Spectral theory for a class of periodically perturbed unbounded Jacobi matrices: elementary methods

Jan Janas

Insitute of Mathematics, Polish Academy of Sciences
ul. sw. Tomasza 30, 31-027 Krakow, Poland

Serguei Naboko

Department of Mathematical Physics, Insitute of Physics
St. Petersburg University, Ulianovkaia 1
198904 St. Petergoff, St. Petersburg, Russia

Gunter Stolz

Department of Mathematics, University of Alabama at Birmingham
CH 452, Birmingham, AL 35294-1170, USA

Dedicated to Norrie Everitt on the occasion of his 80th birthday

Abstract

We use elementary methods to give a full characterization of the spectral properties of unbounded Jacobi matrices with zero diagonal and off-diagonal entries of the type $\lambda_n = n^\alpha + c_n$, where $1/2 < \alpha \leq 1$ and c_n is a real periodic sequence. The spectral properties depend strongly on the parity of the minimal period of c_n. The methods used are asymptotic diagonalization techniques, including the finite difference version of Levinson's theorem, subordinacy theory, and the variational principle.

Keywords: Jacobi Matrices, Spectral Theory

1 Introduction

Let J be a Jacobi matrix acting in $l^2(\mathbb{N})$ by the formula

$$(Ju)(n) = \lambda_{n-1}u(n-1) + \lambda_n u(n+1),\ n \geq 1$$

with the condition $u(0) = 0$. In this paper we shall consider the case of weights $\lambda_n = n^\alpha + c_n$, $\alpha > 0$, where c_n is a periodic sequence of real numbers such that $\lambda_n > 0$ for all n. As is seen from the methods of proof, essentially all of our results remain the same if finitely many entries of J are changed. Denote by T the minimal period of c_n. In the special case of $T = 2$ and $\alpha = 1$ the Jacobi matrix with weights $n + c_n$ has been studied (from the spectral point of view) by M. Moszynski in [M] and by J. Dombrowski and S. Pedersen in [D–P]. We shall study the general case of arbitrary T and α. In the case $\alpha \leq 1$ we know that J is essentially selfadjoint on the finite vectors (with finitely many coordinates different from zero) due to the Carleman condition: $\sum_k \lambda_k^{-1} = \infty$, see Theorem 1.3 in [Ber, p. 504].

J is not essentially selfadjoint on the finite vectors if $\alpha > 1$. This also follows from the results in [Ber] (note that J is a bounded perturbation of the Jacobi matrix with weights n^α and that the latter is not essentially selfadjoint due to Theorem 1.5 of [Ber, p. 507]). Thus all selfadjoint extensions of J have discrete spectrum (see the Corollary from [Ber, p. 528]). Therefore we assume in the following that $\alpha \leq 1$.

One of our main goals here is to provide elementary techniques which demonstrate the recent discovery that Jacobi matrices of the type considered here provide examples of unbounded Jacobi matrices with a spectral gap. This topic is now under very active investigation and new results appear in rapid succession. It is not our aim to survey the most general results, for which we rather provide references.

The case $T = 2$ will be treated separately in Theorem 1.3 below, where our methods will allow for arbitrary $\alpha \in (0, 1)$ (but we will exclude $\alpha = 1$). In this case one can also study the spectral structure of Jacobi operators with zero diagonal and n^α replaced by a more general "regular" sequence j_n tending to infinity as n goes to infinity, see [DJMP].

For general period T we will state our results for $\alpha \in (\frac{1}{2}, 1]$. One can obtain the same results for any positive α but the analysis is more complicated (and has been done in the recent paper by Janas and Moszynski [JM2]). Nevertheless, the elementary methods used here are strong enough to uncover the spectral structure of J. We shall use the so called grouping in blocks approach as already applied successfully in our earlier paper [JN1]. This approach, when combined with the discrete version of the Levinson theorem (with l^1 perturbation), will allow to find the asymptotics of generalized eigenvectors of J at infinity (see Section 4). Knowledge of the asymptotics of generalized eigenvectors and the Gilbert-Pearson theory of subordinacy is sufficient to determine the character of the spectrum of J. It will be shown that the spectral properties of J depend crucially on the parity of T. The asymptotics of generalized eigenvectors of J (found below in the case of even T) will also be used in a forthcoming paper to demonstrate the sharpness of Combes-Thomas type estimates for the resolvent of general Jacobi operators [JNS].

We will give proofs of the following results.

Theorem 1.1 *Let J be the Jacobi matrix with weights*

$$\lambda_n = n^\alpha + c_n,\ \alpha \in \left(\frac{1}{2},\ 1\right], \tag{1.1}$$

where c_n is a periodic sequence of real numbers with minimal odd period such that $\lambda_n > 0$ for all n. Then the spectrum of J is purely absolutely continuous in $\mathbb{R}$.

In the proof Theorem 1.1 as well as in the statement and proof of the following Theorem 1.2 we will use the abbreviation

$$a := \sum_{s=1}^{L} (c_{2s} - c_{2s-1}).$$

Theorem 1.2 *If λ_n is given by (1.1) and $T = 2L$ is the even minimal period, then $\sigma(J)$ is purely absolutely continuous in $\mathbb{R} \setminus \left[-\frac{|a|}{L}, \frac{|a|}{L}\right]$ and is discrete in $\left(-\frac{|a|}{L}, \frac{|a|}{L}\right)$. Moreover, $0 \in \sigma_p(J)$ if and only if $a > 0$.*

Theorem 1.3 *In the case $T = 2$ and for any $\alpha \in (0, 1)$, the spectrum of J in $(-|c_1 - c_2|, |c_1 - c_2|)$ is finite, i.e. no accumulation of eigenvalues appears at $\pm|c_1 - c_2|$.*

In fact, the result of Theorem 1.3 also holds for $\alpha = 1$ as was proven in [D–P] and [M] with different methods. We exclude it from our presentation here as the elementary method of proof which we provide in Section 5 (using quadratic forms) would become technically less transparent in this case.

Acknowledgements: This work was supported through NATO Collaborative Linkage Grant PST.CLG.976441. The work of G. S. was also supported through NSF Grant DMS 0070343.

2 Preliminaries

We start with some preparatory computations which will be used in the next sections. Fix $\lambda \in \mathbb{R}$. A complex sequence $u(n)$ $(n \geq 1)$ satisfying the equation

$$(Ju)(n) = \lambda u(n),\ n \geq 2 \tag{2.1}$$

is called a generalized eigenvector for J and λ. Following a standard procedure we rewrite the scalar system (2.1) in the vector form

$$\widetilde{u}(n+1) = B(n)\widetilde{u}(n), \tag{2.2}$$

here $\widetilde{u}(n) := \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}$ and $B(n) := \begin{pmatrix} 0 & 1 \\ -\lambda_{n-1}\lambda_n^{-1} & \lambda\lambda_n^{-1} \end{pmatrix}$ is the transfer matrix. Due to the periodic perturbation the entries of $B(n)$ are not sufficiently regular to directly apply the well

known methods used already in our previous papers, [JN2], [JN1], [S]. However the products of $B(k)$ of length T behave more regularly and so we define

$$C(n) := \prod_{s=0}^{T-1} B(nT+s). \tag{2.3}$$

Now consider the equation

$$x(n+1) = C(n)x(n),\ n \geq 1. \tag{2.4}$$

Note that for any solution $x(\cdot)$ of (2.4) there is a unique generalized eigenvector $u(\cdot)$ for J and λ satisfying $\widetilde{u}(nT) = x(n)$. It is easy to check that for $\alpha \leq 1$

$$B(n) = E + n^{-\alpha}\begin{pmatrix} 0 & 0 \\ c_n - c_{n-1} & \lambda \end{pmatrix} + \alpha n^{-1}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + O\left(n^{-2\alpha}\right),$$

where $E := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In what follows $O\left(n^{-2\alpha}\right)$ terms can be different in various formulas (but we shall keep the notation to stress their order of decay). As usual it is useful to work with pairs of the $B(k)$'s.

$$\begin{aligned} B(k+1)B(k) &= -I + \alpha k^{-1} I + k^{-\alpha}\begin{pmatrix} c_k - c_{k-1} & \lambda \\ -\lambda & c_{k+1} - c_k \end{pmatrix} + O\left(k^{-2\alpha}\right) \\ &= -\left(1 - \alpha(k+1)^{-1}\right)\left[I - k^{-\alpha}\begin{pmatrix} c_k - c_{k-1} & \lambda \\ -\lambda & c_{k+1} - c_k \end{pmatrix} + O\big(k^{-2\alpha}\big)\right]. \end{aligned} \tag{2.5}$$

In particular, (2.5) will be used below to find a suitable form of $C(n)$ given by (2.3).

Although the spectral analysis of J looks similar for odd and even T, we shall deal with each case separately because there are some important differences. Below, $[x]$ stands for the integer part of x.

3 Odd case

Proof of Theorem 1.1. Let $T = 2L + 1$. The asymptotic behaviour of solutions of (2.2) is reduced to the asymptotic analysis of the products $\prod_{n=1}^{N} C(n)$. Thus we start by finding a suitable form of $C(n)$.

Using (2.5) we can write

$$\begin{aligned} C(n) &= \prod_{s=1}^{L}\big(B(nT+2s)B(nT+2s-1)\big)\cdot B(nT) \\ &= \prod_{s=1}^{L}\Bigg[-\left(1-\alpha(nT+2s)^{-1}\right) \\ &\quad \left(I-(nT+2s-1)^{-\alpha}\begin{pmatrix} c_{2s-1}-c_{2s-2} & \lambda \\ -\lambda & c_{2s}-c_{2s-1}\end{pmatrix}+O\left(n^{-2\alpha}\right)\right)\Bigg]\cdot B(nT) \\ &= (-1)^L\left(1-\alpha(nT)^{-1}\right)^L\left[I-(nT)^{-\alpha}\begin{pmatrix}-a & b\\ -b & a\end{pmatrix}+O\left(n^{-2\alpha}\right)\right]\cdot B(nT) \\ &= (-1)^L\left(1-\alpha(nT)^{-1}\right)^L E(n), \end{aligned}$$

where $\Delta_n = c_T - c_{T-1} + \alpha(nT)^{\alpha-1}$, $b=\lambda L$ and

$$E(n) := E + (nT)^{-\alpha}\begin{pmatrix} b & a \\ a+\Delta_n & \lambda+b\end{pmatrix} + O\left(n^{-2\alpha}\right).$$

In the above computations we have replaced $1-\alpha(nT+2s)^{-1}$ by $1-\alpha(nT)^{-1}$ and $(nT+2s-1)^{-\alpha}$ by $(nT)^{-\alpha}$, for $s=1,\dots,L$, which is correct up to $O\left(n^{-2\alpha}\right)$.

Note that $E(\cdot)$ belongs to D^1 (the class of bounded sequences of bounded variation) and $E(n)\to E$, as $n\to\infty$. Applying Lemma 1.7 of [JM1] we obtain a sequence of invertible matrices S_n in D^1 and a sequence of diagonal matrices $\{D_n\}\in D^1$ such that $E(n)=S_nD_nS_n^{-1}$ whenever $n\geq n_0$ for some positive integer n_0. Since $\{S_n\}\in D^1$ and the norms of S_n^{-1} are uniformly bounded we have

$$\begin{aligned} \prod_{n=n_0}^{N}\left\|S_{n+1}^{-1}S_n\right\| &= \prod_{n=n_0}^{N}\left\|S_{n+1}^{-1}\left[S_{n+1}-(S_{n+1}-S_n)\right]\right\| \\ &\leq \prod_{n=n_0}^{N}\left(1+\left\|S_{n+1}^{-1}\right\|\,\|S_{n+1}-S_n\|\right)\leq M, \end{aligned}$$

for some $M>0$ and all $N\geq n_0$.

It follows (due to the fact that D_n has complex conjugate eigenvalues and thus $\|D_n\|^2 = |\det D_n|$) that

$$\begin{aligned} \left\|\prod_{n=n_0}^{N}E(n)\right\|^2 &\leq \|S_N\|^2\left\|S_{n_0}^{-1}\right\|^2\prod_{n=n_0}^{N-1}\left\|S_{n+1}^{-1}S_n\right\|^2\prod_{n=n_0}^{N}|\det D_n| \\ &\leq M_1\prod_{n=n_0}^{N}|\det E(n)|, \end{aligned} \tag{3.1}$$

for some $M_1 > 0$. Since $C(n) = (-1)^L(1-\alpha(nT)^{-1})^L E(n)$, using (3.1) we have (for $NT \le n < (N+1)T$ and s_0 sufficiently large)

$$\begin{aligned}
\left\| \prod_{k=T}^{n} B(k) \right\|^2 &= \left\| B(n)\dots B(NT) \prod_{s=1}^{N-1} C(s) \right\|^2 \\
&\le M_2 \left\| \prod_{s=1}^{N-1} C(s) \right\|^2 \le M_3 \prod_{s=s_0}^{N-1} |\det C(s)| \\
&= M_3 \prod_{k=s_0T}^{NT-1} |\det B(k)| \\
&\le M_4 \prod_{k=2}^{n} |\det B(k)| \,,
\end{aligned}$$

for some positive constants M_2, M_3 and M_4.

But $\det B(k) = \lambda_{k-1}/\lambda_k$ and the above estimates imply that for any solution $\widetilde{u}$ of (2.2), $\|\widetilde{u}(n)\|^2 \le \text{const}\, \lambda_n^{-1}$. Using the generalized Behncke-Stolz Lemma, see [JN1], it follows that (2.1) has no subordinate solutions. Finally, by applying Gilbert-Kahn-Pearson theory [KP] we conclude that J is purely absolutely continuous in $\mathbb{R}$. The proof of Theorem 1.1 is complete.

We emphasize that in the odd case the analysis is simple because the matrices $E(n)$ are small real perturbations of the real matrix E with elliptic spectrum (complex conjugate eigenvalues). Hence the spectrum of $E(n)$ is again elliptic and this leads to the conclusion about pure absolute continuity of the spectrum of J.

4 Even case

Proof of Theorem 1.2. If the minimal period T is even, i.e. $T = 2L$, then the spectral structure of J is more complicated. Again, let λ be real. By repeating the computation in pairs of transfer matrices we have

$$\begin{aligned}
C(n) &:= \prod_{s=0}^{T-1} B(nT+s) \\
&= (-1)^L \left(1-\alpha(nT)^{-1}\right)^L \\
&\quad \prod_{j=0}^{L-1} \left[I - (nT)^{-\alpha} \begin{pmatrix} c_{2j} - c_{2j-1} & \lambda \\ -\lambda & c_{2j+1} - c_{2j} \end{pmatrix} + O\left(n^{-2\alpha}\right) \right] \\
&= d_n \left[I - (nT)^{-\alpha} \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} + O\left(n^{-2\alpha}\right) \right], \qquad (4.1)
\end{aligned}$$

where $d_n := (-1)^L(1-\alpha(nT)^{-1})^L$, $b = \lambda L$.

Denote by $A := \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$, such that $\sigma(A) = \{z_+, -z_+\}$, where $z_+ = (a^2 - b^2)^{1/2}$. Provided that $a^2 - b^2$ is different from zero, we may diagonalize A through $A = SD_1S^{-1}$, where $D_1 =$

diag $(z_+, -z_+)$. It is clear that the properties of generalized eigenvectors of J depend strongly on the sign of $a^2 - b^2$.

Assume first that

i) $a^2 - b^2 < 0$.

Then z_+ is pure imaginary. We have

$$\left\| \prod_{n=1}^{N} C(n) \right\|^2 = \left| \prod_{n=1}^{N} d_n \right|^2 \left\| S \prod_{n=1}^{N} \left[I - (nT)^{-\alpha} D_1 + O\left(n^{-2\alpha}\right) \right] S^{-1} \right\|^2. \tag{4.2}$$

But

$$I - (nT)^{-\alpha} D_1 + O\left(n^{-2\alpha}\right) = \exp\left[-(nT)^{-\alpha} D_1\right] \left[I + O\left(n^{-2\alpha}\right) \right].$$

Therefore the right hand side of (4.2) can be estimated from above by

$$C_1 \left(N^{-\frac{\alpha}{T}}\right)^{2L} \prod_{n=1}^{N} \left(1 + O\left(n^{-2\alpha}\right)\right) \leq C_2 N^{-\alpha},$$

where $C_1, C_2 > 0$. This is obvious because the matrices $\exp[-(nT)^{-\alpha} D_1]$ are unitary.

Hence, again $\|\widetilde{u}(n)\| \leq M \lambda_n^{-1}$ and so (2.1) has no subordinate solutions. Consequently J is purely absolutely continuous in $\mathbb{R} \setminus \left[\frac{-|a|}{L}, \frac{|a|}{L}\right]$.

In the case

ii) $a^2 - b^2 > 0$

the analysis is more complicated (the matrices $\exp[-(nT)^{-\alpha} D_1]$ are no longer unitary!). Moreover, the matrices $C(n)$ have the form:

$$C(n) = d_n S \left[I - n^{-\alpha} D + O\left(n^{-2\alpha}\right) \right] S^{-1}, \tag{4.3}$$

where $D = \mathrm{diag}\,(\eta, -\eta)$, $T^{-\alpha} A = S D S^{-1}$, and $\eta = (a^2 - b^2)^{\frac{1}{2}} T^{-\alpha}$.

Since the identity has a double root we can not apply the standard version of the discrete Levinson theorem, see [E]. However, we can use the idea of grouping in blocks (developed in [JN1]) to find the asymptotics of $\widetilde{u}(n)$. As a matter of fact, this is a new version of the grouping in blocks method. It will be based on the "accumulation" of slight differences between two eigenvalues of $C(n)$ over sufficiently long blocks of indices, eventually allowing us to apply the standard version of the discrete Levinson theorem.

Assume first that $\alpha < 1$. Let $\mathbb{N} = \bigcup_{l=1}^{\infty} (N_{l-1}, N_l]$ be the decomposition of natural numbers into an infinite union of intervals $(N_{l-1}, N_l]$ satisfying

$$\sum_{k=N_{l-1}+1}^{N_l} k^{-\alpha} = 1 + \varepsilon_l, \tag{4.4}$$

where $\varepsilon_l = O(N_l^{-\alpha})$ for large l. One can choose $N_0 = 0$ and $N_l = [s_\alpha l^{\frac{1}{1-\alpha}}]$ for $l \geq 1$, where $s_\alpha := (1-\alpha)^{\frac{1}{1-\alpha}}$. Therefore, $\varepsilon_l = O(l^{-\frac{\alpha}{1-\alpha}})$ is summable (remember that $\alpha > \frac{1}{2}$).

Note that

$$\begin{aligned} C(n) &= d_n S\left[I - n^{-\alpha}D + O\left(n^{-2\alpha}\right)\right] S^{-1} \\ &= d_n S\left[\exp\left(-n^{-\alpha}D\right) + O\left(n^{-2\alpha}\right)\right] S^{-1} \\ &= d_n S\left[I + O\left(n^{-2\alpha}\right)\right] \exp\left(-n^{-\alpha}D\right) S^{-1} \end{aligned} \tag{4.5}$$

Using (4.5) we can write (note that S does not depend on k)

$$\begin{aligned} \prod_{N_{l-1}+1}^{N_l} C(k) &= \prod_{N_{l-1}+1}^{N_l} \left\{d_k S\left[I + O\left(k^{-2\alpha}\right)\right] \exp\left[-k^{-\alpha}D\right] S^{-1}\right\} \\ &= S \prod_{N_{l-1}+1}^{N_l} \left[I + O\left(k^{-2\alpha}\right)\right] \prod_{N_{l-1}+1}^{N_l} \left(d_k \exp\left[-k^{-\alpha}D\right]\right) S^{-1}, \end{aligned} \tag{4.6}$$

the last equality holds due to the uniform boundedness of $\sum_{k=N_{l-1}+1}^{N_l} k^{-\alpha}$ in l.

Moreover, direct computation shows that

$$\prod_{k=N_{l-1}+1}^{N_l} \left(I + O\left(k^{-2\alpha}\right)\right) = I + O\left(l^{-\frac{\alpha}{1-\alpha}}\right) \tag{4.7}$$

(because $N_l - N_{l-1} \sim s_\alpha(l^{\frac{1}{1-\alpha}} - (l-1)^{\frac{1}{1-\alpha}}) \sim s_\alpha(1-\alpha)^{-1} l^{\frac{\alpha}{1-\alpha}}$ for large l).

Combining (4.4), (4.6) and (4.7) we have

$$\begin{aligned} \prod_{n=1}^{N_s} C(n) &= \prod_{l=1}^{s} \left(\prod_{k=N_{l-1}+1}^{N_l} C(k) \right) \\ &= S\left(\prod_{n=1}^{N_s} d_n\right) \prod_{l=1}^{s} \left[\left(I + O\left(l^{-\frac{\alpha}{1-\alpha}}\right)\right) \exp\left(-\left(1+\varepsilon_l\right) D\right)\right] S^{-1}. \end{aligned} \tag{4.8}$$

But

$$\left(I + O\left(l^{-\frac{\alpha}{1-\alpha}}\right)\right) \exp\left(-\left(1+\varepsilon_l\right) D\right) = \exp\left(-\left(1+\varepsilon_l\right) D\right) + W_l,$$

where $||W_l|| = O(l^{-\frac{\alpha}{1-\alpha}})$.

Thus if we set $n = N_s$ in (2.4), then for all $s \geq 1$,

$$x(N_s+1) = S\left(\prod_{n=1}^{N_s} d_n\right) \prod_{l=1}^{s} \left(\exp(-(1+\varepsilon_l)D) + W_l\right) S^{-1} x(1). \tag{4.9}$$

Setting

$$w(s) := \left(\prod_{n=1}^{N_s} d_n\right)^{-1} S^{-1} x(N_s+1)$$

for $s \geq 1$ and $w(0) = S^{-1}x(1)$, then (4.9) becomes equivalent to

$$w(s) = \prod_{l=1}^{s} \left(\exp(-(1+\varepsilon_l)D) + W_l\right) w(0), \qquad s \geq 1. \tag{4.10}$$

Applying the discrete Levinson theorem (see [E]) we obtain a solution w_1 of (4.10) such that

$$w_1(s) = \prod_{l=1}^{s} \exp(-\eta(1+\varepsilon_l))(e_1 + o(1))$$

for large s.

Thus the solution x_1 of (2.4) with $x_1(1) = Sw_1(0)$ satisfies

$$\begin{aligned} x_1(N_s+1) &= \left(\prod_{n=1}^{N_s} d_n\right) S w_1(s) \\ &= S\left(\prod_{n=1}^{N_s} d_n\right) \exp\left(-\eta \sum_{k=1}^{N_s} k^{-\alpha}\right)(e_1 + o(1)). \end{aligned} \tag{4.11}$$

One can extend formula (4.11) to arbitrary $n \in (N_{s-1}, N_s]$. Indeed, using the argument from (4.6) we have

$$\begin{aligned} &C(n)C(n-1)\dots C(N_{s-1}+1) \\ &= S \prod_{k=N_{s-1}+1}^{n} d_k \exp\left[-\sum_{N_{s-1}+1}^{n} k^{-\alpha} D\right](I + o(1))\, S^{-1}, \end{aligned} \tag{4.12}$$

as $s \to \infty$.

Combining (4.11) and (4.12) we obtain

$$x_1(n+1) = S \prod_{k=1}^{n} \left[(-1)^L \left(1 - \frac{\alpha}{2k}\right)\right] \exp\left[-\eta \sum_{k=1}^{n} k^{-\alpha}\right](e_1 + o(1)) \tag{4.13}$$

as $n \to \infty$. In particular $\{x_1(n)\} \in l^2$.

There is a unique generalized eigenvector $\widetilde{u}_1$ for J and λ with $\widetilde{u}_1(nT) = x_1(n)$ for all n. Thus $\{\widetilde{u}_1(nT)\} \in l^2$ and, as $\sup_n \|B(n)\| < \infty$, the whole sequence $\{\widetilde{u}_1(n)\}$ is in l^2. Since λ is an arbitrary point in $(-|a|/L, |a|/L)$, it follows that the spectrum of J in this interval is pure point.

In the case $\alpha = 1$, the grouping in blocks approach is also efficient but we have to choose integers N_l to satisfy the condition: $\sum_{s=N_{l-1}}^{N_l} s^{-1} = 1 + \varepsilon_l$, where $\{\varepsilon_l\} \in l^1$ (put $N_l = [e^l]$, then $\varepsilon_l = O(e^{-l})$).

By repeating the reasoning given for $\alpha < 1$ above it is clear that $\sigma(J)$ is pure point in the gap $\left(-\frac{|a|}{L}, \frac{|a|}{L}\right)$ also in the case $\alpha = 1$.

Finally, for $\lambda = 0$ direct computation shows that $0 \in \sigma_p(J)$ if and only if $a > 0$. The proof of Theorem1.2 is now complete.

Remark 4.1 *Whether the boundary points $\lambda = \pm\frac{|a|}{L}$ do belong to $\sigma_p(J)$ is an open question (we conjecture they do not).*

Remark 4.2 *In the next section we shall prove that for $T = 2$ (which is much easier than the general even case) the point spectrum in $(-|c_1 - c_2|, |c_1 - c_2|)$ is finite.*

Remark 4.3 *As a matter of fact, L. Silva proved (private communication) by using a uniform version of a discrete Levinson type theorem due to him that (in the case $a \neq 0$, $T = 2L$) $\sigma_p(J)$ is discrete in $\left(-\frac{|a|}{L}, \frac{|a|}{L}\right)$.*

5 Case $T = 2$

In this section we will prove Theorem 1.3, namely that for period $T = 2$ the number of eigenvalues in $(-|a|/L, |a|/L)$ is finite. We do not know the answer to this question for general $T = 2L$.

The idea of proof is simple. We want to prove that for some $N > 0$ and all

$$f = (0, \ldots, 0, f_N, f_{N+1}, \ldots) \in D(J)$$

it holds that

$$\|Jf\|^2 \geq (c_1 - c_2)^2 \|f\|^2. \tag{5.1}$$

Then, by the variational principle, the estimate (5.1) forces the number of eigenvalues of J in $(-|c_1 - c_2|, |c_1 - c_2|)$ to be less than N. The proof of (5.1) is straightforward but tedious. We shall try to be brief.

First note that J^2 preserves the subspaces M_e (M_o) generated by even (odd) canonical vectors e_s.

Denote $\varphi_k := e_{2k}$. Let $f \in M_e$. Then $f = \sum_s f_s \varphi_s$.

Suppose that $f \in D(J^2)$. We have

$$\begin{aligned} \|Jf\|^2 &= \left(J^2 f, f\right) \\ &= \sum_{n \geq 1} \left[\left(\lambda_{2n-1}^2 + \lambda_{2n}^2\right) |f_n|^2 + \lambda_{2n-2}\lambda_{2n-1} f_{n-1} \overline{f}_n \right. \\ &\qquad \left. + \lambda_{2n}\lambda_{2n+1} f_{n+1} \overline{f}_n \right]. \end{aligned} \tag{5.2}$$

But for any $c\delta > 0$ and $n > \delta$

$$\begin{aligned} &\lambda_{2n+1}\lambda_{2n} \left| f_{n+1} \overline{f}_n \right| \\ &\leq \frac{1}{2} \left\{ \lambda_{2n}\lambda_{2n+1} \left[\left(1 - \frac{\delta}{n+1}\right) |f_n|^2 + \left(1 - \frac{\delta}{n+1}\right)^{-1} |f_{n+1}|^2 \right] \right\} \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} &\lambda_{2n-1}\lambda_{2n-2} \left| f_{n-1} \overline{f}_n \right| \\ &\leq \frac{1}{2} \left\{ \lambda_{2n-2}\lambda_{2n-1} \left[\left(1 - \frac{\delta}{n}\right) |f_{n-1}|^2 + \left(1 - \frac{\delta}{n}\right)^{-1} |f_n|^2 \right] \right\}. \end{aligned} \tag{5.4}$$

After inserting into (5.2) it follows that the series in (5.2) can be estimated from below by a "diagonal" series with coefficients of $|f_n|^2$ given by

$$\lambda_{2n-1}^2+\lambda_{2n}^2-\left[\lambda_{2n-2}\lambda_{2n-1}\left(1-\frac{\delta}{n}\right)^{-1}+\lambda_{2n}\lambda_{2n+1}\left(1-\frac{\delta}{n+1}\right)\right].$$

For this expression we can write $[\ldots]=A_n(\delta)+O(n^{2\alpha-3})$, where

$$\begin{aligned}A_n(\delta) &:= \lambda_{2n-2}\lambda_{2n-1}n^{-2}\delta^2+\left(\lambda_{2n-1}\lambda_{2n-2}n^{-1}-\lambda_{2n}\lambda_{2n+1}n^{-1}+\lambda_{2n}\lambda_{2n+1}n^{-2}\right)\delta\\ &\quad+\lambda_{2n-2}\lambda_{2n-1}+\lambda_{2n}\lambda_{2n+1}\end{aligned}$$

Now choosing $\delta=\alpha-\frac{1}{2}$ (which is asymtotically close to the position of the minimum of $A_n(\delta)$ as $n\to\infty$) we can write

$$[\ldots]=(2n)^{2\alpha}\left\{I_1(n)\left(1+\frac{\delta}{n}+\frac{\delta^2}{n^2}\right)+I_2(n)\left(1-\frac{\delta}{n}+\frac{\delta}{n^2}\right)\right\}+O\left(n^{2\alpha-3}\right), \tag{5.5}$$

where

$$\begin{aligned}I_1(n) &:= \left[\left(1-\alpha n^{-1}+c_2(2n)^{-\alpha}+\frac{1}{2}\alpha(\alpha-1)n^{-2}\right)\right]\times\\ &\quad\left[\left(1-\alpha(2n)^{-1}+c_1(2n)^{-\alpha}+\frac{1}{2}\alpha(\alpha-1)(2n)^{-2}\right)\right],\\ I_2(n) &:= \left(1+c_2(2n)^{-\alpha}\right)\left[\left(1+\alpha(2n)^{-1}+\frac{1}{2}\alpha(\alpha-1)(2n)^{-2}+c_1(2n)^{-\alpha}\right)\right].\end{aligned}$$

We also find that

$$\begin{aligned}&\left(\lambda_{2n-1}(2n)^{-\alpha}\right)^2+\left(\lambda_{2n}(2n)^{-\alpha}\right)^2\\ &\quad=\ 2+2\left(c_1+c_2\right)(2n)^{-\alpha}+\left(c_1^2+c_2^2\right)(2n)^{-2\alpha}-\alpha n^{-1}\\ &\qquad-2\alpha c_1\left[(2n)^{\alpha}2n\right]^{-1}+\frac{1}{4}\left(2\alpha^2-\alpha\right)n^{-2}+O\left(n^{-2-\alpha}\right).\end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6) we find after a number of cancellations that the coefficient at $|f_n|^2$ is bounded from below by

$$(c_1-c_2)^2+dn^{2\alpha-2}+O\left(n^{\alpha-2}\right),$$

with positive $d=4^{\alpha-1}(1-\alpha)^2$.

This proves the desired estimate of $\|Jf\|^2$ for $f\in M_e$.

In the case $f\in M_0$ one could repeat similar estimates but we shall use the following simple observation.

Let U^e (resp. U^o) be the canonical isometries mapping l^2 onto M_e (resp. M_o) and given by $U^e e_n=e_{2n}$ (resp. $U^o e_n=e_{2n-1}$).

If $(Au)(n):=\lambda_{2n-1}u(n)+\lambda_{2n}u(n+1)$ then direct computation shows that

$$U^oA^*A=J^2U^o\quad\text{and}\quad U^eAA^*=J^2U^e \tag{5.7}$$

(here we consider A as an operator in l^2). Since A^*A and AA^* are unitary equivalent on the orthogonal complements of their null spaces (of finite dimension), the relation (5.7) implies the same equivalence of

$$J^2 \big| M_e \quad \text{and} \quad J^2 \big| M_o.$$

This completes the proof of Theorem 1.3.

Remark 5.1 *Note that much simpler calculations of similar type allow to prove the discreteness of the spectrum in the gap with possible accumulation of eigenvalues at the boundary points only. The above more precise considerations showed that accumulations do not occur. Moreover, in the recent work of J. Dombrowski [D] it was shown (in the case* $0 < \alpha < 1$*) that* $(-|c_1 - c_2|, |c_1 - c_2|)$ *has no non-zero eigenvalues provided that all weights* λ_n *are positive.*

References

[Ber] Ju. M. Berezanskii, *Expansions in Eigenfunctions of Selfadjoint Operators*, Vol. 17, Translations of Mathematical Monographs, American Math. Society, Providence, RI, 1968

[D] J. Dombrowski, private communication, 2003.

[DJMP] J. Dombrowski, J. Janas, M. Moszynski, S. Pedersen, *Spectral gap resulting from periodic perturbations of a class of Jacobi operators*, Preprint, 2003.

[D–P] J. Dombrowski, S. Pedersen, *Spectral transition parameters for a class of Jacobi matrices*, Studia Math. (2002), 217–229.

[E] S. N. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, New York, Inc., 1999.

[JM1] J. Janas, M. Moszyński, *Alternative approaches to the absolute continuity of Jacobi matrices with monotonic weights*, Int. Eq. Oper. Th. 43 (2002), 397–416.

[JM2] J. Janas, M. Moszyński, *Asymptotics of solutions of difference equations and spectral properties of some Jacobi matrices*, J. Approx. Th. 120 (2003), 309–336.

[JN1] J. Janas, S. Naboko, *Jacobi matrices with power-like weights – grouping in blocks approach*, J. Funct. Anal. 166 (1999), 218–243.

[JN2] J. Janas, S. Naboko, *Multithreshold spectral phase transitions examples for a class of Jacobi matrices*, OT Advances and Applications 124 (2001), 267–285.

[JNS] J. Janas, S. Naboko and G. Stolz, in preparation

[KP] S. Khan and D.B. Pearson, Subordinacy and spectral theory for infinite matrices, Helv. Phys. Acta 65 (1992), 505–527.

[M] M. Moszyński, *Spectral properties of some Jacobi matrices with double weights*, (to appear in J. Math. Anal. Appl.).

[S] G. Stolz, *Spectral Theory for Slowly Oscillating Potentials I. Jacobi Matrices*, Manuscripta Math. 84 (1994), 245–260.