2+2v^2)^{\delta/2}}\\ &\le ce^{-\alpha|a|}/|a|^{2-\delta}.\end{aligned} \tag{120}$$

Estimates (117)- (120) imply (116). □

Lemma 13 *Suppose $b(x)$ is a function satisfying $|b(x)| \le ce^{-m|x|}$ for some $m > 1$, and $e(x)$ is a bounded function with asymptotic behaviour $e(x) = c_1 e^{-|x|}/\sqrt{|x|}(1+O(1/|x|))$ as $|x| \to \infty$. Fix $z \in \mathbb{R}^2$ and set*

$$I(z) := \int_{\mathbb{R}^2} b(x)e(x-z)dx.$$

Then

$$I(z) = c_1 e^{-|z|}/\sqrt{|z|}\int_{\mathbb{R}^2} e^{x\cdot z/|z|}b(x)dx[1+O(1/|z|)] \tag{121}$$

as $|z| \to \infty$.

Proof: Choose α with $1/m < \alpha < 1$. Let $D_{\alpha|z|}$ denote the disk of radius $\alpha|z|$ about the origin. We have

$$|\int_{\mathbb{R}^2\setminus D_{\alpha|z|}} b(x)e(x-z)dx| \le c\int_{\alpha|z|}^\infty e^{-mr}rdr \le c|z|e^{-m\alpha|z|} << e^{-|z|}/|z|^p \tag{122}$$

for any p. On $D_{\alpha|z|}$, $|x| << |z|$, and we Taylor expand:

$$|z-x|^{-1/2} = |z|^{-1/2}[1+O(|x|/|z|)],$$

$$e^{-|z-x|} = e^{-|z|}e^{x\cdot z/|z|}[1+O(|x|^2/|z|)],$$

yields

$$\int_{D_{\alpha|z|}} b(x)e(x-z)dx = e^{-|z|}/\sqrt{|z|}\int_{D_{\alpha|z|}} e^{x\cdot z/|z|}b(x)[1+O(|x|^2/|z|)]. \quad (123)$$

Estimates (122) and (123) together yield (121). $\square$

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